NUMERICAL SOLUTION OF HIGHER ORDER BOUNDARY VALUE PROBLEMS BY PETROV-GALERKIN METHOD WITH DIFFERENT ORDERS OF B-SPLINES

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S. MALLISHWAR REDDY
(ROLL NO: 701318)

Under the Supervision of
PROF. K.N.S. KASI VISWANADHAM

DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY
WARANGAL - 506 004 INDIA
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CERTIFICATE

This is to certify that the thesis entitled **NUMERICAL SOLUTION OF HIGHER ORDER BOUNDARY VALUE PROBLEMS BY PETROV-GALERKIN METHOD WITH DIFFERENT ORDERS OF B-SPLINES** *submitted to National Institute of Technology, Warangal, INDIA is the bonafide research work done by* **Mr. S.Mallishwar Reddy** *under my supervision. The contents of the thesis have not been submitted elsewhere for the award of any degree.*

Prof. K.N.S. Kasi Viswanadham
(Supervisor)
Department of Mathematics
National Institute of Technology
Warangal - 506 004 INDIA.

Date:

Place: Warangal

Dedicated to

my wife

Smt. Saritha

and

My Parents

DECLARATION

This is to certify that the work presented in the thesis entitled **Numerical solution of higher order boundary value problems by Petrov-Galerkin method with different orders of B-splines** *is a bonafide work done by me under the supervision of* **Prof. K.N.S. Kasi Viswanadham** *and was not submitted elsewhere for the award of any degree.*

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(Roll No: 701318)

Date:

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(S. Mallishwar Reddy)

Abstract

In this thesis, various orders of higher order boundary value problems have been solved with the combination of B-splines as basis functions as well as weight functions by Petrov-Galerkin method. In Petrov-Galerkin method, the basis functions, which constitute a basis for the approximation space under consideration, have been redefined into a new set of basis functions which vanish on the boundary where the given set of boundary conditions or most of the boundary conditions prescribed and also the weight functions have been redefined into a new set of weight functions which in number match with the number of redefined basis functions. The various orders of higher boundary value problems have been solved by Petrov-Galerkin method with redefined set of basis functions and the redefined set of weight functions. The solution to a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique. Several numerical examples of linear and nonlinear boundary value problems have been considered for testing the efficiency of the proposed Petrov-Galerkin method.

Contents

Chapter 1

Introduction

Two point boundary value problems manifest themselves in almost all branches of Science, Engineering and Technology, for instance, boundary layer theory in fluid mechanics, heat power transmission theory, space technology and also control and optimization theory, to cite only a few. Most of the real problems are defined on domains that are geometrically complex, many have different boundary conditions on different portions of the boundary. The solutions to these equations would be exact and closed form solutions to the problems are available in rare cases. However, due to complexities in the geometry, properties and boundary conditions in most of the real world problems, an exact solution or analytical solution can not be obtained in a reasonable amount of time. The engineers or designers want to obtain the solutions of such problems in a short period of time. That means they want to obtain the approximate solutions in a reasonable time frame with reasonable effort. The availability of high speed digital computers has made it possible to undertake such a task, when the chosen approximation method involves computation.

We have some of the numerical methods like Finite Difference Methods, Spectral Methods, Finite Element Methods, Boundary Element Methods and Finite Volume Methods etc. Finite Element Method, although somewhat more difficult than other methods from computer programming point of view, have certain inherent advantages. Using Finite Element Method, the boundary value problem can be solved with ease even in complicated domains. The schemes also give better results than the other methods.

The fundamental idea of the Finite Element Method is the replacement of continuous functions by piecewise approximations, usually polynomials. In Finite Element Method the approximate solution can be written as a linear combination of a set of functions which constitute a basis for the approximation space under consideration. Finite Element Analysis typically employs Lagrangian or Hermite interpolation functions. These functions are easy to implement and can provide sufficient accuracy. However, they are relatively inefficient and can be expensive computationally. Furthermore, if a smooth solution with continuity of higher-order derivatives is desired these interpolation functions become even less efficient.

In the past few decades, the Finite Element Method has been developed into a key indispensable technology in the modelling and simulation of various physical or engineering systems. In the development of an advanced physical or engineering system, engineers or designers have to go through a very rigorous process of modelling, simulation, visualization, analysis, designing, prototyping, testing and finally, fabrication or construction. Such techniques are related to modelling and simulation play an important role in building advanced physical or engineering systems and therefore the applications of the Finite Element Method have been involved in getting the solutions of such problems.

The modern use of finite elements really started in the field of structural engineering. Later, this method has been used to discuss low velocity flow phenomena in Stokes equations, flow phenomena in weather prediction, blood flows using Navier-Stokes equations, magnetic field around a coil from Maxwell equations, pollution from transport equation, wave phenomena, acoustics, seismic waves from wave equation, displacement and stress of the elastic bodies from Cauchy-Navier elasticity equation, wave motion of an electron arbitrary armed the proton at the origin from Schrodinger equation, heat conduction, pollution, osmosis, diffusion through cell membrane, gravitation, ground water flow, electrostatics from Poisson equation.

The Finite Element Method involves variational methods like Rayleigh-Ritz method, Collocation method, Galerkin method, Petrov-Galerkin method, Least Squares method etc. When the given differential operator is self-adjoint and positive definite, then only one can use the Rayleigh-Ritz method to find the approximate solution to the given differential equation [29]. The Collocation method seeks an approximate solution by requiring the residual to be identically zero at n selected points(Collocated points), where n is the number of basis functions in the basis [89]. The selection of the points is crucial in obtaining a well conditioned system of equations and ultimately in obtaining an accurate solution. In Galerkin method, the residual of approximation is made orthogonal to the basis functions.

The Least Squares method can be treated as a special case of Petrov-Galerkin method [89]. In Petrov-Galerkin method, the residual of approximation is made orthogonal to the weight functions. The weight functions are chosen from another linear independent set which is different from the test space. The number of weight functions should match in number with the number of basis functions. When we use Petrov-Galerkin method, a weak form of approximate solution for a given differential equation exists and is unique under appropriate conditions [16, 65] irrespective of the properties of a given differential operator. Further, the weak solution also tends to the classical solution of the given differential equation provided sufficient attention is given to the boundary conditions [75]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. In view of this, in this thesis we intend to present the use of Petrov-Galerkin method to solve higher order boundary value problems with various orders of B-splines.

1.1 B-splines

Splines play an important role in approximation and geometric modeling. They are widely used in data fitting, Computer Aided Design (CAD), automated manufacturing (CAM) and computer graphics. B-splines were first defined by Schoenberg [104]. This early work revealed that splines possess powerful approximation properties. The B-splines are one kind of spline polynomials and these B-splines form a basis for spline polynomial space under consideration. B-splines represent a piecewise polynomial curve that typically provide a better curve fit than other interpolation polynomial curves.

B-spline curves were created as an improvement over Bezier curves in the $1970's$. This effort produced splines which contain local support [100]. Since B-splines have local support, the shape of a particular segment of the curve can be altered without affecting the overall curve. Cox and de Boor [22] discovered a recursive formula for the definition of B-spline basis functions of any order.

Kadalbajoo and Agarwal have used a fitted mesh B-spline method for solving singularly perturbed boundary value problems [44]. Brown et al. focused on the accuracy of B-spline finite element approximation to a PDE [18]. Caglar et al. compared B-spline approximation with the usual finite element, finite difference and finite volume methods for a two-point boundary value problem [20]. Results show lower maximum error norms for B-spline approximation than all other methods.

There have been various applications of numerical methods using splines. Pullman and Schaff performed analysis of a cross-ply laminate with a circular hole using a 3D spline variational method [91]. It was determined that the spline variational method can reduce the number of degrees of freedom by a factor of 3-5 while maintaining interlaminar stress distributions comparable to ordinary finite elements.

Mizusawa has applied a spline element method to analyze the bending of skew plates [78]. It was also observed that use of high-order splines and a mesh grading technique were effective in improvement of accuracy. Leung and Au applied spline finite elements to beams and plates [67]. In this paper, the advantages of splines including computational efficiency, flexibility in modeling different boundary conditions and the variation diminishing property of splines is noted.

Kong and Cheung have applied spline finite strip analysis to shear-deformable plates [63]. This type of analysis was proposed to study thick laminated composite plates. The use of cubic B-splines allowed for the flexibility to meet various continuity requirements for classical plate analysis. Kutluay and Esen used a B-spline finite element method to analyze a thermistor problem involving electrical conductivity [64]. Aksan has also used quadratic B-splines to approximate the solution of a 1D non-linear Burgers' equation [12]. The Burgers' equation was converted to a set of non-linear ordinary differential equations. Each equation was then solved by means of a quadratic B-spline finite element method.

Soliman [126] solved Burgers' equation by cubic B-spline Galerkin approach. The numerical solutions of heat problem and one-dimensional hyperbolic problem have been obtained by cubic B-splines collocation method by Duygu Donmez Demir and Necdet Bildik, Christopher et al. [31, 26] respectively. Mittal and Jain [77, 76] have used cubic B-spline collocation method to solve a nonlinear parabolic partial differential equations with Neumann boundary conditions and nonlinear Burgers' equation respectively.

Feng and Xio [33] used quartic B-splines collocation to solve fifth boundary value problems. Mingzhu et al. [74] obtained numerical solution of linear sixth order boundary value problems by using quartic B-splines technique. Geyiklia and Karakocb [34] applied Subdomain Finite Element Method with quartic B-Splines for the solution of Modified Equal Width Wave (MEW) equation.

Saka [101] used quintic B-spline Finite Element Method for solving the nonlinear Schrodinger equation. Siddiqi and Arshed [108] developed quintic B-spline collocation method for the solution of fourth order parabolic partial differential equations. Ali et al. [13] obtained approximation of the Korteweg-de Vries Burgers' equation (KdVB) equation by the quintic B-spline differential quadrature method. Rasoul and Reza [97] obtained numerical solution of the Rosenau equation using quintic collocation B-spline method.

Battal and Turabi [105] obtained numerical solution of the MEW equation using sextic B-splines. Ghazala [40] has obtained the solution of the system of fifth order boundary value problem using sextic splines. Reza Mohammadi [98] developed sextic B-spline collocation method for the solution of Euler-Bernoulli beam models.

Talaat and Aly, Turabi and Battal, Marzieh and Nazemi [133, 73, 135] developed septic B-spline collocation method for the numerical solution of KdVB, MEW equation and one-dimensional hyperbolic telegraph equation respectively.

In these days, one finds many researchers are using different finite element methods for solving initial and boundary value problems. In most cases, the solution is a smooth function which is a piecewise polynomial. This has led several investigators to develop algorithms for the solution of initial and boundary value problems which are based on the finite element methods. In Finite Element Method, most of the researchers use Lagrange and Hermite functions as basis functions. They are C^0 and C^1 functions. While using these basis functions, one may get the desired accuracy with more number of intervals in the space variable domain. m^{th} order B-splines are C^{m-1} functions. When these splines are used as basis functions, the approximate solution can be obtained with minimum number of intervals in the space variable domain. B-spline Finite Element Method produces solution with less error than the standard Finite Element Method. Therefore, in this thesis, we have proposed and illustrated Petrov-Galerkin method with combination of B-splines as basis functions and weight functions for solving various orders of two point boundary value problems.

When the chosen approximation satisfies all the prescribed boundary conditions or most of the prescribed boundary conditions, it gives better approximate results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where all the boundary conditions or most of the boundary conditions are prescribed. The chosen approximation has been written as a linear combination of the redefined set of basis functions along with the non-homogeneous part function which takes care of the boundary conditions where the redefined basis functions vanish. So, in this thesis we developed a Petrov-Galerkin method with different orders of B-splines as basis functions to solve various orders of higher order boundary value problems.

The recurrence formula for the numerical evaluation of cubic B-splines was discovered by Cox [28] and de Boor [23] and it was conveniently modified from the notation point of view by Kasi Viswanadham [45, 49, 46, 47, 48]. In a similar analogue quartic, quintic, sextic and septic B-splines are defined in this thesis. The definitions of the above mentioned B-splines are described in the following sections.

1.1.1 Cubic B-splines

The existence of cubic spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) $c = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of cubic B-splines. Introduce six additional knots x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2} and x_{n+3} such that

$$x_{-3} < x_{-2} < x_{-1} < x_0 \text{ and } x_n < x_{n+1} < x_{n+2} < x_{n+3}.$$

Now the cubic B-splines, given in[28, 23], are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-2}^{i+2} \frac{(x_r - x)_+^3}{\pi'(x_r)}, & \text{if } x \in [x_{i-2}, x_{i+2}] \\ 0, & \text{otherwise.} \end{cases}$$

where

$$(x_r - x)_+^3 = \begin{cases} (x_r - x)^3, & \text{if } x_r \geq x \\ 0, & \text{if } x_r \leq x \end{cases}$$

and $\pi(x) = \prod_{r=i-2}^{i+2} (x - x_r)$.

It can be shown the set $\{B_{-1}(x), B_0(x), \cdots, B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_3(\pi)$ of cubic polynomial splines [71]. Schoenberg [104] has proved that the cubic B-splines are the unique non zero splines of smallest compact support with knots at

$$x_{-3} < x_{-2} < x_{-1} < x_0 < x_1 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3}.$$

1.1.2 Quartic B-splines

The existence of the quartic spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) $c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the quartic B-splines. Introduce eight additional knots $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} such that

$$x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 \text{ and } x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4}.$$

Now the quartic B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-2}^{i+3} \frac{(x_r - x)_+^4}{\pi'(x_r)}, & \text{if } x \in [x_{i-2}, x_{i+3}] \\ 0, & \text{otherwise.} \end{cases}$$

where

$$(x_r - x)_+^4 = \begin{cases} (x_r - x)^4, & \text{if } x_r \geq x \\ 0, & \text{if } x_r \leq x \end{cases}$$

and $\pi(x) = \prod_{r=i-2}^{i+3} (x - x_r)$.

Here the set $\{B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_4(\pi)$ of fourth degree polynomial splines [71]. The quartic B-splines are the unique non-zero splines of smallest compact support with knots at

$$x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4}.$$

1.1.3 Quintic B-splines

The existence of quintic spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) $c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the quintic B-splines. Introduce ten additional knots x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} and x_{n+5} such that

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 \text{ and } x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5}.$$

Now the quintic B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \displaystyle\sum_{r=i-3}^{i+3} \frac{(x_r - x)_+^5}{\pi'(x_r)}, & \text{for } x \in [x_{i-3}, x_{i+3}] \\ 0, & \text{otherwise.} \end{cases}$$

where

$$(x_r - x)_+^5 = \begin{cases} (x_r - x)^5, & \text{for } x_r \geq x \\ 0, & \text{for } x_r \leq x \end{cases}$$

and

$$\pi(x) = \prod_{r=i-3}^{i+3} (x - x_r).$$

Here the set $\{B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_5(\pi)$ of fifth degree polynomial splines [71]. The quintic B-splines are the unique nonzero splines of smallest compact support with the knots at

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5}.$$

1.1.4 Sextic B-splines

The existence of the sixth degree spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) $c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the sixth degree B-splines. Introduce twelve additional knots x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5} and x_{n+6} such that

$$x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$
$$\text{and } x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6}.$$

Now the sixth degree B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \displaystyle\sum_{r=i-3}^{i+4} \frac{(x_r - x)_+^6}{\pi'(x_r)}, & \text{for } x \in [x_{i-3}, x_{i+4}] \\ 0, & \text{otherwise.} \end{cases}$$

where

$$(x_r - x)_+^6 = \begin{cases} (x_r - x)^6, & \text{for } x_r \geq x \\ 0, & \text{for } x_r \leq x \end{cases}$$

and

$$\pi(x) = \prod_{r=i-3}^{i+4} (x - x_r).$$

Here the set $\{B_{-3}(x)$, $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $\ldots$, $B_n(x)$, $B_{n+1}(x)$, $B_{n+2}(x)\}$ forms a basis for the space $S_6(\pi)$ of sixth degree polynomial splines [71]. The sixth degree

B-splines are the unique non zero splines of smallest compact support with knots at

$$x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots$$
$$< x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6}.$$

1.1.5 Septic B-splines

The existence of the seventh degree spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) $c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the septic B-splines. Introduce fourteen additional knots x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6} and x_{n+7} such that

$$x_{-7} < x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$
$$\text{and } x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6} < x_{n+7}.$$

Now the seventh degree B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \displaystyle\sum_{r=i-4}^{i+4} \frac{(x_r - x)_+^7}{\pi'(x_r)}, & \text{for } x \in [x_{i-4}, x_{i+4}] \\ 0, & \text{otherwise.} \end{cases}$$

where

$$(x_r - x)_+^7 = \begin{cases} (x_r - x)^7, & \text{for } x_r \geq x \\ 0, & \text{for } x_r \leq x \end{cases}$$

and

$$\pi(x) = \prod_{r=i-4}^{i+4} (x - x_r).$$

Here the set $\{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_n(x), B_{n+1}(x), B_{n+2}(x), B_{n+3}(x)\}$ forms a basis for the space $S_7(\pi)$ of seventh degree polynomial splines [71]. The septic B-splines are the unique non zero splines of smallest compact support with knots at

$$x_{-7} < x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \dots$$
$$< x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6} < x_{n+7}.$$

1.2 Description of the Thesis

The thesis has been divided into five chapters. Chapter-1 is introductory in nature and deals with the definitions of a complete set of cubic B-splines, quartic B-splines, quintic B-splines, sextic B-splines and septic B-splines [71, 104, 23]. A brief review of the available literature dealing with numerical solutions of higher order boundary value problems with B-splines as basis functions has been presented.

In Chapter 2, we consider a Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions for solving a general fourth order and fifth order boundary value problems. Chapter 3 deals with a Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions to solve a general fifth order and sixth order boundary value problems. Chapter 4 concerns itself with solving a general sixth order, seventh order, eighth order, ninth order and tenth order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions. The thesis ends with chapter 5, where we present the main conclusions of the thesis and scope for further research.

In all the chapters 2, 3 and 4, the basis functions are redefined into a new set of basis functions which vanish on the boundary where all the boundary conditions or most of the boundary conditions are prescribed and the weight functions are also redefined into a new set of weight functions which in number match with the number of basis functions. The approximate solution has been written as a linear combination of the redefined set of basis functions along with the non-homogeneous part function which takes care of the boundary conditions where the redefined basis functions vanish.

In each chapter, several numerical examples of linear and nonlinear problems are presented to test the efficiency of the method developed. The solution of a nonlinear problem has been obtained as the limit of a sequence of solutions of linear problems generated by quasilinearization technique [15]. The numerical results obtained by the proposed method are compared with the exact solutions available in the literature and it is seen that the approximate solutions obtained by the proposed method are in good agreement with the exact solutions.

In a nut-shell, the Petrov-Galerkin method with B-splines have been shown to be accurate. The method is easy to implement to solve a given two point boundary value problem. All the computational results presented in this thesis have been computed in the Computational Laboratory of Department of Mathematics, National Institute of Technology Warangal, INDIA. We have used FORTRAN - 90 programming to develop the software packages with B-splines for obtaining the solution of a given boundary value problem.

Chapter 2

Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions

2.1 Petrov-Galerkin method for solving a general fourth order boundary value problem with cubic B-splines as basis functions and quintic B-splines as weight functions

2.1.1 Introduction

In this section, we consider a general fourth order linear boundary value problem given by

$$a_0(x)y^{(4)}(x) + a_1(x)y'''(x) + a_2(x)y''(x) + a_3(x)y'(x) + a_4(x)y(x) = b(x),\ c < x < d \tag{2.1.1}$$

subject to the boundary conditions

$$y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1 \tag{2.1.2}$$

where $A_0,\ C_0,\ A_1,\ C_1$, are finite real constants and $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x),\ a_4(x)$ and $b(x)$ are all continuous functions defined on the interval $[c,\ d]$.

The fourth order boundary value problems occur in a number of areas of applied mathematics, among which are fluid mechanics, elasticity and quantum mechanics as well as in science and engineering. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [9]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on fourth order boundary value problems by using different methods for numerical solutions. Papamichael and Worsey [90] developed the solution of a special case of linear fourth order boundary value problems by cubic spline method. Agarwal and Chow [10] presented the solution of nonlinear fourth order boundary value problems by the Picard's iterative method and the quasilinear iterative method. Taiwo and Evans [132] developed perturbed collocation method to solve a general linear fourth order boundary value problem. Wazawz [8] presented modified decomposition method to solve a special case of fourth order boundary value problems. Waleed and Luis [139] developed decomposition method to solve fourth order boundary value problems. Erturk and Momani [137] presented a numerical comparison between differential transform method and the Adomian decomposition method for solving fourth-order boundary value problems. Momani and Noor [106] presented a numerical comparison between the Differential transform method, Adomian decomposition, and Homotopy perturbation method for solving a fourth-order boundary value problem. Samuel and Sinkala [103] developed higher order B-spline collocation method to solve fourth order boundary value problems. Syed and Noor [131], Noor and Syed [86] developed Homotopy perturbation method and Variational iteration technique respectively for the solution of fourth order boundary value problems. Ahniyaz et al. [11] developed Sinc-Galerkin method to solve a general linear fourth order boundary value problem. Manoj and Pankaj [72], Ramadan et al. [93], Pankaj et al. [128] and Ghazala and Amin [38] presented the solution of a special case of linear fourth order boundary value problems by spline techniques. Kasi Viswanadham et al. [52], Kasi Viswanadham and Sreenivasulu [59] developed Galerkin methods with quintic B-splines and cubic B-splines respectively to solve a general fourth order boundary value problem. Rashidinia and Ghasemi [94], Kasi Viswanadham and Showri Raju [53] have developed B-spline collocation method, cubic B-spline collocation method respectively to solve a general fourth order boundary value problem. So far, fourth order boundary value problems have not been solved by using Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions. Therefore in this section, we try to present a simple Petrov-Galerkin method using cubic B-splines as basis functions and quintic B-splines as weight functions to solve a general fourth order boundary value problem of type (2.1.1)-(2.1.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

2.1.2 Description of the Method

Divide the space variable domain $[c, d]$ of the system (2.1.1)-(2.1.2) into n subintervals by means of $n+1$ distinct grid points x_0, x_1, ..., x_n such that

$$c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d.$$

Introduce six additional knots x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2} and x_{n+3} such that

$$x_{-2} - x_{-3} = x_{-1} - x_{-2} = x_0 - x_{-1} = x_1 - x_0$$
$$x_{n+3} - x_{n+2} = x_{n+2} - x_{n+1} = x_{n+1} - x_n = x_n - x_{n-1}.$$

To solve the boundary value problem (2.1.1) subject to boundary conditions (2.1.2) by the Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions which are described in sections 1.1.1 and 1.1.3

respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-1}^{n+1} \alpha_j B_j(x) \tag{2.1.3}$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are cubic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of cubic B-splines $\{B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x)\}$, the basis functions $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. The procedure for redefining the basis functions is as follows.

Using the definition of cubic B-splines described in section 1.1.1 and the Dirichlet boundary conditions of (2.1.2), we get the approximate solution for $y(x)$ at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-1}^{1} \alpha_j B_j(x_0) = A_0 \tag{2.1.4}$$

$$y(d) = y(x_n) = \sum_{j=n-1}^{n+1} \alpha_j B_j(x_n) = C_0 \tag{2.1.5}$$

Eliminating α_{-1}, α_{n+1} from the equations (2.1.3), (2.1.4) and (2.1.5), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j P_j(x) \tag{2.1.6}$$

where

$$w(x) = \frac{A_0}{B_{-1}(x_0)} B_{-1}(x) + \frac{C_0}{B_{n+1}(x_n)} B_{n+1}(x)$$

and

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-1}(x_0)} B_{-1}(x), & \text{for} \quad j = 0, 1 \\ B_j(x), & \text{for} \quad j = 2, 3, \cdots, n-2 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+1}(x_n)} B_{n+1}(x), & \text{for} \quad j = n-1, n \end{cases} \tag{2.1.7}$$

The new set of basis functions in the approximation $y(x)$ is $\{\ P_j(x), j = 0, 1, \ldots, n\}$. Here $w(x)$ takes care of given set of the Dirichlet boundary conditions and $P_j(x)'s$ vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation is $n+1$, where as the number of weight functions is $n+5$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-2}^{n+2} \beta_j R_j(x) \tag{2.1.8}$$

where $R_j(x)$'s are quintic B-splines defined in section 1.1.3 and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of the given boundary conditions of (2.1.2). That means $v(x)$, defined in (2.1.8), satisfies the conditions

$$v(c) = 0, \quad v(d) = 0, \quad v'(c) = 0, \quad v'(d) = 0 \tag{2.1.9}$$

Using the definition of quintic B-splines described in section 1.1.3 and applying the boundary conditions (2.1.9) to (2.1.8), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-2}^{2} \beta_j R_j(x_0) = 0 \tag{2.1.10}$$

$$v(d) = v(x_n) = \sum_{j=n-2}^{n+2} \beta_j R_j(x_n) = 0 \tag{2.1.11}$$

$$v'(c) = v'(x_0) = \sum_{j=-2}^{2} \beta_j R'_j(x_0) = 0 \tag{2.1.12}$$

$$v'(d) = v'(x_n) = \sum_{j=n-2}^{n+2} \beta_j R'_j(x_n) = 0 \tag{2.1.13}$$

Eliminating β_{-2}, β_{-1}, β_{n+1} and β_{n+2} from the equations (2.1.8) and (2.1.10) to (2.1.13), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=0}^{n} \beta_j T_j(x) \tag{2.1.14}$$

where

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S'_j(x_0)}{S'_{-1}(x_0)} S_{-1}(x), & j = 0, 1, 2 \\ S_j(x), & j = 3, 4, \ldots, n-3 \\ S_j(x) - \dfrac{S'_j(x_n)}{S'_{n+1}(x_n)} S_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{2.1.15}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-2}(x_0)} R_{-2}(x), & j = -1, 0, 1, 2 \\ R_j(x), & j = 3, 4, \ldots, n-3 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+2}(x_n)} R_{n+2}(x), & j = n-2, n-1, n, n+1 \end{cases}$$

Now the new set of basis functions for the approximation $v(x)$ is $\{ T_j(x), j = 0, 1, \ldots, n\}$. Here $T_j(x)$'s and their first order derivatives vanish on the boundary. Let us take $T_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined cubic basis functions $P_j(x)$'s defined in (2.1.7) and the redefined quintic weight functions $T_j(x)$'s defined in (2.1.15) match in number.

Applying the Petrov-Galerkin method to (2.1.1) with the redefined set of cubic basis functions $\{P_j(x), j = 0, 1, \ldots, n\}$ and the redefined set of quintic weight functions $\{ T_j(x), j = 0, 1, \ldots, n\}$, we get

$$\int_{x_0}^{x_n} \Big[a_0(x)y^{(4)}(x) + a_1(x)y'''(x) + a_2(x)y''(x) + a_3(x)y'(x) + a_4(x)y(x)\Big]T_i(x)\,dx$$
$$= \int_{x_0}^{x_n} b(x)T_i(x)\,dx \quad \text{for} \quad i = 0, 1, \ldots, n. \quad (2.1.16)$$

Integrating by parts the first three terms on the left hand side of (2.1.16) and after applying the boundary conditions prescribed in (2.1.2), we get

$$\int_{x_0}^{x_n} a_0(x)T_i(x)y^{(4)}(x)dx = \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} C_1 - \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_0} A_1$$
$$- \int_{x_0}^{x_n} \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]y'(x)\,dx \quad (2.1.17)$$

$$\int_{x_0}^{x_n} a_1(x)T_i(x)y'''(x)(x)dx = \int_{x_0}^{x_n} \frac{d^2}{dx^2}\Big[a_1(x)T_i(x)\Big]y'(x)\,dx \quad (2.1.18)$$

$$\int_{x_0}^{x_n} a_2(x)T_i(x)y''(x)dx = -\int_{x_0}^{x_n} \frac{d}{dx}\Big[a_2(x)T_i(x)\Big]y'(x)\,dx \quad (2.1.19)$$

Substituting (2.1.17), (2.1.18) and (2.1.19) in (2.1.16) and using the approximation for $y(x)$ given in (2.1.6) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \quad (2.1.20)$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \Bigg\{\Bigg[-\frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big] + \frac{d^2}{dx^2}\Big[a_1(x)T_i(x)\Big]$$
$$-\frac{d}{dx}\Big[a_2(x)T_i(x)\Big] + a_3(x)T_i(x)\Bigg] P_j^{'}(x) + a_4(x)T_i(x)P_j(x)\Bigg\}\,dx$$
$$\text{for} \quad i = 0, 1, 2, \ldots, n; \quad j = 0, 1, 2, \ldots, n. \quad (2.1.21)$$

$\mathbf{B} = [b_i];$

$$
\begin{aligned}
b_i = \int_{x_0}^{x_n} \Bigg\{ & b(x)T_i(x) + \left[\frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big] - \frac{d^2}{dx^2}\Big[a_1(x)T_i(x)\Big] \right. \\
& \left. + \frac{d}{dx}\Big[a_2(x)T_i(x)\Big] - a_3(x)T_i(x) \right] w'(x) - a_4(x)T_i(x)w(x) \Bigg\}\ dx \\
& - \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} C_1 + \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_0} A_1 \\
& \text{for} \quad i = 0, 1, 2, \ldots, n. \qquad (2.1.22)
\end{aligned}
$$

and $\alpha = [\alpha_0\ \alpha_1\ \ldots \alpha_n]^T$.

2.1.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix **A** is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x)\, dx$ and $r_j(x)$ are the cubic B-spline basis functions or their derivatives, $v_i(x)$ are the quintic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-2}, x_{j+2}) \cap (x_{i-3}, x_{i+3}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 5-point Gauss-Legendre quadrature formula. Thus the stiff matrix **A** is a nine diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

2.1.4 Numerical Examples

To demonstrate the applicability of the proposed method for solving the fourth order boundary value problems of the type (2.1.1) and (2.1.2), we considered three linear and four nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 2.1.1. *Consider the linear boundary value problem*

$$y^{(4)} + 4y = 1, \quad -1 \le x \le 1 \tag{2.1.23}$$

subject to $y(-1) = 0,\ y(1) = 0,\ y'(-1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)},\ y'(1) = \frac{\sin 2 - \sinh 2}{4(\cosh 2 + \cos 2)}$.
The exact solution for the above problem is

$$y = .25\left[1 - 2\frac{\sinh 1 \sin 1 \sinh x \sin x + \cosh 1 \cos 1 \cosh x \cos x}{\cos 2 + \cosh 2}\right].$$

The proposed method is tested on this problem where the domain $[-1, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2.1.1. The maximum absolute error obtained by the proposed method is* 2.413988×10^{-6}.

Example 2.1.2. *Consider the linear boundary value problem*

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x, \quad 0 < x < 1 \tag{2.1.24}$$

subject to $y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -e$.

The exact solution for the above problem is $y = x(1 - x)e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2.1.2. The maximum absolute error obtained by the proposed method is* 6.765127×10^{-6}.

Example 2.1.3. *Consider the linear boundary value problem*

$$y^{(4)} - y'' - y = e^x(x - 3), \quad 0 < x < 1 \tag{2.1.25}$$

subject to $y(0) = 1,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = -e$.

The exact solution for the above problem is $y = (1-x)e^x$.

The proposed method is tested on this problem where the domain $[0,1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2.1.3. The maximum absolute error obtained by the proposed method is* 4.500151×10^{-6}.

Example 2.1.4. *Consider the nonlinear boundary value problem*

$$y^{(4)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \quad 0 < x < 1 \quad (2.1.26)$$

subject to $y(0) = 0,\ y(1) = 1,\ y'(0) = 0,\ y'(1) = 1.$

The exact solution for the above problem is $y = x^5 - 2x^4 + 2x^2$.

The nonlinear boundary value problem (2.1.26) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(4)}_{(n+1)} - [2y_{(n)}]y_{(n+1)} = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 - [y_{(n)}]^2,$$
$$n = 0, 1, 2, \ldots \quad (2.1.27)$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = 1,\ y'_{(n+1)}(0) = 0,\ y'_{(n+1)}(1) = 1.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.1.27)*. The obtained numerical results for this problem are presented in Table 2.1.4. The maximum absolute error obtained by the proposed method is* 1.120567×10^{-5}.

Example 2.1.5. *Consider the nonlinear boundary value problem*

$$y^{(4)} = \sin x + \sin^2 x - [y'']^2, \quad 0 < x < 1 \quad (2.1.28)$$

subject to $y(0) = 0,\ y(1) = \sin 1,\ y'(0) = 1,\ y'(1) = \cos 1.$

The exact solution for the above problem is $y = \sin x$.

The nonlinear boundary value problem (2.1.28) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(4)}_{(n+1)} + [2y''_{(n)}]y''_{(n+1)} = \sin x + \sin^2 x + [y''_{(n)}]^2, \quad n = 0, 1, 2, ... \qquad (2.1.29)$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = \sin 1,\ y_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = \cos 1.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y. *The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.1.29). *The obtained numerical results for this problem are presented in Table 2.1.5. The maximum absolute error obtained by the proposed method is* 5.334616×10^{-6}.

Example 2.1.6. *Consider the nonlinear boundary value problem*

$$y^{(4)} - 6e^{-4y} = -12(1 + x)^{-4}, \quad 0 < x < 1 \qquad (2.1.30)$$

subject to $y(0) = 0,\ \ y(1) = \ln 2,\ \ y'(0) = 1,\ \ y'(1) = 0.5.$

The exact solution for the above problem is $y = \ln(1 + x)$.

The nonlinear boundary value problem (2.1.30) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(4)}_{(n+1)} + [24e^{-4y_{(n)}}]y_{(n+1)} = -12(1+x)^{-4} + e^{-4y_{(n)}}[6+24y_{(n)}], \quad n = 0, 1, 2, ... \qquad (2.1.31)$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = \ln 2,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = 0.5.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y. *The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.1.31). *The obtained numerical results for this problem are presented* *in Table 2.1.6. The maximum absolute error obtained by the proposed method is* 5.125999×10^{-6}.

Example 2.1.7. *Consider the nonlinear boundary value problem*

$$y^{(4)} + \frac{x^2}{1+y^2} = -72(1-5x+5x^2) + \frac{x^2}{1+(x-x^2)^6}, \quad 0<x<1 \tag{2.1.32}$$

subject to $y(0)=0,\ y(1)=0,\ y'(0)=0,\ y'(1)=0.$

The exact solution for the above problem is $y = x^3(1-x)^3$.

The nonlinear boundary value problem (2.1.32) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} y^{(4)}_{(n+1)} - \frac{2x^2 y_{(n)}}{(1+[y_{(n)}]^2)^2} y_{(n+1)} &= \frac{x^2}{1+(x-x^2)^6} - 72(1-5x+5x^2) \\ &\quad - \frac{2x^2[y_{(n)}]^2}{(1+[y_{(n)}]^2)^2} - \frac{x^2}{1+[y_{(n)}]^2}, \quad n=0,1,2,... \end{aligned} \tag{2.1.33}$$

subject to $y_{(n+1)}(0)=0,\ y_{(n+1)}(1)=0,\ y'_{(n+1)}(0)=0,\ y'_{(n+1)}(1)=0.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y. *The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.1.33). *The obtained numerical results for this problem are presented in Table 2.1.7. The maximum absolute error obtained by the proposed method is* 4.966976×10^{-6}.

x	Absolute error by the proposed method
-0.8	1.907349E-06
-0.6	2.175570E-06
-0.4	2.048910E-06
-0.2	2.168119E-06
0.0	2.160668E-06
0.2	2.220273E-06
0.4	2.183020E-06
0.6	2.413988E-06
0.8	2.022833E-06

Table 2.1.1: Numerical results for the Example 2.1.1.

x	Absolute error by the proposed method
0.1	8.642673E-07
0.2	9.387732E-07
0.3	8.344650E-07
0.4	1.192093E-06
0.5	2.115965E-06
0.6	3.784895E-06
0.7	4.827976E-06
0.8	6.765127E-06
0.9	6.288290E-06

Table 2.1.2: Numerical results for the Example 2.1.2.

x	Absolute error by the proposed method
0.1	8.344650E-07
0.2	4.768372E-07
0.3	1.788139E-07
0.4	1.072884E-06
0.5	2.503395E-06
0.6	4.112720E-06
0.7	4.410744E-06
0.8	4.500151E-06
0.9	3.188848E-06

Table 2.1.3: Numerical results for the Example 2.1.3.

x	Absolute error by the proposed method
0.1	3.879890E-06
0.2	1.497567E-06
0.3	2.533197E-06
0.4	5.781651E-06
0.5	8.285046E-06
0.6	9.894371E-06
0.7	1.037121E-05
0.8	1.120567E-05
0.9	9.357929E-06

Table 2.1.4: Numerical results for the Example 2.1.4.

x	Absolute error by the proposed method
0.1	6.929040E-07
0.2	2.041459E-06
0.3	3.606081E-06
0.4	4.947186E-06
0.5	5.334616E-06
0.6	4.947186E-06
0.7	4.172325E-06
0.8	2.801418E-06
0.9	1.132488E-06

Table 2.1.5: Numerical results for the Example 2.1.5.

x	Absolute error by the proposed method
0.1	1.937151E-07
0.2	1.579523E-06
0.3	3.278255E-06
0.4	4.678965E-06
0.5	5.125999E-06
0.6	4.678965E-06
0.7	3.933907E-06
0.8	2.622604E-06
0.9	1.013279E-06

Table 2.1.6: Numerical results for the Example 2.1.6.

x	Absolute error by the proposed method
0.1	4.939735E-06
0.2	2.135988E-06
0.3	8.568168E-07
0.4	2.083369E-06
0.5	2.671033E-06
0.6	1.993962E-06
0.7	7.431954E-07
0.8	2.217479E-06
0.9	4.966976E-06

Table 2.1.7: Numerical results for the Example 2.1.7.

2.2 Petrov-Galerkin method for solving a general fifth order boundary value problem with cubic B-splines as basis functions and quintic B-splines as weight functions

2.2.1 Introduction

In this section, we developed a Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions for getting the numerical solution of a general linear fifth order boundary value problem

$$a_0(x)y^{(5)}(x) + a_1(x)y^{(4)}(x) + a_2(x)y'''(x) + a_3(x)y''(x) + a_4(x)y'(x) + a_5(x)y(x) = b(x), \quad c < x < d \quad (2.2.1)$$

subject to the boundary conditions

$$y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \quad y''(c) = A_2 \quad (2.2.2)$$

where A_0, C_0, A_1, C_1, A_2 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

The fifth order boundary value problems occur in the mathematical modelling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences [30, 21]. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [9]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on fifth order boundary value problems by using different methods for numerical solutions. Wazwaz [4] developed the solution of special type of fifth order boundary value problems by using the modified Adomian decomposition method. Siddiqi et al. [121], Siddiqi and Ghazala [123] presented the solution of a special case of linear fifth order boundary value problems by using quartic spline functions and sextic spline functions techniques respectively. Azam et al. [85], Siddiqi and Ghazala [122], Siddiqi et al. [124] and Rashidinia et al. [95] have presented the solution of a special case of linear fifth order boundary value problems by using non polynomial spline functions. Noor and Syed [82] applied the Homotopy perturbation method for solving fifth order boundary value problems. Caglar and Caglar [19] presented the Local polynomial regression method to solve the special case of fifth order boundary value problems. Gamel [80] presented the solution of fifth order boundary value problems by Sinc-Galerkin method. Noor and Syed [84], Zhao [141] have developed the solution of fifth order boundary value problems by variational iteration method. Lamnii et al. [66] developed the sextic spline collocation method to solve a special case of fifth order boundary value problems. Kasi Viswanadham and Showri Raju [54] developed the quartic B-spline collocation method to solve a general fifth order boundary value problem. Kasi Viswanadham and Murali [51] presented quintic B-spline Galerkin method to solve a special case of fifth order boundary value problem. Kasi Viswanadham and Sreenivasulu [58] developed quartic B-spline Galerkin method to solve a general fifth order boundary value problem. Syam and Ahili [87] developed a solution of singularly perturbed fifth order boundary value problems by Adomian decomposition method. So far, fifth order boundary value problems have not been solved by using Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions. Therefore in this section, we try to present a simple Petrov-Galerkin method using cubic B-splines as basis functions and quintic B-splines as weight functions to solve the fifth order boundary value problem of type (2.2.1)-(2.2.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

2.2.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (2.2.1)-(2.2.2) into n subintervals as described in section 2.1.2. To solve the boundary value problem (2.2.1)-(2.2.2) by Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions which are described in sections 1.1.1 and 1.1.3

respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-1}^{n+1} \alpha_j B_j(x) \tag{2.2.3}$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are cubic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of cubic B-splines $\{B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x)\}$, the basis functions $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. The procedure for redefining the basis functions is as follows.

Using the definition of cubic B-splines described in section 1.1.1, the Dirichlet boundary conditions of (2.2.2) and proceeding as in section 2.1.2, we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j P_j(x) \tag{2.2.4}$$

where

$$w(x) = \frac{A_0}{B_{-1}(x_0)} B_{-1}(x) + \frac{C_0}{B_{n+1}(x_n)} B_{n+1}(x)$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-1}(x_0)} B_{-1}(x), & j = 0, 1 \\ B_j(x), & j = 2, 3, ..., n-2 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+1}(x_n)} B_{n+1}(x), & j = n-1, n \end{cases} \tag{2.2.5}$$

The new set of basis functions in the approximation $y(x)$ is $\{ P_j(x), j = 0, 1, \ldots, n\}$. Here $w(x)$ takes care of given set of the Dirichlet boundary conditions and $P_j(x)'s$ vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation is $n+1$, where as the number of weight functions is $n+5$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-2}^{n+2} \beta_j R_j(x) \tag{2.2.6}$$

where $R_j(x)$'s are quintic B-splines defined in section 1.1.3 and here we assume that above approximation $v(x)$ satisfies the corresponding homogeneous boundary conditions of the Dirichlet and Neumann boundary conditions of (2.2.2). Now Proceeding as in section 2.1.2, we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=0}^{n} \beta_j T_j(x) \tag{2.2.7}$$

where

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S_j'(x_0)}{S_{-1}'(x_0)} S_{-1}(x), & j = 0,1,2 \\ S_j(x), & j = 3,4,\ldots,n-3 \\ S_j(x) - \dfrac{S_j'(x_n)}{S_{n+1}'(x_n)} S_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{2.2.8}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-2}(x_0)} R_{-2}(x), & j = -1,0,1,2 \\ R_j(x), & j = 3,4,\ldots,n-3 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+2}(x_n)} R_{n+2}(x), & j = n-2, n-1, n, n+1 \end{cases}$$

Let us take $\{\, T_j(x), j = 0,1,\ldots,n\}$ as the set of weight functions for the prescribed Petrov-Galerkin method. Here $T_j(x)$'s and their derivatives vanish on the boundary. Here the redefined cubic basis functions $P_j(x)$'s defined in (2.2.5) and the redefined quintic weight functions $T_j(x)$'s defined in (2.2.8) match in number.

Applying the Petrov-Galerkin method to (2.2.1) with the redefined set of cubic basis functions $\{P_j(x), j = 0, 1, \ldots, n\}$ and the redefined set of quintic weight functions $\{ T_j(x), j = 0, 1, \ldots, n\}$, we get

$$\int_{x_0}^{x_n} \Big[a_0(x)y^{(5)}(x) + a_1(x)y^{(4)}(x) + a_2(x)y'''(x) + a_3(x)y''(x) + a_4(x)y'(x) + a_5(x)y(x)\Big]T_i(x)\,dx = \int_{x_0}^{x_n} b(x)T_i(x)\,dx \quad \text{for} \quad i = 0, 1, \ldots, n. \tag{2.2.9}$$

Integrating by parts the first three terms on the left hand side of (2.2.9) and after applying the boundary conditions prescribed in (2.2.2), we get

$$\int_{x_0}^{x_n} a_0(x)T_i(x)y^{(5)}(x)dx = \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} y''(x_n) - \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_0} A_2 - \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_n} C_1 + \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_0} A_1 + \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_0(x)T_i(x)\Big]y'(x)\,dx \tag{2.2.10}$$

$$\int_{x_0}^{x_n} a_1(x)T_i(x)y^{(4)}(x)dx = \int_{x_0}^{x_n} \frac{d^2}{dx^2}\Big[a_1(x)T_i(x)\Big]y''(x)\,dx \tag{2.2.11}$$

$$\int_{x_0}^{x_n} a_2(x)T_i(x)y'''(x)dx = -\int_{x_0}^{x_n} \frac{d}{dx}\Big[a_2(x)T_i(x)\Big]y''(x)\,dx \tag{2.2.12}$$

Substituting (2.2.10), (2.2.11) and (2.2.12) in (2.2.9) and using the approximation for $y(x)$ given in (2.2.4) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{2.2.13}$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \left\{ \left[\frac{d^2}{dx^2}\Big[a_1(x)T_i(x)\Big] - \frac{d}{dx}\Big[a_2(x)T_i(x)\Big] + a_3(x)T_i(x)\right] P_j''(x) + \left[\frac{d^4}{dx^4}\Big[a_0(x)T_i(x)\Big] + a_4(x)T_i(x)\right] P_j'(x) + a_5(x)T_i(x)P_j(x)\right\} dx + \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} P_j''(x_n) \text{ for } \quad i = 0,1,2,\ldots,n; \quad j = 0,1,2,\ldots,n. \quad (2.2.14)$$

$\mathbf{B} = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \left\{ b(x)T_i(x) - \left[\frac{d^2}{dx^2}\Big[a_1(x)T_i(x)\Big] - \frac{d}{dx}\Big[a_2(x)T_i(x)\Big] + a_3(x)T_i(x)\right] w''(x) - \left[\frac{d^4}{dx^4}\Big[a_0(x)T_i(x)\Big] + a_4(x)T_i(x)\right] w'(x) - a_5(x)T_i(x)w(x)\right\} dx - \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} w''(x_n) + \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_0} A_2 + \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_n} C_1 - \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_0} A_1 \quad \text{for} \quad i = 0,1,2,\ldots,n. \quad (2.2.15)$$

and $\alpha = [\alpha_0 \; \alpha_1 \; \ldots \alpha_n]^T$.

2.2.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x)\, dx$ and $r_j(x)$ are the cubic B-spline basis functions or their derivatives, $v_i(x)$ are the quintic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-2}, x_{j+2}) \cap (x_{i-3}, x_{i+3}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 5-point Gauss-Legendre quadrature formula. Thus the stiff matrix $\mathbf{A}$ is a nine diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

2.2.4 Numerical Examples

To demonstrate the applicability of the proposed method for solving the fifth order boundary value problems of the type (2.2.1) and (2.2.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 2.2.1. *Consider the linear boundary value problem*

$$y^{(5)} + xy = (1-x)cosx - 5sinx + xsinx - x^2 sinx, \quad 0 < x < 1 \tag{2.2.16}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 1,\ y'(1) = -sin1,\ y''(0) = -2.$

The exact solution for the above problem is $y = (1-x)sinx$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2.2.1. The maximum absolute error obtained by the proposed method is* 2.369285×10^{-6}.

Example 2.2.2. *Consider the linear boundary value problem*

$$y^{(5)}+y^{(4)}+e^{-2x}y = e^{-x}\Big[-4e^{2x}(-3+x)cosx-(1-x+4e^{2x}(5+2x))sinx\Big], \quad 0 \le x \le 1 \tag{2.2.17}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = -1,\ y'(1) = esin1,\ y''(0) = 0.$

The exact solution for the above problem is $y = e^x(x-1)sinx$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2.2.2. The maximum absolute error obtained by the proposed method is* 6.347895×10^{-6}.

Example 2.2.3. *Consider the linear boundary value problem*

$$y^{(5)} + (x-2)y^4 + 2y''' - (x^2+2x-1)y'' + (2x^2+4x)y' - 2x^2y$$
$$= 4e^x cosx - 2x^4 + 4x^3 + 6x^2 - 4x + 2, \quad 0 < x < 1 \quad (2.2.18)$$

subject to $y(0) = 0,\ y(1) = 1 + 2esin1,\ y'(0) = 2,\ y'(1) = 2e(sin1 + cos1) + 2,$
$y''(0) = 6.$

The exact solution for the above problem is $y = 2e^x sinx + x^2$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2.2.3. The maximum absolute error obtained by the proposed method is* 3.015995×10^{-5}.

Example 2.2.4. *Consider the nonlinear boundary value problem*

$$y^{(5)} + 24e^{-5y} = \frac{48}{(1+x)^5}, \quad 0 \le x \le 1 \quad (2.2.19)$$

subject to $y(0) = 0,\ y(1) = ln2,\ y'(0) = 1,\ y'(1) = 0.5,\ y''(0) = -1.$

The exact solution for the above problem is $y = ln(1 + x)$.

The nonlinear boundary value problem (2.2.19) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(5)}_{(n+1)} - 120e^{-5y_{(n)}}y_{(n+1)} = \frac{48}{(1+x)^5} - 120y_{(n)}e^{-5y_{(n)}} - 24e^{-5y_{(n)}},$$
$$n = 0, 1, 2, ... \quad (2.2.20)$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = ln2,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = 0.5,$
$y''_{(n+1)}(0) = -1.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.2.20)*. The obtained numerical results for this problem are given in Table 2.2.4. The maximum absolute error obtained by the proposed method is* 1.639128×10^{-6}.

Example 2.2.5. *Consider the nonlinear boundary value problem*

$$y^{(5)} + [y']^2 e^{4y} - 4y^2 e^{y''} + e^{2x}[y''']^2 = 32e^{-2x}, \quad 0 \le x \le 1 \qquad (2.2.21)$$

subject to $y(0) = 1,\ y(1) = e^{-2},\ y'(0) = -2,\ y'(1) = -2e^{-2},\ y''(0) = 4.$

The exact solution for the above problem is $y = e^{-2x}$.

The nonlinear boundary value problem (2.2.21) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} & y^{(5)}_{(n+1)} + 2e^{2x} y'''_{(n)} y'''_{(n+1)} - 4y^2_{(n)} e^{y''_{(n)}} y''_{(n+1)} + 2y'_{(n)} e^{4y_{(n)}} y'_{(n+1)} \\ & \quad + [4y'^2_{(n)} e^{4y_{(n)}} - 8y_{(n)} e^{y''_{(n)}}] y_{(n+1)} = e^{2x}[y'''_{(n)}]^2 + 4y^2_{(n)} e^{y''_{(n)}} (1 - y''_{(n)}) + [y'_{(n)}]^2 e^{4y_{(n)}} \\ & \qquad + 32e^{-2x} + (4[y'_{(n)}]^2 e^{4y_{(n)}} - 8y_{(n)} e^{y''_{(n)}}) y_{(n)}, \quad n = 0, 1, 2, \ldots \quad (2.2.22) \end{aligned}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e^{-2},\ y'_{(n+1)}(0) = -2,\ y'_{(n+1)}(1) = -2e^{-2},$
$y''_{(n+1)}(0) = 4.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.2.22)*. The obtained numerical results for this problem are given in Table 2.2.5. The maximum absolute error obtained by the proposed method is* $3.695488E \times 10^{-6}$.

Example 2.2.6. *Consider the nonlinear boundary value problem*

$$y^{(5)} + y^{(4)} + e^{-2x}y^2 = 2e^x + 1, \quad 0 < x < 1 \tag{2.2.23}$$

subject to $y(0) = 1,\ y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1.$

The exact solution for the above problem is $y = e^x$

The nonlinear boundary value problem (2.2.23) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(5)}_{(n+1)} + y^{(4)}_{(n+1)} + 2e^{-2x}y_{(n)}y_{(n+1)} = 2e^x + e^{-2x}y^2_{(n)} + 1, \quad n = 0, 1, 2, ... \tag{2.2.24}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = e,\ y''_{(n+1)}(0) = 1.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (2.2.24)*. The obtained numerical results for this problem are given in Table 2.2.6. The maximum absolute error obtained by the proposed method is* 6.675720×10^{-6}.

x	Absolute error by proposed method
0.1	1.594424E-06
0.2	2.980232E-08
0.3	2.041459E-06
0.4	5.513430E-07
0.5	2.369285E-06
0.6	4.917383E-07
0.7	2.086163E-06
0.8	1.490116E-08
0.9	1.393259E-06

Table 2.2.1: Numerical results for the Example 2.2.1.

x	Absolute error by proposed method
0.1	2.980232E-07
0.2	4.857779E-06
0.3	1.281500E-06
0.4	6.347895E-06
0.5	1.728535E-06
0.6	6.198883E-06
0.7	2.384186E-07
0.8	3.844500E-06
0.9	2.518296E-06

Table 2.2.2: Numerical results for the Example 2.2.2.

x	Absolute error by proposed method
0.1	1.099706E-05
0.2	2.145767E-06
0.3	2.557039E-05
0.4	1.347065E-05
0.5	3.015995E-05
0.6	1.311302E-05
0.7	2.169609E-05
0.8	9.536743E-07
0.9	1.049042E-05

Table 2.2.3: Numerical results for the Example 2.2.3.

x	Absolute error by proposed method
0.1	6.556511E-07
0.2	4.619360E-07
0.3	4.470348E-07
0.4	1.609325E-06
0.5	1.162291E-06
0.6	1.639128E-06
0.7	9.536743E-07
0.8	1.311302E-06
0.9	0.000000E+00

Table 2.2.4: Numerical results for the Example 2.2.4.

x	Absolute error by proposed method
0.1	3.457069E-06
0.2	1.668930E-06
0.3	3.695488E-06
0.4	1.668930E-06
0.5	2.861023E-06
0.6	2.682209E-07
0.7	1.609325E-06
0.8	5.513430E-07
0.9	1.206994E-06

Table 2.2.5: Numerical results for the Example 2.2.5.

x	Absolute error by proposed method
0.1	1.907349E-06
0.2	2.145767E-06
0.3	4.529953E-06
0.4	5.125999E-06
0.5	6.675720E-06
0.6	5.125999E-06
0.7	5.960464E-06
0.8	3.337860E-06
0.9	1.668930E-06

Table 2.2.6: Numerical results for the Example 2.2.6.

Chapter 3

Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions

3.1 Petrov-Galerkin method for solving a general fifth order boundary value problem with quartic B-splines as basis functions and sextic B-splines as weight functions

This section is an extension of 2.2. It mainly focuses on the effect of using quartic B-splines as basis functions and sextic B-splines as weight functions in Petrov-Galerkin method for solving a general fifth order boundary value problem.

3.1.1 Introduction

Consider a general linear fifth order boundary value problem

$$a_0(x)y^{(5)}(x) + a_1(x)y^{(4)}(x) + a_2(x)y'''(x) + a_3(x)y''(x) + a_4(x)y'(x) + a_5(x)y(x) \\ = b(x), \quad c < x < d \quad (3.1.1)$$

subject to the boundary conditions

$$y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \quad y''(c) = A_2 \qquad (3.1.2)$$

where $A_0,\ C_0,\ A_1,\ C_1,\ A_2$ are finite real constants and $a_0(x), a_1(x), a_2(x), a_3(x), a_4(x),\ a_5(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

3.1.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (3.1.1)-(3.1.2) into n subintervals by means of $n+1$ distinct grid points $x_0,\ x_1,\ \ldots,\ x_n$ such that

$$c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d.$$

Introduce eight additional knots $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_{n+1},\ x_{n+2},\ x_{n+3}$ and x_{n+4} such that

$$x_{-3} - x_{-4} = x_{-2} - x_{-3} = x_{-1} - x_{-2} = x_0 - x_{-1} = x_1 - x_0$$
$$x_{n+4} - x_{n+3} = x_{n+3} - x_{n+2} = x_{n+2} - x_{n+1} = x_{n+1} - x_n = x_n - x_{n-1}.$$

To solve the boundary value problem (3.1.1)-(3.1.2) by Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions which are described in sections 1.1.2 and 1.1.4 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+1} \alpha_j B_j(x) \qquad (3.1.3)$$

where $\alpha_j{}'s$ are the nodal parameters to be determined and $B_j(x)$'s are quartic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quartic B-splines $\{B_{-2}(x),\ B_{-1}(x),\ B_0(x),\ B_1(x),\ B_2(x),\ \dots\ ,B_{n-1}(x),$ $B_n(x),\ B_{n+1}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. The procedure for redefining the basis functions is as follows.

Using the definition of quartic B-splines described in section 1.1.2 and the Dirichlet boundary conditions of (3.1.2), we get the approximation for $y(x)$ at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-2}^{1} \alpha_j B_j(x_0) = A_0 \tag{3.1.4}$$

$$y(d) = y(x_n) = \sum_{j=n-2}^{n+1} \alpha_j B_j(x_n) = C_0 \tag{3.1.5}$$

Eliminating α_{-2} and α_{n+1} from the equations (3.1.3), (3.1.4) and (3.1.5), we get

$$y(x) = w(x) + \sum_{j=-1}^{n} \alpha_j P_j(x) \tag{3.1.6}$$

where

$$w(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+1}(x_n)} B_{n+1}(x)$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1 \\ B_j(x), & j = 2, 3, ..., n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+1}(x_n)} B_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{3.1.7}$$

The new set of basis functions in the approximation $y(x)$ is $\{\ P_j(x), j = -1, 0, \dots,$ $n\}$. Here $w(x)$ takes care of given set of the Dirichlet boundary conditions and $P_j(x)$'s vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $y(x)$ defined in (3.1.6) is $n+2$, where as the number of weight functions is $n+6$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-3}^{n+2} \beta_j R_j(x) \tag{3.1.8}$$

where $R_j(x)$'s are sextic B-splines defined in section 1.1.4 and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of the Dirichlet and Neumann boundary conditions of (3.1.2). That means $v(x)$, defined in (3.1.8), satisfies the conditions

$$v(c) = 0, \quad v(d) = 0, \quad v'(c) = 0, \quad v'(d) = 0 \tag{3.1.9}$$

Using the definition of sextic B-splines described in section 1.1.4 and applying the boundary conditions (3.1.9) to (3.1.8), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-3}^{2} \beta_j R_j(x_0) = 0 \tag{3.1.10}$$

$$v(d) = v(x_n) = \sum_{j=n-3}^{n+2} \beta_j R_j(x_n) = 0 \tag{3.1.11}$$

$$v'(c) = v'(x_0) = \sum_{j=-3}^{2} \beta_j R'_j(x_0) = 0 \tag{3.1.12}$$

$$v'(d) = v'(x_n) = \sum_{j=n-3}^{n+2} \beta_j R'_j(x_n) = 0 \tag{3.1.13}$$

Eliminating β_{-3}, β_{-2}, β_{n+1} and β_{n+2} from the equations (3.1.8) and (3.1.10) to (3.1.13), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=-1}^{n} \beta_j T_j(x) \tag{3.1.14}$$

where

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S'_j(x_0)}{S'_{-2}(x_0)} S_{-2}(x), & j = -1, 0, 1, 2 \\ S_j(x), & j = 3, 4, \ldots, n-4 \\ S_j(x) - \dfrac{S'_j(x_n)}{S'_{n+1}(x_n)} S_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases} \tag{3.1.15}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-3}(x_0)} R_{-3}(x), & j = -2, -1, 0, 1, 2 \\ R_j(x), & j = 3, 4, \ldots, n-4 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+2}(x_n)} R_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases}$$

Now the new set of basis functions for the approximation $v(x)$ is $\{ T_j(x), j = -1, 0, 1, \ldots, n\}$. Here $T_j(x)$'s and their first order derivatives vanish on the boundary. Let us take $T_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined quartic basis functions $P_j(x)$'s defined in (3.1.7) and the redefined sextic weight functions $T_j(x)$'s defined in (3.1.15) match in number.

Applying the Petrov-Galerkin method to (3.1.1) with the redefined set of quartic basis functions $\{P_j(x), j = -1, 0, 1, \ldots, n\}$ and the redefined set of sextic weight functions $\{ T_j(x), j = -1, 0, 1, \ldots, n\}$, we get

$$\int_{x_0}^{x_n} \big[a_0(x)y^{(5)}(x) + a_1(x)y^{(4)}(x) + a_2(x)y'''(x) + a_3(x)y''(x) + a_4(x)y'(x) + a_5(x)y(x)\big] T_i(x)\, dx = \int_{x_0}^{x_n} b(x) T_i(x)\, dx \quad \text{for} \quad i = -1, 0, 1, \ldots, n. \tag{3.1.16}$$

Integrating by parts the first two terms on the left hand side of (3.1.16) and after applying the boundary conditions prescribed in (3.1.2), we get

$$\int_{x_0}^{x_n} a_0(x)T_i(x)y^{(5)}(x)dx = \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} y''(x_n) - \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_0} A_2 - \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_n} C_1 + \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_0} A_1 + \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_0(x)T_i(x)\Big]y'(x)\,dx \quad (3.1.17)$$

$$\int_{x_0}^{x_n} a_1(x)T_i(x)y^{(4)}(x)dx = -\int_{x_0}^{x_n} \frac{d}{dx}\Big[a_1(x)T_i(x)\Big]y'''(x)\,dx \quad (3.1.18)$$

Substituting (3.1.17) and (3.1.18) in (3.1.16) and using the approximation for $y(x)$ given in (3.1.6) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \quad (3.1.19)$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \left\{ \left[-\frac{d}{dx}\Big[a_1(x)T_i(x)\Big] + a_2(x)T_i(x)\right] P_j^{'''}(x) + a_3(x)T_i(x)P_j^{''}(x) + \left[\frac{d^4}{dx^4}\Big[a_0(x)T_i(x)\Big] + a_4(x)T_i(x)\right] P_j^{'}(x) + a_5(x)T_i(x)P_j(x) \right\} dx + \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} P_j^{''}(x_n)$$

$$\text{for} \quad i = -1, 0, 1, 2, \ldots, n; \quad j = -1, 0, 1, 2, \ldots, n. \quad (3.1.20)$$

$\mathbf{B} = [b_i];$

$$b_i = \int_{x_0}^{x_n} \left\{ b(x)T_i(x) + \left[\frac{d}{dx}\Big[a_1(x)T_i(x)\Big] - a_2(x)T_i(x) \right] w'''(x) - a_3(x)T_i(x)w''(x) \right.$$
$$\left. - \left[\frac{d^4}{dx^4}\Big[a_0(x)T_i(x)\Big] + a_4(x)T_i(x) \right] w'(x) - a_5(x)T_i(x)w(x) \right\} dx$$
$$- \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_n} w''(x_n) + \frac{d^2}{dx^2}\Big[a_0(x)T_i(x)\Big]_{x_0} A_2 + \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_n} C_1$$
$$- \frac{d^3}{dx^3}\Big[a_0(x)T_i(x)\Big]_{x_0} A_1 \quad \text{for} \quad i = -1, 0, 1, 2, \ldots, n. \quad (3.1.21)$$

and $\alpha = [\alpha_{-1}\ \alpha_0\ \ldots \alpha_n]^T$.

3.1.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x)r_j(x)Z(x)\,dx$ and $r_j(x)$ are the quartic B-spline basis functions or their derivatives, $v_i(x)$ are the sextic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+2}) \cap (x_{i-4}, x_{i+3}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 6-point Gauss-Legendre quadrature formula. Thus the stiff matrix $\mathbf{A}$ is a eleven diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

3.1.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the fifth order boundary value problems of the type (3.1.1) and (3.1.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 3.1.1. *Consider the linear boundary value problem*

$$y^{(5)} + xy = (1 - x)cosx - 5sinx + xsinx - x^2 sinx, \quad 0 < x < 1 \tag{3.1.22}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 1,\ y'(1) = -sin1,\ y''(0) = -2.$

The exact solution for the above problem is $y = (1 - x)sinx$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3.1.1. The maximum absolute error obtained by the proposed method is* 6.586313×10^{-6}.

Example 3.1.2. *Consider the linear boundary value problem*

$$y^{(5)}+y^{(4)}+e^{-2x}y = e^{-x}\Big[-4e^{2x}(-3+x)cosx-(1-x+4e^{2x}(5+2x))sinx\Big], \quad 0 \le x \le 1 \tag{3.1.23}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = -1,\ y'(1) = esin1,\ y''(0) = 0.$

The exact solution for the above problem is $y = e^x(x - 1)sinx$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3.1.2. The maximum absolute error obtained by the proposed method is* 6.914139×10^{-6}.

Example 3.1.3. *Consider the linear boundary value problem*

$$\begin{aligned} y^{(5)} + (x - 2)y^4 + 2y''' - (x^2 + 2x - 1)y'' + (2x^2 + 4x)y' - 2x^2 y \\ = 4e^x cosx - 2x^4 + 4x^3 + 6x^2 - 4x + 2, \quad 0 < x < 1 \end{aligned} \tag{3.1.24}$$

subject to $y(0) = 0,\ \ y(1) = 1 + 2esin1,\ \ y'(0) = 2,\ \ y'(1) = 2e(sin1 + cos1) + 2,$
$y''(0) = 6.$

The exact solution for the above problem is $y = 2e^x sinx + x^2$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3.1.3. The maximum absolute error obtained by the proposed method is* 7.152557×10^{-6}.

Example 3.1.4. *Consider the nonlinear boundary value problem*

$$y^{(5)} + 24e^{-5y} = \frac{48}{(1+x)^5}, \quad 0 \le x \le 1 \tag{3.1.25}$$

subject to $y(0) = 0,\ y(1) = ln2,\ y'(0) = 1,\ y'(1) = 0.5,\ y''(0) = -1.$

The exact solution for the above problem is $y = ln(1 + x)$.

The nonlinear boundary value problem (3.1.25) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(5)}_{(n+1)} - 120e^{-5y_{(n)}}y_{(n+1)} = \frac{48}{(1+x)^5} - 120y_{(n)}e^{-5y_{(n)}} - 24e^{-5y_{(n)}},$$
$$n = 0, 1, 2, ... \tag{3.1.26}$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = ln2,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = 0.5,$
$y''_{(n+1)}(0) = -1.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y. *The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (3.1.26). *The obtained numerical results for this problem are given in Table 3.1.4. The maximum absolute error obtained by the proposed method is* 6.198883×10^{-6}.

Example 3.1.5. *Consider the nonlinear boundary value problem*

$$y^{(5)} + [y']^2 e^{4y} - 4y^2 e^{y''} + e^{2x}[y''']^2 = 32e^{-2x}, \quad 0 \le x \le 1 \tag{3.1.27}$$

subject to $y(0) = 1,\ y(1) = e^{-2},\ y'(0) = -2,\ y'(1) = -2e^{-2},\ y''(0) = 4.$

The exact solution for the above problem is $y = e^{-2x}$.

The nonlinear boundary value problem (3.1.27) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} y^{(5)}_{(n+1)} &+ 2e^{2x} y'''_{(n)} y'''_{(n+1)} - 4y^2_{(n)} e^{y''_{(n)}} y''_{(n+1)} + 2y'_{(n)} e^{4y_{(n)}} y'_{(n+1)} \\ &+ [4y'^2_{(n)} e^{4y_{(n)}} - 8y_{(n)} e^{y''_{(n)}}] y_{(n+1)} = e^{2x}[y'''_{(n)}]^2 + 4y^2_{(n)} e^{y''_{(n)}}(1 - y''_{(n)}) + [y'_{(n)}]^2 e^{4y_{(n)}} \\ &+ 32e^{-2x} + (4[y'_{(n)}]^2 e^{4y_{(n)}} - 8y_{(n)} e^{y''_{(n)}}) y_{(n)}, \quad n = 0, 1, 2, ... \end{aligned} \tag{3.1.28}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e^{-2},\ y'_{(n+1)}(0) = -2,\ y'_{(n+1)}(1) = -2e^{-2},$
$y''_{(n+1)}(0) = 4.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (3.1.28). *The obtained numerical results for this problem are given in Table 3.1.5. The maximum absolute error obtained by the proposed method is* 7.659197×10^{-6}.

Example 3.1.6. *Consider the nonlinear boundary value problem*

$$y^{(5)} + y^{(4)} + e^{-2x} y^2 = 2e^x + 1, \quad 0 < x < 1 \tag{3.1.29}$$

subject to $y(0) = 1,\ y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1.$

The exact solution for the above problem is $y = e^x$.

The nonlinear boundary value problem (3.1.29) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(5)}_{(n+1)} + y^{(4)}_{(n+1)} + 2e^{-2x} y_{(n)} y_{(n+1)} = 2e^{x} + e^{-2x} y^{2}_{(n)} + 1, \quad n = 0, 1, 2, ... \qquad (3.1.30)$$

subject to $y_{(n+1)}(0) = 1$, $y_{(n+1)}(1) = e$, $y'_{(n+1)}(0) = 1$, $y'_{(n+1)}(1) = e$, $y''_{(n+1)}(0) = 1$.

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (3.1.30)*. The obtained numerical results for this problem are given in Table 3.1.6. The maximum absolute error obtained by the proposed method is* 4.649162×10^{-6}.

x	Absolute error by proposed method
0.1	2.160668E-07
0.2	6.556511E-07
0.3	2.667308E-06
0.4	4.649162E-06
0.5	6.154180E-06
0.6	6.586313E-06
0.7	5.930662E-06
0.8	4.008412E-06
0.9	1.572073E-06

Table 3.1.1: Numerical results for the Example 3.1.1.

x	Absolute error by proposed method
0.1	2.980232E-08
0.2	8.940697E-08
0.3	2.205372E-06
0.4	4.470348E-06
0.5	6.169081E-06
0.6	6.914139E-06
0.7	6.645918E-06
0.8	4.917383E-06
0.9	1.952052E-06

Table 3.1.2: Numerical results for the Example 3.1.2.

x	Absolute error by proposed method
0.1	2.130866E-06
0.2	2.980232E-06
0.3	7.748604E-07
0.4	0.000000E+00
0.5	2.264977E-06
0.6	7.152557E-06
0.7	5.722046E-06
0.8	3.099442E-06
0.9	4.291534E-06

Table 3.1.3: Numerical results for the Example 3.1.3.

x	Absolute error by proposed method
0.1	7.450581E-08
0.2	5.066395E-07
0.3	2.831221E-06
0.4	4.917383E-06
0.5	6.198883E-06
0.6	6.169081E-06
0.7	6.020069E-06
0.8	4.410744E-06
0.9	1.788139E-06

Table 3.1.4: Numerical results for the Example 3.1.4.

x	Absolute error by proposed method
0.1	1.370907E-06
0.2	2.026558E-06
0.3	2.622604E-06
0.4	4.887581E-06
0.5	6.884336E-06
0.6	7.659197E-06
0.7	6.496906E-06
0.8	4.261732E-06
0.9	1.281500E-06

Table 3.1.5: Numerical results for the Example 3.1.5.

x	Absolute error by proposed method
0.1	2.503395E-06
0.2	4.649162E-06
0.3	1.549721E-06
0.4	2.384186E-07
0.5	3.576279E-07
0.6	4.768372E-07
0.7	1.668930E-06
0.8	1.668930E-06
0.9	4.768372E-07

Table 3.1.6: Numerical results for the Example 3.1.6.

3.2 Petrov-Galerkin method for solving a general sixth order boundary value problem with quartic B-splines as basis functions and sextic B-splines as weight functions

3.2.1 Introduction

In this section, we developed a Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions for getting the numerical solution of a general linear sixth order boundary value problem

$$a_0(x)y^{(6)}(x) + a_1(x)y^{(5)}(x) + a_2(x)y^{(4)}(x) + a_3(x)y'''(x) + a_4(x)y''(x) + a_5(x)y'(x) + a_6(x)y(x) = b(x), \quad c < x < d \quad (3.2.1)$$

subject to the boundary conditions

$$y(c) = A_0,\ y(d) = C_0,\ y'(c) = A_1,\ y'(d) = C_1,\ y''(c) = A_2,\ y''(d) = C_2 \quad (3.2.2)$$

where $A_0,\ C_0,\ A_1,\ C_1,\ A_2,\ C_2$ are finite real constants and $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x),\ a_4(x),\ a_5(x),\ a_6(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

The sixth order boundary value problems occur in astrophysics [134]. Chandrasekhar [24] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is of sixth order. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [9]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on sixth order boundary value problems by using different methods for numerical solutions. Wazwaz [7] developed the solution of special type of sixth order boundary value problems by using the modified Adomian decomposition method. Huan [43] presented varaiational approach technique to solve a special case of sixth order boundary value problems. Noor et al. [81] presented the variational iteration principle to solve a special case of sixth order boundary value problems after transforming the given differential equation into a system of integral equations. Ghazala and Siddiqi [39], Ramadan et al. [92] presented the solution of a special case of sixth order boundary value problems by using non-polynomial spline functions and septic non-polynomial spline functions respectively. Siddiqi et al. [107], Siddiqi and Ghazala [112] developed quintic spline funtions and septic spline functions techniques to solve a special case of linear sixth order boundary value problems respectively. Lamnii et al. [3], Kasi Viswanadham and Showri Raju [56] developed septic spline collocation and quintic B-spline collocation methods to solve sixth order boundary value problems respectively. Loghmani and Ahmadinia [69] used sixth degree B-spline functions to construct an approximation solution for sixth order boundary value problems. Waleed [140] presented Adomian decomposition method with Green's function to solve a special case of sixth order boundary value problems. Liang and Jefferey [127] presented Homotopy analysis method to solve a parameterized sixth order boundary value problems for large parameter values. Kasi Viswanadham and Murali Krishna [50] developed septic B-spline Collocation method to solve a special case of sixth order boundary value problems. Kasi Viswanadham and Sreenivasulu [61] developed quintic B-spline Galerkin method to solve a general sixth order boundary value problem. So far, sixth order boundary value problems have not been solved by using Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions. Therefore in this section, we try to present a simple Petrov-Galerkin method using quartic B-splines as basis functions and sextic B-splines as weight functions to solve the sixth order boundary value problem of type (3.2.1)-(3.2.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

3.2.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (3.2.1)-(3.2.2) into n subintervals as described in section 3.1.2. To solve the boundary value problem (3.2.1)-(3.2.2) by the Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions which are described in sections 1.1.2 and 1.1.4 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+1} \alpha_j B_j(x) \tag{3.2.3}$$

where $\alpha_j{}' s$ are the nodal parameters to be determined and $B_j(x)$'s are quartic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quartic B-splines $\{B_{-2}(x),\ B_{-1}(x),\ B_0(x),\ B_1(x),\ B_2(x),\ \ldots\ ,B_{n-1}(x), B_n(x), B_{n+1}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and Neumann type of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of quartic B-splines described in section 1.1.2, the Dirichlet and Neumann boundary conditions of (3.2.2), we get the approximation for $y(x)$ at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-2}^{1} \alpha_j B_j(x_0) = A_0 \tag{3.2.4}$$

$$y(d) = y(x_n) = \sum_{j=n-2}^{n+1} \alpha_j B_j(x_n) = C_0 \tag{3.2.5}$$

$$y'(c) = y'(x_0) = \sum_{j=-2}^{1} \alpha_j B_j'(x_0) = A_1 \tag{3.2.6}$$

$$y'(d) = y'(x_n) = \sum_{j=n-2}^{n+1} \alpha_j B_j'(x_n) = C_1 \tag{3.2.7}$$

Eliminating α_{-2}, α_{-1}, α_n and α_{n+1} from the equations (3.2.3) to (3.2.7), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n-1} \alpha_j Q_j(x) \tag{3.2.8}$$

where

$$w(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-1}'(x_0)} P_{-1}(x) + \frac{C_1 - w_1'(x_n)}{P_n'(x_n)} P_n(x)$$

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+1}(x_n)} B_{n+1}(x)$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-1}'(x_0)} P_{-1}(x), & j = 0, 1 \\ P_j(x), & j = 2, 3, \ldots, n-3 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_n'(x_n)} P_n(x), & j = n-2, n-1 \end{cases} \tag{3.2.9}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1 \\ B_j(x), & j = 2, 3, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+1}(x_n)} B_{n+1}(x), & j = n-2, n-1, n \end{cases}$$

The new set of basis functions in the approximation $y(x)$ is $\{ Q_j(x), j = 0, 1, \ldots, n-1\}$. Here $w(x)$ takes care of given set of the Dirichlet and Neumann type of boundary conditions and $Q_j(x)$'s and its first order derivatives vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $y(x)$ defined in (3.2.8) is n, where as the number of weight functions is $n + 6$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-3}^{n+2} \beta_j R_j(x) \tag{3.2.10}$$

where $R_j(x)$'s are sextic B-splines defined in section 1.1.4 and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of the given boundary conditions of (3.2.2). That means $v(x)$, defined in (3.2.10), satisfies the conditions

$$v(c) = 0, \quad v(d) = 0, \quad v'(c) = 0, \quad v'(d) = 0, \quad v''(c) = 0, \quad v''(d) = 0 \tag{3.2.11}$$

Using the definition of sextic B-splines described in section 1.1.4 and applying the boundary conditions (3.2.11) to (3.2.10), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-3}^{2} \beta_j R_j(x_0) = 0 \tag{3.2.12}$$

$$v(d) = v(x_n) = \sum_{j=n-3}^{n+2} \beta_j R_j(x_n) = 0 \tag{3.2.13}$$

$$v'(c) = v'(x_0) = \sum_{j=-3}^{2} \beta_j R'_j(x_0) = 0 \tag{3.2.14}$$

$$v'(d) = v'(x_n) = \sum_{j=n-3}^{n+2} \beta_j R'_j(x_n) = 0 \tag{3.2.15}$$

$$v''(c) = v''(x_0) = \sum_{j=-3}^{2} \beta_j R''_j(x_0) = 0 \tag{3.2.16}$$

$$v''(d) = v''(x_n) = \sum_{j=n-3}^{n+2} \beta_j R''_j(x_n) = 0 \tag{3.2.17}$$

Eliminating β_{-3}, β_{-2}, β_{-1}, β_n, β_{n+1} and β_{n+2} from the equations (3.2.10) and (3.2.12) to (3.2.17), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=0}^{n-1} \beta_j V_j(x) \tag{3.2.18}$$

where

$$V_j(x) = \begin{cases} T_j(x) - \dfrac{T_j''(x_0)}{T_{-1}''(x_0)} T_{-1}(x), & j = 0, 1, 2 \\ T_j(x), & j = 3, 4, \ldots, n-4 \\ T_j(x) - \dfrac{T_j''(x_n)}{T_n''(x_n)} T_n(x), & j = n-3, n-2, n-1 \end{cases} \tag{3.2.19}$$

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S_j'(x_0)}{S_{-2}'(x_0)} S_{-2}(x), & j = -1, 0, 1, 2 \\ S_j(x), & j = 3, 4, \ldots, n-4 \\ S_j(x) - \dfrac{S_j'(x_n)}{S_{n+1}'(x_n)} S_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-3}(x_0)} R_{-3}(x), & j = -2, -1, 0, 1, 2 \\ R_j(x), & j = 3, 4, \ldots, n-4 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+2}(x_n)} R_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases}$$

Now the new set of basis functions for the approximation $v(x)$ is $\{ V_j(x), j = 0, 1, \ldots, n-1\}$. Here $V_j(x)$'s and its first and second order derivatives vanish on the boundary. Let us take $V_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined quartic basis functions $Q_j(x)$'s defined in (3.2.9) and the redefined sextic weight functions $V_j(x)$'s defined in (3.2.19) match in number.

Applying the Petrov-Galerkin method to (3.2.1) with the redefined set of quartic basis functions $\{Q_j(x), j = 0, 1, \ldots, n-1\}$ and the redefined set of sextic weight functions $\{ V_j(x), j = 0, 1, \ldots, n-1\}$, we get

$$\int_{x_0}^{x_n} \big[a_0(x)y^{(6)}(x) + a_1(x)y^{(5)}(x) + a_2(x)y^{(4)}(x) + a_3(x)y'''(x) + a_4(x)y''(x) + a_5(x)y'(x) + a_6(x)y(x)\big] V_i(x)\, dx = \int_{x_0}^{x_n} b(x)V_i(x)\, dx \quad \text{for} \quad i = 0, 1, \ldots, n-1. \tag{3.2.20}$$

Integrating by parts the first three terms on the left hand side of (3.2.20) and after applying the boundary conditions prescribed in (3.2.2), we get

$$\int_{x_0}^{x_n} a_0(x)V_i(x)y^{(6)}(x)dx = -\frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} C_2 + \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_0} A_2 + \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big] y''(x)\, dx \tag{3.2.21}$$

$$\int_{x_0}^{x_n} a_1(x)V_i(x)y^{(5)}(x)dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big] y''(x)\, dx \tag{3.2.22}$$

$$\int_{x_0}^{x_n} a_2(x)V_i(x)y^{(4)}(x)dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3}\Big[a_2(x)V_i(x)\Big] y'(x)\, dx \tag{3.2.23}$$

Substituting (3.2.21), (3.2.22) and (3.2.23) in (3.2.20) and using the approximation for $y(x)$ given in (3.2.8) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{3.2.24}$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \Bigg\{ a_3(x)V_i(x)Q_j'''(x) + \Bigg[\frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big] - \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big] + a_4(x)V_i(x)\Bigg] Q_j''(x) + \Bigg[-\frac{d^3}{dx^3}\Big[a_2(x)V_i(x)\Big] + a_5(x)V_i(x)\Bigg] Q_j'(x) + a_6(x)V_i(x)Q_j(x)\Bigg\}\, dx$$

$$\text{for} \quad i = 0, 1, 2, \ldots, n-1, \quad j = 0, 1, 2, \ldots, n-1. \tag{3.2.25}$$

$\mathbf{B} = [b_i];$

$$
\begin{aligned}
b_i = \int_{x_0}^{x_n} \Bigg\{ b(x)V_i(x) - a_3(x)V_i(x)w'''(x) - \Bigg[\frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big] \\
- \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big] + a_4(x)V_i(x)\Bigg] w''(x) \\
- \Bigg[-\frac{d^3}{dx^3}\Big[a_2(x)V_i(x)\Big] + a_5(x)V_i(x)\Bigg] w'(x) - a_6(x)V_i(x)w(x) \Bigg\} \, dx \\
+ \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} C_2 - \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} A_2 \\
\text{for} \quad i = 0, 1, 2, \ldots, n-1. \quad (3.2.26)
\end{aligned}
$$

and $\alpha = [\alpha_0 \; \alpha_1 \; \ldots \alpha_{n-1}]^T$.

3.2.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x)\, dx$ and $r_j(x)$ are the quartic B-spline basis functions or their derivatives, $v_i(x)$ are the sextic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+2}) \cap (x_{i-4}, x_{i+3}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 6-point Gauss-Legendre quadrature formula. Thus the stiff matrix $\mathbf{A}$ is a eleven diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

3.2.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the sixth order boundary value problems of the type (3.2.1) and (3.2.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 3.2.1. *Consider the linear boundary value problem*

$$y^{(6)} + e^{-x}y = -720 + (x - x^2)^3 e^{-x}, \quad 0 < x < 1 \tag{3.2.27}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = 0,\ y''(0) = 0,\ y''(1) = 0.$

The exact solution for the above problem is $y = x^3(1 - x)^3$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3.2.1. The maximum absolute error obtained by the proposed method is* 4.861504×10^{-7}.

Example 3.2.2. *Consider the linear boundary value problem*

$$y^{(6)} + y''' + y'' - y = (-15x^2 + 78x - 114)e^{-x}, \quad 0 < x < 1 \tag{3.2.28}$$

subject to $y(0) = 0,\ y(1) = \frac{1}{e},\ y'(0) = 0,\ y'(1) = \frac{2}{e},\ y''(0) = 0,\ y''(1) = \frac{1}{e}.$

The exact solution for the above problem is $y = x^3 e^{-x}$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3.2.2. The maximum absolute error obtained by the proposed method is* 2.667308×10^{-6}.

Example 3.2.3. *Consider the linear boundary value problem*

$$\begin{aligned} sinx\ y^{(6)} + cosx\ y^{(5)} + x^2 y^{(4)} + (1 + sinx)y \\ = (2sinx + cosx + x^2 + 1)e^x, \quad 0 < x < 1 \end{aligned} \tag{3.2.29}$$

subject to $y(0) = 1,\ y(1) = e,\ y'(0) = 1,\ y'(1) = e, y''(0) = 1, y''(1) = e.$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3.2.3. The maximum absolute error obtained by the proposed method is 5.316734×10^{-5}.

Example 3.2.4. *Consider the nonlinear boundary value problem*

$$y^{(6)} + e^{-x}y^2 = e^{-x} + e^{-3x}, \quad 0 < x < 1 \tag{3.2.30}$$

subject to $y(0) = 1,\ y(1) = \frac{1}{e},\ y'(0) = -1,\ y'(1) = -\frac{1}{e},\ y''(0) = 1,\ y''(1) = \frac{1}{e}$.

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (3.2.30) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(6)}_{(n+1)} + 2e^{-x}y_{(n)}y_{(n+1)} = e^{-x}y^2_{(n)} + e^{-x} + e^{-3x}, \quad n = 0, 1, 2, ... \tag{3.2.31}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = \frac{1}{e},\ y'_{(n+1)}(0) = -1,\ y'_{(n+1)}(1) = -\frac{1}{e},$
$y''_{(n+1)}(0) = 1,\ y''_{(n+1)}(1) = \frac{1}{e}$.

Here $y_{(n+1)}$ is the $(n + 1)^{th}$ approximation for y. The domain $[0, 1]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (3.2.31) *. The obtained numerical results for this problem are given in Table 3.2.4. The maximum absolute error obtained by the proposed method is* 3.516674×10^{-6}.

Example 3.2.5. *Consider the nonlinear boundary value problem*

$$y^{(6)} = e^x y^3, \quad 0 < x < 1 \tag{3.2.32}$$

subject to $y(0) = 1,\ y(1) = e^{-\frac{1}{2}},\ y'(0) = -\frac{1}{2},\ y'(1) = -\frac{1}{2}e^{-\frac{1}{2}},$
$y''(0) = \frac{1}{4},\ y''(1) = \frac{1}{4}e^{-\frac{1}{2}}.$

The exact solution for the above problem is $y = e^{-\frac{x}{2}}$.

The nonlinear boundary value problem (3.2.32) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(6)}_{(n+1)} - 3e^x y^2_{(n)} y_{(n+1)} = -2e^x y^3_{(n)} \quad n = 0, 1, 2, ... \tag{3.2.33}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e^{-\frac{1}{2}},\ y'_{(n+1)}(0) = -\frac{1}{2},\ y'_{(n+1)}(1) = -\frac{1}{2}e^{-\frac{1}{2}},$
$y''_{(n+1)}(0) = \frac{1}{4}, y''_{(n+1)}(1) = \frac{1}{4}e^{-\frac{1}{2}}.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for y. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (3.2.33) *. The obtained numerical results for this problem are given in Table 3.2.5. The maximum absolute error obtained by the proposed method is* 4.053116×10^{-6}.

Example 3.2.6. *Consider the nonlinear boundary value problem*

$$y^{(6)} - 20e^{-36y} = -40(1 + x)^{-6}, \quad 0 < x < 1 \tag{3.2.34}$$

subject to $y(0) = 0,\ y(1) = \frac{ln2}{6},\ y'(0) = \frac{1}{6},\ y'(1) = \frac{1}{12},\ y''(0) = -\frac{1}{6},\ y''(1) = -\frac{1}{24}.$

The exact solution for the above problem is $y = \frac{ln(1+x)}{6}$.

The nonlinear boundary value problem (3.2.34) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(6)}_{(n+1)} + 720e^{-36y_{(n)}}y_{(n+1)} = 720e^{-36y_{(n)}}y_{(n)} + 20e^{-36y_{(n)}} - 40(1+x)^{-6}, \quad n = 0, 1, 2, ... \tag{3.2.35}$$

subject to $y_{(n+1)}(0) = 0$, $y_{(n+1)}(1) = \frac{ln2}{6}$, $y'_{(n+1)}(0) = \frac{1}{6}$, $y'_{(n+1)}(1) = \frac{1}{12}$,
$y''_{(n+1)}(0) = -\frac{1}{6}$, $y''_{(n+1)}(1) = -\frac{1}{24}$.

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y. *The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (3.2.35). *The obtained numerical results for this problem are given in Table 3.2.6. The maximum absolute error obtained by the proposed method is* 3.874302×10^{-7}.

x	Absolute error by proposed method
0.1	1.945882E-07
0.2	3.194436E-07
0.3	1.285225E-07
0.4	2.738088E-07
0.5	2.719462E-07
0.6	3.939494E-07
0.7	3.054738E-07
0.8	4.861504E-07
0.9	1.018634E-07

Table 3.2.1: Numerical results for the Example 3.2.1.

x	Absolute error by proposed method
0.1	9.615906E-08
0.2	3.338791E-07
0.3	9.369105E-07
0.4	1.758337E-06
0.5	2.443790E-06
0.6	2.667308E-06
0.7	2.101064E-06
0.8	1.311302E-06
0.9	5.066395E-07

Table 3.2.2: Numerical results for the Example 3.2.2.

x	Absolute error by proposed method
0.1	1.609325E-05
0.2	3.969669E-05
0.3	5.316734E-05
0.4	4.804134E-05
0.5	3.302097E-05
0.6	1.680851E-05
0.7	6.675720E-06
0.8	9.536743E-07
0.9	9.536743E-07

Table 3.2.3: Numerical results for the Example 3.2.3.

x	Absolute error by proposed method
0.1	4.172325E-07
0.2	8.344650E-07
0.3	8.940697E-07
0.4	1.072884E-06
0.5	2.801418E-06
0.6	3.516674E-06
0.7	2.592802E-06
0.8	1.490116E-06
0.9	4.470348E-07

Table 3.2.4: Numerical results for the Example 3.2.4.

x	Absolute error by proposed method
0.1	4.768372E-07
0.2	1.728535E-06
0.3	4.053116E-06
0.4	3.755093E-06
0.5	2.622604E-06
0.6	1.370907E-06
0.7	8.344650E-07
0.8	5.960464E-08
0.9	2.384186E-07

Table 3.2.5: Numerical results for the Example 3.2.5.

x	Absolute error by proposed method
0.1	2.980232E-08
0.2	1.657754E-07
0.3	3.874302E-07
0.4	3.799796E-07
0.5	2.384186E-07
0.6	5.960464E-08
0.7	7.450581E-09
0.8	6.705523E-08
0.9	6.705523E-08

Table 3.2.6: Numerical results for the Example 3.2.6.

Chapter 4

Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions

4.1 Petrov-Galerkin method for solving a general sixth order boundary value problem with quintic B-splines as basis functions and septic B-splines as weight functions

This section is an extension of 3.2. It mainly focuses on the effect of using quintic B-splines as basis functions and septic B-splines as weight functions in Petrov-Galerkin method for solving a general sixth order boundary value problem.

4.1.1 Introduction

Consider a general linear sixth order boundary value problem

$$a_0(x)y^{(6)}(x) + a_1(x)y^{(5)}(x) + a_2(x)y^{(4)}(x) + a_3(x)y'''(x) + a_4(x)y''(x) + a_5(x)y'(x) + a_6(x)y(x) = b(x), \quad c < x < d \quad (4.1.1)$$

subject to the boundary conditions

$$y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \quad y''(c) = A_2, \quad y''(d) = C_2 \quad (4.1.2)$$

where A_0, C_0, A_1, C_1, A_2, C_2 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$, $a_6(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

4.1.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (4.1.1)-(4.1.2) into n subintervals by means of $n+1$ distinct grid points x_0, x_1, ..., x_n such that

$$c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d.$$

Introduce ten additional knots x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} and x_{n+5} such that

$$x_{-4} - x_{-5} = x_{-3} - x_{-4} = x_{-2} - x_{-3} = x_{-1} - x_{-2} = x_0 - x_{-1} = x_1 - x_0$$

$$x_{n+5} - x_{n+4} = x_{n+4} - x_{n+3} = x_{n+3} - x_{n+2} = x_{n+2} - x_{n+1} = x_{n+1} - x_n = x_n - x_{n-1}.$$

To solve the boundary value problem (4.1.1)-(4.1.2) by the Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions which are described in sections 1.1.3 and 1.1.5 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \quad (4.1.3)$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are quintic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and Neumann type of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of quintic B-splines described in section 1.1.3, the Dirichlet and Neumann boundary conditions of (4.1.2), we get the approximation for $y(x)$ at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-2}^{2} \alpha_j B_j(x_0) = A_0 \tag{4.1.4}$$

$$y(d) = y(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(x_n) = C_0 \tag{4.1.5}$$

$$y'(c) = y'(x_0) = \sum_{j=-2}^{2} \alpha_j B'_j(x_0) = A_1 \tag{4.1.6}$$

$$y'(d) = y'(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B'_j(x_n) = C_1 \tag{4.1.7}$$

Eliminating α_{-2}, α_{-1}, α_{n+1} and α_{n+2} from the equations (4.1.3) to (4.1.7), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j Q_j(x) \tag{4.1.8}$$

where

$$w(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-1}'(x_0)} P_{-1}(x) + \frac{C_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x)$$

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-1}'(x_0)} P_{-1}(x), & j = 0, 1, 2 \\ P_j(x), & j = 3, 4, \ldots, n-3 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{4.1.9}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1, 2 \\ B_j(x), & j = 3, 4, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & j = n-2, n-1, n, n+1 \end{cases}$$

The new set of basis functions in the approximation $y(x)$ is $\{\ Q_j(x), j = 0, 1, \ldots, n\}$. Here $w(x)$ takes care of given set of the Dirichlet and Neumann type of boundary conditions and $Q_j(x)$'s and its first order derivatives vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $y(x)$ defined in (4.1.8) is $n+1$, where as the number of weight functions is $n + 7$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-3}^{n+3} \beta_j R_j(x) \tag{4.1.10}$$

where $R_j(x)$'s are septic B-splines defined in section 1.1.5 and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of the given boundary conditions of (4.1.2). That means $v(x)$, defined in (4.1.10), satisfies the conditions

$$v(c) = 0, \quad v(d) = 0, \quad v'(c) = 0, \quad v'(d) = 0, \quad v''(c) = 0, \quad v''(d) = 0 \tag{4.1.11}$$

Using the definition of septic B-splines described in section 1.1.5 and applying the boundary conditions (4.1.11) to (4.1.10), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-3}^{3} \beta_j R_j(x_0) = 0 \tag{4.1.12}$$

$$v(d) = v(x_n) = \sum_{j=n-3}^{n+3} \beta_j R_j(x_n) = 0 \tag{4.1.13}$$

$$v'(c) = v'(x_0) = \sum_{j=-3}^{3} \beta_j R'_j(x_0) = 0 \tag{4.1.14}$$

$$v'(d) = v'(x_n) = \sum_{j=n-3}^{n+3} \beta_j R'_j(x_n) = 0 \tag{4.1.15}$$

$$v''(c) = v''(x_0) = \sum_{j=-3}^{3} \beta_j R''_j(x_0) = 0 \tag{4.1.16}$$

$$v''(d) = v''(x_n) = \sum_{j=n-3}^{n+3} \beta_j R''_j(x_n) = 0 \tag{4.1.17}$$

Eliminating β_{-3}, β_{-2}, β_{-1}, β_{n+1}, β_{n+2} and β_{n+3} from the equations (4.1.10) and (4.1.12) to (4.1.17), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=0}^{n} \beta_j V_j(x) \tag{4.1.18}$$

where

$$V_j(x) = \begin{cases} T_j(x) - \dfrac{T''_j(x_0)}{T''_{-1}(x_0)} T_{-1}(x), & j = 0, 1, 2, 3 \\ T_j(x), & j = 4, 5, \ldots, n-4 \\ T_j(x) - \dfrac{T''_j(x_n)}{T''_{n+1}(x_n)} T_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases} \tag{4.1.19}$$

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S'_j(x_0)}{S'_{-2}(x_0)} S_{-2}(x), & j = -1, 0, 1, 2, 3 \\ S_j(x), & j = 4, 5, \ldots, n-4 \\ S_j(x) - \dfrac{S'_j(x_n)}{S'_{n+2}(x_n)} S_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-3}(x_0)} R_{-3}(x), & j = -2, -1, 0, 1, 2, 3 \\ R_j(x), & j = 4, 5, \ldots, n-4 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+3}(x_n)} R_{n+3}(x), & j = n-3, n-2, n-1, n, n+1, n+2 \end{cases}$$

Now the new set of basis functions for approximation $v(x)$ is $\{ V_j(x), j = 0, 1, \ldots, n\}$. Here $V_j(x)$'s and its first and second order derivatives vanish on the boundary. Let us take $V_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined quintic basis functions $Q_j(x)$'s defined in (4.1.9) and the redefined septic weight functions $V_j(x)$'s defined in (4.1.19) match in number.

Applying the Petrov-Galerkin method to (4.1.1) with the redefined set of quintic basis functions $\{Q_j(x), j = 0, 1, \ldots, n\}$ and the redefined set of septic weight functions $\{ V_j(x), j = 0, 1, \ldots, n\}$, we get

$$\int_{x_0}^{x_n} \Big[a_0(x)y^{(6)}(x) + a_1(x)y^{(5)}(x) + a_2(x)y^{(4)}(x) + a_3(x)y'''(x) + a_4(x)y''(x) + a_5(x)y'(x) + a_6(x)y(x)\Big] V_i(x)\, dx = \int_{x_0}^{x_n} b(x)V_i(x)\, dx \quad \text{for} \quad i = 0, 1, \ldots, n. \tag{4.1.20}$$

Integrating by parts the first two terms on the left hand side of (4.1.20) and after applying the boundary conditions prescribed in (4.1.2), we get

$$\int_{x_0}^{x_n} a_0(x)V_i(x)y^{(6)}(x)dx = -\frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} C_2 + \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_0} A_2 + \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big] y''(x)\, dx \tag{4.1.21}$$

$$\int_{x_0}^{x_n} a_1(x)V_i(x)y^{(5)}(x)dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big]y''(x)\ dx \qquad (4.1.22)$$

Substituting (4.1.21) and (4.1.22) in (4.1.20) and using the approximation for $y(x)$ given in (4.1.8) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \qquad (4.1.23)$$

where $\mathbf{A} = [a_{ij}]$;

$$\begin{aligned} a_{ij} = \int_{x_0}^{x_n} & \Big\{a_2(x)V_i(x)Q_j^{(4)}(x) + a_3(x)V_i(x)Q_j^{'''}(x) \\ & + \left[\frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big] - \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big] + a_4(x)V_i(x)\right] Q_j^{''}(x) \\ & + a_5(x)V_i(x)Q_j^{'}(x) + a_6(x)V_i(x)Q_j(x)\Big\}\ dx \\ & \text{for} \quad i = 0,1,2,\ldots,n, \quad j = 0,1,2,\ldots,n. \end{aligned} \qquad (4.1.24)$$

$\mathbf{B} = [b_i]$;

$$\begin{aligned} b_i = \int_{x_0}^{x_n} & \{b(x)V_i(x) - a_2(x)V_i(x)w^{(4)}(x) - a_3(x)V_i(x)w'''(x) \\ & - \left[\frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big] - \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big] + a_4(x)V_i(x)\right] w''(x) \\ & - a_5(x)V_i(x)w'(x) - a_6(x)V_i(x)w(x)\}\ dx \\ & + \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} C_2 - \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} A_2 \\ & \text{for} \quad i = 0,1,2,\ldots,n. \end{aligned} \qquad (4.1.25)$$

and $\alpha = [\alpha_0\ \alpha_1\ \ldots \alpha_n]^T$.

4.1.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix **A** is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x)\, dx$ and $r_j(x)$ are the quintic B-spline basis functions or their derivatives, $v_i(x)$ are the septic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+3}) \cap (x_{i-4}, x_{i+4}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiff matrix **A** is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

4.1.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the sixth order boundary value problems of the type (4.1.1) and (4.1.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 4.1.1. *Consider the linear boundary value problem*

$$y^{(6)} - 4y^{(4)} + 2y'' + xy = (5 + 2x - x^2)e^x, \quad 0 < x < 1 \tag{4.1.26}$$

subject to $y(0) = 1,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = -e,\ y''(0) = -1,\ y''(1) = -2e.$

The exact solution for the above problem is $y = (1 - x)e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.1.1. The maximum absolute error obtained by the proposed method is* 1.382828×10^{-6}.

Example 4.1.2. *Consider the linear boundary value problem*

$$y^{(6)} + y^{(5)} + sinx\, y^{(4)} + xy = (2 + sinx + x)e^x, \quad 0 < x < 1 \tag{4.1.27}$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e.$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.1.2. The maximum absolute error obtained by the proposed method is* 1.716614×10^{-5}.

Example 4.1.3. *Consider the linear boundary value problem*

$$y^{(6)} + y''' + y'' - y = (-15x^2 + 78x - 114)e^{-x}, \quad 0 < x < 1 \tag{4.1.28}$$

subject to $y(0) = 0,\ y(1) = \frac{1}{e},\ \ y'(0) = 0,\ y'(1) = \frac{2}{e}, y''(0) = 0, y''(1) = \frac{1}{e}.$

The exact solution for the above problem is $y = x^3 e^{-x}$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.1.3. The maximum absolute error obtained by the proposed method is* 2.533197×10^{-6}.

Example 4.1.4. *Consider the nonlinear boundary value problem*

$$y^{(6)} + e^{-x}y^2 = e^{-x} + e^{-3x}, \quad 0 < x < 1 \tag{4.1.29}$$

subject to $y(0) = 1,\ y(1) = \frac{1}{e},\ y'(0) = -1,\ y'(1) = -\frac{1}{e},\ y''(0) = 1,\ y''(1) = \frac{1}{e}.$

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (4.1.29) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(6)}_{(n+1)} + 2e^{-x}y_{(n)}y_{(n+1)} = e^{-x}y^2_{(n)} + e^{-x} + e^{-3x}, \quad n = 0, 1, 2, ... \tag{4.1.30}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = \frac{1}{e},\ y'_{(n+1)}(0) = -1,\ y'_{(n+1)}(1) = -\frac{1}{e},$
$y''_{(n+1)}(0) = 1,\ y''_{(n+1)}(1) = \frac{1}{e}.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.1.30)*. Numerical results for this problem are given in Table 4.1.4. The maximum absolute error obtained by the proposed method is* 2.563000×10^{-6}.

Example 4.1.5. *Consider the nonlinear boundary value problem*

$$y^{(6)} + y'y^{(5)} - \pi^3 \sin(\pi x)y''' + yy'' + \pi^2 y^2 = -\pi^6 \cos(\pi x), \qquad 0 < x < 1 \tag{4.1.31}$$

subject to $y(0) = 1,\ y(1) = -1,\ y'(0) = 0,\ y'(1) = 0,\ y''(0) = -\pi^2, y''(1) = \pi^2.$

The exact solution for the above problem is $y = \cos(\pi x)$.

The nonlinear boundary value problem (4.1.31) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} y^{(6)}_{(n+1)} + y'_{(n)}y^{(5)}_{(n+1)} - \pi^3 \sin(\pi x)y'''_{(n+1)} + y_{(n)}y''_{(n+1)} + y^{(5)}_{(n)}y'_{(n+1)} + (2\pi^2 y_{(n)} + y''_{(n)})y_{(n+1)} \\ = y_{(n)}y''_{(n)} + \pi^2 y^2_{(n)} + y'_{(n)}y^{(5)}_{(n)} - \pi^6 \cos(\pi x), \qquad n = 0, 1, 2, ... \end{aligned} \tag{4.1.32}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = -1,\ y'_{(n+1)}(0) = 0,\ y'_{(n+1)}(1) = 0,$
$y''_{(n+1)}(0) = -\pi^2,\ y''_{(n+1)}(1) = \pi^2.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of a linear problems* (4.1.32)*. The obtained numerical results for this problem are presented* *in Table 4.1.5. The maximum absolute error obtained by the proposed method is* 3.288842×10^{-05}.

Example 4.1.6. *Consider the nonlinear boundary value problem*

$$y^{(6)} - 20e^{-36y} = -40(1+x)^{-6}, \quad 0 < x < 1 \tag{4.1.33}$$

subject to $y(0) = 0,\ y(1) = \frac{ln2}{6},\ y'(0) = \frac{1}{6},\ y'(1) = \frac{1}{12},\ y''(0) = -\frac{1}{6},\ y''(1) = -\frac{1}{24}.$

The exact solution for the above problem is $y = \frac{ln(1+x)}{6}$.

The nonlinear boundary value problem (4.1.33) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(6)}_{(n+1)} + 720e^{-36y_{(n)}}y_{(n+1)} = 720e^{-36y_{(n)}}y_{(n)} + 20e^{-36y_{(n)}} - 40(1+x)^{-6}, \quad n = 0, 1, 2, 3, ... \tag{4.1.34}$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = \frac{ln2}{6},\ y'_{(n+1)}(0) = \frac{1}{6},\ y'_{(n+1)}(1) = \frac{1}{12},$
$y''_{(n+1)}(0) = -\frac{1}{6},\ y''_{(n+1)}(1) = -\frac{1}{24}.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y. *The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.1.34). *Numerical results for this problem are given in Table 4.1.6. The maximum absolute error obtained by the proposed method is* 6.780028×10^{-7}.

x	Absolute error by proposed method
0.1	1.192093E-07
0.2	3.278553E-06
0.3	8.404255E-06
0.4	1.186132E-05
0.5	1.382828E-05
0.6	1.275539E-05
0.7	8.046627E-06
0.8	3.367662E-06
0.9	1.341105E-07

Table 4.1.1: Numerical results for the Example 4.1.1.

x	Absolute error by proposed method
0.1	1.311302E-06
0.2	9.536743E-07
0.3	4.768372E-07
0.4	1.072884E-06
0.5	7.510185E-06
0.6	1.549721E-05
0.7	1.716614E-05
0.8	1.406670E-05
0.9	9.775162E-06

Table 4.1.2: Numerical results for the Example 4.1.2.

x	Absolute error by proposed method
0.1	3.114110E-08
0.2	1.401640E-07
0.3	1.527369E-07
0.4	3.352761E-08
0.5	9.238720E-07
0.6	2.115965E-06
0.7	2.533197E-06
0.8	2.190471E-06
0.9	1.639128E-06

Table 4.1.3: Numerical results for the Example 4.1.3.

x	Absolute error by proposed method
0.1	7.748604E-07
0.2	4.768372E-07
0.3	1.132488E-06
0.4	8.940697E-07
0.5	1.490116E-06
0.6	2.563000E-06
0.7	2.413988E-06
0.8	1.728535E-06
0.9	1.102686E-06

Table 4.1.4: Numerical results for the Example 4.1.4.

x	Absolute error by proposed method
0.1	4.172325E-07
0.2	9.536743E-06
0.3	2.193451E-05
0.4	3.221631E-05
0.5	3.288842E-05
0.6	2.413988E-05
0.7	1.323223E-05
0.8	3.933907E-06
0.9	7.748604E-07

Table 4.1.5: Numerical results for the Example 4.1.5.

x	Absolute error by proposed method
0.1	1.303852E-08
0.2	1.676381E-08
0.3	6.705523E-08
0.4	1.192093E-07
0.5	3.725290E-07
0.6	6.780028E-07
0.7	6.854534E-07
0.8	5.066395E-07
0.9	3.427267E-07

Table 4.1.6: Numerical results for the Example 4.1.6.

4.2 Petrov-Galerkin method for solving a general seventh order boundary value problem with quintic B-splines as basis functions and septic B-splines as weight functions

In this section, we developed a Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions for getting the numerical solution of a general linear seventh order boundary value problem.

4.2.1 Introduction

Consider a general seventh order linear boundary value problem

$$a_0(x)y^{(7)}(x) + a_1(x)y^{(6)}(x) + a_2(x)y^{(5)}(x) + a_3(x)y^{(4)}(x) + a_4(x)y'''(x) \\ + a_5(x)y''(x) + a_6(x)y'(x) + a_7(x)y(x) = b(x),\ c < x < d \quad (4.2.1)$$

subject to the boundary conditions

$$y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \quad y''(c) = A_2, \\ y''(d) = C_2, \quad y'''(c) = A_3 \quad (4.2.2)$$

where $A_0,\ C_0,\ A_1,\ C_1,\ A_2,\ C_2,\ A_3$ are finite real constants and $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x),\ a_4(x),\ a_5(x),\ a_6(x),\ a_7(x)$ and $b(x)$ are all continuous functions defined on the interval $[c,\ d]$.

The seventh order boundary value problems generally arise in modelling induction motors with two rotor circuits. The induction motor behaviour is represented by a fifth order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed. This is done under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. The behaviour of such models is shown as seventh order boundary value problems [99]. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [9]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on seventh order boundary value problems by using different methods for numerical solutions. Siddiqi et al. [113] developed the solution of special type of seventh order boundary value problems by using differential transformation method. Siddiqi et al. [114] presented the variational iteration principle to solve a special case of seventh order boundary value problems after transforming the given differential equation into a system of integral equations. Siddiqi and Iftikhar [115] presented the variational iteration technique for the solution of seventh order boundary value problems by using He's polynomials. Siddiqi and Iftikhar [117] discussed Adomian decomposition method to solve the seventh order boundary value problems. Siddiqi and Iftikhar [118] discussed the numerical solution of higher order boundary value problems by using homotopy analysis method. Siddiqi and Iftikhar [119] dealt with variation of parameters method to solve a special case of seventh order boundary value problems. Siddiqi and Iftikhar [120] presented variational iteration homotopy perturbation method to solve the seventh order boundary value problems, where the variational iteration homotopy perturbation method is formulated by coupling of variational iteration method and homotopy perturbation method. Mustafa and Ali [88], Ghazala and Rehman [37] got the solution of a special case of seventh order boundary value problems by using reproducing kernel Hilbert space method and reproducing kernel method respectively. So far, seventh order boundary value problems have not been solved by using Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions. Therefore in this section, we try to present a simple Petrov-Galerkin method using quintic B-splines as basis functions and septic B-splines as weight functions to solve a general seventh order boundary value problem of type (4.2.1)-(4.2.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

4.2.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (4.2.1)-(4.2.2) into n subintervals as described in section 4.1.2.

To solve the boundary value problem (4.2.1)-(4.2.2) by the Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions which are described in sections 1.1.3 and 1.1.5 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{4.2.3}$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are quintic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines $\{B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $\ldots$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$, $B_{n+2}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet and Neumann type of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of quintic B-splines described in section 1.1.3, the Dirichlet and Neumann boundary conditions of (4.2.2) and proceeding as in section 4.1.2, we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j Q_j(x) \tag{4.2.4}$$

where

$$w(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-1}'(x_0)} P_{-1}(x) + \frac{C_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x)$$

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-1}'(x_0)} P_{-1}(x), & j = 0, 1, 2 \\ P_j(x), & j = 3, 4, \ldots, n-3 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & j = n-2, n-1, n \end{cases} \tag{4.2.5}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1, 2 \\ B_j(x), & j = 3, 4, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & j = n-2, n-1, n, n+1 \end{cases}$$

The new set of basis functions in the approximation $y(x)$ is $\{ Q_j(x), j = 0, 1, \ldots, n\}$. Here $w(x)$ takes care of given set of the Dirichlet and Neumann type of boundary conditions and $Q_j(x)$'s and its first order derivatives vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $y(x)$ defined in (4.2.4) is $n+1$, where as the number of weight functions is $n+7$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-3}^{n+3} \beta_j R_j(x) \tag{4.2.6}$$

where $R_j(x)$'s are septic B-splines defined in section 1.1.5 and here we assume that above approximation $v(x)$ satisfies the corresponding homogeneous boundary conditions of the Dirichlet, Neumann and second order derivatives boundary conditions of (4.2.2). Now proceeding as in section 4.1.2, we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=0}^{n} \beta_j V_j(x) \tag{4.2.7}$$

where

$$V_j(x) = \begin{cases} T_j(x) - \dfrac{T_j''(x_0)}{T_{-1}''(x_0)} T_{-1}(x), & j = 0, 1, 2, 3 \\ T_j(x), & j = 4, 5, \ldots, n-4 \\ T_j(x) - \dfrac{T_j''(x_n)}{T_{n+1}''(x_n)} T_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases} \tag{4.2.8}$$

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S_j'(x_0)}{S_{-2}'(x_0)} S_{-2}(x), & j = -1, 0, 1, 2, 3 \\ S_j(x), & j = 4, 5, \ldots, n-4 \\ S_j(x) - \dfrac{S_j'(x_n)}{S_{n+2}'(x_n)} S_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j(x_0)}{R_{-3}(x_0)} R_{-3}(x), & j = -2, -1, 0, 1, 2, 3 \\ R_j(x), & j = 4, 5, \ldots, n-4 \\ R_j(x) - \dfrac{R_j(x_n)}{R_{n+3}(x_n)} R_{n+3}(x), & j = n-3, n-2, n-1, n, n+1, n+2 \end{cases}$$

Now the new set of basis functions for approximation $v(x)$ is $\{ V_j(x), j = 0, 1, \ldots, n\}$. Here $V_j(x)$'s and its first and second order derivatives vanish on the boundary. Let us take $V_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined quintic basis functions $Q_j(x)$'s defined in (4.2.5) and the redefined septic weight functions $V_j(x)$'s defined in (4.2.8) match in number.

Applying the Petrov-Galerkin method to (4.2.1) with the redefined set of quintic basis functions $\{Q_j(x), j = 0, 1, \ldots, n\}$ and the redefined set of septic weight functions $\{ V_j(x), j = 0, 1, \ldots, n\}$, we get

$$\int_{x_0}^{x_n} \Big[a_0(x)y^{(7)}(x) + a_1(x)y^{(6)}(x) + a_2(x)y^{(5)}(x) + a_3(x)y^{(4)}(x) + a_4(x)y'''(x)$$

$$+ a_5(x)y''(x) + a_6(x)y'(x) + a_7(x)y(x)\Big] V_i(x)\, dx = \int_{x_0}^{x_n} b(x)V_i(x)\, dx$$

$$\text{for} \quad i = 0, 1, \ldots, n. \quad (4.2.9)$$

Integrating by parts the first three terms on the left hand side of (4.2.9) and after applying the boundary conditions prescribed in (4.2.2), we get

$$\int_{x_0}^{x_n} a_0(x)V_i(x)y^{(7)}(x)dx = -\frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} y'''(x_n) + \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_0} A_3$$

$$+ \frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big]_{x_n} C_2 - \frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big]_{x_0} A_2$$

$$- \int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_0(x)V_i(x)\Big] y''(x)\, dx \quad (4.2.10)$$

$$\int_{x_0}^{x_n} a_1(x)V_i(x)y^{(6)}(x)dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big] y'''(x)\, dx \quad (4.2.11)$$

$$\int_{x_0}^{x_n} a_2(x)V_i(x)y^{(5)}(x)dx = \int_{x_0}^{x_n} \frac{d^2}{dx^2}\Big[a_2(x)V_i(x)\Big] y'''(x)\, dx \quad (4.2.12)$$

Substituting (4.2.10), (4.2.11) and (4.2.12) in (4.2.9) and using the approximation for $y(x)$ given in (4.2.4) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \quad (4.2.13)$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \Bigg\{ a_3(x)V_i(x)Q_j^{(4)}(x) + \Bigg[-\frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big]$$
$$+\frac{d^2}{dx^2}\Big[a_2(x)V_i(x)\Big] + a_4(x)V_i(x)\Bigg] Q_j^{'''}(x) + \Bigg[-\frac{d^5}{dx^5}\Big[a_0(x)V_i(x)\Big] + a_5(x)V_i(x)\Bigg] Q_j^{''}(x)$$
$$+a_6(x)V_i(x)Q_j^{'}(x) + a_7(x)V_i(x)Q_j(x)\Bigg\} \, dx - \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} Q_j^{'''}(x_n)$$
$$\text{for} \quad i = 0, 1, 2, \ldots, n, \quad j = 0, 1, 2, \ldots, n. \quad (4.2.14)$$

$\mathbf{B} = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \Bigg\{ b(x)V_i(x) - a_3(x)V_i(x)w^{(4)}(x) - \Bigg[-\frac{d^3}{dx^3}\Big[a_1(x)V_i(x)\Big]$$
$$+\frac{d^2}{dx^2}\Big[a_2(x)V_i(x)\Big] + a_4(x)V_i(x)\Bigg] w^{'''}(x) - \Bigg[-\frac{d^5}{dx^5}\Big[a_0(x)V_i(x)\Big] + a_5(x)V_i(x)\Bigg] w^{''}(x)$$
$$-a_6(x)V_i(x)w^{'}(x) - a_7(x)V_i(x)w(x)\Bigg\} \, dx + \frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_n} w^{'''}(x_n)$$
$$-\frac{d^3}{dx^3}\Big[a_0(x)V_i(x)\Big]_{x_0} A_3 - \frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big]_{x_n} C_2 + \frac{d^4}{dx^4}\Big[a_0(x)V_i(x)\Big]_{x_0} A_2$$
$$\text{for} \quad i = 0, 1, 2, \ldots, n. \quad (4.2.15)$$

and $\alpha = [\alpha_0 \; \alpha_1 \; \ldots \alpha_n]^T$.

4.2.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x) \, dx$ and $r_j(x)$ are the quintic B-spline basis functions or their derivatives, $v_i(x)$ are the septic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+3}) \cap (x_{i-4}, x_{i+4}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiff matrix $\mathbf{A}$ is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

4.2.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the seventh order boundary value problems of the type (4.2.1) and (4.2.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 4.2.1. *Consider the linear boundary value problem*

$$y^{(7)} + y = -(35 + 12x + 12x^2)e^x, \quad 0 < x < 1 \tag{4.2.16}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 1,\ y'(1) = -e,$
$y''(0) = 0,\ y''(1) = -4e,\ y'''(0) = -3.$

The exact solution for the above problem is $y = x(1 - x)e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.2.1. The maximum absolute error obtained by the proposed method is* 6.735325×10^{-6}.

Example 4.2.2. *Consider the linear boundary value problem*

$$y^{(7)} - xy = (x^2 - 2x - 6)e^x, \quad 0 \leq x \leq 1 \tag{4.2.17}$$

subject to $y(0) = 1, y(1) = 0,\ y'(0) = 0,\ y'(1) = -e,$
$y''(0) = -1,\ y''(1) = -2e,\ y'''(0) = -2.$

The exact solution for the above problem is $y = (1 - x)e^x$.

The proposed method is tested on this problem where the domain $[0,1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.2.2. The maximum absolute error obtained by the proposed method is* 4.905462×10^{-5}.

Example 4.2.3. *Consider the linear boundary value problem*

$$y^{(7)} + sinx\ y^{(4)} + cosx\ y''' + (1-x)y = (2 + sinx + cosx - x)e^x, \quad 0 < x < 1 \qquad (4.2.18)$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,\ y'''(0) = 1.$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain $[0,1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.2.3. The maximum absolute error obtained by the proposed method is* 5.269051×10^{-5}.

Example 4.2.4. *Consider the nonlinear boundary value problem*

$$y^{(7)} - yy' = e^{-2x}(2 + e^x(x-8) - 3x + x^2), \quad 0 \leq x \leq 1 \qquad (4.2.19)$$

subject to $y(0) = 1, y(1) = 0,\ y'(0) = -2,\ y'(1) = -e^{-1},$
$y''(0) = 3,\ y''(1) = 2e^{-1},\ y'''(0) = -4.$

The exact solution for the above problem is $y = (1-x)e^{-x}$.

The nonlinear boundary value problem (4.2.19) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(7)}_{(n+1)} - y_{(n)}y'_{(n+1)} - y'_{(n)}y_{(n+1)} = e^{-2x}(2 + e^x(x-8) - 3x + x^2) - y_{(n)}y'_{(n)}, \quad n = 0, 1, 2, ... \qquad (4.2.20)$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = 0,\ y'_{(n+1)}(0) = -2,\ y'_{(n+1)}(1) = -e^{-1},$
$y''_{(n+1)}(0) = 3,\ y''(1)_{(n+1)} = 2e^{-1},\ y'''_{(n+1)}(0) = -4.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y. *The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.2.20). *Numerical results for this problem are given in Table 4.2.4. The maximum absolute error obtained by the proposed method is* 1.746416×10^{-5}.

Example 4.2.5. *Consider the nonlinear boundary value problem*

$$y^{(7)}+y^{(4)}-e^{y}y=e^{x}((12-4x+e^{-e^{x}(x-1)cosx}(x-1))cosx-8(5+x)sinx), \quad 0<x<1 \tag{4.2.21}$$

subject to $y(0)=1,\ y(1)=0,\ y'(0)=0,\ y'(1)=-ecos1,$
$y''(0)=-2,\ y''(1)=-2ecos1+2esin1, y'''(0)=-2.$

The exact solution for the above problem is $y=e^{x}(1-x)sinx$.

The nonlinear boundary value problem (4.2.21) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} y^{(7)}_{(n+1)}+y^{(4)}_{(n+1)}-e^{y_{(n)}}(1+y_{(n)})y_{(n+1)} &= e^{x}((12-4x+e^{-e^{x}(x-1)cosx}(x-1)cosx \\ &\quad -8(5+x)sinx)-e^{y_{(n)}}y^{2}_{(n)} \quad n=0,1,2,... \end{aligned} \tag{4.2.22}$$

subject to $y_{(n+1)}(0)=1,\ y_{(n+1)}(1)=0,\ y'_{(n+1)}(0)=0,\ y'_{(n+1)}(1)=-ecos1,$
$y''_{(n+1)}(0)=-2, y''_{(n+1)}(1)=-2ecos1+2esin1, y'''_{(n+1)}(0)=-2.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y. *The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.2.22). *Numerical results for this problem are given in Table 4.2.5. The maximum absolute error obtained by the proposed method is* 2.980232×10^{-5}.

Example 4.2.6. *Consider the nonlinear boundary value problem*

$$y^{(7)}+siny\ y^{(4)}+e^{y}y''=e^{x}(1+sin(e^{x})+e^{e^{x}}), \quad 0<x<1 \tag{4.2.23}$$

subject to $y(0)=1,\ y(1)=e,\ y'(0)=1,\ y'(1)=e,\ y''(0)=1,\ y''(1)=e,$
$y'''(0)=1.$

The exact solution for the above problem is $y = e^x$.

The nonlinear boundary value problem (4.2.23) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} & y^{(7)}_{(n+1)} + sin(y_{(n)})y^{(4)}_{(n+1)} + e^{y_{(n)}}y''_{(n+1)} + (cos(y_{(n)})y^{(4)}_{(n)} + e^{y_{(n)}}y''_{(n)})y_{(n+1)} \\ & \quad = (cos(y_{(n)})y^{(4)}_{(n)} + e^{y_{(n)}}y''_{(n)})y_{(n)} + e^x(1 + sin(e^x) + e^{e^x}), \quad n = 0, 1, 2, ... \end{aligned} \quad (4.2.24)$$

subject to $y_{(n+1)}(0) = 1$, $y_{(n+1)}(1) = e$, $y'_{(n+1)}(0) = 1$, $y'_{(n+1)}(1) = e$, $y''_{(n+1)}(0) = 1$, $y''_{(n+1)}(1) = e$, $y'''_{(n+1)}(0) = 1$.

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y. *The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.2.24). *Numerical results for this problem are given in Table 4.2.6. The maximum absolute error obtained by the proposed method is* 2.849102×10^{-5}.

x	Absolute error by proposed method
0.1	1.415610E-07
0.2	6.407499E-07
0.3	2.920628E-06
0.4	4.410744E-06
0.5	6.735325E-06
0.6	6.407499E-06
0.7	3.665686E-06
0.8	3.278255E-07
0.9	1.430511E-06

Table 4.2.1: Numerical results for the Example 4.2.1.

x	Absolute error by proposed method
0.1	6.556511E-07
0.2	9.596348E-06
0.3	2.580881E-05
0.4	4.059076E-05
0.5	4.905462E-05
0.6	4.571676E-05
0.7	3.087521E-05
0.8	1.317263E-05
0.9	1.326203E-06

Table 4.2.2: Numerical results for the Example 4.2.2.

x	Absolute error by proposed method
0.1	2.384186E-06
0.2	1.204014E-05
0.3	3.457069E-05
0.4	4.410744E-05
0.5	5.269051E-05
0.6	4.005432E-05
0.7	2.121925E-05
0.8	1.192093E-06
0.9	2.622604E-06

Table 4.2.3: Numerical results for the Example 4.2.3.

x	Absolute error by proposed method
0.1	8.344650E-07
0.2	6.377697E-06
0.3	1.281500E-05
0.4	1.686811E-05
0.5	1.746416E-05
0.6	1.436472E-05
0.7	8.881092E-06
0.8	3.449619E-06
0.9	3.911555E-07

Table 4.2.4: Numerical results for the Example 4.2.4.

x	Absolute error by proposed method
0.1	3.576279E-07
0.2	4.947186E-06
0.3	1.436472E-05
0.4	2.515316E-05
0.5	2.980232E-05
0.6	2.753735E-05
0.7	1.963973E-05
0.8	9.894371E-06
0.9	2.294779E-06

Table 4.2.5: Numerical results for the Example 4.2.5.

x	Absolute error by proposed method
0.1	1.192093E-06
0.2	9.059906E-06
0.3	9.536743E-06
0.4	2.205372E-05
0.5	2.300739E-05
0.6	2.849102E-05
0.7	2.503395E-05
0.8	2.336502E-05
0.9	1.907349E-06

Table 4.2.6: Numerical results for the Example 4.2.6.

4.3 Petrov-Galerkin method for solving a general eighth order boundary value problem with quintic B-splines as basis functions and septic B-splines as weight functions

In this section, we developed a Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions for getting the numerical solution of a general linear eighth order boundary value problem.

4.3.1 Introduction

Consider a general linear eighth order boundary value problem

$$a_0(x)y^{(8)}(x)+a_1(x)y^{(7)}(x)+a_2(x)y^{(6)}(x)+a_3(x)y^{(5)}(x)+a_4(x)y^{(4)}(x)+a_5(x)y'''(x) \\ + a_6(x)y''(x) + a_7(x)y'(x) + a_8(x)y(x) = b(x), \qquad c < x < d \quad (4.3.1)$$

subject to the boundary conditions

$$\begin{aligned} y(c) &= A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \\ y''(c) &= A_2, \quad y''(d) = C_2, \quad y'''(c) = A_3, \quad y'''(d) = C_3 \end{aligned} \qquad (4.3.2)$$

where $A_0,\ C_0,\ A_1,\ C_1,\ A_2,\ C_2,\ A_3,\ C_3$ are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$, $a_6(x)$, $a_7(x)$, $a_8(x)$ and $b(x)$ are all continuous functions defined on the interval $[c, d]$.

Generally, this type of eighth order boundary value problems arise in the study of astrophysics, hydrodynamics and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, applied mathematics, engineering and applied physics. The boundary value problems of higher order differential equations have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. The literature on the numerical solutions of eighth order boundary value problems is very rare. Chandra sekhar [24] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in, when this instability is an ordinary convection the ordinary differential equation is sixth order, when the instability sets in as overstability, it is modeled by an eighth order ordinary differential equation. The existence and uniqueness of the solution for these type of problems are discussed in Agarwal [9]. Over the years, many researchers have worked on eighth order boundary value problems by using different methods for numerical solutions.

Boutayed and Twizell [17] developed a family of finite difference methods for the solution of special nonlinear eighth order boundary value problems by writing the eighth order differential equation as a system of four second order differential equations. Siddiqi and Twizell [109] presented the solution of a special case of linear eighth order boundary value problems by using sextic spline functions. Rashidinia et al. [96] developed non-polynomial spline techniques to solve a special case of linear eighth order boundary value problems. Liu and Wu [68] developed Differential quadrature solutions for a special case of linear eighth order boundary value problems. Ghazala and Siddiqi [35] discussed the solution for a special case of linear eighth order boundary value problems by using nonic spline functions. Golbabai and Javidi [41] discussed homotopy perturbation method for the solution of eighth order boundary value problems. Mladen [79] dealt with the solution of a special case of eighth order boundary value problems by using a modified Adomian decomposition method. Noor and Sayed [83] developed the variational iteration decomposition method to solve a special case of linear eighth order boundary value problems. Haq et al. [125] presented the optimal homotopy asymptotic method for the solution of eighth order boundary value problems. Kasi Viswanadham and Showri raju [55] developed a quintic B-spline collocation method to solve a general eighth order boundary value problem. Costabile and Napoli [27] presented the solution of eighth order boundary value problems with Bernoulli boundary conditions by using collocation method. Ghazala and Rehman [36] developed the solution of eighth order boundary value problems by using reproducing kernel space method. Kasi Viswanadham and Sreenivasulu [60] developed the quintic B-spline Galerkin method to solve a general eighth order boundary value problem. So far, eighth order boundary value problems have not been solved by using Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions.

Therefore in this section, we try to present a simple Petrov-Galerkin method using quintic B-splines as basis functions and septic B-splines as weight functions to solve the eighth order boundary value problem of type (4.3.1)-(4.3.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

4.3.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (4.3.1)-(4.3.2) into n subintervals as described in section 4.1.2. To solve the boundary value problem (4.3.1)-(4.3.2) by the Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions which are described in sections 1.1.3 and 1.1.5 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{4.3.3}$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are quintic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, Neumann and second order derivative type of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of quintic B-splines described in section 1.1.3, the Dirichlet, Neumann and second order derivative boundary condition of (4.3.2), we get the approximate solution at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-2}^{2} \alpha_j B_j(x_0) = A_0 \tag{4.3.4}$$

$$y(d) = y(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(x_n) = C_0 \tag{4.3.5}$$

$$y'(c) = y'(x_0) = \sum_{j=-2}^{2} \alpha_j B_j'(x_0) = A_1 \tag{4.3.6}$$

$$y'(d) = y'(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j'(x_n) = C_1 \tag{4.3.7}$$

$$y''(c) = y''(x_0) = \sum_{j=-2}^{2} \alpha_j B_j''(x_0) = A_2 \tag{4.3.8}$$

$$y''(d) = y''(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j''(x_n) = C_2 \tag{4.3.9}$$

Eliminating α_{-2}, α_{-1}, α_0, α_n, α_{n+1} and α_{n+2} from the equations (4.3.4) to (4.3.9), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=1}^{n-1} \alpha_j R_j(x) \tag{4.3.10}$$

where

$$w(x) = w_2(x) + \frac{A_2 - w_2''(x_0)}{Q_0''(x_0)} Q_0(x) + \frac{C_2 - w_2''(x_n)}{Q_n''(x_n)} Q_n(x)$$

$$w_2(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{Q_{-1}'(x_0)} Q_{-1}(x) + \frac{C_1 - w_1'(x_n)}{Q_{n+1}'(x_n)} Q_{n+1}(x)$$

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$$

$$R_j(x) = \begin{cases} Q_j(x) - \dfrac{Q''_j(x_0)}{Q''_0(x_0)}Q_0(x), & j = 1, 2 \\ Q_j(x), & j = 3, 4, \ldots, n-3 \\ Q_j(x) - \dfrac{Q''_j(x_n)}{Q''_n(x_n)}Q_n(x), & j = n-2, n-1 \end{cases} \tag{4.3.11}$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P'_j(x_0)}{P'_{-1}(x_0)}P_{-1}(x), & j = 0, 1, 2 \\ P_j(x), & j = 3, 4, \ldots, n-3 \\ P_j(x) - \dfrac{P'_j(x_n)}{P'_{n+1}(x_n)}P_{n+1}(x), & j = n-2, n-1, n \end{cases}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)}B_{-2}(x), & j = -1, 0, 1, 2 \\ B_j(x), & j = 3, 4, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)}B_{n+2}(x), & j = n-2, n-1, n, n+1 \end{cases}$$

The new set of basis functions in the approximation $y(x)$ is $\{ R_j(x), j = 1, 2, \ldots, n-1\}$. Here $w(x)$ takes care of given set of the Dirichlet, Neumann and second order type of boundary conditions and $R_j(x)'s$ and its first and second order derivatives vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation is $n-1$, where as the number of weight functions is $n+7$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-3}^{n+3} \beta_j S_j(x) \tag{4.3.12}$$

where $S_j(x)$'s are septic B-splines defined in section 1.1.5 and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of (4.3.2). That means $v(x)$, defined in (4.3.12), satisfies the conditions

$$v(c) = 0, \quad v(d) = 0, \quad v'(c) = 0, \quad v'(d) = 0,$$
$$v''(c) = 0, \quad v''(d) = 0, \quad v'''(c) = 0, \quad v'''(d) = 0 \quad (4.3.13)$$

Using the definition of septic B-splines described in section 1.1.5 and applying the boundary conditions (4.3.13) to (4.3.12), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-3}^{3} \beta_j S_j(x_0) = 0 \quad (4.3.14)$$

$$v(d) = v(x_n) = \sum_{j=n-3}^{n+3} \beta_j S_j(x_n) = 0 \quad (4.3.15)$$

$$v'(c) = v'(x_0) = \sum_{j=-3}^{3} \beta_j S'_j(x_0) = 0 \quad (4.3.16)$$

$$v'(d) = v'(x_n) = \sum_{j=n-3}^{n+3} \beta_j S'_j(x_n) = 0 \quad (4.3.17)$$

$$v''(c) = v''(x_0) = \sum_{j=-3}^{3} \beta_j S''_j(x_0) = 0 \quad (4.3.18)$$

$$v''(d) = v''(x_n) = \sum_{j=n-3}^{n+3} \beta_j S''_j(x_n) = 0 \quad (4.3.19)$$

$$v'''(c) = v'''(x_0) = \sum_{j=-3}^{3} \beta_j S'''_j(x_0) = 0 \quad (4.3.20)$$

$$v'''(d) = v'''(x_n) = \sum_{j=n-3}^{n+3} \beta_j S'''_j(x_n) = 0 \quad (4.3.21)$$

Eliminating β_{-3}, β_{-2}, β_{-1}, β_0, β_n, β_{n+1}, β_{n+2} and β_{n+3} from the equations (4.3.12) and (4.3.14) to (4.3.21), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=1}^{n-1} \beta_j \hat{V}_j(x) \quad (4.3.22)$$

where

$$\hat{V}_j(x) = \begin{cases} V_j(x) - \dfrac{V_j'''(x_0)}{V_0'''(x_0)} V_0(x), & j = 1, 2, 3 \\ V_j(x), & j = 4, 5, \ldots, n-4 \\ V_j(x) - \dfrac{V_j'''(x_n)}{V_n'''(x_n)} V_n(x), & j = n-3, n-2, n-1 \end{cases} \tag{4.3.23}$$

$$V_j(x) = \begin{cases} U_j(x) - \dfrac{U_j''(x_0)}{U_{-1}''(x_0)} U_{-1}(x), & j = 0, 1, 2, 3 \\ U_j(x), & j = 4, 5, \ldots, n-4 \\ U_j(x) - \dfrac{U_j''(x_n)}{U_{n+1}''(x_n)} U_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases}$$

$$U_j(x) = \begin{cases} T_j(x) - \dfrac{T_j'(x_0)}{T_{-2}'(x_0)} T_{-2}(x), & j = -1, 0, 1, 2, 3 \\ T_j(x), & j = 4, 5, \ldots, n-4 \\ T_j(x) - \dfrac{T_j'(x_n)}{T_{n+2}'(x_n)} T_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases}$$

$$T_j(x) = \begin{cases} S_j(x) - \dfrac{S_j(x_0)}{S_{-3}(x_0)} S_{-3}(x), & j = -2, -1, 0, 1, 2, 3 \\ S_j(x), & j = 4, 5, \ldots, n-4 \\ S_j(x) - \dfrac{S_j(x_n)}{S_{n+3}(x_n)} S_{n+3}(x), & j = n-3, n-2, n-1, n, n+1, n+2 \end{cases}$$

Now the new set of basis functions for the approximation $v(x)$ is $\{\hat{V}_j(x), j = 1, 2, \ldots, n-1\}$. Here $\hat{V}_j(x)$'s and its first, second and third order derivatives vanish on the boundary. Let us take $\hat{V}_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined quintic basis functions $R_j(x)$'s defined in (4.3.11) and the redefined septic weight functions $\hat{V}_j(x)$'s defined in (4.3.23) match in number.

Applying the Petrov-Galerkin method to (4.3.1) with the redefined set of quintic basis functions $\{R_j(x), j = 1, 2, \ldots, n-1\}$ and the redefined set of septic weight functions $\{\hat{V}_j(x), j = 1, 2, \ldots, n-1\}$, we get

$$\int_{x_0}^{x_n} \big[a_0(x)y^{(8)}(x) + a_1(x)y^{(7)}(x) + a_2(x)y^{(6)}(x) + a_3(x)y^{(5)}(x) + a_4(x)y^{(4)}(x)$$
$$+ a_5(x)y'''(x) + a_6(x)y''(x) + a_7(x)y'(x) + a_8(x)y(x)\big]\hat{V}_i(x)\, dx$$
$$= \int_{x_0}^{x_n} b(x)\hat{V}_i(x)\, dx \quad \text{for} \quad i = 1, 2, \ldots, n-1. \quad (4.3.24)$$

Integrating by parts the first four terms on the left hand side of (4.3.22) and after applying the boundary conditions prescribed in (4.3.2), we get

$$\int_{x_0}^{x_n} a_0(x)\hat{V}_i(x)y^{(8)}(x)dx = \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} C_3 - \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_0} A_3$$
$$- \int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]y'''(x)\, dx \quad (4.3.25)$$

$$\int_{x_0}^{x_n} a_1(x)\hat{V}_i(x)y^{(7)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_1(x)\hat{V}_i(x)\Big]y'''(x)\, dx \quad (4.3.26)$$

$$\int_{x_0}^{x_n} a_2(x)\hat{V}_i(x)y^{(6)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_2(x)\hat{V}_i(x)\Big]y''(x)\, dx \quad (4.3.27)$$

$$\int_{x_0}^{x_n} a_3(x)\hat{V}_i(x)y^{(5)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_3(x)\hat{V}_i(x)\Big]y'(x)\, dx \quad (4.3.28)$$

Substituting (4.3.25) to (4.3.28) in (4.3.24) and using the approximation for $y(x)$ given in (4.3.10) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \quad (4.3.29)$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \Big\{ a_4(x)\hat{V}_i(x)R_j^{(4)}(x) + \Big[-\frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big] + \frac{d^4}{dx^4}\Big[a_1(x)\hat{V}_i(x)\Big]$$

$$+a_5(x)\hat{V}_i(x)\Big] R_j'''(x) + \Big[\frac{d^4}{dx^4}\Big[a_2(x)\hat{V}_i(x)\Big] + a_6(x)\hat{V}_i(x)\Big] R_j''(x)$$

$$+ \Big[\frac{d^4}{dx^4}\Big[a_3(x)\hat{V}_i(x)\Big] + a_7(x)\hat{V}_i(x)\Big] R_j'(x) + a_8(x)\hat{V}_i(x)R_j(x)\Big\} \, dx$$

$$\text{for} \quad i = 1, 2, \ldots, n-1, \quad j = 1, 2, \ldots, n-1. \quad (4.3.30)$$

$\mathbf{B} = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \Big\{ b(x)\hat{V}_i(x) - a_4(x)\hat{V}_i(x)w^{(4)}(x)$$

$$- \Big[-\frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big] + \frac{d^4}{dx^4}\Big[a_1(x)\hat{V}_i(x)\Big] + a_5(x)\hat{V}_i(x)\Big] w'''(x)$$

$$- \Big[\frac{d^4}{dx^4}\Big[a_2(x)\hat{V}_i(x)\Big] + a_6(x)\hat{V}_i(x)\Big] w''(x)$$

$$- \Big[\frac{d^4}{dx^4}\Big[a_3(x)\hat{V}_i(x)\Big] + a_7(x)\hat{V}_i(x)\Big] w'(x) - a_8(x)\hat{V}_i(x)w(x)\Big\} \, dx$$

$$- \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} C_3 + \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_0} A_3$$

$$\text{for} \quad i = 1, 2, \ldots, n-1. \quad (4.3.31)$$

and $\alpha = [\alpha_1 \; \alpha_2 \; \ldots \alpha_{n-1}]^T$.

4.3.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x)r_j(x)Z(x)\, dx$ and $r_j(x)$ are the quintic B-spline basis functions or their derivatives, $v_i(x)$ are the septic B-spline weight functions or their

derivatives.
It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+3}) \cap (x_{i-4}, x_{i+4}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiff matrix $\mathbf{A}$ is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

4.3.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the eighth order boundary value problems of the type (4.3.1) and (4.3.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 4.3.1. *Consider the linear boundary value problem*

$$y^{(8)} + y^{(7)} + 2y^{(6)} + 2y^{(5)} + 2y^{(4)} + 2y''' + 2y'' + y' + y$$
$$= 14cosx - 16sinx - 4xsinx, \quad 0 < x < 1 \quad (4.3.32)$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = -1,\ y'(1) = 2sin1,$
$y''(0) = 0,\ y''(1) = 4cos1 + 2sin1, y'''(0) = 7,\ y'''(1) = 6cos1 - 6sin1.$

The exact solution for the above problem is $y = (x^2 - 1)sinx$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 4.3.1. The maximum absolute error obtained by the proposed method is* 2.235174×10^{-6}.

Example 4.3.2. *Consider the linear boundary value problem*

$$y^{(8)} + xy = -(48 + 15x + x^3)e^x, \quad 0 < x < 1 \quad (4.3.33)$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 1,\ y'(1) = -e,\ y''(0) = 0,\ y''(1) = -4e,$
$y'''(0) = -3,\ y'''(1) = -9e.$

The exact solution for the above problem is $y = x(1-x)e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 4.3.2. The maximum absolute error obtained by the proposed method is* 7.569790×10^{-6}.

Example 4.3.3. *Consider the linear boundary value problem*

$$y^{(8)} + sinx\ y^{(5)} + (1-x^2)y^{(4)} + y = (3 + sinx - x^2)e^x, \quad 0 < x < 1 \qquad (4.3.34)$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,$
$y'''(0) = 1,\ y'''(1) = e.$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 4.3.3. The maximum absolute error obtained by the proposed method is* 4.875660×10^{-5}.

Example 4.3.4. *Consider the nonlinear boundary value problem*

$$y^{(8)} = 7!(e^{-8y} - \frac{2}{(1+x)^8}), \quad 0 < x < e^{\frac{1}{2}} - 1 \qquad (4.3.35)$$

subject to $y(0) = 0, y(e^{\frac{1}{2}} - 1) = \frac{1}{2},\ y'(0) = 1,\ y'(e^{\frac{1}{2}} - 1) = e^{-\frac{1}{2}},$
$y''(0) = -1,\ y''(e^{\frac{1}{2}} - 1) = -\frac{1}{e},\ y'''(0) = 2,\ y'''(e^{\frac{1}{2}} - 1) = 2e^{-\frac{3}{2}}.$

The exact solution for the above problem is $y = ln(1 + x)$.

The nonlinear boundary value problem (4.3.35) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(8)}_{(n+1)} + 8!e^{-8y_{(n)}}y_{(n+1)} = e^{-8y_{(n)}}(8!y_{(n)} + 7!) - \frac{2.7!}{(1+x)^8}, \quad n = 0, 1, 2, ... \qquad (4.3.36)$$

subject to $y_{(n+1)}(0) = 0, y_{(n+1)}(e^{\frac{1}{2}} - 1) = \frac{1}{2},\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(e^{\frac{1}{2}} - 1) = e^{-\frac{1}{2}},$
$y''_{(n+1)}(0) = -1,\ y''_{(n+1)}(e^{\frac{1}{2}} - 1) = -\frac{1}{e},\ y'''_{(n+1)}(0) = 2,\ y'''_{(n+1)}(e^{\frac{1}{2}} - 1) = 2e^{-\frac{3}{2}}.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0, e^{\frac{1}{2}} - 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.3.36)*. Numerical results for this problem are shown in Table 4.3.4. The maximum absolute error obtained by the proposed method is* 8.225441×10^{-6}.

Example 4.3.5. *Consider the nonlinear boundary value problem*

$$y^{(8)} + e^{-x}y^2 = e^{-x} + e^{-3x}, \quad 0 < x < 1 \tag{4.3.37}$$

subject to $y(0) = 1, y(1) = e^{-1},\ y'(0) = -1,\ y'(1) = -e^{-1},\ y''(0) = 1,\ y''(1) = e^{-1},$
$y'''(0) = -1,\ y'''(1) = -e^{-1}.$

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (4.3.37) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(8)}_{(n+1)} + 2e^{-x}y_{(n)}y_{(n+1)} = e^{-x}y^2_{(n)} + e^{-x} + e^{-3x} \quad n = 0, 1, 2, ... \tag{4.3.38}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e^{-1},\ y'_{(n+1)}(0) = -1,\ y'_{(n+1)}(1) = -e^{-1},$
$y''_{(n+1)}(0) = 1, y''_{(n+1)}(1) = e^{-1}, y'''_{(n+1)}(0) = -1, y'''_{(n+1)}(1) = -e^{-1}.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.3.38)*. Numerical results for this problem are shown in Table 4.3.5. The maximum absolute error obtained by the proposed method is* 8.702278×10^{-6}.

Example 4.3.6. *Consider the nonlinear boundary value problem*

$$y^{(8)} + siny\, y''' = (1 + sin(e^x))e^x, \quad 0 < x < 1 \tag{4.3.39}$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,$
$y'''(0) = 1,\ y'''(1) = e.$

The exact solution for the above problem is $y = e^x$.

The nonlinear boundary value problem (4.3.39) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} y^{(8)}_{(n+1)} + sin(y_{(n)})y'''_{(n+1)} + cos(y_{(n)})y'''_{(n)}y_{(n+1)} &= (1 + sin(e^x))e^x \\ &+ cos(y_{(n)})y'''_{(n)}y_{(n)}, \quad n = 0, 1, 2, ... \end{aligned} \tag{4.3.40}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = e,$
$y''_{(n+1)}(0) = 1,\ y''_{(n+1)}(1) = e,\ y'''_{(n+1)}(0) = 1, \quad y'''_{(n+1)}(1) = e.$

Here $y_{(n+1)}$ *is the* $(n + 1)^{th}$ *approximation for* y*. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.3.40)*. Numerical results for this problem are shown in Table 4.3.6. The maximum absolute error obtained by the proposed method is* 1.931190×10^{-5}.

x	Absolute error by proposed method
0.1	2.086163E-07
0.2	2.086163E-07
0.3	2.980232E-08
0.4	2.980232E-07
0.5	5.364418E-07
0.6	1.102686E-06
0.7	2.115965E-06
0.8	2.235174E-06
0.9	1.281500E-06

Table 4.3.1: Numerical results for the Example 4.3.1.

x	Absolute error by proposed method
0.1	3.129244E-07
0.2	1.236796E-06
0.3	2.861023E-06
0.4	5.275011E-06
0.5	6.824732E-06
0.6	7.569790E-06
0.7	7.301569E-06
0.8	5.215406E-06
0.9	2.399087E-06

Table 4.3.2: Numerical results for the Example 4.3.2.

x	Absolute error by proposed method
0.1	8.344650E-07
0.2	4.053116E-06
0.3	1.573563E-05
0.4	2.801418E-05
0.5	4.112720E-05
0.6	4.875660E-05
0.7	4.458427E-05
0.8	2.932549E-05
0.9	1.287460E-05

Table 4.3.3: Numerical results for the Example 4.3.3.

x	Absolute error by proposed method
6.487213E-02	3.501773E-07
1.297443E-01	4.768372E-07
1.946164E-01	4.470348E-07
2.594885E-01	1.147389E-06
3.243607E-01	3.188848E-06
3.892328E-01	5.960464E-06
4.541049E-01	8.225441E-06
4.541049E-01	5.036592E-06
5.838492E-01	2.473593E-06

Table 4.3.4: Numerical results for the Example 4.3.4.

x	Absolute error by proposed method
0.1	6.556511E-07
0.2	8.940697E-07
0.3	3.993511E-06
0.4	6.556511E-06
0.5	8.702278E-06
0.6	8.702278E-06
0.7	6.258488E-06
0.8	3.248453E-06
0.9	1.281500E-06

Table 4.3.5: Numerical results for the Example 4.3.5.

x	Absolute error by proposed method
0.1	2.503395E-06
0.2	8.940697E-06
0.3	1.561642E-05
0.4	1.823902E-05
0.5	8.821487E-06
0.6	7.510185E-06
0.7	1.883507E-05
0.8	1.931190E-05
0.9	1.168251E-05

Table 4.3.6: Numerical results for the Example 4.3.6.

4.4 Petrov-Galerkin method for solving a general ninth order boundary value problem with quintic B-splines as basis functions and septic B-splines as weight functions

In this section, we developed a Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions for getting the numerical solution of a general linear ninth order boundary value problem.

4.4.1 Introduction

Consider a general linear ninth order boundary value problem

$$a_0(x)y^{(9)}(x)+a_1(x)y^{(8)}(x)+a_2(x)y^{(7)}(x)+a_3(x)y^{(6)}(x)+a_4(x)y^{(5)}(x)+a_5(x)y^{(4)}(x)$$
$$+ a_6(x)y'''(x) + a_7(x)y''(x) + a_8(x)y'(x) + a_9(x)y(x) = b(x),\ c < x < d \quad (4.4.1)$$

subject to the boundary conditions

$$y(c) = A_0,\quad y(d) = C_0,\quad y'(c) = A_1,\quad y'(d) = C_1,\quad y''(c) = A_2,\quad y''(d) = C_2,$$
$$y'''(c) = A_3,\quad y'''(d) = C_3,\quad y^{(4)}(c) = A_4 \quad (4.4.2)$$

where $A_0,\ C_0,\ A_1,\ C_1,\ A_2,\ C_2,\ A_3,\ C_3,\ A_4$ are finite real constants and $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x),\ a_4(x),\ a_5(x),\ a_6(x),\ a_7(x),\ a_8(x),\ a_9(x)$ and $b(x)$ are all continuous functions defined on the interval $[c,\ d]$.

The ninth-order boundary value problems are known to arise in the study of astrophysics, hydrodynamic and hydromagnetic stability [24]. A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydromagnetic stability [24]. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [9]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on ninth-order boundary value problems by using different methods for numerical solutions. Chawla and Katti [25] developed a finite difference scheme for the solution of a special case of nonlinear higher order two point boundary value problems. Wazwaz [5] developed the solution of a special type of higher order boundary value problems by using the modified Adomian decomposition method. Hassan and Erturk [2] provided solution of different types of linear and nonlinear higher order boundary value problems by using Differential transformation method. Tauseef and Ahmet [129] presented the solution of ninth and tenth order boundary value problems by using homotopy perturbation method without any discretization, linearization or restrictive assumptions. Tauseef and Ahmet [130] developed modified variational method for solving ninth and tenth order boundary value problems introducing He's polynomials in the correction functional. Jafar and Shirin [42] presented homotopy perturbation method for solving the boundary value problems of higher order by reformulating them as an equivalent system of integral equations. Tawfiq and Yassien [70] developed semi-analytic technique for the solution of higher order boundary value problems using two-point oscillatory interpolation to construct polynomial solution. Hossain and Islam [14] presented the Galerkin method with Legendre polynomials as basis functions for the solution of odd higher order boundary value problems. Samir [102] developed spectral collocation method for the solution of m^{th} order boundary value problems with help of Tchebychev polynomials by converting the given differential equation into a system of first order boundary value problems. So far, ninth order boundary value problems have not been solved by using Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions. Therefore in this section, we try to present a simple Petrov-Galerkin method using quintic B-splines as basis functions and septic B-splines as weight functions to solve the ninth order boundary value problem of type (4.4.1)-(4.4.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

4.4.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (4.4.1)-(4.4.2) into n subintervals as described in section 4.1.2. To solve the boundary value problem (4.4.1)-(4.4.2) by the Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions which are described in sections 1.1.3 and 1.1.5 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-3}^{n+2} \alpha_j B_j(x) \tag{4.4.3}$$

where $\alpha_j^{'}s$ are the nodal parameters to be determined and $B_j(x)$'s are quintic B-spline basis functions.

Proceeding as in section 4.3.2 and applying the Petrov-Galerkin method to (4.4.1) with the redefined set of quintic basis functions $\{R_j(x), j = 1, 2, \ldots, n-1\}$ defined in (4.3.11) and with the redefined set of septic weight functions $\{\hat{V}_j(x), j = 1, 2, \ldots, n-1\}$ defined in (4.3.23), we get

$$\begin{aligned}\int_{x_0}^{x_n} \Big[a_0(x)y^{(9)}(x) + a_1(x)y^{(8)}(x) + a_2(x)y^{(7)}(x) + a_3(x)y^{(6)}(x) + a_4(x)y^{(5)}(x)\\ + a_5(x)y^{(4)}(x) + a_6(x)y'''(x) + a_7(x)y''(x) + a_8(x)y'(x)\\ + a_9(x)y(x)\Big]\hat{V}_i(x)\, dx = \int_{x_0}^{x_n} b(x)\hat{V}_i(x)\, dx \quad \text{for} \quad i = 1, 2, \ldots, n-1.\end{aligned} \tag{4.4.4}$$

Integrating by parts the first five terms on the left hand side of (4.4.4) and after applying the boundary conditions prescribed in (4.4.2), we get

$$\begin{aligned}\int_{x_0}^{x_n} a_0(x)\hat{V}_i(x)y^{(9)}(x)dx = \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} y^{(4)}(x_n) - \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_0} A_4\\ - \frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} C_3 + \frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_0} A_3\\ + \int_{x_0}^{x_n} \frac{d^6}{dx^6}\Big[a_0(x)\hat{V}_i(x)\Big]y'''(x)\, dx\end{aligned} \tag{4.4.5}$$

$$\int_{x_0}^{x_n} a_1(x)\hat{V}_i(x)y^{(8)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_1(x)\hat{V}_i(x)\Big]y^{(4)}(x)\ dx \tag{4.4.6}$$

$$\int_{x_0}^{x_n} a_2(x)\hat{V}_i(x)y^{(7)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_2(x)\hat{V}_i(x)\Big]y'''(x)\ dx \tag{4.4.7}$$

$$\int_{x_0}^{x_n} a_3(x)\hat{V}_i(x)y^{(6)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_3(x)\hat{V}_i(x)\Big]y''(x)\ dx \tag{4.4.8}$$

$$\int_{x_0}^{x_n} a_4(x)\hat{V}_i(x)y^{(5)}(x)dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4}\Big[a_4(x)\hat{V}_i(x)\Big]y'(x)\ dx \tag{4.4.9}$$

Substituting (4.4.5) to (4.4.9) in (4.4.4) and using the approximation for $y(x)$ given in (4.3.10) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{4.4.10}$$

where $\mathbf{A} = [a_{ij}]$;

$$\begin{aligned}
a_{ij} = \int_{x_0}^{x_n} \Bigg\{ & \Bigg[\frac{d^4}{dx^4}\Big[a_1(x)\hat{V}_i(x)\Big] + a_5(x)\hat{V}_i(x)\Bigg] R_j^{(4)}(x) \\
& + \Bigg[\frac{d^6}{dx^6}\Big[a_0(x)\hat{V}_i(x)\Big] + \frac{d^4}{dx^4}\Big[a_2(x)\hat{V}_i(x)\Big] + a_6(x)\hat{V}_i(x)\Bigg] R_j^{'''}(x) \\
& + \Bigg[\frac{d^4}{dx^4}\Big[a_3(x)\hat{V}_i(x)\Big] + a_7(x)\hat{V}_i(x)\Bigg] R_j^{''}(x) \\
& + \Bigg[\frac{d^4}{dx^4}\Big[a_4(x)\hat{V}_i(x)\Big] + a_8(x)\hat{V}_i(x)\Bigg] R_j^{'}(x) + a_9(x)\hat{V}_i(x)R_j(x)\Bigg\}\ dx \\
& + \frac{d^4}{dx^4}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} R_j^{(4)}(x_n) \quad \text{for} \quad i = 1, 2, \ldots, n-1, \quad j = 1, 2, \ldots, n-1.
\end{aligned} \tag{4.4.11}$$

$\mathbf{B} = [b_i];$

$$
\begin{aligned}
b_i = \int_{x_0}^{x_n} \Bigg\{ b(x)\hat{V}_i(x) - \Bigg\{ \left[\frac{d^4}{dx^4}\left[a_1(x)\hat{V}_i(x)\right] + a_5(x)\hat{V}_i(x) \right] w^{(4)}(x) \\
+ \left[\frac{d^6}{dx^6}\left[a_0(x)\hat{V}_i(x)\right] + \frac{d^4}{dx^4}\left[a_2(x)\hat{V}_i(x)\right] + a_6(x)\hat{V}_i(x) \right] w'''(x) \\
+ \left[\frac{d^4}{dx^4}\left[a_3(x)\hat{V}_i(x)\right] + a_7(x)\hat{V}_i(x) \right] w''(x) \\
+ \left[\frac{d^4}{dx^4}\left[a_4(x)\hat{V}_i(x)\right] + a_8(x)\hat{V}_i(x) \right] w'(x) + a_9(x)\hat{V}_i(x)R_j(x) \Bigg\}\Bigg\} \, dx \\
- \frac{d^4}{dx^4}\left[a_0(x)\hat{V}_i(x)\right]_{x_n} w^{(4)}(x_n) + \frac{d^4}{dx^4}\left[a_0(x)\hat{V}_i(x)\right]_{x_0} A_4 + \frac{d^5}{dx^5}\left[a_0(x)\hat{V}_i(x)\right]_{x_n} C_3 \\
- \frac{d^5}{dx^5}\left[a_0(x)\hat{V}_i(x)\right]_{x_0} A_3 \quad \text{for} \quad i = 1, 2, \ldots, n-1. \quad (4.4.12)
\end{aligned}
$$

and $\alpha = [\alpha_1 \ \alpha_2 \ \ldots \alpha_{n-1}]^T$.

4.4.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x)\, dx$ and $r_j(x)$ are the quintic B-spline basis functions or their derivatives, $v_i(x)$ are the septic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+3}) \cap (x_{i-4}, x_{i+4}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiff matrix $\mathbf{A}$ is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

4.4.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the ninth order boundary value problems of the type (4.4.1) and (4.4.2), we considered three linear and two nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 4.4.1. *Consider the linear boundary value problem*

$$y^{(9)} - y = -9e^x, \quad 0 < x < 1 \tag{4.4.13}$$

subject to $y(0) = 1,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = -e,\ y''(0) = -1,\ y''(1) = -2e,$
$y'''(0) = -2,\ y'''(1) = -3e,\ y^{(4)}(0) = -3.$

The exact solution for the above problem is $y = (1 - x)e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 4.4.1. The maximum absolute error obtained by the proposed method is* 9.179115×10^{-6}.

Example 4.4.2. *Consider the linear boundary value problem*

$$y^{(9)} + sinx\ y^{(4)} + y = (2 + sinx)e^x, \quad 0 < x < 1 \tag{4.4.14}$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,$
$y'''(0) = 1,\ y'''(1) = e,\ y^{(4)}(0) = 1.$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 4.4.2. The maximum absolute error obtained by the proposed method is* 1.764297×10^{-5}.

Example 4.4.3. *Consider the linear boundary value problem*

$$y^{(9)} + y^{(7)} + xy^{(4)} + y''' + sinx\ y' + y$$
$$= 5xsinx - cosx + x^2cosx - xsin^2x + sinxcosx + xcosx, \quad 0 < x < 1 \quad (4.4.15)$$

subject to $y(0) = 0, y(1) = cos1,\ y'(0) = 1,\ y'(1) = cos1 - sin1,$
$y''(0) = 0,\ y''(1) = -2sin1 - cos1,\ y'''(0) = -3,\ y'''(1) = -3cos1 + sin1,$
$y^{(4)}(0) = 0.$

The exact solution for the above problem is $y = xcosx$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 4.4.3. The maximum absolute error obtained by the proposed method is* 2.324581×10^{-6}.

Example 4.4.4. *Consider the nonlinear boundary value problem*

$$y^{(9)} + e^y y''' + y'y = (1 + e^{e^x} + e^x)e^x, \quad 0 < x < 1 \quad (4.4.16)$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,$
$y'''(0) = 1,\ y'''(1) = e,\ y^{(4)}(0) = 1.$

The exact solution for the above problem is $y = e^x$.

The nonlinear boundary value problem (4.4.16) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(9)}_{(n+1)} + e^{y_{(n)}} y'''_{(n+1)} + y_{(n)} y'_{(n+1)} + (y'''_{(n)} e^{y_{(n)}} + y'_{(n)}) y_{(n+1)}$$
$$= (1 + e^{e^x} + e^x)e^x + (y'''_{(n)} e^{y_{(n)}} + y'_{(n)}) y_{(n)}, \quad n = 0, 1, 2, ... \quad (4.4.17)$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = e,$
$y''_{(n+1)}(0) = 1, y''_{(n+1)}(1) = e,\ y'''_{(n+1)}(0) = 1, y'''_{(n+1)}(1) = e,\ y^{(4)}_{(n+1)}(0) = 1.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.4.17)*. The obtained numerical results for this problem are given in Table 4.4.4. The maximum absolute error obtained by the proposed method is* 1.716614×10^{-5}.

Example 4.4.5. *Consider the nonlinear boundary value problem*

$$y^{(9)} - y'y^2 = cos^3x, \quad 0 < x < 1 \tag{4.4.18}$$

subject to $y(0) = 0, y(1) = sin1,\ y'(0) = 1,\ y'(1) = cos1,\ y''(0) = 0,\ y''(1) = -sin1,$ $y'''(0) = -1,\ y'''(1) = -cos1,\ y^{(4)}(0) = 0.$

The exact solution for the above problem is $y = sinx$.

The nonlinear boundary value problem (4.4.18) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(9)}_{(n+1)} - y^2_{(n)}y'_{(n+1)} - 2y_{(n)}y'_{(n)}y_{(n+1)} = cos^3x - 2y^2_{(n)}y'_{(n)} \quad n = 0, 1, 2, ... \tag{4.4.19}$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = sin1,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = cos1,$
$y''_{(n+1)}(0) = 0, y''_{(n+1)}(1) = -sin1,\ y'''_{(n+1)}(0) = -1,\ y'''_{(n+1)}(1) = -cos1,\ y^{(4)}_{(n+1)}(0) = 0.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for* y*. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.4.19)*. The obtained numerical results for this problem are given in Table 4.4.5. The maximum absolute error obtained by the proposed method is* 5.662441×10^{-6}.

x	Absolute error by proposed method
0.1	1.072884E-06
0.2	5.960464E-07
0.3	3.278255E-06
0.4	7.152557E-06
0.5	9.179115E-06
0.6	6.616116E-06
0.7	1.311302E-06
0.8	2.086163E-06
0.9	1.892447E-06

Table 4.4.1: Numerical results for the Example 4.4.1.

x	Absolute error by proposed method
0.1	1.549721E-06
0.2	6.198883E-06
0.3	9.894371E-06
0.4	1.299381E-05
0.5	5.125999E-06
0.6	7.629395E-06
0.7	1.764297E-05
0.8	1.716614E-05
0.9	1.120567E-05

Table 4.4.2: Numerical results for the Example 4.4.2.

x	Absolute error by proposed method
0.1	2.458692E-07
0.2	7.003546E-07
0.3	1.430511E-06
0.4	2.324581E-06
0.5	1.668930E-06
0.6	2.086163E-07
0.7	1.430511E-06
0.8	1.609325E-06
0.9	1.072884E-06

Table 4.4.3: Numerical results for the Example 4.4.3.

x	Absolute error by proposed method
0.1	1.788139E-06
0.2	5.960464E-06
0.3	1.072884E-05
0.4	1.573563E-05
0.5	9.894371E-06
0.6	3.457069E-06
0.7	1.525879E-05
0.8	1.716614E-05
0.9	1.120567E-05

Table 4.4.4: Numerical results for the Example 4.4.4.

x	Absolute error by proposed method
0.1	1.862645E-07
0.2	7.301569E-07
0.3	9.834766E-07
0.4	1.221895E-06
0.5	8.344650E-07
0.6	3.874302E-06
0.7	5.662441E-06
0.8	4.887581E-06
0.9	2.861023E-06

Table 4.4.5: Numerical results for the Example 4.4.5.

4.5 Petrov-Galerkin method for solving a general tenth order boundary value problem with quintic B-splines as basis functions and septic B-splines as weight functions

In this section, we developed a Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions for getting the numerical solution of a general linear tenth order boundary value problem.

4.5.1 Introduction

Consider a general linear tenth order boundary value problem

$$a_0(x)y^{(10)}(x)+a_1(x)y^{(9)}(x)+a_2(x)y^{(8)}(x)+a_3(x)y^{(7)}(x)+a_4(x)y^{(6)}(x)+a_5(x)y^{(5)}(x)$$
$$+a_6(x)y^{(4)}(x)+a_7(x)y'''(x)+a_8(x)y''(x)+a_9(x)y'(x)+a_{10}(x)y(x) = b(x),\ c < x < d \tag{4.5.1}$$

subject to the boundary conditions

$$y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \quad y''(c) = A_2, \quad y''(d) = C_2,$$
$$y'''(c) = A_3, \quad y'''(d) = C_3, \quad y^{(4)}(c) = A_4, \quad y^{(4)}(d) = C_4 \tag{4.5.2}$$

where $A_0,\ C_0,\ A_1,\ C_1,\ A_2,\ C_2,\ A_3,\ C_3,\ A_4,\ C_4$ are finite real constants and $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x),\ a_4(x),\ a_5(x),\ a_6(x),\ a_7(x),\ a_8(x),\ a_9(x),\ a_{10}(x)$ and $b(x)$ are all continuous functions defined on the interval $[c,\ d]$.

A class of characteristic-value problems of high order (as high as twenty four) are known to arise in hydrodynamic and hydromagnetic stability [24]. Tenth-order differential equations govern the physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When this instability sets in as ordinary convection, it is modelled by a tenth-order ordinary differential equation [24]. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [9]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on tenth order boundary value problems by using different methods for numerical solutions. Twizell and Boutayeb [136] developed finite difference techniques for the solution of eighth, tenth and twelfth order boundary value problems. Siddiqi and Twizell [116], Siddiqi and Ghazala [110] presented the solution of a special case of linear tenth order boundary value problems by using tenth order and eleventh order spline functions respectively. Wazwaz [6] dealt with modified Adomian decomposition for the solution of tenth and twelfth order boundary value problems. Siddiqi and Ghazala [111] discussed the solution of a special case of linear tenth order boundary value problems by using non-polynomial spline techniques. Erturk and Shaher [138] presented differential transform method for the solution of tenth order boundary value problems. Geng and Li [32], Abbasbandy and Shirzdi [1] discussed the solution of a special case of tenth order boundary value problems by using variational iteration techniques respectively. Kasi Viswanadham and Showri Raju [57] developed quintic B-spline collocation method to solve a general tenth order boundary value problem. Kasi Viswanadham and Sreenivasulu [62] developed the quintic B-spline Galerkin method to solve a general tenth order boundary value problem. So far, tenth order boundary value problems have not been solved by using Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions. Therefore in this section, we try to present a simple Petrov-Galerkin method using quintic B-splines as basis functions and septic B-splines as weight functions to solve the tenth order boundary value problem of type (4.5.1)-(4.5.2). The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [15].

4.5.2 Description of the method

Divide the space variable domain $[c, d]$ of the system (4.5.1)-(4.5.2) into n subintervals as described in section 4.1.2.

To solve the boundary value problem (4.5.1)-(4.5.2) by the Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions which are described in sections 1.1.3 and 1.1.5 respectively, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{4.5.3}$$

where α_j's are the nodal parameters to be determined and $B_j(x)$'s are quintic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$, the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, Neumann, second order derivative and third order derivative type of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of 1.1.3, the Dirichlet, Neumann, second order derivative and third order derivative boundary conditions of (4.5.2), we get the approximate solution at the boundary points as

$$A_0 = y(c) = y(x_0) = \sum_{j=-2}^{2} \alpha_j B_j(x_0) \tag{4.5.4}$$

$$C_0 = y(d) = y(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(x_n) \tag{4.5.5}$$

$$A_1 = y'(c) = y'(x_0) = \sum_{j=-2}^{2} \alpha_j B_j'(x_0) \tag{4.5.6}$$

$$C_1 = y'(d) = y'(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j'(x_n) \tag{4.5.7}$$

$$A_2 = y''(c) = y''(x_0) = \sum_{j=-2}^{2} \alpha_j B_j''(x_0) \tag{4.5.8}$$

$$C_2 = y''(d) = y''(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j''(x_n) \tag{4.5.9}$$

$$A_3 = y''''(c) = y'''(x_0) = \sum_{j=-2}^{2} \alpha_j B_j'''(x_0) \tag{4.5.10}$$

$$C_3 = y'''(d) = y'''(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j'''(x_n) \tag{4.5.11}$$

Eliminating α_{-2}, α_{-1}, α_0, α_1, α_{n-1}, α_n, α_{n+1} and α_{n+2} from the equations (4.5.3) to (4.5.11), we get

$$y(x) = w(x) + \sum_{j=2}^{n-2} \alpha_j S_j(x) \tag{4.5.12}$$

where

$$w(x) = w_3(x) + \frac{A_3 - w_3'''(x_0)}{R_1'''(x_0)} R_1(x) + \frac{C_3 - w_3'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x)$$

$$w_3(x) = w_2(x) + \frac{A_2 - w_2''(x_0)}{Q_0''(x_0)} Q_0(x) + \frac{C_2 - w_2''(x_n)}{Q_n''(x_n)} Q_n(x)$$

$$w_2(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-1}'(x_0)} P_{-1}(x) + \frac{C_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x)$$

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j'''(x_0)}{R_1'''(x_0)} R_1(x), & j = 2 \\ R_j(x), & j = 3, 4, \ldots, n-3 \\ R_j(x) - \dfrac{R_j'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x), & j = n-2 \end{cases} \tag{4.5.13}$$

$$R_j(x) = \begin{cases} Q_j(x) - \dfrac{Q_j''(x_0)}{Q_0''(x_0)} Q_0(x), & j = 1, 2 \\ Q_j(x), & j = 3, 4, \ldots, n-3 \\ Q_j(x) - \dfrac{Q_j''(x_n)}{Q_n''(x_n)} Q_n(x), & j = n-2, n-1 \end{cases}$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-1}'(x_0)} P_{-1}(x), & j = 0, 1, 2 \\ P_j(x), & j = 3, 4, \ldots, n-3 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & j = n-2, n-1, n \end{cases}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1, 0, 1, 2 \\ B_j(x), & j = 3, 4, \ldots, n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & j = n-2, n-1, n, n+1 \end{cases}$$

The new set of basis functions in the approximation $y(x)$ is $\{\, S_j(x), j = 2, 3, \ldots, n-2\}$. Here $w(x)$ takes care of the given set of Dirichlet, Neumann, second order derivative and third order derivative type of boundary conditions and $S_j(x)'s$ and its first, second and third order derivatives vanish on the boundary. In Petrov-Galerkin method, the number of basis functions in the approximation should match with the number of weight functions. Here the number of basis functions in the approximation for $y(x)$ defined in (4.5.12) is $n-3$, where as the number of weight functions is $n+7$. So, there is a need to redefine the weight functions into a new set of weight functions which in number match with the number of basis functions. The procedure for redefining the weight functions is as follows.

Let us write the approximation for $v(x)$ as

$$v(x) = \sum_{j=-3}^{n+3} \beta_j \hat{B}_j(x) \tag{4.5.14}$$

where $\hat{B}_j(x)$'s are septic B-splines defined in section 1.1.5 and here we assume that above approximation $v(x)$ satisfies corresponding homogeneous boundary conditions of (4.5.2). That means $v(x)$, defined in (4.5.14), satisfies the conditions

$$\begin{aligned} &v(c) = 0, \quad v(d) = 0, \quad v'(c) = 0, \quad v'(d) = 0, \\ &\quad v''(c) = 0, \quad v''(d) = 0, \quad v'''(c) = 0, \quad v'''(d) = 0, \quad v^{(4)}(c) = 0, \quad v^{(4)}(d) = 0 \end{aligned} \tag{4.5.15}$$

Using the definition of septic B-splines defined in section 1.1.5 and applying the boundary conditions (4.5.15) to (4.5.14), we get the approximate solution at the boundary points as

$$v(c) = v(x_0) = \sum_{j=-3}^{3} \beta_j \hat{B}_j(x_0) = 0 \tag{4.5.16}$$

$$v(d) = v(x_n) = \sum_{j=n-3}^{n+3} \beta_j \hat{B}_j(x_n) = 0 \tag{4.5.17}$$

$$v'(c) = v'(x_0) = \sum_{j=-3}^{3} \beta_j \hat{B}'_j(x_0) = 0 \tag{4.5.18}$$

$$v'(d) = v'(x_n) = \sum_{j=n-3}^{n+3} \beta_j \hat{B}'_j(x_n) = 0 \tag{4.5.19}$$

$$v''(c) = v''(x_0) = \sum_{j=-3}^{3} \beta_j \hat{B}''_j(x_0) = 0 \tag{4.5.20}$$

$$v''(d) = v''(x_n) = \sum_{j=n-3}^{n+3} \beta_j \hat{B}''_j(x_n) = 0 \tag{4.5.21}$$

$$v'''(c) = v'''(x_0) = \sum_{j=-3}^{3} \beta_j \hat{B}'''_j(x_0) = 0 \tag{4.5.22}$$

$$v'''(d) = v'''(x_n) = \sum_{j=n-3}^{n+3} \beta_j \hat{B}_j'''(x_n) = 0 \tag{4.5.23}$$

$$v^{(4)}(c) = v^{(4)}(x_0) = \sum_{j=-3}^{3} \beta_j \hat{B}_j^{(4)}(x_0) = 0 \tag{4.5.24}$$

$$v^{(4)}(d) = v^{(4)}(x_n) = \sum_{j=n-3}^{n+3} \beta_j \hat{B}_j^{(4)}(x_n) = 0 \tag{4.5.25}$$

Eliminating β_{-3}, β_{-2}, β_{-1}, β_0, β_1, β_{n-1}, β_n, β_{n+1}, β_{n+2} and β_{n+3} from the equations (4.5.14) and (4.5.16) to (4.5.25), we get the approximation for $v(x)$ as

$$v(x) = \sum_{j=2}^{n-2} \beta_j \hat{V}_j(x) \tag{4.5.26}$$

where

$$\hat{V}_j(x) = \begin{cases} \hat{U}_j(x) - \dfrac{\hat{U}_j^{(4)}(x_0)}{\hat{U}_1^{(4)}(x_0)}\hat{U}_1(x), & j = 2, 3 \\ \hat{U}_j(x), & j = 4, 5, \ldots, n-4 \\ \hat{U}_j(x) - \dfrac{\hat{U}_j^{(4)}(x_n)}{\hat{U}_{n-1}^{(4)}(x_n)}\hat{U}_{n-1}(x), & j = n-3, n-2 \end{cases} \tag{4.5.27}$$

$$\hat{U}_j(x) = \begin{cases} V_j(x) - \dfrac{V_j'''(x_0)}{V_0'''(x_0)}V_0(x), & j = 1, 2, 3 \\ V_j(x), & j = 4, 5, \ldots, n-4 \\ V_j(x) - \dfrac{V_j'''(x_n)}{V_n'''(x_n)}V_n(x), & j = n-3, n-2, n-1, \end{cases}$$

$$V_j(x) = \begin{cases} U_j(x) - \dfrac{U_j''(x_0)}{U_{-1}''(x_0)}U_{-1}(x), & j = 0, 1, 2, 3 \\ U_j(x), & j = 4, 5, \ldots, n-4 \\ U_j(x) - \dfrac{U_j''(x_n)}{U_{n+1}''(x_n)}U_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases}$$

$$U_j(x) = \begin{cases} T_j(x) - \dfrac{T_j'(x_0)}{T_{-2}'(x_0)} T_{-2}(x), & j = -1, 0, 1, 2, 3 \\ T_j(x), & j = 4, 5, \dots, n-4 \\ T_j(x) - \dfrac{T_j'(x_n)}{T_{n+2}'(x_n)} T_{n+2}(x), & j = n-3, n-2, n-1, n, n+1 \end{cases}$$

$$T_j(x) = \begin{cases} \hat{B}_j(x) - \dfrac{\hat{B}_j(x_0)}{\hat{B}_{-3}(x_0)} \hat{B}_{-3}(x), & j = -2, -1, 0, 1, 2, 3 \\ \hat{B}_j(x), & j = 4, 5, \dots, n-4 \\ \hat{B}_j(x) - \dfrac{\hat{B}_j(x_n)}{\hat{B}_{n+3}(x_n)} \hat{B}_{n+3}(x), & j = n-3, n-2, n-1, n, n+1, n+2 \end{cases}$$

Now the new set of basis functions for the approximation $v(x)$ is $\{\ \hat{V}_j(x), j = 2, 3, \dots, n-2\}$. Here $\hat{V}_j(x)$'s and its first, second, third and fourth order derivatives vanish on the boundary. Let us take $\hat{V}_j(x)$'s as weight functions for the prescribed Petrov-Galerkin method. Here the redefined quintic basis functions $S_j(x)$'s defined in (4.5.13) and the redefined septic weight functions $\hat{V}_j(x)$'s defined in (4.5.27) match in number.

Applying the Petrov-Galerkin method to (4.5.1) with the redefined set of quintic basis functions $\{S_j(x), j = 2, 3, \dots, n-2\}$ and the redefined set of septic weight functions $\{\ \hat{V}_j(x), j = 2, 3, \dots, n-2\}$, we get

$$\begin{aligned} \int_{x_0}^{x_n} \big[& a_0(x) y^{(10)}(x) + a_1(x) y^{(9)}(x) + a_2(x) y^{(8)}(x) + a_3(x) y^{(7)}(x) + a_4(x) y^{(6)}(x) \\ & + a_5(x) y^{(5)}(x) + a_6(x) y^{(4)}(x) + a_7(x) y'''(x) + a_8(x) y''(x) + a_9(x) y'(x) \\ & + a_{10}(x) y(x) \big] \hat{V}_i(x)\, dx = \int_{x_0}^{x_n} b(x) \hat{V}_i(x)\, dx \\ & \text{for} \quad i = 2, 3, \dots, n-2. \quad (4.5.28) \end{aligned}$$

Integrating by parts the first six terms on the left hand side of (4.5.28) and after applying the boundary conditions prescribed in (4.5.2), we get

$$\int_{x_0}^{x_n} a_0(x)\hat{V}_i(x)y^{(10)}(x)dx = -\frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} C_4 + \frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_0} A_4 + \int_{x_0}^{x_n} \frac{d^6}{dx^6}\Big[a_0(x)\hat{V}_i(x)\Big] y^{(4)}(x)\ dx \quad (4.5.29)$$

$$\int_{x_0}^{x_n} a_1(x)\hat{V}_i(x)y^{(9)}(x)dx = -\int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_1(x)\hat{V}_i(x)\Big] y^{(4)}(x)\ dx \quad (4.5.30)$$

$$\int_{x_0}^{x_n} a_2(x)\hat{V}_i(x)y^{(8)}(x)dx = -\int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_2(x)\hat{V}_i(x)\Big] y'''(x)\ dx \quad (4.5.31)$$

$$\int_{x_0}^{x_n} a_3(x)\hat{V}_i(x)y^{(7)}(x)dx = -\int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_3(x)\hat{V}_i(x)\Big] y''(x)\ dx \quad (4.5.32)$$

$$\int_{x_0}^{x_n} a_4(x)\hat{V}_i(x)y^{(6)}(x)dx = -\int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_4(x)\hat{V}_i(x)\Big] y'(x)\ dx \quad (4.5.33)$$

$$\int_{x_0}^{x_n} a_5(x)\hat{V}_i(x)y^{(5)}(x)dx = -\int_{x_0}^{x_n} \frac{d^5}{dx^5}\Big[a_5(x)\hat{V}_i(x)\Big] y(x)\ dx \quad (4.5.34)$$

Substituting (4.5.29) to (4.5.34) in (4.5.28) and using the approximation for $y(x)$ given in (4.5.12) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \quad (4.5.35)$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \Bigg\{ \left[\frac{d^6}{dx^6}\Big[a_0(x)\hat{V}_i(x)\Big] - \frac{d^5}{dx^5}\Big[a_1(x)\hat{V}_i(x)\Big] + a_6(x)\hat{V}_i(x) \right] S_j^{(4)}(x)$$
$$+ \left[-\frac{d^5}{dx^5}\Big[a_2(x)\hat{V}_i(x)\Big] + a_7(x)\hat{V}_i(x) \right] S_j'''(x)$$
$$+ \left[-\frac{d^5}{dx^5}\Big[a_3(x)\hat{V}_i(x)\Big] + a_8(x)\hat{V}_i(x) \right] S_j''(x) + \left[-\frac{d^5}{dx^5}\Big[a_4(x)\hat{V}_i(x)\Big] + a_9(x)\hat{V}_i(x) \right] S_j'(x)$$
$$+ \left[-\frac{d^5}{dx^5}\Big[a_5(x)\hat{V}_i(x)\Big] + a_{10}(x)\hat{V}_i(x) \right] S_j(x) \Bigg\}\, dx$$
$$\text{for} \quad i = 2, 3, \ldots, n-2, \quad j = 2, 3, \ldots, n-2. \quad (4.5.36)$$

$\mathbf{B} = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \Bigg\{ b(x)\hat{V}_i(x) - \left[\frac{d^6}{dx^6}\Big[a_0(x)\hat{V}_i(x)\Big] - \frac{d^5}{dx^5}\Big[a_1(x)\hat{V}_i(x)\Big] + a_6(x)\hat{V}_i(x) \right] w^{(4)}(x)$$
$$- \left[-\frac{d^5}{dx^5}\Big[a_2(x)\hat{V}_i(x)\Big] + a_7(x)\hat{V}_i(x) \right] w'''(x)$$
$$- \left[-\frac{d^5}{dx^5}\Big[a_3(x)\hat{V}_i(x)\Big] + a_8(x)\hat{V}_i(x) \right] w''(x) - \left[-\frac{d^5}{dx^5}\Big[a_4(x)\hat{V}_i(x)\Big] + a_9(x)\hat{V}_i(x) \right] w'(x)$$
$$- \left[-\frac{d^5}{dx^5}\Big[a_5(x)\hat{V}_i(x)\Big] + a_{10}(x)\hat{V}_i(x) \right] w(x) \Bigg\}\, dx$$
$$+ \frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_n} C_4 - \frac{d^5}{dx^5}\Big[a_0(x)\hat{V}_i(x)\Big]_{x_0} A_4$$
$$\text{for} \quad i = 2, 3, \ldots, n-2. \quad (4.5.37)$$

and $\alpha = [\alpha_2\ \alpha_3\ \ldots \alpha_{n-2}]^T$.

4.5.3 Solution procedure to find the nodal parameters

A typical integral element in the matrix **A** is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} v_i(x) r_j(x) Z(x)\, dx$ and $r_j(x)$ are the quintic B-spline basis functions or their derivatives, $v_i(x)$ are the septic B-spline weight functions or their derivatives.

It may be noted that $I_m = 0$ if $(x_{j-3}, x_{j+3}) \cap (x_{i-4}, x_{i+4}) \cap (x_m, x_{m+1}) = \emptyset$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiff matrix **A** is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using the band matrix solution package.

4.5.4 Numerical Results

To demonstrate the applicability of the proposed method for solving the tenth order boundary value problems of the type (4.5.1) and (4.5.2), we considered three linear and three nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 4.5.1. *Consider the linear boundary value problem*

$$y^{(10)} + xy = -(80 + 19x + x^3)e^x, \quad 0 < x < 1 \tag{4.5.38}$$

subject to $y(0) = 0,\ y(1) = 0,\ y'(0) = 1,\ y'(1) = -e,\ y''(0) = 0,\ y''(1) = -4e,$ $y'''(0) = -3,\ y'''(1) = -9e,\ y^{(4)}(0) = -8,\ y^{(4)}(1) = -16e.$

The exact solution for the above problem is $y = x(1-x)e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.5.1. The maximum absolute error obtained by the proposed method is* 2.825260×10^{-5}.

Example 4.5.2. *Consider the linear boundary value problem*

$$y^{(10)} - (x^2 - 2)y = 10cosx - (x-1)^2(x+1)sinx, \quad -1 \le x \le 1 \tag{4.5.39}$$

subject to $y(-1) = 2sin1, y(1) = 0,\ y'(-1) = -2cos1 - sin1,\ y'(1) = sin1,$
$y''(-1) = 2cos1 - 2sin1,\ y''(1) = 2cos1,\ y'''(-1) = 2cos1 + 3sin1,\ y'''(1) = -3sin1,$
$y^{(4)}(-1) = -4cos1 + 2sin1,\ y^{(4)}(1) = -4cos1.$

The exact solution for the above problem is $y = (x-1)sinx$.

The proposed method is tested on this problem where the domain $[-1, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.5.2. The maximum absolute error obtained by the proposed method is* 1.725554×10^{-5}.

Example 4.5.3. *Consider the linear boundary value problem*

$$y^{(10)} + y^{(9)} + sinx\ y^{(4)} + cosx\ y''' + x^2 y = (2 + sinx + cosx + x^2)e^x, \quad 0 < x < 1 \tag{4.5.40}$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,$
$y'''(0) = 1,\ y'''(1) = e,\ y^{(4)}(0) = 1,\ y^{(4)}(1) = e.$

The exact solution for the above problem is $y = e^x$.

The proposed method is tested on this problem where the domain $[0, 1]$ *is divided into 10 equal subintervals. Numerical results for this problem are given in Table 4.5.3. The maximum absolute error obtained by the proposed method is* 8.916855×10^{-5}.

Example 4.5.4. *Consider the nonlinear boundary value problem*

$$y^{(10)} + e^{-x}y^2 = e^{-x} + e^{-3x}, \quad 0 < x < 1 \tag{4.5.41}$$

subject to $y(0) = 1, y(1) = e^{-1},\ y'(0) = -1,\ y'(1) = -e^{-1},\ y''(0) = 1,\ y''(1) = e^{-1},$
$y'''(0) = -1,\ y'''(1) = -e^{-1},\ y^{(4)}(0) = 1,\ y^{(4)}(1) = e^{-1}.$

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (4.5.41) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$y^{(10)}_{(n+1)} + 2e^{-x}y_{(n)}y_{(n+1)} = e^{-x}y^2_{(n)} + e^{-x} + e^{-3x}, \quad n = 0, 1, 2, ... \tag{4.5.42}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e^{-1},\ y'_{(n+1)}(0) = -1,\ y'_{(n+1)}(1) = -e^{-1}$,
$y''_{(n+1)}(0) = 1, y''_{(n+1)}(1) = e^{-1},\ y'''_{(n+1)}(0) = -1, y'''_{(n+1)}(1) = -e^{-1}$,
$y^{(4)}_{(n+1)}(0) = 1, y^{(4)}_{(n+1)}(1) = e^{-1}$.

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for y. The domain* $[0, 1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.5.42). *Numerical results for this problem are given in Table 4.5.4. The maximum absolute error obtained by the proposed method is* 2.998114×10^{-5}.

Example 4.5.5. *Consider the nonlinear boundary value problem*

$$y^{(10)} = \frac{14175}{4}(x + y + 1)^{11}, \quad 0 \le x \le 1 \tag{4.5.43}$$

subject to $y(0) = 0, y(1) = 0,\ y'(0) = -0.5,\ y'(1) = 1,\ y''(0) = 0.5,\ y''(1) = 4$,
$y'''(0) = 0.75,\ y'''(1) = 12,\ y^{(4)}(0) = 1.5,\ y^{(4)}(1) = 48$.

The exact solution for the above problem is $y = \frac{2}{2-x} - x - 1$.

The nonlinear boundary value problem (4.5.43) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} y^{(10)}_{(n+1)} - \frac{14175}{4}(x + y_{(n)} + 1)^{10}y_{(n+1)} \\ = \frac{14175}{4}(x + y_{(n)} + 1)^{10}(1 + x - 10y_{(n)}) \quad n = 0, 1, 2, ... \end{aligned} \tag{4.5.44}$$

subject to $y_{(n+1)}(0) = 0,\ y_{(n+1)}(1) = 0,\ y'_{(n+1)}(0) = -0.5,\ y'_{(n+1)}(1) = 1$,

$y''_{(n+1)}(0) = 0.5, y''_{(n+1)}(1) = 4,\ y'''_{(n+1)}(0) = 0.75, y'''_{(n+1)}(1) = 12,$

$y^{(4)}_{(n+1)}(0) = 1.5, y^{(4)}_{(n+1)}(1) = 48.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for y. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.5.44)*. Numerical results for this problem are given in Table 4.5.5. The maximum absolute error obtained by the proposed method is* 6.258488×10^{-6}.

Example 4.5.6. *Consider the nonlinear boundary value problem*

$$y^{(10)} + y^{(9)} + y^2 y^{(4)} + cosy\ y' = (2 + e^{2x} + cos(e^x))e^x, \quad 0 < x < 1 \qquad (4.5.45)$$

subject to $y(0) = 1, y(1) = e,\ y'(0) = 1,\ y'(1) = e,\ y''(0) = 1,\ y''(1) = e,$
$y'''(0) = 1,\ y'''(1) = e,\ y^{(4)}(0) = 1,\ y^{(4)}(1) = e.$

The exact solution for the above problem is $y = e^x$.

The nonlinear boundary value problem (4.5.45) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [15] as*

$$\begin{aligned} & y^{(10)}_{(n+1)} + y^{(9)}_{(n+1)} + y^2_{(n)} y^{(4)}_{(n+1)} + cos(y_{(n)}) y'_{(n+1)} + (2y_{(n)} y^{(4)}_{(n)} - sin(y_{(n)}) y'_{(n)}) y_{(n+1)} \\ & \quad = (2y_{(n)} y^{(4)}_{(n)} - sin(y_{(n)}) y'_{(n)}) y_{(n)} + (2 + e^{2x} + cos(e^x))e^x, \quad n = 0, 1, 2, ... \qquad (4.5.46) \end{aligned}$$

subject to $y_{(n+1)}(0) = 1,\ y_{(n+1)}(1) = e,\ y'_{(n+1)}(0) = 1,\ y'_{(n+1)}(1) = e,$
$y''_{(n+1)}(0) = 1,\ y''_{(n+1)}(1) = e,\ y'''_{(n+1)}(0) = 1, y'''_{(n+1)}(1) = e,$
$y^{(4)}_{(n+1)}(0) = 1, y^{(4)}_{(n+1)}(1) = e.$

Here $y_{(n+1)}$ *is the* $(n+1)^{th}$ *approximation for y. The domain* $[0,1]$ *is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems* (4.5.46)*. The obtained numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is* 8.428097×10^{-5}.

x	Absolute error by proposed method
0.1	4.172325E-07
0.2	3.233552E-06
0.3	1.123548E-05
0.4	2.232194E-05
0.5	2.825260E-05
0.6	2.512336E-05
0.7	1.624227E-05
0.8	7.450581E-06
0.9	2.518296E-06

Table 4.5.1: Numerical results for the Example 4.5.1.

x	Absolute error by proposed method
-0.8	7.152557E-07
-0.6	6.973743E-06
-0.4	1.382828E-05
-0.2	1.725554E-05
0.0	1.461588E-05
0.2	6.839633E-06
0.4	3.427267E-07
0.6	2.250075E-06
0.8	1.624227E-06

Table 4.5.2: Numerical results for the Example 4.5.2.

x	Absolute error by proposed method
0.1	2.861023E-06
0.2	1.585484E-05
0.3	4.673004E-05
0.4	8.201599E-05
0.5	8.916855E-05
0.6	5.972385E-05
0.7	1.788139E-05
0.8	7.152557E-06
0.9	9.775162E-06

Table 4.5.3: Numerical results for the Example 4.5.3.

x	Absolute error by proposed method
0.1	1.609325E-06
0.2	7.867813E-06
0.3	1.960993E-05
0.4	2.998114E-05
0.5	2.872944E-05
0.6	1.639128E-05
0.7	3.337860E-06
0.8	2.175570E-06
0.9	1.460314E-06

Table 4.5.4: Numerical results for the Example 4.5.4.

x	Absolute error by proposed method
0.1	1.490116E-07
0.2	9.238720E-07
0.3	2.965331E-06
0.4	5.558133E-06
0.5	6.258488E-06
0.6	4.455447E-06
0.7	1.817942E-06
0.8	1.788139E-07
0.9	1.192093E-07

Table 4.5.5: Numerical results for the Example 4.5.5.

x	Absolute error by proposed method
0.1	2.980232E-06
0.2	1.597404E-05
0.3	4.577637E-05
0.4	7.903576E-05
0.5	8.428097E-05
0.6	5.435944E-05
0.7	1.430511E-05
0.8	8.106232E-06
0.9	9.775162E-06

Table 4.5.6: Numerical results for the Example 4.5.6.

Chapter 5

Conclusions and Scope for further research

Conclusions

In this thesis, we have developed Petrov-Galerkin method with the combination of different orders of B-splines as basis functions and weight functions to solve higher order boundary value problems. In the method the basis functions are redefined into a new set of basis functions which vanish on the boundary where all the boundary conditions or most of the boundary conditions are prescribed and the weight functions are also redefined into a new set of weight functions which in number match with the number of basis functions. The approximate solution has been written as a linear combination of the redefined set of basis functions along with the non-homogeneous part function which takes care of the boundary conditions where the redefined basis functions vanish. We have approximated the solution with a B-spline polynomial whose order is less than the order of the given differential equation. The proposed method is applied to solve several linear and nonlinear boundary value problems numerically with moderate value of step size to test the efficiency. The solution of a nonlinear problem has been obtained as the limit of a sequence of solutions of the linear problems generated by quasilinearization technique. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this thesis is to present simple and accurate method to solve higher order boundary value problems.

Scope for further research

Time dependent problems can be solved with B-splines as basis functions. The method can be extended to solve two dimensional problems by using tensor probuct B-splines.

Bibliography

[1] A. Abbasbandy and A. Shirzadi: "The variational iteration method for a class of tenth-order boundary value problems", *International Journal of Industrial Mathematics*, **2** (2010) pp. 29–36.

[2] I. H. Abdel-Halim Hassan and Vedat Suat Erturk: "Solutions of different types of the linear and nonlinear higher order boundary value problems by differential transformation method", *European Journal of Pure and Applied Mathematics*, **2** (2009)(3) pp. 426–447.

[3] Abdellah Lamini, Hamid Mraoui, Driss Sbibih, Ahmed Tijini and Ahmed Zidna: "Spline Collocation method for solving linear sixth order boundary value problems", *International Journal of Computer Mathematics*, **85** (2008)(11) pp. 1673–1684.

[4] Abdul-Majid Wazwaz : "The numerical solution of fifth order boundary value problems by the decompostion method", *Journal of Computational and Applied Mathematics*, **136** (2001) pp. 259–270.

[5] Abdul-Majid Wazwaz: "Approximate solutions to boundary value problems of higher order by the modified decomposition method", *Computers and Mathematics with Applications*, **40** (2000) pp. 679–691.

[6] Abdul-Majid Wazwaz: "The modified Adomian decomposition method for solving linear and nonlinear boundary value problems of tenth order and twelfth order", *International Journal of Nonlinear Sciences and Numerical Simulation*, **1** (2000) pp. 17–24.

[7] Abdul-Majid Wazwaz: "The numerical solution of sixth-order boundary value problems by the modified decomposition method", *Applied Mathematics and Computation*, **118** (2001) pp. 311–325.

[8] Abdul-Majid Wazwaz: "The numerical solution of special fourth-order boundary value problems by modified decompostion method", *International Journal of Computer Mathematics*, **79** (2002) pp. 345–356.

[9] R. P. Agarwal: *Boundary Value Problems for Higher Order Differential Equations*, World Scientific (1986).

[10] R. P. Agarwal and M. Chow: "Iterative methods for a fourth order boundary value problem", *Journal of Computational and Applied Mathematics*, **10** (1984) pp. 203–217.

[11] Ahniyaz Nurmuhammad, Mayinurmuhammad and Masatake Mori: "Sinc-Galerkin method based on the DE transformation for the boundary value problem of fourth-order ode", *Journal of Computational and Applied Mathematics*, **206** (2007) pp. 17–26.

[12] E. Aksan: "Quadratic B-spline finite element method for numerical solution of the Burgers' equation", *Applied Mathematics and Computation*, **174** (2006) pp. 884–896.

[13] Ali Bashan, Seydi Battal Gazi Karakoc and Turabi Geyikli: "Approximation of the KdVB equation by the quintic B-spline differential quadrature method", *Kuwait Journal of Science*, **42** (2015)(2) pp. 67–92.

[14] Md. Bellal Hossain and Md. Shafiqul Islam: "A novel numerical approach for odd higher order boundary value problems", *Mathematical Theory and Modeling*, **4** (2014)(5) pp. 1–11.

[15] R. E. Bellman and R. E. Kalaba: *Quasilinearization and Nonlinear Boundary Value Problems*, American Elsevier, New York (1965).

[16] L. Bers, F. John and M. Schecheter: *Partial Differential Equations*, John Wiley Inter Science, New York (1964).

[17] A. Boutayeb and E.H. Twizell: "Finite difference methods for the solution of special eighth-order boundary value problems", *International Journal of Computer Mathematics*, **48** (1993) pp. 63–75.

[18] J. Brown, M. Bloor, M. S. Bloor and M. Wilson: "The accuracy of B-spline finite element approximations to PDE surfaces", *Computer Methods in Applied Mechanics and Engineering*, **158** (1998) pp. 221–234.

[19] H. Caglar and N. Caglar: "Solution of fifth-order boundary value problems by using local polynomial regression", *Applied Mathematics and Computation*, **186** (2007) pp. 952–956.

[20] H. Caglar, N. Caglar and K. Elfaituri: "B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems", *Applied Mathematics and Computation*, **175** (2006) pp. 72–79.

[21] H. N. Caglar, S. H. Caglar and E. H. Twizell: "The numerical solution of fifth order boundary value problems with sixth degree B-spline functions", *Applied Mathematics Letters*, **12** (1999) pp. 25–30.

[22] Carl de Boor: "On the calculation with B-splines", *Journal of Approximation Theory*, **6** (1972) pp. 50–62.

[23] Carl de Boor: *A Practical Guide to Splines*, Springer-Verlag (2001).

[24] S. Chandrasekhar: *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford (Reprinted: Dover Books, New York, 1981) (1961).

[25] M. M. Chawla and C. P. Katti: "Finite difference methods for two-point boundary value problems involving high order differential equations", *BIT*, **19** (1979) pp. 27–33.

[26] Christopher G. Provatidis and Stelios K. Isidorou: "Solution of one-dimensional hyperbolic problems using cubic B-splines collocation", *International Journal of Computer Science and Application*, **1** (2012) pp. 12–18.

[27] F. Costabile and A. Napoli: "Numerical solution of higher order Bernoulli boundary value problems", *Journal of Applied Mathematics*, **Article id: 276585** (2014) pp. 1–9.

[28] M. G. Cox: "The numerical evaluation of B-splines", *IMA Journal of Applied Mathematics*, **10** (1972) pp. 134–149.

[29] A. J. Davies: *The Finite Element Method: A first approach*, Clarandon Press, Oxford (2010).

[30] A. R. Davies, A. Karageorghis and T. N. Phillips: "Spectral Galerkin methods for the primary two point boundary value problem in modelling viscoelastic flow", *International Journal of Numerical Methods*, **26** (1988) pp. 647–662.

[31] Duygu Dnmez Demir and Necdet Bildik: "The numerical solution of heat problem using cubic B-splines", *Applied Mathematics*, **2** (2012) pp. 131–135.

[32] Fazhan Geng and Xiuying Li: "Variational iteration method for solving linear tenth-order boundary value problems", *Mathematical Sciences*, **3** (2009) pp. 161–172.

[33] Feng-Gong Lang and Xiao-Ping Xu: "Quartic B-spline collocation method for fifth order boundary value problems", *Computing*, **92** (2011) pp. 365–378.

[34] T. Geyiklia and S. B. G. Karakocb: "Subdomain finite element method with quartic B-splines for the modified equal width wave equation", *Computational Mathematics and Mathematical Physics*, **55** (2015)(3) pp. 410–421.

[35] Ghazala Akram and Shahid S.Siddiqi: "Nonic spline solutions of eighth-order boundary value problems", *Applied Mathematics and Computation*, **182** (2006) pp. 829–845.

[36] Ghazala Akram and Hamood Ur Rehman: "Numerical solution of eighth order boundary value problems in reproducing kernel space", *Numer Algor*, **62** (2013) pp. 527–540.

[37] Ghazala Akram and Hamood Ur Rehman: "Numerical solution of seventh-order boundary value problems using the reproducing kernel space", *Research Journal of Applied Sciences, Engineering and Technology*, **7** (2014)(4) pp. 892–896.

[38] Ghazala Akram and Nadia Amin: "Solution of a fourth order singularity perturbed boundary value problem using quintic spline", *International Mathematical Forum*, **7**(44).

[39] Ghazala Akram and Shahid S. Siddiqi: "Solution of sixth-order boundary value problems by using non-polynomial spline technique", *Applied Mathematics and Computation*, **181** (2006) pp. 708–720.

[40] Ghazala Akram: "Solution of the system of fifth order boundary value problem using sextic spline", *Journal of the Egyptian Mathematical Society*, **23** (2015) pp. 406–409.

[41] A. Golbaba and M. Javidi: "Homotopy perturbation method for solving eighth-order boundary value problems", *Applied Mathematics and Computation*, **191** (2007) pp. 334–346.

[42] Jafar Saberi-Nadjafi and Shirin Zahmatkesh: "Homotopy perturbation method for solving higher order boundary value problems", *Applied Mathematical and Computational Sciences*, **1** (2010)(2) pp. 199–224.

[43] Ji-Huan He: "Variational approach to the sixth-order boundary value problems", *Applied Mathematics and Computation*, **143** (2003) pp. 537–538.

[44] M. Kadalbajoo and V. Agarwal: "Fitted mesh B-spline method for solving a class of singular singularly perturbed boundary value problems", *International Journal of Computer Mathematics*, **82** (2005)(1) pp. 67–76.

[45] K. N. S. Kasi Viswanadham: *Finite Element Method for Time Dependent Problems with B-splines*, Ph.D. thesis, Indian Institute of Technology, Bombay, India (1990).

[46] K. N. S. Kasi Viswanadham: "A finite element method for driven square cavity flow problem with B-splines", in "Proceedings of 5^{th} National conference of Indian Society of Industrial and Applied Mathematics", pp. 142–147, Maulana Azad College of Technology, Bhopal, India (1998).

[47] K. N. S. Kasi Viswanadham: "Galerkin method for solving a class of singular perturbation problems with B-splines", *International Journal of Logic Based Intelligent Systems*, **1** (2007) pp. 1–13.

[48] K. N. S. Kasi Viswanadham: "Numerical solution of some singular boundary value problems by Galerkin method with B-splines", *Int. J. Sci. Comput.*, **1** (2007) pp. 105–113.

[49] K. N. S. Kasi Viswanadham and Rao S. Koneru: "Finite element method for one dimensional and two dimensional time dependent problems with B-splines", *Computer Methods in Applied Mechanics and Engineering*, **108** (1993) pp. 201–222.

[50] K. N. S. Kasi Viswanadham and P. Murali Krishna: "Septic B-spline collocation method for sixth-order boundary value problems", *ARPN Journal: Journal of Engineering ana Applications*, **5**(7).

[51] K. N. S. Kasi Viswanadham and P. Murali Krishna: "Quintic B-spline Galerkin method for fifth-order boundary value problems", *ARPN Journal of Engineering and Applied Sciences*, **5** (2010)(2) pp. 74–77.

[52] K. N. S. Kasi Viswanadham, P. Murali Krishna and Rao S. Koneru: "Numerical solutions of fourth order boundary value problems by Galerkin method with quintic B-splines", *Internatinal Journal of Nonlinear Science*, **10** (2010) pp. 222–230.

[53] K. N. S. Kasi Viswanadham and Y. Showri Raju: "Cubic B-spline collocation method for fourth-order boundary value problems", *International Journal of Nonlinear Science*, **14** (2012) pp. 336–344.

[54] K. N. S. Kasi Viswanadham and Y. Showri Raju: "Quartic B-spline Collocation method for fifth order boundary value problems", *International Journal of Computer Applications*, **43** (2012)(13) pp. 1–6.

[55] K. N. S. Kasi Viswanadham and Y. Showri Raju: "Quintic B-spline Collocation method for eighth order boundary value problems", *Advances in Computational Mathematics and its Applications*, **1** (2012) pp. 47–52.

[56] K. N. S. Kasi Viswanadham and Y. Showri Raju: "Quintic B-spline Collocation method for sixth order boundary value problems", *Global Journal of Researches in Engineering Numerical Methods*, **12** (2012) pp. 1–9.

[57] K. N. S. Kasi Viswanadham and Y. Showri Raju: "Quintic B-spline Collocation method for tenth order boundary value problems", *International Journal of Computer Applications*, **51** (2012) pp. 7–13.

[58] K. N. S. Kasi Viswanadham and Sreenivasulu Ballem: "Numerical solution of fifth-order boundary value problems by Galerkin method with quartic B-splines", *International Journal of Computer Applications*, **77** (2013)(17) pp. 7–12.

[59] K. N. S. Kasi Viswanadham and Sreenivasulu Ballem: "Numerical solution of fourth order boundary value problems by Galerkin method with cubic B-splines", *International Journal of Engineering Science and Innovative Technology*, **2** (2013) pp. 41–53.

[60] K. N. S. Kasi Viswanadham and Sreenivasulu Ballem: "Numerical solution of eighth order boundary value problems by Galerkin method with quintic B-splines", *International Journal of Computer Applications*, **89** (2014) pp. 7–13.

[61] K. N. S. Kasi Viswanadham and Sreenivasulu Ballem: "Numerical solution of sixth order boundary value problems by Galerkin method with quintic B-splines", *International Journal for Mathematical Sciences and Engineering Applications*, **8** (2014)(2) pp. 371–385.

[62] K. N. S. Kasi Viswanadham and Sreenivasulu Ballem: "Numerical solution of tenth order boundary value problems by Galerkin method with quintic B-splines", *International Journal of Applied Mathematics and Statistical Sciences*, **3** (2014) pp. 17–30.

[63] J. Kong and Y. Cheung: "Application of the spline finite strip to the analysis of shear-deformable plates", *Computers and Structures*, **46** (1993)(6) pp. 985–988.

[64] S. Kutluay and A. Esen: "A B-spline finite element method for the thermistor problem with the modified electrical conductivity", *Applied Mathematics and Computation*, **156** (2004) pp. 621–632.

[65] J. L. Lions and E. Magenes: *Non-Homogeneous Boundary Value Problems and Applications*, SpringerVerlag, Berlin (1972).

[66] A. Lamnii, H. Mraoui, D. Sbibih and A. Tijini: "Sextic spline solution of fifth order boundary value problems", *Mathematics and Computers Simulation*, **77** (2008) pp. 237–246.

[67] A. Leung and F. Au: "Spline finite elements for beam and plate", *Computers and Structures*, **37** (1990)(5) pp. 717–729.

[68] G. R. Liu and T. Wu: "Differential quadrature solutions of eighth order boundary value problems", *Journal of Computational and Applied Mathematics*, **145** (2002) pp. 223–235.

[69] G. B. Loghmani and M. Ahmadinia: "Numerical solution of sixth order boundary value problem with sixth degree B-spline functions", *Applied Mathematics and Computation*, **186** (2007) pp. 992–999.

[70] Luma N.M.Tawfiq and Samaher M.Yassien: "Solution of higher order boundary value problems using semi-analytic technique", *Ibn Al-Haitham Journal for Pure and Applied Science*, **26** (2013)(1) pp. 281–291.

[71] P. M. Prenter: *Splines and Variational Methods*, John-Wiley and Sons, New York (1989).

[72] Manoj Kumar and Pankaj Kumar Srivastava: "Computational techniques for solving differential equations by cubic, quintic and sextic splines", *International Journal for Computational Methods in Engineering Science and Mechanics*, **10** (2009) pp. 108–115.

[73] Marzieh Dosti and Alireza Nazemi: "Septic B-spline collocation method for solving one-dimensional hyperbolic telegraph equation", *World Academy of Science, Engineering and Technology*, **56** (2011) pp. 1085–1089.

[74] L. Mingzhu Li and Qiang Ma: "The numerical solution of linear sixth order boundary value problems with quartic B-splines", *Journal of Applied Mathematics*, **Article ID 962165** (2013) pp. 1–7.

[75] A. R. Mitchel and R. Wait: *The Finite Element Method in Partial Differential Equations*, John Wiley and Sons, London (1977).

[76] R. C. Mittal and R. K. Jain: "Numerical solutions of nonlinear Burgers' equation with modified cubic B-splines collocation method", *Applied Mathematics and Computation*, **218**.

[77] R. C. Mittal and R. K. Jain: "Cubic B-splines collocation method for solving nonlinear parabolic partial differential equations with neumann boundary conditions", *Commun Nonlinear Sci Numer Simulat*, **17** (2012) pp. 4616–4625.

[78] T. Mizusawa: "Application of the spline element method to analyze the bending of skew plates", *Computers and Structures*, **53** (1994)(2) pp. 439–448.

[79] Mladen Mestrovic: "The modified decomposition method for eighth-order boundary value problems", *Applied Mathematics and computation*, **188** (2007) pp. 1437–1444.

[80] Mohamed El-Gamel: "Sinc and the numerical solution of fifth order boundary value problems", *Applied Mathematics and Computation*, **187** (2007) pp. 1417–1433.

[81] Muhammad Aslam Noor, Khalida Inayat Noor and Syed Tauseef Mohyd-Din: "Variational iteration method for solving sixth order boundary value problems", *Commun Nonlinear Sci Numer Simulat*, **14** (2009) pp. 2571–2580.

[82] Muhammad Aslam Noor and Syed Tauseef Mohyud-Din: "An efficient algorithm for solving fifth-order boundary value problems", *Mathematical and Computer Modelling*, **45** (2007) pp. 954–964.

[83] Muhammad Aslam Noor and Syed Tauseef Mohyud-Din: "Variational iteration decomposition method for solving eighth-order boundary value problems", *Differential equations and Nonlinear Mechanics*, **Article id: 19529** (2007) pp. 1–16.

[84] Muhammad Aslam Noor and Syed Tauseef Mohyud-Din: "Variational iteration method for fifth order boundary value problems using He's ploynomials", *Mathematical Problems in Engineering*, **Article ID: 9547912** (2008).

[85] Muhammad Azam Khan, Siraj-ul-Islam, Ikram A.Tirmizi, E. H. Twizell and Saadat Ashraf: "A class of methods based on non-polynomial spline functions for the solution of a special fifth order boundary value problems", *Journal of Mathematical Analysis and Applications*, **321** (2006) pp. 651–660.

[86] Muhammed Aslam Noor and Syed Tauseef Mohyud-Din: "Variational iteration technique for solving higher order boundary value problems", *Applied Mathematics and Computation*, **189** (2007) pp. 1929–1942.

[87] Muhammed I. Syam and Basem S.Ahili: "Numerical solution of singularly perturbed fifth order two point boundary value problem", *Applied Mathematics and Computation*, **170** (2005) pp. 1085–1094.

[88] Mustafa Inc and Ali Akgul: "Numerical solution of seventh-order boundary value problems by a novel method", *Abstract and Applied Analysis*, **Article Id 745287** (2014) pp. 1–9.

[89] J. N. Reddy: *An Introduction to the Finite Element Method*, Tata Mc-Graw Hill Publishing Company Limited, 3^{rd} Edition, New Delhi (2005).

[90] N. Papamichael and A. J. Worsey: "A cubic spline method for the solution a linear fourth order two point boundary value problem", *Journal of Computational and Applied Mathematics*, **7** (1981) pp. 187–189.

[91] D. Pullman and J. Schaff: "A comparison of 3-d spline variational and finite element solutions for a cross-ply laminate with a circular hole", *Mechanics of Composite Materials and Structures*, **5** (1998) pp. 309–325.

[92] M. A. Ramadan, I. F. Lashien and W. K. Zahra: "A class of methods based on septic non-polynomial spline function for the solution of sixth-order two-point boundary value problems", *International Journal of Computer Mathematics*, **85** (2008)(5) pp. 759–770.

[93] M. A. Ramadan, I. F. Lashien and W. K. Zahra: "Quintic nonpolynomial spline solutions for fourth order two-point boundary value problem", *Communications in Nonlinear Science and Numerical Simulation*, **14** (2009) pp. 1105–1114.

[94] J. Rashidinia and M. Ghasemi: "B-spline collocation for solution of two-point boundary value problems", *Journal of Computational and Applied Mathematics*, **235** (2011) pp. 2325–2342.

[95] J. Rashidinia, R. Jalilian and K. Farajeyam: "Spline approximate solution of fifth order boundary value problem", *Applied Mathematics and Computation*, **192** (2007) pp. 107–112.

[96] J. Rashidinia, R. Jalilian and K. Farajeyan: "Spline approximate solution of eighth-order boundary value problems", *International Journal of Computer Mathematics*, **86** (2009) pp. 1319–1333.

[97] Rasoul Abazari and Reza Abazari: "Numerical solution of the rosenau equation using quintic collocation B-spline method", *Iranian Journal of Science & Technology*, **39** (2015)(A3) pp. 281–288.

[98] Reza Mohammadi: "Sextic B-spline collocation method for solving Euler-Bernoulli beam models", *Applied Mathematics and Computation*, **241** (2014) pp. 151–166.

[99] G. Richards and P. Sarma: "Reduced order models for induction motors with two rotor circuits", *IEEE Transactions on Energy Conversion*, **9** (1994)(4) pp. 673–678.

[100] R. Riesenfeld: *Application of B-spline Approximation to Geometric Problems of Computer Aided Design*, Ph.D. thesis, Syracuse University, Syracuse, New York (1975).

[101] B. Saka: "A quintic B-spline finite-element method for solving the nonlinear Schrodinger's equation", *Physics of Wave Phenomena*, **20** (2012)(2) pp. 107–117.

[102] Samir Kumar Bhowmik: "Tchebychev polynomial approximations for m^{th} order boundary value problems", *International Journal of Pure and Applied Mathematics*, **98** (2015)(1) pp. 45–63.

[103] Samuel Jator and Zachariah sinkala: "A high order B-spline collocation method for linear boundary value problems", *Applied Mathematics and Computation*, **191** (2007) pp. 100–116.

[104] I. J. Schoenberg: *On Spline Functions*, MRC Report 625, University of Wisconsin (1966).

[105] Seydi Battal Gazi Karakoc and Turabi Geyikli: "A numerical solution of the MEW equation using sextic B-splines", *Journal of Advanced Research in Applied Mathematics*, **5** (2013)(3) pp. 51–65.

[106] Shaher Momani and Muhammed Aslam Noor: "Numerical comparision of methods for solving a special fourth-order boundary value problem", *Applied Mathematics and Computation*, **191** (2007) pp. 218–224.

[107] Shahid S. Siddiqi, Ghazala Akram and Saima Nazeer: "Quintic spline solution of linear sixth-order boundary value problems", *Applied Mathematics and Computation*, **189** (2007) pp. 887–892.

[108] Shahid S. Siddiqi and Saima Arshed: "Quintic B-spline for the numerical solution of fourth-order parabolic partial differential equations", *World Applied Sciences Journal*, **23** (2013)(12) pp. 115–122.

[109] Shahid S.Siddiqi and E. H. Twizell: "Spline solutions of linear eighth-order boundary value problems", *Computer Methods in Applied Mechanics and Engineering*, **131** (1996) pp. 309–325.

[110] Shahid S.Siddiqi and Ghazala Akram: "Solutions of tenth order boundary value problems using eleventh degree spline", *Applied Mathematics and Computation*, **185** (2007) pp. 115–127.

[111] Shahid S.Siddiqi and Ghazala Akram: "Solution of 10^{th} order boundary value problems using nonpolynomial spline technique", *Applied Mathematics and Computation*, **190** (2007) pp. 641–651.

[112] Shahid S.Siddiqi and Ghazala Akram: "Septic spline solutions of sixth-order boundary value problems", *Journal of Computational and Applied Mathematics*, **215** (2008) pp. 288–301.

[113] Shahid S.Siddiqi, Ghazala Akram and Muzammal Iftikhar: "Solution of seventh order boundary value problem by differential transformation method", *World Applied Sciences Journal*, **16** (2012)(11) pp. 1521–1526.

[114] Shahid S.Siddiqi, Ghazala Akram and Muzammal Iftikhar: "Solution of seventh order boundary value problems by variational iteration technique", *Applied Mathematical Sciences*, **6** (2012)(94) pp. 4663–4672.

[115] Shahid S.Siddiqi, Ghazala Akram and Muzammal Iftikhar: "Variational iteration method for the solution of seventh-order boundary value problems using He's polynomials", *Journal of the Association of Arab Universities for Basic and Applied Sciences*, (2014) p. http://dx.doi.org/10.1016/j.jaubas.2014.03.001.

[116] Shahid S.Siddiqi, Ghazala Akram and S. Zaheer: "Spline solutions of linear tenth-order boundary value problems", *International Journal of Computer Mathematics*, **68** (19989) pp. 345–362.

[117] Shahid S.Siddiqi and Muzammal Iftikhar: "Solution of seventh-order boundary value problems by the Adomian decomposition method", *http://arxiv.org/abs/1301.3603V1.*

[118] Shahid S.Siddiqi and Muzammal Iftikhar: "Numerical solution of higher order boundary value problems by homotopy analysis method", *Abstract and Applied Analysis*, **Article ID: 427521** (2013) pp. 1–12.

[119] Shahid S.Siddiqi and Muzammal Iftikhar: "Solution of seventh order boundary value problems by variation parameters method", *Research Journal of Applied Sciences, Engineering and Technology*, **5** (2013)(1) pp. 176–179.

[120] Shahid S.Siddiqi and Muzammal Iftikhar: "Variational iteration homotopy perturbation method for the solution of seventh-order boundary value problems", *http://arxiv.org/abs/1310.2915V1*, (2013).

[121] S. S. Siddiqi, Ghazala Akram and Arfa Elahi: "Quartic spline solution of linear fifth order boundary value problems", *Applied Mathematics and Computation*, **196**.

[122] S. S. Siddiqi and Ghazala Akram: "Solution of fifth order boundary value problems using non-polynomial spline technique", *Applied Mathematics and Computation*, **175** (2006) pp. 1574–1581.

[123] S. S. Siddiqi and Ghazala Akram: "Sextic spline solutions of fifth order boundary value problems", *Applied Mathematics Letters*, **20** (2007) pp. 591–597.

[124] S. S. Siddiqi, Ghazala Akram and and Salman A.Malik: "Nonploynomial sextic spline method for the solution along with convergence of linear special case fifth order two-point boundary value problems", *Applied Mathematics and Computation*, **190** (2007) pp. 532–541.

[125] Sirajal Haq, M. Idrees and S. Islam: "Application of optimal homotopy asymptotic method to eighth order initial and boundary value problems", *International Journal of Applied Mathematics and Computation*, **2** (2010) pp. 73–80.

[126] A. A. Soliman: "A galerkin solution for Burgers' equation using cubic B-spline finite elements", *Abstract and Applied Analysis*, **Article ID 527467**.

[127] Songxin Liang and J. David Jefferey: "Approximate solutions to a parameterized sixth order boundary value problem", *Computers and Mathematics with Applications*, **59** (2010) pp. 247–253.

[128] P. K. Srivastava, M. Kumar and R. N. Mohapatra: "Solution of fourth order boundary value problems by numerical algorithms based on nonpolynomial quintic splines", *Journal of Numerical Mathematics and Stochastics*, **4** (2012) pp. 13–25.

[129] Syed Tauseef Mohyud-Din and Ahmet Yildirim: "Solution of tenth and ninth order boundary value problems by homotopy perturbation method", *J.KSIAM*, **14** (2010)(1) pp. 17–27.

[130] Syed Tauseef Mohyud-Din and Ahmet Yildirim: "Solutions of tenth and ninth order boundary value problems by modified variational iteration method method", *Applications and Applied Mathematics: An International Journal (AAM)*, **5** (2010)(1) pp. 11–25.

[131] Syed Tayseef Moyud-Din and Muhammad Aslam noor: "Homotopy perturbation method for solving fourth- order boundary value problems", *Mathematical Problems in Engineering*, **Article ID: 98602** (2007).

[132] A. O. Taiwo and D. J. Evans: "Collocation approximation for fourth-order boundary value problems", *International Journal of Computer Mathematics*, **63** (1997) pp. 57–66.

[133] S. Talaat and Aly El-Danaf: "Septic B-spline method of the Korteweg-de Vries Burgers' equation", *Communications in Nonlinear Science and Numerical Simulation*, **13** (2008) pp. 554–566.

[134] J. Toome, J. R. Zahn, J. Latour and E. A. Spiegel: "Stellar convection theory ii: Single-mode study of the second convection zone in A-type stars", *Astro Physics*, **207** (1976) pp. 545–563.

[135] Turabi Geyikli and Seydi Battal Gazi Karakoc: "Septic B-spline collocation method for the numerical solution of the modified equal width wave equation", *Applied Mathematics*, **2** (2011) pp. 739–749.

[136] E. H. Twizell and A. Boutayeb: "Numerical methods for eighth, tenth and twelfth order eigen value problems arising in thermal instability", *Advances in Computational Mathematics*, **2** (1994) pp. 407–436.

[137] Vedat Suat Erturk and Shaher Momani: "Comparing numerical methods for solving fourth-order boundary value problems", *Applied Mathematics and Computation*, **188** (2007) pp. 1963–1968.

[138] Vedat Suat Erturk and Shaher Momani: "A reliable algorithm for solving tenth-order boundary value problems", *Numer Algor*, **44** (2007) pp. 147–158.

[139] Waleed Al-Hayani and Luis casasus: "Approximate anlytical solution of fourth order boundary value problems", *Numerical Algorithms*, **40** (2005) pp. 67–78.

[140] Waleed-i-Haq: "Adomian decomposition method with green's function for sixth order boundary value problems", *Computers and Mathematics with Applications*, **61** (2011) pp. 1567–1575.

[141] Zhao-Chunwu: "Approximate analytical solutions of fifth order boundary value problems by the variational iteration method", *Computer and Mathematics with Applications*, **58** (2009).

List of papers published / communicated

Papers published

(1) "Numerical solution of fourth order boundary value problems by Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions", *International Journal of Computer Applications*, Vol. 104(6), pp:37–43, 2014.

(2) "Numerical solution of fifth order boundary value problems by Petrov-Galerkin method with cubic B-splines as basis functions and quintic B-splines as weight functions", *International Journal of Computer Science and Electronics Engineering*, Vol. 3(1), pp:87–91, 2015.

(3) "Numerical solution of fifth order boundary value problems by Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions", *International Journal of Engineering Science and Innovative Technology*, Vol. 4(1), pp:161-170, 2015.

(4) "Numerical solution of sixth order boundary value problems by Petrov-Galerkin method with quartic B-splines as basis functions and sextic B-splines as weight functions", *ARPN Journal of Engineering and Applied Sciences*, Vol. 10(10), pp:4720–4727, 2015.

(5) "Numerical solution of sixth order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions", *International Journal of Applied Mathematics & Statistical Sciences*, Vol. 4(3), pp:15–28, 2015.

(6) "Numerical solution of seventh order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions", *International Journal of Computer Applications*, Vol. 122(5), pp:41–47, 2015.

(7) "Numerical solution of ninth order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions", *Procedia Engineering*, Vol. 127, pp:1227–1234, 2015.

Papers Communicated

(1) "Numerical solution of eighth order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions", **Communicated to** *Israel Journal of Mathematics.*

(2) "Numerical solution of tenth order boundary value problems by Petrov-Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions", **Communicated to** *Studies in Applied Mathematics.*