CERTAIN RESULTS ON NONNEGATIVE GENERALIZED INVERSES AND SPLITTINGS OF MATRICES

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Dedicated to

My Parents

My Wife

and

My Teachers

CERTIFICATE

This is to certify that the thesis entitled **"CERTAIN RESULTS ON NON-NEGATIVE GENERALIZED INVERSES AND SPLITTINGS OF MATRICES"**, submitted to the Department of Mathematics, National Institute of Technology, Warangal, is a record of bonafide research work carried out by **Mr. KUSUMA APPI REDDY**, Roll No. 701370, for the award of Degree of Doctor of Philosophy in Mathematics under my supervision. The contents of the thesis have not been submitted elsewhere for the award of any degree or diploma.

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DECLARATION

This is to certify that the work presented in the thesis entitled **"CERTAIN RESULTS ON NONNEGATIVE GENERALIZED INVERSES AND SPLITTINGS OF MATRICES"**, is a bonafide work done by me under the supervision of **Dr. T. KURMAYYA** and has not been submitted elsewhere for the award of any degree.

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ABSTRACT

Keywords: Nonnegative matrix, Group inverse, Moore-Penrose inverse, proper splitting, $B_{\#}$-splitting, double splitting, $\{T, S\}$ splitting, convergence theorem, comparison theorem, Indefinite inner product space, Gram matrix, acute cone.

The main objective of our thesis is to extend some of the results characterizing nonnegativity of classical inverses to nonnegativity of generalized inverses and also to study applications of these extension results in obtaining certain convergence and comparison results of iterative methods which are derived from matrix splittings to solve system of linear equations . There are six chapters in our thesis.

Chapter 1 is introductory. It presents a brief overview of nonnegative matrices, generalized inverses, matrix splittings, monotonicity and indefinite inner product spaces together with some preliminary results associated with these notions that will be used in the subsequent chapters.

Chapter 2 is devoted to study group inverse and Moore-Penrose inverse extensions of certain M-matrix properties. Almost all the results in this chapter are proved using the notion of proper splittings of some type or the other. As applications of these results, some comparison theorems for spectral radii of certain matrices are derived.

Chapter 3 deals with proper double splittings of rectangular matrices. It presents comparison results for the spectral radii of iteration matrices of corresponding iterative schemes which are formulated by using double splittings. The results of this chapter are generalizations of comparison results for ordinary inverses to Moore-Penrose inverses.

In Chapter 4, we consider $\{T, S\}$ splittings of rectangular matrices and derive certain convergence theorems and comparison theorems for iterative schemes. The results of this chapter include outer inverses (or $\{2\}$-inverses) with prescribed range and null space.

In Chapter 5, we deal with indefinite inner product spaces and characterize cone nonnegativity of Moore-Penrose inverses of Gram matrices. This characterization is done by using the acuteness (or obtuseness) of certain closed convex cones.

Finally, we summarize the contents of our thesis and present a list of references.

Contents

LIST OF NOTATIONS

For easy reference, notations that are frequently used in this thesis are given below:

Sets

$\in$	Element membership.
$\subseteq$	Set inclusion.
$\cap$	Set intersection
$\oplus$	Direct sum.
$\perp$	Orthogonal.
$S^{\perp}$	Orthogonal complement of a set S.

Spaces

$\mathbb{R}$	The space of all real numbers.
$\mathbb{R}^n$	The space of all real vectors with n co-ordinates.
$\mathbb{R}^n_+$	The set of all nonnegative vectors in $\mathbb{R}^n$.
$\mathbb{R}^{m\times n}$	The set of all real matrices of order $m \times n$.
$\mathbb{R}^{n\times n}$	The set of all real square matrices of order n.
K	A cone.
K^*	The dual cone of K.
$K^{[*]}$	The dual of a cone K in an indefinite inner product space.

Operators and Matrices

I	The identity operator.
$P_{M,N}$	Projection onto the space M along N.
P_M	Orthogonal projection onto M.
$rank(A)$	Rank of the matrix A.
$ind(A)$	The index of a square matrix A .
$A \geq 0$	All entries of A are nonnegative.
$N(A)$	The null space of A.
$R(A)$	The range space of A.
$\rho(A)$	The spectral radius of A

A^t	The transpose of A.
$\mathcal{R}(A)$	The range space of A with respect to the indefinite matrix product.
$\mathcal{N}(A)$	The null space of A with respect to the indefinite matrix product.
$A^{[*]}$	The adjoint of A in indefinite inner product space.
$A^{[*]} \circ A$	Gram matrix of A with respect to the indefinite matrix product in an indefinite inner product space..
$A^{(2)}$	$\{2\}$-inverse.
$A^{(2)}_{T,S}$	$\{2\}$-inverse with range T and null space S.
$A^{\dagger}$	The Moore-Penrose inverse of A.
$A^{\#}$	The group inverse of A
A^{-1}	The inverse of A
$A^{(\dagger)}$	Moore-Penrose inverse of A in an indefinite inner product space with respect to the usual matrix product
$A^{[\dagger]}$	Moore-Penrose inverse of A in an indefinite inner product space with respect to the indefinite matrix product

Operations

$\langle ., . \rangle$	The usual Euclidean inner product
$[., .]$	Indefinite inner product
$\|\|.\|\|$	A norm

CHAPTER 1

INTRODUCTION

In this chapter we present a brief survey of the relevant literature, some basic definitions and preliminary results that will be used in other chapters of our thesis. We split this chapter into seven sections. In Section 1.1, we introduce notion of nonnegative matrix and discuss well known theorem namely Perron-Frobenius theorem. Also, we present some results related to nonnegative matrices. In Section 1.2, we recall the definition of M-matrix and characterize nonnegativity of inverse of an invertible M-matrix. In Section 1.3, we collect results related to matrix monotonicity and demonstrate how the matrix splittings are useful in its characterization. In Section 1.4, we define various types of generalized inverses and study some of their properties. In Section 1.5, we review some of the important extensions of matrix monotonicity. In Section 1.6, we discuss some basic definitions and fundamental results in an indefinite inner product space. The last section of this chapter summarises the contents of thesis.

1.1 Nonnegative matrices

A matrix $A \in \mathbb{R}^{m\times n}$ is called *nonnegative* if all the entries of A are nonnegative. It is called *positive* if all the entries of A are positive. These two classes of matrices play an important role in many problems of pure and applied mathematics. In particular, these matrices appear in many areas such as Numerical analysis, Graph theory, Economics, Statistics, Optimization and Partial differential equations, to name a few (see [2], [11], [39] and [41]). We next recall the definition of an irreducible matrix which is important in the theory of nonnegative matrices.

Definition 1.1.1. *(Definition 1.15, [69]) "A matrix $A \in \mathbb{R}^{n\times n}$ is reducible if there is a permutation matrix P such that $P^tAP = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$, where the submatrices A_{11}, A_{22} are square. A matrix A is called irreducible if it is not reducible".*

The concepts of reducible and irreducible nonnegative matrices, and directed graphs have connections with the theory of Markov chain (see [32]). In 1907, Perron discovered some remarkable properties of positive square matrices. Later, the work was generalized by Frobenius who extended Perron's results to nonnegative matrices. Since then the theory of nonnegative matrices has been one of the active areas in linear algebra. It has found many applications in various parts of mathematical and physical sciences. We now recall some definitions and fix some notations to discuss salient aspects of the Perron-Frobenius theory.

"The *spectral radius* $\rho(A)$ of $A \in \mathbb{R}^{n\times n}$ is the maximum of the moduli of the eigenvalues of A". As mentioned earlier, a matrix A is called *nonnegative* if all the entries of A are nonnegative; this is denoted by $A \geq 0$. It is called *positive* if all the entries of A are positive; this is denoted by $A > 0$. For $A, B \in \mathbb{R}^{m\times n}$, the notation $A \leq B$ means that $B - A \geq 0$. A vector $x \in \mathbb{R}^n$ is called *nonnegative* and is denoted by $x \geq 0$ if all its coordinates are nonnegative; x is called *positive* if all its coordinates are positive and this will be denoted by $x > 0$. Let $int(\mathbb{R}^n_+)$ denote the set of all interior points of $\mathbb{R}^n_+$. In view of this, if x is positive, sometimes, we also denote that by $x \in int(\mathbb{R}^n_+)$.

Let us now discuss the salient aspects of the Perron-Frobenius theory. Let B be a matrix with all entries positive. Perron showed that $\rho(B)$ is an eigenvalue of B and that it is simple, viz., the eigenspace is one dimensional. He also proved that there exists a *unique positive* vector associated with this eigenvalue which is referred to as the *Perron vector*. Now, let B be a nonnegative matrix with at least one zero entry. Then it is known that $\rho(B)$ is again, an eigenvalue (but could be zero) and that there is an associated eigenvector whose entries are all *nonnegative*. Furthermore, if B is nonnegative and irreducible, then $\rho(B) > 0$, is a simple eigenvalue of B and there exists a positive eigenvector corresponding to $\rho(B)$. For proofs of these statements and other relevant details, we refer to the excellent books [39] and [69].

The following theorem is a part of the Perron-Frobenius theorem which will be used frequently in our thesis.

Theorem 1.1.1. *(Theorem 2.20, [69]) Let A be a real square nonnegative matrix. Then we have the following:*

(i) A has a nonnegative real eigenvalue equal to the spectral radius.

(ii) There exists a nonnegative eigenvector for its spectral radius.

The next result is a part of Frobenius' extension of Perron's theorem.

Theorem 1.1.2. *(Theorem 2.7, [69]) Let A be a real square nonnegative irreducible matrix. Then we have the following.*

(i) A has a positive real eigenvalue equal to the spectral radius.

(ii) There exists a positive eigenvector for its spectral radius.

Next, we present some results connecting nonnegativity of a matrix and its spectral radius.

Lemma 1.1.1. *(Theorem 2.1.11, [11]) "Let $A \geq 0$. Then $\alpha x \leq Ax$, $x \geq 0 \Rightarrow \alpha \leq \rho(A)$ and $Ax \leq \beta x$, $x > 0 \Rightarrow \rho(A) \leq \beta$".*

Theorem 1.1.3. *(Theorem 3.15, [69]) "Let $B \in \mathbb{R}^{n \times n}$ and $B \geq 0$. Then $\rho(B) < 1$ if and only if $(I - B)^{-1}$ exists and $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \geq 0$".*

We conclude this section with the following comparison result for spectral radii of two nonnegative matrices.

Theorem 1.1.4. *(Theorem 2.21, [69]) "Let $A, B \in \mathbb{R}^{n \times n}$. If $A \geq B \geq 0$, then $\rho(A) \geq \rho(B)$".*

1.2 M-matrices

This short section deals with an important class of matrices namely M-matrices. It is proved in the literature that an invertible member of this class always has a nonnegative inverse. The name M-matrix was originally chosen by Alexander Ostrowski in reference to Hermann Minkowski. First, he introduced the class of Z-matrices and then defined the class of M-matrices. The definition of a Z-matrix is as follows.

Definition 1.2.1. *A matrix $A \in \mathbb{R}^{n\times n}$ is called as a Z-matrix if the off-diagonal entries are nonpositive. A Z-matrix A can be written as $A = sI - B$, where $s \geq 0$ and $B \geq 0$.*

The class of Z-matrices is well studied in different contexts such as finite difference methods for partial differential equations, input-output production and growth models in economics, Markov processes in probability and statistics, and linear complementarity problems in operations research. Now using the above definition, Ostrowski in 1937 gave the following definition for M-matrices which is a subclass of Z-matrices.

Definition 1.2.2. *A Z-matrix $A = sI - B$ is called an M-matrix if $s \geq \rho(B)$.*

It is well known that if $s > \rho(B)$ in the representation described above, then A is invertible and $A^{-1} \geq 0$. Thus, each member in the class of invertible M-matrices has a nonnegative inverse. Also, it is pertinent to point out the fact that if A is an invertible M-matrix with the usual representation $A = sI - B$, then $\rho(\frac{1}{s}B) < 1$. In fact, there are many interesting characterizations of invertible M-matrices. The book by Berman and Plemmons [11] records more than fifty equivalent conditions. For our purpose, we recall the following result.

Theorem 1.2.1. *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix with the representation $A = sI - B$. Then the following statements are equivalent:*

(a) *A is invertible and $A^{-1} \geq 0$.*

(b) *There exists x such that all the entries of x and Ax are positive.*

(c) *A is an M-matrix with $s > \rho(B)$.*

1.3 Splittings and monotonicity

A square real matrix A is called *monotone* if $Ax \geq 0$ implies $x \geq 0$. The concept of monotonicity was first proposed by Collatz (see [16], for instance), in connection with the application of finite difference methods for solving elliptic partial differential equations. He showed that a matrix is monotone if and only if it is invertible and the inverse is entrywise nonnegative. A matrix satisfying the later condition is also called an *inverse positive* matrix. Hence, monotonicity of a matrix is equivalent to inverse positivity. The notion of monotonicity has been extended in a great variety of ways. Traditionally, splittings of matrices have been used in studying these extensions. For $A \in \mathbb{R}^{n\times n}$, a decomposition $A = U - V$, where U is nonsingular, is referred to as a *splitting* of A (however, throughout this thesis, for us a splitting simply means a decomposition $A = U - V$). With such a splitting, one associates iterative scheme of the following form:

$$x^{(k+1)} = Hx^{(k)} + c, \tag{1.1}$$

where $H = U^{-1}V$ is called the iteration matrix and $c = U^{-1}b$, for a nonnegative integer k and given an initial vector x^0. It is well known that this sequence converges to the unique solution of the system $Ax = b$ (irrespective of the choice of the initial vector x^0) if and only if $\rho(H) < 1$. Many of the results in the literature show that for certain kinds of splittings $\rho(H) < 1$ if and only if A is monotone (or inverse positive). This establishes a connection between monotonicity and convergence of an iterative scheme. It is well known that standard iterative methods like the Guass-Jacobi, Guass-Seidal and successive over-relaxation methods arise from different choices of U and V. For more details one could refer to the books [11] and [69].

Next, we recall the definitions of regular splitting and weak regular splitting to review some results related to matrix monotonicity.

Definition 1.3.1. *"Let $A \in \mathbb{R}^{n\times n}$. A splitting $A = U - V$ where U is invertible, $U^{-1} \geq 0$ and $V \geq 0$ is called a regular splitting".*

This was proposed by Schroder and Varga [69], among others and it was shown that A is monotone if and only if for any regular splitting $A = U - V$, one has $\rho(U^{-1}V) < 1$.

Definition 1.3.2. *"Let $A \in \mathbb{R}^{n\times n}$. A splitting $A = U - V$ where U is invertible, $U^{-1} \geq 0$ and $U^{-1}V \geq 0$ is called a weak regular splitting".*

This was proposed by Ortega and Rheinboldt [49]. (Clearly, any regular splitting is a weak regular splitting). They showed that A is monotone if and only if for any weak regular splitting $A = U - V$, one has $\rho(U^{-1}V) < 1$. These results again show, the importance of splittings and monotonicity in the study of convergence of iterative schemes of the above form.

Peris extended the Theorem 1.2.1 to a more general class of matrices. These matrices possesses a splitting called a B-splitting defined next.

Definition 1.3.3. *(Definition 1, [51]) A splitting $A = U - V$ of $A \in \mathbb{R}^{n\times n}$ is called a B-splitting if it satisfies the following conditions.*

(i) $U \geq 0$.

(ii) $V \geq 0$.

(iii) $VU^{-1} \geq 0$.

(iv) $Ax, Ux \geq 0 \Longrightarrow x \geq 0$.

Using this splitting Peris obtained the following result.

Theorem 1.3.1. *(Theorem 4, [51]) Let A be a square matrix such that $A = U - V$ is a B-splitting. Then the following conditions are equivalent:*
(a) *A is invertible and $A^{-1} \geq 0$.*
(b) *There exists some $x \geq 0$ and $Ax > 0$.*
(c) $\rho(VU^{-1}) < 1$.

The existence of B-splitting was also studied by Peris.

Theorem 1.3.2. *(Theorem 5, [51]) A^{-1} exists and $A^{-1} \geq 0$ if and only if A allows a B-splitting $A = U - V$ with $\rho(VU^{-1}) < 1$.*

Weber then extended the above results to operators on infinite dimensional spaces. We present the finite dimensional version of his result (Theorem 4, [70]).

Theorem 1.3.3. *"Let $A \in \mathbb{R}^{n \times n}$. Consider the following statements:*
(a) *A^{-1} exists and $A^{-1} \geq 0$.*
(b) $Ax \geq 0 \Longrightarrow x \geq 0$.
(c) $\mathbb{R}^n_+ \subseteq A\mathbb{R}^n_+$.
(d) *There exists $x \in \mathbb{R}^n_+$ and such that $Ax \in int(\mathbb{R}^n_+)$.*
Then we have $(a) \Leftrightarrow (b) \Longrightarrow (c) \Longrightarrow (d)$.
Suppose that A has a B-splitting $A = U - V$. Then each of the above statements is equivalent to the following condition:
(e) $\rho(VU^{-1}) < 1$*".*

The results which we have discussed till now in this section are related to convergence of the iterative scheme 1.1. On the other hand, if the matrix A has two decompositions or splittings then the comparison of the spectral radius of the corresponding iteration matrices, is an important problem in analyzing the iterative scheme $x^{(k+1)} = Hx^{(k)} + c$ of the system $Ax = b$. It is well known that the iterative scheme with smaller spectral radius will converge faster. The comparison of asymptotic rates of convergence of the iterative schemes induced by two splittings of a given matrix has been studied by many authors. For details of these results one could refer to ([6] to [11], [20], [36], [65], [69], [75], [76] and [77]).

Now, we turn our focus on to double splittings of a matrix. For $A \in \mathbb{R}^{n\times n}$, a decomposition $A = P - R + S$, where P is nonsingular, is called double splitting of A. This notion was introduced by *Woźnicki* [74]. Analogous to the earlier discussion, with such a splitting, the following iterative scheme was formulated to solve $Ax = b$:

$$x^{(k+1)} = P^{-1}Rx^{(k)} - P^{-1}Sx^{(k-1)} + P^{-l}b, \qquad k = 1, 2, 3... \tag{1.2}$$

Following the idea of Golub and Varga [22] , *Woźnicki* wrote equation (1.2) in the following equivalent form:

$$\begin{pmatrix} x^{k+1} \\ x^{k} \end{pmatrix} = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^{k} \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ 0 \end{pmatrix},$$

where I is the identity matrix. Then, it was shown that the iterative method (1.2) converges to the unique solution of $Ax = b$ for all initial vectors x^0 , x^1 if and only if the spectral radius of the iteration matrix

$$W = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix}$$

is less than one, that is $\rho(W) < 1$.

In recent years, several comparison theorems have been proved for double splittings of matrices. We briefly review few of them here. First, let us recall the definitions of regular and weak regular double splittings.

Definition 1.3.4. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Then, the double splitting $A = P - R + S$ is*
(i) regular double splitting if $P^{-1} \geq 0$, $R \geq 0$ and $-S \geq 0$.
(ii) weak regular double splitting if $P^{-1} \geq 0$, $P^{-1}R \geq 0$ and $-P^{-1}S \geq 0$.

Shen and Huang [60] have considered regular and weak regular double splittings of a monotone matrix or Hermitian positive definite matrix and obtained some comparison theorems. Miao and Zheng [40] have obtained comparison theorem for the spectral radii of matrices obtained from double splitting of different monotone matrices. Song and Song [64] have studied convergence and comparison theorems for nonnegative double splittings of a real square nonsingular matrices. Li and Wu [34] have obtained some comparison theorems for double splittings of a matrix.

Throughout this section, we have considered real nonsingular matrices and characterized the matrix monotonicity and discussed convergence and comparison of iterative schemes derived from various matrix splittings. In Section 1.5, we will collect extensions of these results for singular matrices and rectangular matrices. For that, we need various types of generalized inverses. Hence, we collect results related to these notions in the next section.

1.4 Generalized inverses

If the given matrix A is not invertible then we look for generalized inverse of A. The concept of generalized inverse has been studied by several mathematicians. E. H. Moore [46] was the first to give an explicit definition of the generalized inverse of an arbitrary matrix. This definition was given by Moore in an abstract published in the Bulletin of the American Mathematical Society in 1920, whose significance was not realized for the concept remain undeveloped for decades there after. He studied some of its main properties and its applications to linear system of equations. He also showed that it is unique (see [46] and [47]). He obtained the following theorem.

Theorem 1.4.1. *(29.3, part 1, P.174, [47]) "For every matrix $A \in \mathbb{R}^{m\times n}$, there exists a matrix $X : R(A) \to R(A^t)$ such that $AX = P_{R(A)}$ and $XA = P_{R(A^t)}$".*

However, this work did not come to prominence during the period 1920-1950. Unaware of Moore's work Penrose [50] rediscovered the concept of generalized inverse (or pseudo inverse) of a matrix. The equivalence of the Moore's definition and Penrose's definition was pointed out by Rado [52]. Various important contributions in 1950's were also made by Bjerhammer, Greville, Rao and others. For a history of generalized inverses one can refer to an excellent survey by Ried [59], Ben-Israel [4], and the historical note in the book by Ben-Israel and Greville [5]. For a detailed study of generalized inverses and applications, we mention the books by Rao and Mitra [58], Groetsch [23], Campbell and Meyer [12] and Ben-Israel and Greville [5]. The Penrose definition [50] for the generalized inverse of a given matrix is presented next.

Definition 1.4.1. *For a given matrix $A \in \mathbb{R}^{m\times n}$, the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying $AXA = A$, $XAX = X$, $(AX)^t = AX$ and $(XA)^t = XA$ is called the Moore-Penrose inverse of A and it is denoted by $A^\dagger$.*

The Moore-Penrose inverse always exists and is unique. There are also other equivalent definitions for $A^\dagger$ [23]. A definition of $A^\dagger$ (Definition (D-W), p-5), [23] that was proved by Desoer and Whalen(1963) is given next. This will be used in some of the proofs in our thesis.

Theorem 1.4.2. *(Theorem 2.2.2, [23]) For $A \in \mathbb{R}^{m\times n}$, $A^\dagger$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying*

$$XAx = x \ for \ all \ x \in R(A^t)$$

$$and \qquad Xy = 0 \ for \ all \ y \in N(A^t).$$

Next, we discuss two other important generalized inverses which are defined only for square matrices. Let us first recall the index of a square matrix which is used in both the definitions.

Definition 1.4.2. *The index of a square matrix A is the least nonnegative integer k such that $rank(A^{k+1}) = rank(A^k)$. It is denoted by $ind(A)$.*

Definition 1.4.3. *Let $A \in \mathbb{R}^{n\times n}$. The matrix $X \in \mathbb{R}^{n\times n}$ satisfying $A = AXA$, $X = XAX$ and $AX = XA$ is called the the group inverse of A.*

For a given matrix $A \in \mathbb{R}^{n\times n}$, the group inverse need not exists always. If it exists, then it is unique and is usually denoted by$A^\#$. The name group inverse has come due to the fact that the positive powers of A and $A^\#$ together with the projector $AA^\#$ form an abelian group under multiplication, (with $A^\#$ being the inverse of A) and was named by I. Endelyi in 1967.

The following result gives existence conditions for the group inverse. We refer the reader to [5] for the proofs.

Proposition 1.4.1. *Let $A \in \mathbb{R}^{n\times n}$. Then $A^\#$ exists if A satisfies any one of the following conditions:*

(a) $R(A) \cap N(A) = 0$.

(b) $N(A) = N(A^2)$.

(c) $R(A) = R(A^2)$.

(d) $rank(A) = rank(A^2)$.

(e) *The index of A equal to 1.*

Another equivalent definition for $A^{\#}$ is given below.

Theorem 1.4.3. *(Definition 41, page 163, [5]) "Let $A \in \mathbb{R}^{n\times n}$. If $X \in \mathbb{R}^{n\times n}$ satisfies*

$$XAx = x \text{ for all } x \in R(A)$$

$$\text{and} \qquad Xy = 0 \text{ for all } y \in N(A)$$

then $X = A^{\#}$."

Let L and M be complementary subspaces of $\mathbb{R}^n$. Then the projection of $\mathbb{R}^n$ on L along M will be denoted by $P_{L,M}$. If, in addition, L and M are orthogonal then we denote this by P_L. Some of the well known properties of $A^\dagger$ and $A^{\#}$ which will be frequently used in our thesis, are: $R(A^t) = R(A^\dagger)$, $N(A^t) = N(A^\dagger)$, $AA^\dagger = P_{R(A)}$, $A^\dagger A = P_{R(A^t)}$; $R(A) = R(A^{\#})$, $N(A) = N(A^{\#})$ and $AA^{\#} = P_{R(A),N(A)} = A^{\#}A$. In particular, if $x \in R(A^t)$ then $x = A^\dagger Ax$ and if $x \in R(A)$ then $x = A^{\#}Ax$.

We now present another generalized inverse called the Drazin inverse (named after M.P. Drazin [19] who studied this notion in associative rings) which exists for all square matrices.

Definition 1.4.4. *Let $A \in \mathbb{R}^{n\times n}$ be of index k. Then the Drazin inverse of A is the unique matrix $A^D \in \mathbb{R}^{n\times n}$ satisfying $A^k = A^kA^DA$, $A^D = A^DAA^D$ and $AA^D = A^DA$.*

It is easy to note that the Drazin inverse of a matrix with index 1 is equal to the group inverse. An equivalent definition for A^D is given below.

Theorem 1.4.4. *(Definition 41, page 163, [5]) "Let* $A \in \mathbb{R}^{n\times n}$*. If* $X \in \mathbb{R}^{n\times n}$ *satisfies*

$$XAx = x \ for \ all \ x \in R(A^k)$$

$$and \qquad Xy = 0 \ for \ all \ y \in N(A^k)$$

then $X = A^D$*".*

In this thesis, we will use one more generalized inverse called outer inverse with prescribed range and null space. First we recall the definition of $\{2\}$-inverse.

Definition 1.4.5. *For* $A \in \mathbb{R}^{m\times n}$ *the matrix* $X \in \mathbb{R}^{n\times m}$ *satisfying the equation* $XAX = X$ *is called* $\{2\}$*-inverse (or outer inverse) of* A*. It always exists and is denoted by* $A^{(2)}$*.*

This generalized inverse is used in the iterative methods for solving the non-linear equations and the applications to statistics. In particular, $\{2\}$-inverse plays an important role in stable approximation of ill posed problems and in linear and non-linear problems involving rank deficient generalized inverse. Nashed [48] presented Kantorovich-type analysis for Newton-like methods for singular operator equations using outer inverses.

Let T and S be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then for $A \in \mathbb{R}^{m\times n}$, the matrix $A^{(2)}_{T,S} \in \mathbb{R}^{n\times m}$ denotes $\{2\}$-inverse with range T and null space S.
The following theorem guarantees the existence and uniqueness of $A^{(2)}_{T,S}$ for a given matrix $A \in \mathbb{R}^{m\times n}$.

Theorem 1.4.5. *(Theorem 14, [5]) Let* $A \in \mathbb{R}^{m\times n}$ *be of rank* r*, let* T *be a subspace of* $\mathbb{R}^n$ *of dimension* $s \leq r$ *and let* S *be a subspace of* $\mathbb{R}^m$ *of dimension* $m - s$*. Then* $A^{(2)}_{T,S}$ *exists and unique if and only if* $AT \oplus S = \mathbb{R}^m$*, where* $AT \oplus S$ *denotes the direct sum of subspaces* AT *and* S *of* $\mathbb{R}^m$*.*

The next result gives the relation between $\{2\}$-inverse and the other generalized inverses discussed in this section.

Theorem 1.4.6. *([5], [12]) Let $A \in \mathbb{R}^{m\times n}$. Then*
(i) $A^{\dagger} = A^{(2)}_{R(A^t),N(A^t)}$.
(ii) $A^{D} = A^{(2)}_{R(A^k),N(A^k)}$, *where k is index of the matrix A; in particular, when $k = 1$,* $A^{\#} = A^{(2)}_{R(A),N(A)}$.

In the past thirty years, the subject of computation for $A^{(2)}_{T,S}$ was investigated by numerous authors (see [14], [61], [66], [67], [72] and [73]). Most of these works concentrated on iterative method or approximation to compute $A^{(2)}_{T,S}$.

We conclude this section with the following fundamental results concerning systems of linear equations. These will be rather frequently used in deriving some of our results. We refer the reader to [5] for the proofs.

Lemma 1.4.1. *"Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m}$. Then the system of linear equations $Ax = b$ has a solution if and only if $AA^{\dagger}b = b$. In such a case, the general solution is given by the formula $x = A^{\dagger}b + z$ for some $z \in N(A)$".*

Lemma 1.4.2. *"Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n}$. Then the system of linear equations $Ax = b$ has a solution if and only if $AA^{\#}b = b$. In such a case, the general solution is given by the formula $x = A^{\#}b + z$ for some $z \in N(A)$".*

1.5 Extensions of monotonicity to rectangular matrices

We review here some of the important extensions of monotonicity by using proper splittings of a matrix. Mangasarian [35] called a rectangular matrix A to be monotone if $Ax \geq 0 \Rightarrow x \geq 0$. He showed, using the duality theorem of linear programming, that A is monotone if and only if A has a nonnegative left inverse. Berman and Plemmons generalized the concept of monotonicity in several ways, in a series of papers, where they studied their relationships with nonnegativity of generalized inverses. The book by Berman and Plemmons [11] documents these results. Several applications are also studied there.

Let us recall the following result that collects two characterizations for the nonnegativity of the two generalized inverses, viz., the Moore-Penrose inverse and the group inverse. These were proved in (Theorem 2) [8] and (Theorem 1) [10], respectively.

Theorem 1.5.1. *Let $A \in \mathbb{R}^{n\times n}$. Then the following hold.*
(a) $A^\dagger \geq 0$ if and only if

$$Ax \in \mathbb{R}^n_+ + N(A^t) \text{ and } x \in R(A^t) \Longrightarrow x \geq 0.$$

(b) $A^\#$ exists and $A^\# \geq 0$ if and only if

$$Ax \in \mathbb{R}^n_+ + N(A) \text{ and } x \in R(A) \Longrightarrow x \geq 0.$$

It is helpful to observe that $A^\dagger \geq 0$ and $A^\# \geq 0$ are extensions of $A^{-1} \geq 0$ to singular matrices, whereas the second parts of statements (a) and (b) above are generalizations of the implication $Ax \geq 0 \Rightarrow x \geq 0$.

1.5.1 Proper splittings

The notion of proper splitting of matrices has proved to be an important tool in the study of nonnegativity of generalized inverses (or generalized monotonicity). Let us recall this briefly.

Definition 1.5.1. *"A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper splitting if $R(A) = R(U)$ and $N(A) = N(U)$".*

This notion was introduced by Berman and Plemmons [9]. Analogous to the invertible case, with such a splitting, one associates an iterative sequence $x^{(k+1)} = Hx^{(k)} + c$, where (this time) $H = U^\dagger V$ is (again) called the iteration matrix and $c = U^\dagger b$, for a nonnegative integer k. The initial vector x^0 however, cannot be chosen arbitrarily; it must not belong to $N(V)$. Once again, it is well known that this sequence converges to the unique solution of the system $Ax = b$ (irrespective of the choice of the initial vector x^0) if and only if $\rho(H) < 1$. For details, refer to [11]. Recently, Mishra ([15] and [42]) extended the convergence theory of alternating iterations and obtained comparison results for such iterations using proper splittings.

Berman and Plemmons [9] proved the following results which collects some of the properties of a proper splitting.

Theorem 1.5.2. *(Theorem 1, [9]) "Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{n\times n}$. Then*

(i) $AA^\dagger = UU^\dagger$ and $A^\dagger A = U^\dagger U$.

(ii) $A = U(I - U^\dagger V)$.

(iii) $I - U^\dagger V$ is invertible.

(iv) $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$".

The following result provides necessary and sufficient conditions for the convergence of iterative methods to solve $Ax = b$.

Theorem 1.5.3. *(Theorem 3, [9]) "Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$ such that $U^\dagger \geq 0$ and $U^\dagger V \geq 0$. Then the following are equivalent:*

(i) $A^\dagger \geq 0$.

(ii) $A^\dagger V \geq 0$.

(iii) $\rho(U^\dagger V) = \dfrac{\rho(A^\dagger V)}{1 + \rho(A^\dagger V)} < 1$".

It is known that [7] any $A \in \mathbb{R}^{m\times n}$ of rank r has a factorization of the form

$$A = P \begin{pmatrix} I \\ C \end{pmatrix} A_{11} \begin{pmatrix} I & B \end{pmatrix} Q,$$

where A_{11} and I are of order $r \times r$, P and Q are permutation matrices of order m and n, $C \in \mathbb{R}^{m-r\times r}$ and $B \in \mathbb{R}^{r\times n-r}$. The next result shows how to construct proper splittings using the above factorization.

Theorem 1.5.4. *(Theorem 1, [7]) Let $A \in \mathbb{R}^{m\times n}$ of rank r be factorized as above. Then $A = U - V$ is a proper splitting if and only if $U = P \begin{pmatrix} I \\ C \end{pmatrix} M_{11} \begin{pmatrix} I & B \end{pmatrix} Q$, where M_{11} is a nonsingular matrix of order $r \times r$*

Using this theorem a simpler method of constructing a proper splitting is then stated.

Theorem 1.5.5. *Let $A \in \mathbb{R}^{n\times n}$ and $A = FG$ be a full-rank factorization. Then $A = U - V$ is a proper splitting if and only if $U = FWG$ for some nonsingular W. In this case $\rho(U^{\dagger}V) = \rho(I - W^{-1})$.*

The next result is the group inverse analogue of the above result.

Theorem 1.5.6. *Let $A \in \mathbb{R}^{m\times n}$ and $A = FG$ be a full-rank factorization. Suppose that $U^{\#}$ exists. Then $A = U - V$ is a proper splitting if and only if $U = FWG$ for some nonsingular W. In this case $\rho(U^{\#}V) = \rho(I - W^{-1})$.*

Jena et al. [28] extended the notion of regular and weak regular splittings to rectangular matrices and derived some results which characterize nonnegativity of generalized inverses. We next discuss proper double splittings of a matrix.

1.5.2 Proper double splittings

A double splitting $A = P - R + S$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper double splitting* if $R(A) = R(P)$ and $N(A) = N(P)$. Again, consider the following rectangular linear system,

$$Ax = b, \tag{1.3}$$

where $A \in \mathbb{R}^{m\times n}$ (this time A need not be nonsingular), $b \in \mathbb{R}^{m\times 1}$ is a given vector and $x \in \mathbb{R}^{n\times 1}$ is an unknown vector. Similar to the nonsingular case, if we use proper double splitting $A = P - R + S$ to solve (1.3), it leads to the following iterative scheme:

$$x^{(k+1)} = P^{\dagger}Rx^{(k)} - P^{\dagger}Sx^{(k-1)} + P^{\dagger}b, \text{ where } k = 1, 2, ... \tag{1.4}$$

Motivated by $Wo\acute{z}nicki's$ [74] idea , equation (1.4) can be written as

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} P^{\dagger}R & -P^{\dagger}S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^{\dagger}b \\ 0 \end{pmatrix}.$$

If we denote, $X^{k+1} = \begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix}$, $W = \begin{pmatrix} P^{\dagger}R & -P^{\dagger}S \\ I & 0 \end{pmatrix}$, $X^k = \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix}$ and $B = \begin{pmatrix} P^{\dagger}b \\ 0 \end{pmatrix}$, then we get

$$X^{(k+1)} = WX^{(k)} + B, \qquad k = 1, 2... \tag{1.5}$$

Then, it can be shown that the iterative method (1.5) converges to the unique least square solution of minimum norm of (1.3) if and only if $\rho(W) < 1$.

So far, several convergence and comparison theorems for double splittings have been proved. Jena et al. [28] and Mishra [43] have introduced the notions of proper regular double splittings and proper weak regular double splittings and derived some comparison theorems. Recently, Alekha kumar and Mishra [1] have considered proper nonnegative double splittings of nonnegative matrix and derived certain comparison theorems.

Next, we present literature on indefinite inner product spaces.

1.6 Indefinite inner product spaces

The concept of indefinite inner product space first appeared in a paper of Dirac on quantum field theory [17]. Soon afterwards, Pontrajagin gave the first mathematical interpretation for an indefinite inner product space. Several authors renewed the attempts of Dirac and Pauli in indefinite inner product space to quantum field theory (see [24] and [37]). The past two decades have seen many an investigation into linear transformations in an indefinite inner product space in various directions.

Rodman investigated the behavior of nonnegative invariant subspaces in an indefinite inner product space and applied these results to the problems concerning factorization of nonnegative matrix polynomials and solution of the algebraic Riccati equation.

We now recall the definition of indefinite inner product space before moving to further results.

Definition 1.6.1. *Let N be a real symmetric matrix of order $n \times n$ such that $N = N^{-1}$. Such a matrix N is called a weight. An indefinite inner product in $\mathbb{R}^n$ is defined by $[x, y] = \langle x, Ny \rangle$ for all $x, y \in \mathbb{R}^n$, where $\langle ., . \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^n$. A space with an indefinite inner product is called an indefinite* *inner product space.*

Throughout this section, $\mathbb{R}^m$ and $\mathbb{R}^n$ represent indefinite inner product spaces with the corresponding weights M and N, respectively.

Sun and Wei [68] used the term weighted conjugate transpose for adjoint. The definition of adjoint relative to weights N and M is defined defined as follows.

Definition 1.6.2. *Let $A \in \mathbb{R}^{m\times n}$, where $\mathbb{R}^{m\times n}$ denotes the set of all real matrices of order $m \times n$. The adjoint $A^{[*]}$ of A (relative to weights N, M) is defined by $A^{[*]} = NA^*M$, where A^* stands for the transpose of A.*

The next result gives the properties of adjoint.

Theorem 1.6.3. *(Proposition 2.1.3, [30]) Let A, B and C be real matrices of order $m \times p$, $p \times n$ and $m \times p$, respectively. Then*
(i) $(A^{[*]})^{[*]} = A$.
(ii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iii) $(A + C)^{[*]} = A^{[*]} + C^{[*]}$.
(iv) If $\mathbb{R}^n$ is an indefinite inner product spaces with weight N, then $N^{[*]} = N$.
(v) $I_n^{[*]} = I_n$, *where I_n denotes the identity matrix of order n.*

Kalman [29] gave a characterization for the existence of generalized inverses in an arbitrary field. Mehl and Rodman [38] remarked that there is no systematic study of generalized inverse in an indefinite inner product space. Kamaraj and Sivakumar [30] defined Moore-Penrose inverse in an indefinite inner product space and proved existence and uniqueness of Moore-Penrose inverse.

Now, we recall the definition of Moore-Penrose inverse in an indefinite inner product space.

Definition 1.6.1. *For a given matrix $A \in \mathbb{R}^{m\times n}$, the matrix $X \in \mathbb{R}^{n\times m}$ (if it exists) is called the Moore-Penrose inverse of A if it satisfies the equations $AXA = A$, $XAX = X$, $(AX)^{[*]} = AX$ and $(XA)^{[*]} = XA$, and it is denoted by $A^{(\dagger)}$. If A is invertible then $A^{(\dagger)} = A^{-1}$.*

In the literature, usually $A^{[\dagger]}$ is used to denote More-Penrose inverse of A in an indefinite inner product space with respect to usual matrix product. But, we reserve this notation till Chapter 4, to denote More-Penrose inverse of a matrix A in an indefinite inner product spaces with respect to indefinite matrix product.

We know that Moore-Penrose inverse of each matrix always exists and is unique in Euclidean case. But, this is not true in indefinite inner product space. Our next result is for existence and uniqueness of Moore-Penrose inverse in an indefinite inner product space.

Theorem 1.6.4. *(Theorem 2.2.6, [30]) Let $A \in \mathbb{R}^{m\times n}$. Then $A^{(\dagger)}$ exists if and only if $rank(A) = rank(AA^{[*]}) = rank(A^{[*]}A)$. If $A^{(\dagger)}$ exists, then it is unique.*

We now collect some properties and basic results of $A^{(\dagger)}$

Theorem 1.6.5. *(Proposition 2.4.1, [30]) Let $A \in \mathbb{R}^{m\times n}$. If $A^{(\dagger)}$ exists then $(A^{[*]})^{(\dagger)} = (A^{(\dagger)})^{[*]}$.*

Theorem 1.6.6. *(Proposition 2.4.3, [30]) Let $A \in \mathbb{R}^{m\times n}$ of full-column rank. Then $A^{(\dagger)}$ exists iff $A^{[*]}A$ is invertible. In this case, $A^{(\dagger)} = (A^{[*]}A)^{-1}A^{[*]}$.*

Theorem 1.6.7. *(Theorem 2.4.7, [30]) Let $A \in \mathbb{R}^{m\times n}$. If $A^{(\dagger)}$ exists then $(AA^{[*]})^{(\dagger)}$ and $(A^{[*]}A)^{(\dagger)}$ exist. In this case, $(AA^{[*]})^{(\dagger)} = (A^{[*]})^{(\dagger)}A^{(\dagger)}$ and $(A^{[*]}A)^{(\dagger)} = A^{(\dagger)}(A^{[*]})^{(\dagger)}$.*

Theorem 1.6.8. *(Corollary 2.4.8, [30]) Let $A \in \mathbb{R}^{m\times n}$ such that $A^{(\dagger)}$ exists. Then $A^{(\dagger)} = A^{[*]}(AA^{[*]})^{(\dagger)} = (A^{[*]}A)^{(\dagger)}A^{[*]}$.*

Kamaraj and Sivakumar (see [31] and [63]) studied spectral theorem for normal matrices and characterization of *-isomorphism in indefinite inner product spaces. While one studies matrices in an indefinite inner product space, the usual matrix multiplication is employed. This gives rise to a mismatch when one computes the inner product of vectors. To rectify this deficiency, the authors of [55] defined a new matrix product namely *indefinite matrix product*. Its definition is the following.

Definition 1.6.9. *Let A and B be $m \times n$ and $n \times l$ real matrices, respectively. Let N be an arbitrary but fixed weight matrix of order $n \times n$. An indefinite matrix product of A and B (relative to N) is defined by $A \circ B = ANB$.*

Note that for $N = I$ the above product becomes the usual matrix product. For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times l}$, it easily follows that $(A \circ B)^{[*]} = B^{[*]} \circ A^{[*]}$. We refer the reader to [55] for the detailed study of properties of this product.

Using this indefinite matrix product quite a few results for matrices in the setting of a real Euclidean space were obtained in the setting of indefinite inner product spaces with a feature that one could obtain the results in the Euclidean space as particular cases. This aspect was exemplified in [55] in connection with the proof of the existence of Moore-Penrose inverses, in [56] in the proof of the Farkas lemma and nonnegativity of the Moore-Penrose inverse of Gram operators in [57]. This new matrix product proved fruitful in other considerations as well. Let us cite a few results in this regard. The author in [25] studied EP matrices with respect to the new multiplication and obtained characterizations. A host of questions on nonnegative generalized inverses were considered in the work [26]. He also considered the reverse order law and obtained necessary and sufficient conditions for this law to hold. Relationships with certain matrix partial orders were also obtained [27]. Finally, the author of [53] again considered EP matrices and extended many results of [25].

1.7 Outline of the Thesis

In Chapter 2, the first main result generalizes the result of Fan (Lemma 2 in [21]) who showed that if $A-I$ is an invertible M-matrix, then (A is invertible and) the matrix $I - A^{-1}$ is also an invertible M-matrix. In Theorem 2.3.1, we obtain an extension of this result for the group inverse. We obtain two consequences of this result. The first one is an extension of Fan's result for inverse positive matrices and the second one is the result that motivated the generalization of Fan's result which we proved. Barker (Theorem 10, Proposition 7, Proposition 8, Proposition 9 and Proposition 11 in [3]) studied generalizations of M-matrix properties to matrices which allow splittings of certain types. We prove extensions of his results in Theorem 2.3.2, Theorem 2.3.3 and Theorem 2.3.4. Once again, these results involve the group inverse. Theorem 2.3.5 presents an analogue of Theorem 2.1.3 for matrices that possess a $B_{\#}$-splitting. Converse of Theorem 2.3.5 need not be true. However, we have shown that the converse can be recovered in the presence of an additional condition, in Theorem 2.3.6. The last set of results concern extensions of the corresponding results of [20]. To derive these theorems, we use the notion of a $B_{\#}$-splitting. As an application of this, certain comparison theorems are proved in Theorem 2.4.3 and Theorem 2.4.4, extending the corresponding results of [20].

In Chapter 3, we generalize the comparison results of Shen and Huang (Theorem 3.1 and Theorem 3.2 in [60]) from square nonsingular matrices to rectangular matrices and from classical inverses to Moore-Penrose inverses, in Theorem 3.3.1 and Theorem 3.3.2. Infact, we consider two double splittings $A = P_1 - R_1 + S_1$ and $A = P_2 - R_2 + S_2$ of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$ and derive two comparison theorems for the spectral radii of corresponding iteration matrices.

In Chapter 4, the first main result (Lemma 4.3.1) gives a relation between eigenvalues of certain matrices obtain from $\{T, S\}$ splitting of a rectangular matrix. This result is a generalization of Lemma 2.6 in [44]. Theorem 4.3.1 and Theorem 4.3.2 are convergence results for $\{T, S\}$ splitting of a rectangular matrix. These results generalizes the Lemma 3.4 and Lemma 3.5 in [43]. Theorem 4.4.1 and Theorem 4.4.2 are comparison results for $\{T, S\}$ splitting which generalize Theorem 3.2 and Theorem 3.3 in [28].

In Chapter 5, we deal with indefinite inner product spaces and we characterize cone nonnegativity of the Moore-Penrose inverse of Gram matrices in terms of obtuseness or acuteness of certain cones (Theorem 5.3.2), generalizing the Sivakumar's result [62] from finite dimensional real Euclidean space to indefinite inner product space.

Finally, we summarize the contents of our thesis and present a list of references.

CHAPTER 2

GROUP INVERSE EXTENSIONS OF CERTAIN M-MATRIX PROPERTIES

2.1 Introduction

In this chapter, we focus mainly on group inverse extensions of results on M-matrices. Also, we state Moore-Penrose inverse extensions of these results (without proofs) at the end of the chapter. As mentioned in the introduction, a Z-matrix $A = sI - B$, where $s \geq 0$ and $B \geq 0$, is called an M-matrix if $s \geq \rho(B)$. It is well known that if $s > \rho(B)$ in the representation described above, then A is invertible and $A^{-1} \geq 0$. A matrix satisfying the later condition is also called an inverse positive matrix. In fact, there are many interesting characterizations of invertible M-matrices. The book by Berman and Plemmons [11] records more than fifty equivalent conditions. We recall that a square real matrix A is called *monotone* if $Ax \geq 0$ implies $x \geq 0$. This notion was first proposed by Collatz [16]. He showed that a matrix is monotone if and only if it is invertible and the inverse is entrywise nonnegative. Hence monotonicity is equivalent to inverse positivity. Also, we make a note that monotonicity (or inverse positivity) is one of the M-matrix properties. We generalize this M-matrix property for the group inverse. Our results in this chapter are group inverse extensions of some of the results of [3] and [20].

Let us review the results which provide a motivation to the results of this chapter. We begin with a nice result of Fan [21] which concerns the M-matrix property of an invertible matrix of the type $I - A^{-1}$.

Theorem 2.1.1. *(Lemma 2, [21]) Let $A - I$ be an invertible M-matrix. Then A is invertible and the matrix $I - A^{-1}$ is also an invertible M-matrix.*

We next mention the work of Barker [3], who considered regular splittings and completely regular splittings of a matrix and considered several extensions of the properties of M-matrices. One notable contribution in this work is the use of cones in place of the nonnegative orthant of the real Euclidean space. Irreducibility and imprimitivity of matrices were also studied in that work. We recall the following result from that work for the present purpose.

Theorem 2.1.2. *(Proposition 9, [3]) Let $A = U - V$ be a regular splitting. Then the following statements are equivalent:*
(i) $A^{-1} \geq 0$.
(ii) The real parts of the eigenvalues of $U^{-1}A$ are positive.
(iii) The real eigenvalues of $U^{-1}A$ are positive.

The following definitions are used in theorems to follow.

Definition 2.1.1. *([3]) A splitting $A = U - V$ where U is invertible, $U \geq 0, U^{-1} \geq 0$ and $V \geq 0$ is called a completely regular splitting .*

Definition 2.1.2. *A splitting $A = U - V$ is called:*
(i) weak nonnegative of first type if $U^{-1} \geq 0$ and $U^{-1}V \geq 0$.
(ii) weak nonnegative of second type if $U^{-1} \geq 0$ and $VU^{-1} \geq 0$.
(iii) nonnegative if it is weak nonnegative of both types.

In the next result, the notion of *completely regular splitting* is used to derive a sufficient condition for a matrix to be inverse positive.

Theorem 2.1.3. *(Proposition 11, [3]) If $A = U - V$ is a completely regular splitting and if $U^{-1}V$ or VU^{-1} has an eigenvector $x > 0$ corresponding to an eigenvalue $\lambda < 1$, then $A^{-1} \geq 0$.*

Finally, we present two results of Elsner et.al. [20], who studied comparison results for certain nonnegative splittings and studied their relationships with inverse positive matrices.

Theorem 2.1.4. *(Theorem 3.5, [20]) Assume that $A_1 = U_1 - V$, $A_2 = U_2 - V$ are two weak nonnegative splittings of different types of nonsingular square matrices A_1,*

A_2 with $V \neq 0$. Assume that $A_1 \leq A_2$ (or, equivalently, $U_1 \leq U_2$) and that $A_1^{-1} > 0$, $A_2^{-1} \geq 0$. Then $\rho(U_2^{-1}V) < \rho(U_1^{-1}V) < 1$.

Theorem 2.1.5. *(Theorem 4.2, [20]) Assume that $A_1 = U_1 - V_1$, $A_2 = U_2 - V_2$ are two weak nonnegative splittings of different types of nonsingular square matrices A_1, A_2 with $V_1, V_2 \neq 0$. Assume that $A_1^{-1} - A_2^{-1} > 0$ and that $A_1^{-1} > 0$, $A_2^{-1} \geq 0$.*
(i) If $U_2 - U_1 \leq A_2 - A_1$ then $\rho(U_2^{-1}V_2) < \rho(U_1^{-1}V_1) < 1$.
(ii) If $\rho(U_1^{-1}V_1) > 0$ and $U_1^{-1} - U_2^{-1} \geq A_1^{-1} - A_2^{-1}$, then $\rho(U_1^{-1}V_1) < \rho(U_2^{-1}V_2) < 1$.

The descriptions of the main results of this chapter are as follows: we start with an interesting little result of Fan [21]. In Theorem 2.3.1, we obtain an extension of this result for the group inverse. We obtain two consequences of this result. The first is still an extension of Fan's result for inverse positive matrices. The second is the very result of Fan that motivated the generalization we are proving. We then turn our attention to certain interesting results of Barker [3]. He studied generalizations of M-matrix properties to matrices which allow splittings of certain types, as mentioned above. We prove extensions of his results in Theorem 2.3.2, Theorem 2.3.3 and Theorem 2.3.4. Once again, these results involve the group inverse. The last set of results concern extensions of the corresponding results of [20]. To derive these theorems, we use the notion of $B_{\#}$-splitting. Theorem 2.3.5 presents an analogue of Theorem 2.1.3 for matrices that possess a $B_{\#}$-splitting. We then prove a group inverse analogue of an important result of [20], in Theorem 2.4.1. As an application of this, certain comparison theorems are proved in Theorem 2.4.3 and Theorem 2.4.4, extending the corresponding results of [20].

Most of the definitions that are used in this chapter have already been given in Chapter 1. We include some more definitions and preliminary results in the next section.

2.2 Definitions and preliminary results

We begin this section with classification of proper splittings. Throughout this chapter except in Section 2.5, A is a real square matrix.

Definition 2.2.1. *Let $A \in \mathbb{R}^{n\times n}$. A proper splitting $A = U - V$ is called a pseudo regular splitting if $U^\#$ exists, $U^\# \geq 0$ and $V \geq 0$.*

Definition 2.2.2. *Let $A \in \mathbb{R}^{n\times n}$. A proper splitting $A = U - V$ is called a weak pseudo regular splitting if $U^\#$ exists, $U^\# \geq 0$ and $U^\# V \geq 0$.*

We frequently use the following theorem in proving main results of this chapter.

Theorem 2.2.1. *(Theorem 4.1, [45]) Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{n\times n}$. Suppose that $A^\#$ exists. Then*
(a) $U^\#$ exists.
(b) $AA^\# = UU^\#$ and $A^\# A = U^\# U$.
(c) $A = U(I - U^\# V)$.
(d) $I - U^\# V$ is invertible.
(e) $A^\# = (I - U^\# V)^{-1} U^\#$.

Let us observe that if $A = U - V$ is a proper splitting then $A^\#$ exists if and only if $U^\#$ exists. The following result, characterizing the nonnegativity of the group inverse of A if it has a weak pseudo regular splitting can be considered the group inverse analogue of the result of Berman and Plemmons.

Theorem 2.2.2. *(Theorem 3.5, [71]) Let $A \in \mathbb{R}^{n\times n}$ with index 1. Let $A = U - V$ be a proper splitting of A such that $U^\# \geq 0$ and $U^\# V \geq 0$. Then the following are equivalent.*
(i) $A^\# \geq 0$.

(ii) $A^{\#}V \geq 0$.

(iii) $\rho(U^{\#}V) = \dfrac{\rho(A^{\#}V)}{1+\rho(A^{\#}V)} < 1$.

Next, we define $B_{\#}$-splitting. This was introduced in [45].

Definition 2.2.3. *Let $A \in \mathbb{R}^{n\times n}$. A proper splitting $A = U - V$ is called a $B_{\#}$-splitting if it satisfies the following conditions:*

(i) $U \geq 0$.

(ii) $V \geq 0$.

(iii) $U^{\#}$ exists, $VU^{\#} \geq 0$.

(iv) $Ax, Ux \in \mathbb{R}^n_+ + N(A)$ and $x \in R(A) \Longrightarrow x \geq 0$.

The notion of $B_{\#}$-splitting extends the notion of B-splitting studied by Peris [51]. The next two results were stated in [45].

Theorem 2.2.3. *Let $A \in \mathbb{R}^{n\times n}$. Consider the following statements:*

(a) $A^{\#}$ exists and $A^{\#} \geq 0$.

(b) $Ax \in \mathbb{R}^n_+ + N(A)$ and $x \in R(A) \Longrightarrow x \geq 0$.

(c) $\mathbb{R}^n_+ \subseteq A\mathbb{R}^n_+ + N(A)$.

(d) There exists $x \in \mathbb{R}^n_+$ and $z \in N(A)$ such that $Ax + z > 0$.

Then we have (a) $\Leftrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d).

Suppose that A has a $B_{\#}$-splitting $A = U - V$. Then each of the above statements is equivalent to the following condition:

(e) $\rho(VU^{\#}) < 1$.

Theorem 2.2.4. *Let $A \in \mathbb{R}^{n\times n}$. Suppose that $A^{\#}$ exists, $A^{\#} \geq 0$ and $R(A) \cap int(\mathbb{R}^n_+) \neq \emptyset$. Further, let $A^{\#}A \geq 0$. Then A possesses a $B_{\#}$-splitting $A = U - V$ such that $\rho(VU^{\#}) < 1$.*

2.3 Characterizations of nonnegativity of $A^{\#}$

In this section, we present main results of this chapter. We begin with an extension of the result of Fan, mentioned above.

Theorem 2.3.1. *Let $A \in \mathbb{R}^{n\times n}$ with index 1. Let $F = A - AA^{\#}$ and $G = AA^{\#} - A^{\#}$ be proper splittings of F and G, respectively. Then $F^{\#}$ exists. Furthermore if $AA^{\#} \geq 0$ and $F^{\#} \geq 0$, then $G^{\#}$ exists and $G^{\#} \geq 0$.*

Proof. Since $F = A - AA^{\#}$ and $G = AA^{\#} - A^{\#}$ are proper splittings, it follows that

$$R(F) = R(A) = R(AA^{\#}) = R(G)$$

and

$$N(F) = N(A) = N(AA^{\#}) = N(G).$$

Since $A^{\#}$ exists, the subspaces $R(A)$ and $N(A)$ are complementary; so are $R(F)$ and $N(F)$ so that $F^{\#}$ exists. Since we also have the complementarity of the subspaces $R(G)$ and $N(G)$, it follows that $G^{\#}$ exists. Note that

$$GG^{\#} = P_{R(G),N(G)} = P_{R(A),N(A)} = AA^{\#}$$

and so $GG^{\#} \geq 0$.

Let $u \geq 0$ and $v = G^{\#}u \in R(G) = R(A)$ so that $AA^{\#}v = v$. Then $Gv = GG^{\#}u \geq 0$ as $GG^{\#} \geq 0$. We show that $v \geq Gv$ and so we would have $v \geq 0$, proving that $G^{\#} \geq 0$. By Lemma 1.4.2, we have $u = Gv + w$ for some $w \in N(G)$. Thus we have

$$Gv \in \mathbb{R}^n_+ + N(G) = \mathbb{R}^n_+ + N(F).$$

Let $z = A^{\#}v$. Then $AA^{\#}z = z$ and $Az = AA^{\#}v$ so that

$$Gv = AA^{\#}v - A^{\#}v = Az - z = Az - AA^{\#}z = Fz.$$

So,

$$Fz \in \mathbb{R}^n_+ + N(F) \text{ and } z \in R(A) = R(F).$$

Since $F^\# \geq 0$, by Theorem 1.5.1, we then have $z \geq 0$. So,

$$0 \leq z = A^\# v = AA^\# v - Gv = v - Gv.$$

We have shown that $v \geq Gv$, as required. □

We have the following consequence of Theorem 2.3.1 for matrices with positive inverse.

Corollary 2.3.1. *Let $A \in \mathbb{R}^{n\times n}$ be invertible, $F = A - I$ and $G = I - A^{-1}$. Suppose that F^{-1} exists and $F^{-1} \geq 0$. Then G^{-1} exists and $G^{-1} \geq 0$.*

Proof. Let $Gx = 0$ so that $x = A^{-1}x$. Then $Ax = x$ and so $Fx = 0$, so that $x = 0$. Thus, G is invertible. It now follows that $F = A - I$ and $G = I - A^{-1}$ are (trivial) proper splittings. Theorem 2.3.1 can be applied now to conclude that $G^{-1} \geq 0$. □

In particular, we have the result of Fan.

Corollary 2.3.2. *(Lemma 2, [21]) Let $A - I$ be an invertible M-matrix. Then A is invertible and the matrix $I - A^{-1}$ is also an invertible M-matrix.*

Proof. Let us denote $F = A - I$ and $G = I - A^{-1}$. Since F is an invertible M-matrix, $F = A - I = sI - B$ where $s > \rho(B)$ and $B \geq 0$. So, $F^{-1} \geq 0$ by Theorem 1.2.1. Also, $A = (s + 1)I - B$ and $s + 1 > s > \rho(B)$. This implies that A is invertible and $A^{-1} \geq 0$. Let $Gx = 0$ so that $x = A^{-1}x$. Then $Ax = x$ and so $Fx = 0$, so that $x = 0$. Thus G is invertible. It now follows that F and G satisfies all the conditions of Theorem 2.3.1. So $G^{-1} \geq 0$. Then $\rho(A^{-1}) < 1$ by Theorem 1.2.1. Hence, $G = I - A^{-1}$ is an invertible M-matrix. □

2.3.1 Pseudo regular splittings and nonnegativity of $A^{\#}$

We derive here certain generalizations of the results of [3]. As it was mentioned in the introduction, these are group inverse extensions of results on M-matrices. These are presented in Theorem 2.3.2, Theorem 2.3.3, Theorem 2.3.4 and Theorem 2.3.5.

Theorem 2.3.2. *Let $A \in \mathbb{R}^{n\times n}$ such that $A^{\#}$ exists. Let $A = U - V$ be a pseudo regular splitting. Then the following statements are equivalent:*
(i) $A^{\#} \geq 0$.
(ii) The real part of any nonzero eigenvalue of $U^{\#}A$ is positive.
(iii) Any nonzero real eigenvalue of $U^{\#}A$ is positive.

Proof. $(i) \implies (ii)$: Suppose that $A^{\#} \geq 0$ and $A = U - V$ is a pseudo regular splitting. Then, by Theorem 2.2.2, $\rho(U^{\#}V) < 1$. Let μ be a nonzero eigenvalue of $U^{\#}V$. There exists $0 \neq x$ such that $U^{\#}Vx = \mu x$. Let $x = x^1 + x^2$, where $x^1 \in R(U)$ and $x^2 \in N(U)$. Since $A = U-V$ is a proper splitting, it follows that $N(U) \subseteq N(V)$. So, $U^{\#}Vx^2 = 0$. Consider $U^{\#}Vx = \mu(x^1 + x^2)$. The left hand side vector belongs to $R(U^{\#}) = R(U)$ and so is x^1. Hence, $\mu x^2 = 0$. Since $\mu \neq 0$, we have $x^2 = 0$ and so $x^1 = x \neq 0$. Thus $U^{\#}Vx^1 = \mu x^1$. Also, $U^{\#}Ux = x^1$ and $Ax = Ax^1$. Hence, we have $U^{\#}Ax^1 = U^{\#}Ax = U^{\#}Ux - U^{\#}Vx = x^1 - U^{\#}Vx^1 = (1-\mu)x^1$. So, if μ is a nonzero eigenvalue of $U^{\#}V$, then $1-\mu$ is an eigenvalue of $U^{\#}A$. An entirely similar argument shows that $1-\mu$ is an eigenvalue of $U^{\#}V$, if μ is a nonzero eigenvalue of $U^{\#}A$. So, if μ is a nonzero eigenvalue of $U^{\#}A$, then $|1-\mu| < 1$. This means that $Re\mu > 0$, showing that (ii) holds.
$(ii) \Rightarrow (iii)$: The proof of this part is obvious.
$(iii) \Rightarrow (i)$: Suppose that the nonzero real eigenvalues of $U^{\#}A$ are positive. We must show that $A^{\#} \geq 0$. For this, it is enough to show $\rho(U^{\#}V) < 1$. If $\rho(U^{\#}V) = 0$, then there is nothing to prove. If possible, let $\rho(U^{\#}V) = 1$. Then there exists a nonzero

vector x such that $U^{\#}Vx = x$. Then $x \in R(U^{\#}) = R(U)$ and $UU^{\#}Vx = Ux$. Also, $UU^{\#}Vx = Vx$, since $R(V) \subseteq R(U)$. So, $Vx = Ux$. Therefore, $Ax = Ux - Vx = 0$. Thus, $x \in N(A) = N(U)$, so that $x = 0$, a contradiction. So, $\rho(U^{\#}V) \neq 1$. Since $U^{\#}V \geq 0$, $\rho(U^{\#}V)$ is a non-zero eigenvalue of $U^{\#}V$, by the Perron-Frobenius theorem. Thus, as before, $1 - \rho(U^{\#}V)$ is a nonzero eigenvalue of $U^{\#}A$. So, by hypothesis $1 - \rho(U^{\#}V) > 0$, proving that $\rho(U^{\#}V) < 1$. By Theorem 2.2.2, it now follows that $A^{\#} \geq 0$. □

The following example illustrates Theorem 2.3.2.

Example 2.3.1. *Let* $A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Set* $U = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $U^{\#} = \frac{1}{3}\begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0$, $V \geq 0$, $R(A) = R(U)$ *and* $N(A) = N(U)$. *Therefore,* $A = U - V$ *is a pseudo regular splitting. Also,* $A^{\#} = \frac{1}{2}\begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0$ *and* $U^{\#}A = \frac{1}{3}\begin{pmatrix} 3 & -4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Note that the eigenvalues of* $U^{\#}A$ *are* 0, $\frac{2}{3}$ *and* 1. *So, the non-zero real eigenvalues are positive.*

As mentioned in Section 1.2, if $A \in \mathbb{R}^{n\times n}$ is an M-matrix, then A is inverse positive if and only if there exists a vector $x \in \mathbb{R}^n$ such that both x and Ax are positive vectors. If we define $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) := e^t x,\ x \in \mathbb{R}^n$, where $e \in \mathbb{R}^n$ has all its coordinates equal to 1, then the latter part of the previous statement could be paraphrased as: there exists $x \in \mathbb{R}^n$ such that $h(x)$ and $h(Ax)$ are positive real numbers. In what follows, we generalize this to the nonnegativity of the group inverse, while also extending Theorem 10 of [3] and its converse viz., (part of)

Proposition 7 of [3].

Theorem 2.3.3. *Let $A \in \mathbb{R}^{n\times n}$ be with index 1. Suppose that $R(A) \cap \mathbb{R}^n_+ \neq \{0\}$. Let $A = U - V$ be a proper splitting of A such that $U \geq 0$, $U^{\#} \geq 0$ and $U^{\#}V \geq 0$. Suppose that there exists a linear functional f such that $f(x) \geq 0$ and $f(Ax) > 0$ for every $0 \neq x \in \mathbb{R}^n_+ \cap R(A)$. Then $A^{\#} \geq 0$. Conversely, suppose that $A^{\#} \geq 0$. Then there exists a linear functional f such that $f(x) \geq 0$ for all $0 \neq x \in \mathbb{R}^n_+ \cap R(A)$ and $f(Ax) > 0$.*

Proof. Let us observe that the splitting for A given as above is a weak pseudo regular splitting satisfying the additional condition that $U \geq 0$. We have $U^{\#}V \geq 0$. Let $\rho = \rho(U^{\#}V)$ and let $0 \neq y \geq 0$ be an eigenvector corresponding to ρ so that $U^{\#}Vy = \rho y$. Such a vector exists, by the Perron-Frobenius theorem. If $\rho = 0$, then by Theorem 2.2.2, it follows that $A^{\#} \geq 0$. Suppose that $\rho > 0$. Observe that $y \in R(U^{\#}) = R(A^{\#}) = R(A)$. Thus $y \in \mathbb{R}^n_+ \cap R(A)$. Let g be a linear functional satisfying the hypothesis. Then $g(y) \geq 0$ and $g(Ay) > 0$. We have $U^{\#}Vy = \rho y$ so that upon premultiplying by U we have $UU^{\#}Vy = \rho Uy$. Since $R(V) \subseteq R(U)$, it then follows that $Vy = \rho Uy$ so that $(\rho U - V)y = 0$. If $\rho \geq 1$ we get $\rho U - V \geq U - V = A$ (it is in step where $U \geq 0$ is used). This implies that $0 = (\rho U - V)y \geq Ay$ and so $g(Ay) \leq 0$, since g is nonnegative on $\mathbb{R}^n_+ \cap R(A)$. This is a contradiction and so $\rho(U^{\#}V) < 1$. Again, it follows from Theorem 2.2.2 that $A^{\#} \geq 0$.

To prove the converse, let us suppose that $A^{\#} \geq 0$ and g is a strictly positive linear functional on $\mathbb{R}^n_+$. Clearly, $g(x) := e^t x, x \in \mathbb{R}^n$, where $e \in \mathbb{R}^n$ has all its coordinates equal to 1, is one such functional. Then $g(x) > 0$ for all $0 \neq x \in \mathbb{R}^n_+$. This applies in particular, to all the vectors $0 \neq x \in \mathbb{R}^n_+ \cap R(A)$. Let $0 \neq x^* \geq 0$ and $x^* \in R(A)$. Then $A^{\#}x^* \geq 0$ and $x^* = AA^{\#}x^*$. Set $f = g(A^{\#})$. Then

$$f(Ax^*) = g(A^{\#}Ax^*) = g(AA^{\#}x^*) = g(x^*) > 0,$$

showing that f is the required linear functional. $\square$

Example 2.3.2. *Let* $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$. *Then the index of* A *is* 1 *and* $e \in R(A) \cap \mathbb{R}^3_+$.

Set $U = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. *Then* $R(A) = R(U)$ *and* $N(A) = N(U)$.

Further, $U^{\#} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \geq 0$ *and so* $U^{\#}V \geq 0$. *Note that* $A^{\#} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \geq 0$. *Define* $f(x) = e^t A^{\#} x,\ x \in \mathbb{R}^3$. *Let* $0 \neq x = (x_1, x_2, x_3) \in R(A) \cap \mathbb{R}^3_+$. *Then* $f(x) = \frac{1}{2}(4x_1 + x_3) \geq 0$ *and* $f(Ax) = e^t A^{\#} A x = x_1 + 2x_3 > 0$.

In order to motivate the next result, let us recall the following: Let $A, B \in \mathbb{R}^{m \times n}$ such that $R(A) = R(B)$ and $N(A) = N(B)$. Suppose that $A \leq B$ and $B^{\dagger} \geq 0$. If $int(\mathbb{R}^n_+) \cap \{A\mathbb{R}^n_+ + N(A^t)\} \neq \emptyset$, then $A^{\dagger} \geq B^{\dagger} \geq 0$. The converse also holds. For a proof of this, we refer to (Theorem 3.4) [54]. The next result somewhat resembles the situation mentioned above, without the condition involving the interior. Curiously, there is a reversal of the roles of A and B insofar as the nonnegativity of their group inverses are concerned. It is pertinent to point to the fact that this result is a generalization of a corresponding result for invertible matrices proved in (Proposition 8) [3]. However, the proof technique is completely different from the proof in [3].

Theorem 2.3.4. *Let* $A, B \in \mathbb{R}^{n \times n}$ *where* A *has index* 1. *Suppose that the following hold:*

(a) *A and B have pseudo regular splittings.*

(b) $R(A) = R(B)$, $N(A) = N(B)$ *and* $A \leq B$.

(c) $A^{\#} \geq 0$.

Then $B^{\#}$ *exists and* $A^{\#} \geq B^{\#} \geq 0$.

Proof. Since A and B have pseudo regular splittings, there exist matrices U_A, V_A, U_B and V_B such that $R(A) = R(U_A), N(A) = N(U_A), R(B) = R(U_B)$ and $N(B) = N(U_B)$. Further,

$$A = U_A - V_A \text{ with } U_A^{\#} \geq 0 \text{ and } V_A \geq 0$$

and

$$B = U_B - V_B \text{ with } U_B^{\#} \geq 0 \text{ and } V_B \geq 0.$$

Also, we have $A \leq B = U_B - V_B \leq U_B$. Thus, $U_B - A \geq U_B - B \geq 0$. Set $Z = U_B$ and $W = U_B - A$. Then $R(Z) = R(U_B) = R(B) = R(A)$ and $N(Z) = N(U_B) = N(B) = N(A)$. Thus, $A = Z - W$ is a proper splitting. Further,

$$Z^{\#} = U_B^{\#} \geq 0 \text{ and } Z^{\#}W = U_B^{\#}(U_B - A) \geq 0.$$

This shows that the above proper splitting is also a weak pseudo regular splitting. Since it is given that $A^{\#} \geq 0$, by Theorem 2.2.2, we have

$$1 > \rho(Z^{\#}W) = \rho(U_B^{\#}(U_B - A)) \geq \rho(U_B^{\#}(U_B - B)).$$

Since $U_B - (U_B - B)$ is a pseudo regular splitting of B, again by Theorem 2.2.2, it follows that $B^{\#} \geq 0$.

We have $A \leq B$. Premultiplying by $A^{\#} \geq 0$ and post multiplying by $B^{\#} \geq 0$, we get $A^{\#}AB^{\#} \leq A^{\#}BB^{\#}$. Since $R(B) = R(A)$ and $N(B) = N(A)$, it follows that

$$A^{\#}AB^{\#} = P_{R(A),N(A)}B^{\#} = P_{R(B),N(B)}B^{\#} = B^{\#}BB^{\#} = B^{\#}$$

and

$$A^{\#}BB^{\#} = A^{\#}P_{R(B),N(B)} = A^{\#}P_{R(A),N(A)} = A^{\#}AA^{\#} = A^{\#}.$$

This shows that $A^{\#} \geq B^{\#} \geq 0$, completing the proof. □

The following example illustrates Theorem 2.3.4.

Example 2.3.3. *Let* $A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. *Set* $U_A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $V_A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $U_B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $V_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $U_A^{\#} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \geq 0$, $U_B^{\#} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \geq 0$, $V_A \geq 0$ *and* $V_B \geq 0$. *It can be verified that* $A = U_A - V_A$ *and* $B = U_B - V_B$ *are pseudo regular splittings of* A *and* B*, respectively. Also,* $R(A) = R(B)$, $N(A) = N(B)$, $A \leq B$ *and* $A^{\#} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \geq 0$. *Thus, all the conditions of Theorem 2.3.4 are satisfied. Note that* $B^{\#} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and that* $A^{\#} \geq B^{\#} \geq 0$.

2.3.2 $B_{\#}$-splittings and nonnegativity of $A^{\#}$

We begin this subsection with an extension of Theorem 2.1.3, mentioned in the introduction.

Theorem 2.3.5. *For* $A \in \mathbb{R}^{n \times n}$, *let* $A = U - V$ *be a* $B_{\#}$*-splitting such that no row of* U *is zero. Suppose that there exists* $x > 0$ *such that* $U^{\#}Vx = \lambda x$ *for some* $\lambda < 1$. *Then* $A^{\#} \geq 0$.

Proof. We show that there exists $z \in \mathbb{R}^n_+$ and $w \in N(A)$ such that $Az + w > 0$. It would then follow from (d) of Theorem 2.2.3, that $A^{\#} \geq 0$.

Let $U^{\#}Vx = \lambda x$ for some $\lambda < 1$. Premultiplying with U and by using the fact that $R(V) \subseteq R(U)$, we have $Vx = \lambda Ux$. If $\lambda = 0$, then $Vx = 0$ so that $Ax = Ux$. Since $U \geq 0$ and has no zero row, and $x > 0$, we have $Ux > 0$. Thus, $Ax + w > 0$ by taking $w = 0 \in N(A)$. So, if $\lambda = 0$, then (d) of Theorem 2.2.3 holds. Consider the case $0 < \lambda$ (< 1). Then $x \in R(U^{\#})$ so that $U^{\#}Ux = x$. We have $Ax = U(I - U^{\#}V)x = (1 - \lambda)Ux$. Thus $U^{\#}Ax = (1 - \lambda)U^{\#}Ux = (1 - \lambda)x > 0$. Set $y = U^{\#}Ax > 0$. Then by Lemma 1.4.2, we have $Uy = Ax + z$, for some $z \in N(U^{\#}) = N(A^{\#}) = N(A)$. Again, since U has no zero row, we have $Uy > 0$. Thus, there exists $z \in N(A)$ such that $Ax + z > 0$, as required. □

The following example demonstrates that the converse of the last theorem is not true.

Example 2.3.4. *Let* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. *Then* $A^{\#} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$. *Set* $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. *Then* $R(A) = R(U)$ *and* $N(A) = N(U)$. *Also,* $U \geq 0, V \geq 0, U^{\#} = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $VU^{\#} \geq 0$. *Thus* $A = U - V$ *is a* $B_{\#}$-*splitting. Further,* $U^{\#}V = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ *and the eigenvalues of* $U^{\#}V$ *are* 0 *and*

$\frac{1}{2}$. *The corresponding eigenvectors are* $\begin{pmatrix} k_1 \\ k_2 \\ 0 \end{pmatrix}$ *where* $k_1, k_2 \in \mathbb{R}$ *and* $\begin{pmatrix} 0 \\ k_3 \\ k_3 \end{pmatrix}$, $k_3 \in \mathbb{R}$, *respectively. Thus there is no vector* $x > 0$ *such that* $U^{\#}Vx = \lambda x$ *for any* λ.

However, we show that the converse can be recovered in the presence of an additional condition.

Theorem 2.3.6. *For* $A \in \mathbb{R}^{n \times n}$, *let* $A = U - V$ *be a* $B_{\#}$*-splitting such that no row of* U *is zero. Suppose that either* $U^{\#}V \geq 0$ *and is irreducible or* $U^{\#}V > 0$. *If* $A^{\#} \geq 0$, *then there exists a vector* $x > 0$ *such that* $U^{\#}Vx = \lambda x$ *for some* $\lambda < 1$.

Proof. Suppose that $A^{\#} \geq 0$. Since $A = U - V$ is a $B_{\#}$-splitting, by Theorem 2.2.3 $\rho(U^{\#}V) = \rho(VU^{\#}) < 1$. Also, we have either $U^{\#}V \geq 0$ and is irreducible or $U^{\#}V > 0$. So, by the Perron-Frobenius theory, there exists a unique vector $x > 0$ such that $U^{\#}Vx = \rho(U^{\#}V)x$, proving the result. □

2.4 Comparison results for proper splittings

In this section, we will be concerned with comparison results for the two types of splittings discussed here. In the process we obtain generalizations of the results of [20]. The proof of the first result is very similar to the corresponding result there, to fit into the group inverse frame work. However, we prefer to include the proof for a self-contained discussion. This is mainly used in deriving comparison results, viz., Theorem 2.4.3 and Theorem 2.4.4. We would like to point out that the results in this section are motivated by purely theoretical considerations. In particular, no claim of superiority is made on any splitting over another. Applications of these comparison results to numerical solutions of linear systems are not considered either.

Theorem 2.4.1. *(Extension of Theorem 3.4, [20]) Let $A \in \mathbb{R}^{n\times n}$ with index 1. Suppose that no row or column of A is zero. Suppose also that $A^{\#} \geq 0, R(A) \cap int(\mathbb{R}^n_+) \neq \emptyset$ and $A^{\#}A \geq 0$. Then there exists a $B_{\#}$-splitting $A = U - V$, where the matrices U, V and A are further related by the following statements: There exists $x \in \mathbb{R}^n_+ \cap R(U)$ such that $U^{\#}Vx = \rho(U^{\#}V)x$, and $0 \neq Ax \geq 0$. Moreover, if $VU^{\#}$ is not nilpotent then $0 \neq Vw \geq 0$ for some $w \in \mathbb{R}^n_+ \cap R(U)$.*

Proof. By Theorem 2.2.4, there exists a $B_{\#}$-splitting $A = U - V$. Set $\rho^* = \rho(U^{\#}V)$. Let x be a corresponding eigenvector of $U^{\#}V$ so that $U^{\#}Vx = \rho^* x$. We show that the coordinates of x are all nonnegative or all nonpositive. We have $x \in R(U^{\#}) = R(U)$ so that $UU^{\#}x = U^{\#}Ux = x$. Premultiplying the equation $U^{\#}Vx = \rho^* x$ by U and using the fact that $R(V) \subseteq R(U)$, we get $Vx = \rho^* Ux$. Also, $Vx = VU^{\#}Ux$. Set $z = Ux$. Then $VU^{\#}z = \rho^* z$. If $z = 0$, then $x = 0$, a contradiction. So, z is an eigenvector for the matrix $VU^{\#}$. From Theorem 2.2.4 we have $\rho^* = \rho(U^{\#}V) = \rho(VU^{\#}) < 1$. Note that by the definition of a $B_{\#}$-splitting, we have $VU^{\#} \geq 0$.

First, let us assume that $VU^{\#} > 0$, not just nonnegative. By the result of Perron, there exists a unique positive (Perron) eigenvector y corresponding to the (simple) eigenvalue ρ^* for the positive matrix $VU^{\#}$, i.e., $VU^{\#}y = \rho^* y$. Thus we have $Ux = \alpha y$ for some $0 \neq \alpha \in \mathbb{R}$. Upon premultiplying by $U^{\#}$, we have $x = \alpha U^{\#}y$. Since $N(A) = N(U) = N(U^{\#})$, it follows that no row or column of $U^{\#}$ is zero. Hence $U^{\#}y > 0$. So, either the components of x are all negative or all positive. Replacing x by $-x$, if need be, we have $x > 0$ as well as $Ux > 0$. Thus $x \in int(\mathbb{R}^n_+) \cap R(U)$. Now, we have $Ax = Ux - Vx = (1 - \rho^*)Ux$ and since $\rho^* < 1$ this proves that $Ax > 0$.

To summarize, what we have shown that under the hypotheses of the theorem, if the splitting $A = U - V$ satisfies the assumptions that $VU^{\#} > 0$ then there exists $x \in \mathbb{R}^n_+ \cap R(U)$ such that $U^{\#}Vx = \rho^* x$, where $\rho^* = \rho(U^{\#}V)$.

To complete the proof, we consider the case $VU^{\#} \geq 0$. Let $E \in \mathbb{R}^{n\times n}$ be the matrix with all entries 1. Observe that, as mentioned earlier, since $U^{\#}$ has no zero row or column, it follows that $EU^{\#} > 0$. Let $\| \,.\, \|$ denote any matrix norm. Let α be chosen such that $0 < \alpha < \frac{1}{\|EU^{\#}\|}$. Then the series $\sum_{k=0}^{\infty}(\alpha EU^{\#})^k$ is convergent. We have

$$0 < \sum_{k=0}^{\infty}(\alpha EU^{\#})^k = (I - \alpha EU^{\#})^{-1}.$$

Define $W = (I - \alpha EU^{\#})^{-1}U^{\#}$. Then $W > 0$ and $WUU^{\#} = W$. Note that

$$\begin{aligned} WUA^{\#} &= (I - \alpha EU^{\#})^{-1}U^{\#}UA^{\#} \\ &= (I - \alpha EU^{\#})^{-1}A^{\#}AA^{\#} \\ &= (I - \alpha EU^{\#})^{-1}A^{\#}. \end{aligned}$$

Again, it follows that $WUA^{\#} > 0$. Set $\epsilon_0 = \frac{1}{\|WUA^{\#}\|}$. Let ϵ be chosen such that

$$0 < \epsilon < \epsilon_0.$$

Define $A_\epsilon = A - \epsilon WU$. Then

$$\begin{aligned} A_\epsilon &= A - \epsilon WUU^{\#}U \\ &= A - \epsilon WUA^{\#}A \\ &= (I - \epsilon WUA^{\#})A. \end{aligned}$$

Also, $\| \epsilon WUA^{\#} \| = \epsilon \| WUA^{\#} \| < 1$, so that $I - \epsilon WUA^{\#}$ is invertible. It then follows that $R(A_\epsilon) = R(A) = R(U)$ and $N(A_\epsilon) = N(A) = N(U)$. Thus $A_\epsilon = U - (V + \epsilon WU)$ is a proper splitting. Observe that since $A^{\#}$ exists, the subspaces $R(A)$ and $N(A)$ are complementary. So are $R(A_\epsilon)$ and $N(A_\epsilon)$ and so $A_\epsilon^{\#}$ exists. Next, we show that $A_\epsilon{}^{\#} \geq 0$. First, we show that $A_\epsilon{}^{\#}$ exists. Let $X = A^{\#}(I - \epsilon WUA^{\#})^{-1}$ and $x \in R(A_\epsilon)$. Then

$$XA_\epsilon x = A^\#(I - \epsilon WUA^\#)^{-1}(I - \epsilon WUA^\#)Ax = A^\# Ax = x.$$

For $y \in N(A_\epsilon) = N(A) = N(A^\#)$, we also have

$$Xy = A^\#(I-\epsilon WUA^\#)^{-1}y = A^\# \sum_{k=0}^{\infty}(\epsilon WUA^\#)^k y = A^\# y + \sum_{k=1}^{\infty}(\epsilon WUA^\#)^k y = 0.$$

Hence, $A_\epsilon^\# = A^\#(I - \epsilon WUA^\#)^{-1} = A^\# \sum_{k=0}^{\infty}(\epsilon WUA^\#)^k \geq 0$.

Also, $R(A_\epsilon) \cap int(\mathbb{R}^n_+) = R(A) \cap int(\mathbb{R}^n_+) \neq \emptyset$ and $A_\epsilon^\# A_\epsilon = A^\# A \geq 0$. Further, $(V + \epsilon WU)U^\# = VU^\# + \epsilon WUU^\# = VU^\# + \epsilon W > 0$.

By what we have already shown, there exists $x_\epsilon \in \mathbb{R}^n_+ \cap R(U)$ such that

$$U^\#(V + \epsilon WU)x_\epsilon = \rho(U^\#(V + \epsilon WU))x_\epsilon$$

and $A_\epsilon x_\epsilon > 0$. We may choose x_ϵ such that its 1-norm satisfy $\| x_\epsilon \|_1 = 1$. Set $\epsilon_k = \frac{1}{k}\epsilon_0$. Then the sequence x_{ϵ_k}, being bounded, has a convergent subsequence with limit $0 \neq x \geq 0$. Observe that since

$$U^\#(V + \epsilon_k WU)x_{\epsilon_k} = \rho(U^\#(V + \epsilon_k WU))x_{\epsilon_k},$$

we have in the limit, the equation $U^\# Vx = \rho(U^\# V)x$. We have $Ax \geq 0$, as well. If $Ax = 0$, then $Vx = 0$ and so $x = 0$, a contradiction. Hence $Ax \neq 0$.

Let us prove the last part. We have $Ux = \dfrac{1}{\rho*}Vx$, where we have used $\rho* \neq 0$, since $VU^\#$ is not nilpotent. Thus

$$0 \leq Ax = (U - V)x = U(I - U^\# V)x = (1 - \rho*)Ux = \frac{1 - \rho*}{\rho*}Vx.$$

Note that since $\rho* < 1$, we have $Vx \geq 0$. If $w = \dfrac{1 - \rho*}{\rho*}x$ then $Vw \geq 0$. If $Vw = 0$ then $Ax = 0$, a contradiction. Thus $Vw \neq 0$, completing the proof. □

In the next result, we show that the conclusions of Theorem 2.4.1 follow easily, if we consider a splitting that is stronger than a $B_\#$-splitting.

Theorem 2.4.2. *Let $A \in \mathbb{R}^{n\times n}$ with index 1. Suppose that no row or column of A is zero. Let $A = U - V$ be a proper splitting such that $U \geq 0$, $V \geq 0$ and $U^{\#} \geq 0$. Suppose also that $A^{\#} \geq 0$. Then there exists $x \in \mathbb{R}^n_+ \cap R(U)$ such that $U^{\#}Vx = \rho(U^{\#}V)x$ and $0 \neq Ax \geq 0$. Further, if $VU^{\#}$ is not nilpotent then $0 \neq Vw \geq 0$ for some $w \in \mathbb{R}^n_+ \cap R(U)$.*

Proof. Note that the given splitting is a $B_{\#}$-splitting. Set $\rho^* = \rho(U^{\#}V)$. Since $U^{\#}V \geq 0$, by the Perron-Frobenius theorem there exists a vector $0 \neq x \geq 0$ such that $U^{\#}Vx = \rho^* x$. Premultiplying the equation $U^{\#}Vx = \rho^* x$ by U and using the fact that $R(V) \subseteq R(U)$, we get $Vx = \rho^* Ux$. Now, we have $Ax = Ux - Vx = Ux - \rho^* Ux = (1 - \rho^*)Ux$ and since $\rho^* < 1$ this proves that $Ax \geq 0$. As above, $Ax \neq 0$. The second part may be proved as done earlier. □

Next, we present some applications of Theorem 2.4.1. These are comparison results for the spectral radii of iteration matrices corresponding to two matrices A and B with $A \leq B$. These also extend Theorem 3.5 and Theorem 4.2 in [20].

Theorem 2.4.3. *Let $B \in \mathbb{R}^{n\times n}$ such that B has index 1, $B^{\#} \geq 0$ and no row or column of B is zero. Suppose that $R(B) \cap int(\mathbb{R}^n_+) \neq \emptyset$ and $B^{\#}B \geq 0$. Then there exists a $B_{\#}$-splitting $B = U_B - V$. Let $A \in \mathbb{R}^{n\times n}$ such that A has index 1, $A^{\#} \geq 0$ and let $A = U_A - V$ be a pseudo regular splitting. Further, suppose that $VU_B^{\#}$ is not nilpotent, $U_B^{\#} \geq 0$ and $A \leq B$ with $R(A) = R(B)$ and $N(A) = N(B)$. Then $\rho(U_B^{\#}V) \leq \rho(U_A^{\#}V) < 1$.*

Proof. By Theorem 2.2.4, there exists a $B_{\#}$-splitting $B = U_B - V$. Since $U_B^{\#} \geq 0$, the splitting $B = U_B - V$ is also a pseudo regular splitting. So, $A, B \in \mathbb{R}^{n\times n}$ satisfy all the conditions of Theorem 2.3.4. Therefore $A^{\#} \geq B^{\#} \geq 0$. Since $A^{\#} \geq 0$ and $A = U_A - V$ is a pseudo regular splitting, by Theorem 2.2.2 it follows that $\rho(U_A^{\#}V) < 1$. Similarly, $\rho(U_B^{\#}V) < 1$. Next, we show that $\rho(U_B^{\#}V) \leq \rho(U_A^{\#}V)$. Let us denote

$G_A = A^{\#}V$ and $G_B = B^{\#}V$. Then, again by Theorem 2.2.2, $\rho(U_A^{\#}V) = \dfrac{\rho(G_A)}{1+\rho(G_A)}$ and $\rho(U_B^{\#}V) = \dfrac{\rho(G_B)}{1+\rho(G_B)}$. Since the function $f(t) = \frac{t}{1+t}$ is strictly increasing for $t \geq 0$, it is enough to show that $\rho(G_B) \leq \rho(G_A)$. For this, we consider $B = U_B - V$ that satisfies all the conditions of Theorem 2.4.1. So, there exists a vector $0 \neq x \geq 0$ such that $U_B^{\#}Vx = \rho(U_B^{\#}V)x$ and $0 \neq Vx \geq 0$. Then for the same x,

$$G_A x = A^{\#}Vx \geq B^{\#}Vx = G_B x = \rho(G_B)x.$$

This implies that $\rho(G_B) \leq \rho(G_A)$, by Lemma 1.1.1. □

The following example illustrates Theorem 2.4.3.

Example 2.4.1. *Let* $B = \begin{pmatrix} -1 & 3 & -1 \\ 3 & -2 & 3 \\ -1 & 3 & -1 \end{pmatrix}$, *then* $B^{\#} = \dfrac{1}{14}\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \geq 0$.

Also $R(B) \cap int(\mathbb{R}^3_+) \neq \emptyset$ *and* $B^{\#}B = \dfrac{1}{14}\begin{pmatrix} 7 & 0 & 7 \\ 0 & 14 & 0 \\ 7 & 0 & 7 \end{pmatrix} \geq 0$.

Set $U_B = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$ *and* $V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ *then* $U_B^{\#} = \dfrac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \geq 0$. *So,* $B = U_B - V$ *is a* $B_{\#}$*-splitting. Let* $A = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & -1 \end{pmatrix}$ *then* $A^{\#} = \dfrac{1}{4}\begin{pmatrix} 3 & 4 & 3 \\ 4 & 4 & 4 \\ 3 & 4 & 3 \end{pmatrix} \geq 0$.

Set $U_A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$, *then* $U_A^{\#} = \dfrac{1}{64}\begin{pmatrix} 4 & 16 & 4 \\ 16 & 0 & 16 \\ 4 & 16 & 4 \end{pmatrix} \geq 0$. *So* $A = U_A - V$ *is a* *pseudo regular spliting.* $VU_B^{\#} = \frac{1}{6}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ *is not nilpotent.*

$U_B^{\#}V = \frac{1}{6}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ *and* $U_A^{\#}V = \frac{1}{64}\begin{pmatrix} 8 & 32 & 8 \\ 32 & 0 & 32 \\ 8 & 32 & 8 \end{pmatrix}$.

Observe that $0.4714 = \rho(U_B^{\#}V) \leq \rho(U_A^{\#}V) = 0.8431 < 1$.

Theorem 2.4.4. *Let* $A, B \in \mathbb{R}^{n\times n}$ *with index 1 such that* $A^{\#} - B^{\#} > 0$, $A^{\#} \geq 0$, $B^{\#} \geq 0$ *and no row or column of* A *and* B *is zero. Suppose that* $R(B) \cap int(\mathbb{R}^n_+) \neq \emptyset$ *and* $B^{\#}B \geq 0$. *Then there exists a* $B_{\#}$*-splitting* $B = U_B - V_B$. *Let* $A = U_A - V_A$ *be a pseudo regular splitting. Suppose also that* $U_B^{\#} \geq 0$ *and* $V_AU_A^{\#}$, $V_BU_B^{\#}$ *are not nilpotent.*

(i) *If* $U_B - U_A \leq B - A$ *then* $\rho(U_B^{\#}V_B) \leq \rho(U_A^{\#}V_A) < 1$.

(ii) *If* $U_A^{\#} - U_B^{\#} \geq A^{\#} - B^{\#}$ *then* $\rho(U_A^{\#}V_A) < \rho(U_B^{\#}V_B) < 1$.

Proof. (i) By Theorem 2.2.4, there exists a $B_{\#}$-splitting $B = U_B - V_B$. Clearly $\rho_1 = \rho(U_A^{\#}V_A) < 1$ and $\rho_2 = \rho(U_B^{\#}V_B) < 1$. It remains to show $\rho(U_B^{\#}V_B) \leq \rho(U_A^{\#}V_A)$. Let us denote $G_A = A^{\#}V_A$ and $G_B = B^{\#}V_B$. If $U_B - U_A \leq B - A$ then $V_A \geq V_B$. Applying Theorem 2.4.1 to $B = U_B - V_B$, we get a non-zero vector $x \geq 0$ such that $U_B^{\#}V_Bx = \rho_2 x$. For the same x, we have

$$G_Ax = A^{\#}V_Ax > B^{\#}V_Ax \geq B^{\#}V_Bx = G_Bx = \rho(G_B)x.$$

This implies that $\rho(G_B) \leq \rho(G_A)$, by Lemma 1.1.1.

(ii) Consider

$$U_B^{\#}V_BB^{\#} = U_B^{\#}(U_B - B)B^{\#} = B^{\#} - U_B^{\#},$$

here we have used the fact that $U_B^{\#}U_B = B^{\#}B$. Also, since $U_A^{\#}U_A = A^{\#}A$, one has

$$A^{\#}V_AU_A^{\#} = A^{\#}(U_A - A)U_A^{\#} = A^{\#} - U_A^{\#}.$$

Therefore,

$$U_B^{\#}V_BB^{\#} = B^{\#} - U_B^{\#} \geq A^{\#} - U_A^{\#} = A^{\#}V_AU_A^{\#} \geq 0.$$

Since $V_AU_A^{\#} \geq 0$ and $U_B^{\#}V_B \geq 0$, by the Perron-Frobenius theorem there exist nonzero vectors $x \geq 0$ and $y \geq 0$ such that

$$V_AU_A^{\#}x = \rho_1 x \quad \text{and} \quad y^tU_B^{\#}V_B = \rho_2 y^t.$$

Thus

$$\rho_2 y^tB^{\#}x = y^tU_B^{\#}V_BB^{\#}x \geq y^tA^{\#}V_AU_A^{\#}x = \rho_1 y^tA^{\#}x.$$

Since $A^{\#} > B^{\#}$ and since x and y are both nonzero and $\rho_1 > 0$, we obtain

$$\rho_2 y^tB^{\#}x > \rho_1 y^tB^{\#}x.$$

Therefore, $\rho(U_A^{\#}V_A) < \rho(U_B^{\#}V_B) < 1$. □

2.5 Moore-Penrose inverse extensions of certain M-matrix properties

In this section, we state Moore-Penrose inverse versions of some of the results of Section 2.3 and Section 2.4. Since the proofs of these results are almost similar to the group inverse case, we omit the proofs. The following definition was introduced in [45] to study the nonnegativity of generalized inverses of matrices.

Definition 2.5.1. *Let $A \in \mathbb{R}^{m\times n}$. A proper splitting $A = U - V$ of A is called a $B_{\dagger}$-splitting if it satisfies the following conditions:*

(i) $U \geq 0$.

(ii) $V \geq 0$.

(iii) $VU^{\dagger} \geq 0$.

(iv) $Ax, Ux \in \mathbb{R}^m_+ + N(A)$ *and* $x \in R(A^t) \Longrightarrow x \geq 0$.

The next result provides various equivalent conditions for semimonotonicity, including one involving a $B_{\dagger}$-splitting.

Theorem 2.5.1. *(Theorem 3.8, [45]) Let* $A \in \mathbb{R}^{m\times n}$. *Consider the following statements:*

(a) $A^{\dagger}$ *exists and* $A^{\dagger} \geq 0$.

(b) $Ax \in \mathbb{R}^m_+ + N(A^t)$ *and* $x \in R(A^t) \Longrightarrow x \geq 0$.

(c) $\mathbb{R}^m_+ \subseteq A\mathbb{R}^n_+ + N(A^t)$.

(d) *There exists* $x \in \mathbb{R}^n_+$ *and* $z \in N(A^t)$ *such that* $Ax + z > 0$.

Then we have $(a) \Leftrightarrow (b) \Longrightarrow (c) \Longrightarrow (d)$.

Suppose that A *has a* $B_{\dagger}$*-splitting* $A = U - V$. *Then each of the statements above is equivalent to the following condition:*

(e) $\rho(VU^{\dagger}) < 1$.

2.5.1 Characterizations of nonnegativity of $A^{\dagger}$

The following result is an extension of the result of Barker (Proposition 9, [3]) to Moore-Penrose inverse case.

Theorem 2.5.2. *Let* $A \in \mathbb{R}^{m\times n}$. *Let* $A = U - V$ *be a proper regular splitting. Then the following statements are equivalent:*

(i) $A^{\dagger} \geq 0$.

(ii) *The real part of any nonzero eigenvalue of* $U^{\dagger}A$ *is positive.*

(iii) *Any nonzero real eigenvalue of* $U^{\dagger}A$ *is positive.*

The next result is a generalization of a corresponding result for invertible matrices proved in (Proposition 8) [3].

Theorem 2.5.3. *Let* $A, B \in \mathbb{R}^{m\times n}$. *Suppose that the following hold:*

(a) *A and B have proper regular splittings.*

(b) $R(A) = R(B)$, $N(A) = N(B)$ *and* $A \leq B$.

(c) $A^{\dagger} \geq 0$.

Then $A^{\dagger} \geq B^{\dagger} \geq 0$.

Next, we state an extension of Theorem 2.1.3, mentioned in the introduction.

Theorem 2.5.4. *For* $A \in \mathbb{R}^{m\times n}$, *let* $A = U - V$ *be a* $B_{\dagger}$*-splitting such that no row of* U *is zero. Suppose that there exists* $x > 0$ *such that* $U^{\dagger}Vx = \lambda x$ *for some* $\lambda < 1$. *Then* $A^{\dagger} \geq 0$.

Converse of the theorem above is not true. However, converse can be recovered in the presence of an additional condition.

Theorem 2.5.5. *For* $A \in \mathbb{R}^{m\times n}$, *let* $A = U - V$ *be a* $B_{\dagger}$*-splitting such that no row of* U *is zero. Suppose that either* $U^{\dagger}V \geq 0$ *and is irreducible or* $U^{\dagger}V > 0$. *If* $A^{\dagger} \geq 0$, *then there exists a vector* $x > 0$ *such that* $U^{\dagger}Vx = \lambda x$ *for some* $\lambda < 1$.

2.5.2 Comparison results

In this subsection, we obtain some generalizations of the results of [20]. The first result of this part is used in deriving comparison results, viz., Theorem 2.5.7 and Theorem 2.5.8.

Theorem 2.5.6. *(Extension of Theorem 3.4, [20]) Let* $A \in \mathbb{R}^{m\times n}$. *Suppose that no row or column of* A *is zero. Let* $A = U - V$ *is a proper splitting such that* $U^{\dagger} \geq 0$ *and* $VU^{\dagger} \geq 0$. *Suppose also that* $A^{\dagger} \geq 0$. *Then there exists* $x \in \mathbb{R}^n_+ \cap R(U^t)$ *such that* $U^{\dagger}Vx = \rho(U^{\dagger}V)x$, *and* $0 \neq Ax \geq 0$. *Moreover, if* $VU^{\dagger}$ *is not nilpotent then* $0 \neq Vx \geq 0$ *for some* $x \in \mathbb{R}^n_+ \cap R(U^t)$.

We now present some applications of Theorem 2.5.6. These are comparison results for the spectral radii of iteration matrices corresponding to two matrices A and B with $A \leq B$. Also, these results are generalizations of Theorem 3.5 and Theorem 4.2 in [20].

Theorem 2.5.7. *Let $A = U_A - V$, $B = U_B - V$ are two proper regular splittings of matrices $A, B \in \mathbb{R}^{m \times n}$ with $V \neq 0$. Suppose that $A^{\dagger} \geq 0$ and no row or column of B is zero. Further, suppose that $VU_B^{\dagger}$ is not nilpotent and $A \leq B$ with $R(A) = R(B)$ and $N(A) = N(B)$. Then $\rho(U_B^{\dagger}V) \leq \rho(U_A^{\dagger}V) < 1$.*

Theorem 2.5.8. *Let $A = U_A - V_A$, $B = U_B - V_B$ are two proper regular splittings of matrices $A, B \in \mathbb{R}^{m \times n}$ with $V_A, V_B \neq 0$. Suppose that $A^{\dagger} - B^{\dagger} > 0$, $A^{\dagger} \geq 0$, $B^{\dagger} \geq 0$ and no row or column of A and B is zero. Suppose also that $V_A U_A^{\dagger}$, $V_B U_B^{\dagger}$ are not nilpotent.*

(i) If $U_B - U_A \leq B - A$ then $\rho(U_B^{\dagger}V_B) \leq \rho(U_A^{\dagger}V_A) < 1$.

(ii) If $U_A^{\dagger} - U_B^{\dagger} \geq A^{\dagger} - B^{\dagger}$ then $\rho(U_A^{\dagger}V_A) < \rho(U_B^{\dagger}V_B) < 1$.

CHAPTER 3

COMPARISON RESULTS FOR PROPER DOUBLE SPLITTINGS OF RECTANGULAR MATRICES

3.1 Introduction

In this short chapter, we derive two comparison results (Theorem 3.3.1 and Theorem 3.3.2) for the spectral radii of certain iteration matrices which are induced by double splittings of rectangular matrices. Our results generalize the comparison results of Shen and Huang (Theorem 3.1 and Theorem 3.2 in [60]) in two directions; first, from square nonsingular matrices to rectangular matrices ; secondly, from classical inverses to Moore-Penrose inverses.

A matrix $A \in \mathbb{R}^{m\times n}$ is called semi-monotone (or generalized monotone) if $A^{\dagger} \geq 0$. In this chapter, we consider two double splittings $A = P_1 - R_1 + S_1$ and $A = P_2 - R_2 + S_2$ of a semi-monotone matrix A. Then, as we discussed in Chapter 1, we could formulate two iterative schemes associated with these splittings. In order to study the convergence rate of these iterative schemes, we need to compare the spectral radii of corresponding iteration matrices. This leads us to derive results which give comparison between spectral radii of iteration matrices. It is well known that the iterative scheme with smaller spectral radius will converge faster. In this connection, several comparison results have been proved. We have reviewed some of these results in Chapter 1.

The motivation for results of this chapter comes from the comparison results of Shen and Huang [60]. They considered certain type of double splittings of a monotone matrix and obtained comparison results. We state those results, next.

Theorem 3.1.1. *(Theorem 3.1, [60]) Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a regular double splitting and* $A = P_2 - R_2 + S_2$ *be a weak regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and any one of the following conditions,*

(i) $P_1^{-1}R_1 \geq P_2^{-1}R_2$

(ii) $P_1^{-1}S_1 \geq P_2^{-1}S_2$

holds, then $\rho(W_1) \leq \rho(W_2) < 1$, *where* $W_1 = \begin{pmatrix} P_1^{-1}R_1 & -P_1^{-1}S_1 \\ I & 0 \end{pmatrix}$ *and*

$W_2 = \begin{pmatrix} P_2^{-1}R_2 & -P_2^{-1}S_2 \\ I & 0 \end{pmatrix}$.

In the next theorem, first splitting is weak regular and second one is regular. However, the conclusion is same as conclusion of Theorem 3.1.1.

Theorem 3.1.2. *(Theorem 3.2, [60]) Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a weak regular double splitting and* $A = P_2 - R_2 + S_2$ *be a regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and any one of the following conditions,*

(i) $P_1^{-1}R_1 \geq P_2^{-1}R_2$

(ii) $P_1^{-1}S_1 \geq P_2^{-1}S_2$

holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

In this chapter, we obtain generalization of Theorem 3.1.1 for rectangular matrices, in Theorem 3.3.1. We do this by using the notion of Moore-Penrose inverse of a matrix. In Theorem 3.3.2, we present generalization of Theorem 3.1.2.

3.2 Definitions and preliminary results

We begin this section with definitions of some subclasses of proper double splittings.

Definition 3.2.1. *Let* $A \in \mathbb{R}^{m\times n}$. *A proper double splitting* $A = P - R + S$ *is called*

(i) regular proper double splitting if $P^{\dagger} \geq 0$, $R \geq 0$ *and* $-S \geq 0$.

(ii) weak regular proper double splitting if $P^{\dagger} \geq 0$, $P^{\dagger}R \geq 0$ *and* $-P^{\dagger}S \geq 0$.

The following result is used in deriving certain results of this chapter.

Lemma 3.2.1. *(Lemma 2.2, [60]) Let* $A = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix} \geq 0$ *and* $\rho(B+C) < 1$. *Then,* $\rho(A) < 1$.

The next result gives the relation between the spectral radius of the iteration matrices associated with a single splitting and a double splitting.

Theorem 3.2.1. *(Theorem 4.3, [43]) Let* $A = P - R + S$ *be a weak regular proper double splitting of* $A \in \mathbb{R}^{m\times n}$. *Then* $\rho(W) < 1$ *if and only if* $\rho(U^{\dagger}V) < 1$, *where* $U = P$, $V = R - S$ *and* $W = \begin{pmatrix} P^{\dagger}R & -P^{\dagger}S \\ I & 0 \end{pmatrix}$.

We conclude this section with a convergence theorem for a proper double splitting of a monotone matrix.

Theorem 3.2.2. *(Theorem 3.6, [28]) Let* $A \in \mathbb{R}^{m\times n}$ *such that* $A^{\dagger} \geq 0$. *Let* $A = P - R + S$ *be a weak regular proper double splitting. Then,* $\rho(W) < 1$.

3.3 Comparison results for proper double splittings

This section deals with main results of this chapter. Let $A \in \mathbb{R}^{m\times n}$. Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two proper double splittings of A. Set $W_1 = \begin{pmatrix} P_1^{\dagger}R_1 & -P_1^{\dagger}S_1 \\ I & 0 \end{pmatrix}$ and $W_2 = \begin{pmatrix} P_2^{\dagger}R_2 & -P_2^{\dagger}S_2 \\ I & 0 \end{pmatrix}$. These matrices are called iteration matrices associated with double splittings, as mentioned in Chapter 1.

The following comparison result gives the comparison between $\rho(W_1)$ and $\rho(W_2)$. This comparison is useful to analyse the rate of convergence of the iterative methods formulated from these double splittings, for solving linear system $Ax = b$.

Theorem 3.3.1. *Let* $A \in \mathbb{R}^{m\times n}$ *be such that* $A^{\dagger} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a regular proper double splitting such that* $P_1P_1^{\dagger} \geq 0$ *and let* $A = P_2 - R_2 + S_2$ *be a weak regular proper double splitting. If* $P_1^{\dagger} \geq P_2^{\dagger}$ *and any one of the following conditions,* *(i)* $P_1^\dagger R_1 \geq P_2^\dagger R_2$

(ii) $P_1^\dagger S_1 \geq P_2^\dagger S_2$

holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. Since $A = P_1 - R_1 + S_1$ is a regular proper double splitting of A, by Theorem 3.2.2, we get $\rho(W_1) < 1$. Similarly, $\rho(W_2) < 1$. It remains to show that $\rho(W_1) \leq \rho(W_2)$.

Assume that $\rho(W_1) = 0$. Then the conclusion follows, obviously. So, without loss of generality assume that $\rho(W_1) \neq 0$. Since $A = P_1 - R_1 + S_1$ is a regular proper double splitting, we have $W_1 = \begin{pmatrix} P_1^\dagger R_1 & -P_1^\dagger S_1 \\ I & 0 \end{pmatrix} \geq 0$. Then, by the Perron-Frobenius theorem, there exists a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n}$, $x \geq 0$ and $x \neq 0$ such that $W_1 x = \rho(W_1)x$. This implies that

$$P_1^\dagger R_1 x_1 - P_1^\dagger S_1 x_2 = \rho(W_1) x_1. \tag{3.1}$$

$$x_1 = \rho(W_1) x_2. \tag{3.2}$$

Upon pre multiplying equation (3.1) by P_1 and using equation (3.2), we get

$$[\rho(W_1)]^2 P_1 x_1 = \rho(W_1) P_1 P_1^\dagger R_1 x_1 - P_1 P_1^\dagger S_1 x_1. \tag{3.3}$$

We have $P_1 P_1^\dagger \geq 0$, $R_1 \geq 0$, $-S_1 \geq 0$ and $x_1 \geq 0$. So, by (3.3), $[\rho(W_1)]^2 P_1 x_1 \geq 0$.

Now, again from (3.3),

$$\begin{aligned} 0 &= [\rho(W_1)]^2 P_1 x_1 - \rho(W_1) P_1 P_1^{\dagger} R_1 x_1 + P_1 P_1^{\dagger} S_1 x_1 \\ &\le \rho(W_1) P_1 x_1 - \rho(W_1) P_1 P_1^{\dagger} R_1 x_1 + \rho(W_1) P_1 P_1^{\dagger} S_1 x_1 \\ &= \rho(W_1)[P_1 x_1 - P_1 P_1^{\dagger}(R_1 - S_1) x_1] \\ &= \rho(W_1)[P_1 x_1 - R_1 x_1 + S_1 x_1] \\ &= \rho(W_1) A x_1, \end{aligned}$$

where we have used the facts that $0 < \rho(W_1) < 1$ and $R(R_1 - S_1) \subseteq R(P_1)$. This proves that $Ax_1 \geq 0$.

Also, by using equations (3.1) and (3.2), we get

$$\begin{aligned} W_2 x - \rho(W_1) x &= \begin{pmatrix} P_2^{\dagger} R_2 x_1 - P_2^{\dagger} S_2 x_2 - \rho(W_1) x_1 \\ x_1 - \rho(W_1) x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_2^{\dagger} R_2 - P_1^{\dagger} R_1) x_1 + \frac{1}{\rho(W_1)} (P_1^{\dagger} S_1 - P_2^{\dagger} S_2) x_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \nabla \\ 0 \end{pmatrix}, \end{aligned}$$

where $\nabla = (P_2^{\dagger} R_2 - P_1^{\dagger} R_1) x_1 + \frac{1}{\rho(W_1)} (P_1^{\dagger} S_1 - P_2^{\dagger} S_2) x_1$.

Case(i) Let us assume that $P_1^{\dagger} R_1 \geq P_2^{\dagger} R_2$. Since $0 < \rho(W_1) < 1$, we get $(P_2^{\dagger} R_2 -$

$P_1^\dagger R_1)x_1 \geq \frac{1}{\rho(W_1)}(P_2^\dagger R_2 - P_1^\dagger R_1)x_1$. Then

$$\begin{aligned}\nabla =&(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ \geq& \frac{1}{\rho(W_1)}(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ =& \frac{1}{\rho(W_1)}[(P_2^\dagger(R_2 - S_2)x_1 - P_1^\dagger(R_1 - S_1)x_1] \\ =& \frac{1}{\rho(W_1)}[P_2^\dagger P_2 - P_2^\dagger A - P_1^\dagger P_1 + P_1^\dagger A]x_1 \\ =& \frac{1}{\rho(W_1)}(P_1^\dagger - P_2^\dagger)Ax_1, \end{aligned} \tag{3.4}$$

where we have used the fact that $P_1^\dagger P_1 = P_2^\dagger P_2$. Since $Ax_1 \geq 0$ and $P_1^\dagger \geq P_2^\dagger$, from the above inequality, we get $\nabla \geq 0$. Then, $W_2x - \rho(W_1)x = \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \geq 0$. This implies that $\rho(W_1)x \leq W_2x$. So, by Lemma 1.1.1, $\rho(W_1) \leq \rho(W_2)$. This proves that $\rho(W_1) \leq \rho(W_2) < 1$.

Case(ii) Assume that $P_1^\dagger S_1 \geq P_2^\dagger S_2$. Since $0 < \rho(W_1) < 1$ and $Ax_1 \geq 0$, again we get

$$\begin{aligned}\nabla =&(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ \geq& (P_2^\dagger R_2 - P_1^\dagger R_1)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ =& (P_1^\dagger - P_2^\dagger)Ax_1 \geq 0. \end{aligned}$$

This implies that $W_2x - \rho(W_1)x = \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \geq 0$.

So, again by Lemma 1.1.1, we get $\rho(W_1) \leq \rho(W_2)$. This proves that $\rho(W_1) \leq \rho(W_2) < 1$.

□

The following example shows that the converse of Theorem 3.3.1 is not true.

Example 3.3.1. *Let* $A = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \end{pmatrix}$. *Let* $P_1 = \begin{pmatrix} 5 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $S_1 = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

and $S_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. *Then* $P_1^\dagger = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$, $P_1^\dagger R_1 = \frac{1}{5}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_1^\dagger S_1 =$

$\frac{1}{5}\begin{pmatrix} -2 & -1 & 0 \\ -5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2^\dagger = \frac{1}{6}\begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$, $P_2^\dagger R_2 = \frac{1}{6}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $P_2^\dagger S_2 = \frac{1}{6}\begin{pmatrix} 0 & -2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

It is easy to verify that $A = P_1 - R_1 + S_1$ *is a regular proper double splitting and* $A = P_2 - R_2 + S_2$ *is a weak regular proper double splitting. Also,* $0.9079 = \rho(W_1) \leq \rho(W_2) = 0.9158 < 1$. *However, the conditions* $P_1^\dagger \geq P_2^\dagger$, $P_1^\dagger R_1 \geq P_2^\dagger R_2$ *and* $P_1^\dagger S_1 \geq P_2^\dagger S_2$ *do not hold.*

Corollary 3.3.1. *(Theorem 3.1, [60]) Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a regular double splitting and* $A = P_2 - R_2 + S_2$ *be a weak regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and any one of the following conditions,*

(i) $P_1^{-1} R_1 \geq P_2^{-1} R_2$

(ii) $P_1^{-1} S_1 \geq P_2^{-1} S_2$

holds, then $\rho(W_1) \leq \rho(W_2) < 1$, *where* $W_1 = \begin{pmatrix} P_1^{-1}R_1 & -P_1^{-1}S_1 \\ I & 0 \end{pmatrix}$ *and*

$W_2 = \begin{pmatrix} P_2^{-1}R_2 & -P_2^{-1}S_2 \\ I & 0 \end{pmatrix}$.

Corollary 3.3.2. *Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a regular double splitting and* $A = P_2 - R_2 + S_2$ *be a weak regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and* $R_1 \geq R_2$

hold, then $\rho(W_1) \leq \rho(W_2) < 1$.

The conclusion of Theorem 3.3.1 can also be achieved by replacing a regular proper double splitting $A = P_1 - R_1 + S_1$ with a weak regular proper double splitting; and a weak regular proper double splitting $A = P_2 - R_2 + S_2$ with a regular proper double splitting, in Theorem 3.3.1. The following is the exact statement of this result.

Theorem 3.3.2. *Let* $A \in \mathbb{R}^{m\times n}$ *such that* $e = (1, 1, ..., 1)^t \in R(A)$ *and* $A^\dagger \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a weak regular proper double splitting and let* $A = P_2 - R_2 + S_2$ *be a regular proper double splitting such that* $P_2^\dagger$ *has no zero row and* $P_2P_2^\dagger \geq 0$. *If* $P_1^\dagger \geq P_2^\dagger$ *and any one of the following conditions,*
(i) $P_1^\dagger R_1 \geq P_2^\dagger R_2$
(ii) $P_1^\dagger S_1 \geq P_2^\dagger S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. Since $A = P_1 - R_1 + S_1$ is a weak regular proper double splitting of A, by Theorem 3.2.2, we get $\rho(W_1) < 1$. Similarly, $\rho(W_2) < 1$. It remains to show that $\rho(W_1) \leq \rho(W_2)$.

Let J be an $m \times n$ matrix in which each entry is equal to 1. For given $\epsilon > 0$, set $A_\epsilon = A - \epsilon J$, $R_1(\epsilon) = R_1 + \frac{1}{2}\epsilon J$, $S_1(\epsilon) = S_1 - \frac{1}{2}\epsilon J$, $R_2(\epsilon) = R_2 + \frac{1}{2}\epsilon J$, $S_2(\epsilon) = S_2 - \frac{1}{2}\epsilon J$, $W_1(\epsilon) = \begin{pmatrix} P_1^\dagger R_1(\epsilon) & -P_1^\dagger S_1(\epsilon) \\ I & 0 \end{pmatrix}$ and $W_2(\epsilon) = \begin{pmatrix} P_2^\dagger R_2(\epsilon) & -P_2^\dagger S_2(\epsilon) \\ I & 0 \end{pmatrix}$. We have, $e = (1, 1, ..., 1)^t \in \mathcal{R}(A)$. So, there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $J = AB$. Then $A_\epsilon = A - \epsilon J = (A - \epsilon AB) = (A - \epsilon AA^\dagger AB) = (A - \epsilon AA^\dagger J) = A(I - \epsilon A^\dagger J)$. Now, choose the above ϵ such that $\rho(\epsilon A^\dagger J) < 1$ and $N(A_\epsilon) = N(A)$. Since $\rho(\epsilon A^\dagger J) < 1$, $I - \epsilon A^\dagger J$ is invertible and hence $R(A_\epsilon) = R(A)$. Then $A_\epsilon = A - \epsilon J$ becomes a proper splitting and thus we can conclude that $A_\epsilon = P_1 - R_1(\epsilon) + S_1(\epsilon)$ is a weak regular proper double splitting and $A_\epsilon = P_2 - R_2(\epsilon) + S_2(\epsilon)$ is a regular proper double splitting.

For the same ϵ, define $X = (I - \epsilon A^\dagger J)^{-1} A^\dagger$, we shall prove that X is the Moore-Penrose inverse of A_ϵ. Let $x \in R(A_\epsilon^t)$. Then

$$\begin{aligned} XA_\epsilon x &= (I - \epsilon A^\dagger J)^{-1} A^\dagger (A - \epsilon AA^\dagger J)x \\ &= (I - \epsilon A^\dagger J)^{-1} (A^\dagger Ax - \epsilon A^\dagger AA^\dagger Jx) \\ &= (I - \epsilon A^\dagger J)^{-1} (x - \epsilon A^\dagger Jx) \\ &= x \end{aligned}$$

and for $y \in N(A_\epsilon^t)$, we get

$$Xy = (I - \epsilon A^\dagger J)^{-1} A^\dagger y = 0$$

Hence, by the definition, $A_\epsilon^\dagger = X = (I - \epsilon A^\dagger J)^{-1} A^\dagger$. Also,
$A_\epsilon^\dagger = (I + \epsilon A^\dagger J + \epsilon (A^\dagger J)^2 + ...)A^\dagger \geq 0$. Then $\rho(P_2^\dagger (R_2(\epsilon) - S_2(\epsilon))) < 1$. So, by Lemma 3.2.1, $\rho(W_2(\epsilon)) < 1$.

Clearly, $P_2^\dagger R_2(\epsilon) > 0$ and $-P_2^\dagger S_2(\epsilon) > 0$. So, $W_2(\epsilon) \geq 0$. Then, by the Perron-Frobenius theorem, there exists a vector $x(\epsilon) = \begin{pmatrix} x_1(\epsilon) \\ x_2(\epsilon) \end{pmatrix} \in \mathbb{R}^{2n}$, $x(\epsilon) \geq 0$ and $x(\epsilon) \neq 0$ such that $W_2(\epsilon)x(\epsilon) = \rho(W_2(\epsilon))x(\epsilon)$. This implies,

$$P_2^\dagger R_2(\epsilon) x_1(\epsilon) - P_2^\dagger S_2(\epsilon) x_2(\epsilon) = \rho(W_2(\epsilon)) x_1(\epsilon) \tag{3.5}$$

$$x_1(\epsilon) = \rho(W_2(\epsilon)) x_2(\epsilon). \tag{3.6}$$

If $\rho(W_2(\epsilon)) = 0$ then from equations (3.5) and (3.6), $x(\epsilon) = 0$. This is a contradiction. So, $0 < \rho(W_2(\epsilon)) < 1$. Then by using equations (3.5) and (3.6), as in the proof of the Theorem 3.3.1, we can show that $\rho(W_2(\epsilon))A_\epsilon x_1(\epsilon) \geq 0$. This implies that $A_\epsilon x_1(\epsilon) \geq 0$. Also, from equations (3.5) and (3.6), we get

$$\begin{aligned}
&W_1(\epsilon)x(\epsilon) - \rho(W_2(\epsilon))x(\epsilon)\\
&= \begin{pmatrix} P_1^\dagger R_1(\epsilon)x_1(\epsilon) - P_1^\dagger S_1(\epsilon)x_2(\epsilon) - \rho(W_2(\epsilon))x_1(\epsilon) \\ x_1(\epsilon) - \rho(W_2(\epsilon))x_2(\epsilon) \end{pmatrix}\\
&= \begin{pmatrix} (P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon) \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} \nabla \\ 0 \end{pmatrix},
\end{aligned}$$

where $\nabla = (P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon)$.

Case(i) Assume that $P_1^\dagger R_1 \geq P_2^\dagger R_2$. Since $0 < \rho(W_2(\epsilon)) < 1$, we get that $(P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) \leq \frac{1}{\rho(W_2(\epsilon))}(P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon)$. Therefore,

$$\begin{aligned}
\nabla &\leq \frac{1}{\rho(W_2(\epsilon))}(P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon)\\
&= \frac{1}{\rho(W_2(\epsilon))}[(P_1^\dagger(R_1(\epsilon) - S_1(\epsilon))x_1(\epsilon) - P_2^\dagger(R_2(\epsilon) - S_2(\epsilon))x_1(\epsilon)]\\
&= \frac{1}{\rho(W_2(\epsilon))}[P_1^\dagger P_1 - P_1^\dagger A_\epsilon - P_2^\dagger P_2 + P_2^\dagger A_\epsilon]x_1(\epsilon)\\
&= \frac{1}{\rho(W_2(\epsilon))}(P_2^\dagger - P_1^\dagger)A_\epsilon x_1(\epsilon)
\end{aligned}$$

where we have used the fact that $P_1^\dagger P_1 = P_2^\dagger P_2$. Since $A_\epsilon x_1(\epsilon) \geq 0$ and $P_1^\dagger \geq P_2^\dagger$, we get that $\nabla \leq 0$. Thus, $W_1(\epsilon)x(\epsilon) - \rho(W_2(\epsilon))x(\epsilon) = \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \leq 0$. This implies, $W_1(\epsilon)x(\epsilon) \leq \rho(W_2(\epsilon))x(\epsilon)$. So, by Lemma 1.1.1, $\rho(W_1(\epsilon)) \leq \rho(W_2(\epsilon))$.

Now, from the continuity of eigenvalues, we have

$$\rho(W_1) = \lim_{\epsilon \to \mathbf{0}} \rho(W_1(\epsilon)) \leq \lim_{\epsilon \to \mathbf{0}} \rho(W_2(\epsilon)) = \rho(W_2).$$

Case(ii) Assume that $P_1^\dagger S_1 \geq P_2^\dagger S_2$. We have $\rho(\epsilon A^\dagger J) < 1$. Choose the above ϵ small enough such that

$$P_1^\dagger S_1 - P_2^\dagger S_2 \geq \frac{\epsilon}{2}(P_1^\dagger - P_2^\dagger)J.$$

Since, $P_1^\dagger S_1(\epsilon) \geq P_2^\dagger S_2(\epsilon)$, $A_\epsilon^\dagger \geq 0$ and $0 < \rho(W_2) < 1$, we get

$$\begin{aligned}\nabla &\leq (P_1^\dagger R_1(\epsilon) - P_2^\dagger R_2(\epsilon))x_1(\epsilon) + (P_2^\dagger S_2(\epsilon) - P_1^\dagger S_1(\epsilon))x_1(\epsilon)\\ &= (P_2^\dagger - P_1^\dagger)A_\epsilon x_1(\epsilon) \leq 0.\end{aligned}$$

This implies that $W_1(\epsilon)x(\epsilon) - \rho(W_2(\epsilon))x(\epsilon) = \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \leq 0.$

So, $W_1(\epsilon)x(\epsilon) \leq \rho(W_2(\epsilon))x(\epsilon)$. Then, by Lemma 1.1.1, $\rho(W_1(\epsilon)) \leq \rho(W_2(\epsilon))$. Similar to the proof of case(i), this implies that $\rho(W_1) \leq \rho(W_2)$. □

The following example illustrates Theorem 3.3.2.

Example 3.3.2. *Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *then* $A^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \geq 0.$

Set $P_1 = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. $P_2 = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $S_2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -3 & 0 \end{pmatrix}$. *Then* $P_1^\dagger = \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$, $P_1^\dagger R_1 =$

$\frac{1}{6}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ *and* $P_1^{\dagger}S_1 = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $P_2^{\dagger} = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$, $P_2^{\dagger}R_2 = \frac{1}{8}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ *and* $P_2^{\dagger}S_2 = \frac{1}{8}\begin{pmatrix} -1 & 0 & -1 \\ 0 & -6 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. *Note that* $A = P_1 - R_1 + S_1$ *is a weak regular proper double splitting and* $A = P_2 - R_2 + S_2$ *is a regular proper double splitting. Also,* $e \in R(A)$, $P_2^{\dagger}$ *has no zero row and* $P_2 P_2^{\dagger} \geq 0$. *We can verify that* $P_1^{\dagger} \geq P_2^{\dagger}$, *and* $P_1^{\dagger}R_1 \geq P_2^{\dagger}R_2$. *Hence* $0.7676 = \rho(W_1) \leq \rho(W_2) = 0.8660 < 1$.

The following result is an obvious consequence of Theorem 3.3.2

Corollary 3.3.3. *(Theorem 3.2, [60]) Let* $A^{-1} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a weak regular double splitting and* $A = P_2 - R_2 + S_2$ *be a regular double splitting. If* $P_1^{-1} \geq P_2^{-1}$ *and any one of the following conditions,*
(i) $P_1^{-1}R_1 \geq P_2^{-1}R_2$
(ii) $P_1^{-1}S_1 \geq P_2^{-1}S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

The next result proof is similar to the proof of Theorem 3.3.1. Thus we skip the proof.

Theorem 3.3.3. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $A^{\dagger} \geq 0$. *Let* $A = P_1 - R_1 + S_1$ *be a weak regular proper double splitting and* $A = P_2 - R_2 + S_2$ *be a weak regular proper double splitting. If* $P_1^{\dagger}A \geq P_2^{\dagger}A$ *and any of the following conditions,*
(i) $P_1^{\dagger}R_1 \geq P_2^{\dagger}R_2$
(ii) $P_1^{\dagger}S_1 \geq P_2^{\dagger}S_2$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

CHAPTER 4

$\{T, S\}$ SPLITTINGS OF RECTANGULAR MATRICES REVISITED

4.1 Introduction

In this chapter, we study some more convergence theorems and comparison theorems of iterative methods for solving singular system of linear equations. But, this time we use a matrix splitting called $\{T, S\}$ splitting. This splitting was introduced by Djordjević and Stanimirović [18]. It is a generalization of well known splittings namely index splitting and proper splitting; and it is applicable in characterization and representation of an outer inverse with prescribed range and null space. Also, it is useful to construct iterative methods which produce solution of a given linear system. In this chapter, we will obtain conditions for convergence of these iterative methods, Also, we obtain comparison results to study the rate of convergence of these iterative methods.

Recently, Mishra and Sivakumar [44] considered subclass of proper splitting of matrices and derived few comparison theorems. Also, in another paper Mishra [43] proposed the extension of the nonnegative splitting for rectangular matrices called proper nonnegative splitting (a decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called proper nonnegative splitting if it is a proper splitting such that $U^{\dagger}V \geq 0$) and established different convergence and comparison theorems. Jena et al. [28] obtained several convergence and comparison theorems for proper regular splittings and proper weak regular splittings of rectangular matrices. We review some of those results to provide motivation for results of this chapter.

The following two results are convergence results for nonnegative splittings of a rectangular matrix.

Theorem 4.1.1. *(Lemma 3.4, [43]) Let $A = U - V$ be a proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$ and $A^{\dagger}U \geq 0$ then* $\rho(U^{\dagger}V) = \dfrac{\rho(A^{\dagger}U) - 1}{\rho(A^{\dagger}U)} < 1.$

Theorem 4.1.2. *(Lemma 3.5, [43]) Let $A = U - V$ be a proper nonnegative splitting* *of* $A \in \mathbb{R}^{m\times n}$. *Then* $A^\dagger V \geq 0$ *if and only if* $\rho(U^\dagger V) = \dfrac{\rho A^\dagger V)}{1+\rho(A^\dagger V)} < 1$.

The next two results are comparison results for proper regular splittings. These results were proved by Jena et al. [28].

Theorem 4.1.3. *(Theorem 3.2, [28]) Let* $A = U_1 - V_1 = U_2 - V_2$ *be two proper regular splittings of* $A \in \mathbb{R}^{m\times n}$. *If* $A^\dagger \geq 0$ *and* $V_2 \geq V_1$, *then* $1 > \rho(U_2^\dagger V_2) \geq \rho(U_1^\dagger V_1)$.

Theorem 4.1.4. *(Theorem 3.3, [28]) Let* $A = U_1 - V_1 = U_2 - V_2$ *be two proper regular splittings of* $A \in \mathbb{R}^{m\times n}$. *If* $A^\dagger \geq 0$ *and* $U_1^\dagger \geq U_2^\dagger$, *then* $1 > \rho(U_2^\dagger V_2) \geq \rho(U_1^\dagger V_1)$.

We generalize the above four results for $\{T, S\}$ splittings in Theorem 4.3.1, Theorem 4.3.2, Theorem 4.4.1 and Theorem 4.4.2, respectively. We collect some definitions and preliminary results in the next section.

4.2 $\{T, S\}$ splittings and characterization of $A^{(2)}_{T,S}$

We begin this section with the definition of $\{T, S\}$ splitting which play a key role in this chapter. Then, we collect results characterizing $A^{(2)}_{T,S}$, in terms of these splittings.

Definition 4.2.1. *Let* $A \in \mathbb{R}^{m\times n}$ *be of rank* r, *let* T *be a subspace of* $\mathbb{R}^n$ *of dimension* $s \leq r$ *and let* S *be a subspace of* $\mathbb{R}^m$ *of dimension* $m-s$. *Then the splitting* $A = U-V$ *is called the* $\{T, S\}$ *splitting of* A *if* $UT \oplus S = \mathbb{R}^m$.

Definition 4.2.2. *Let* $A \in \mathbb{R}^{n\times n}$ *with* $k = ind(A)$. *Then the splitting* $A = U - V$ *is called an index splitting of* A *if* $R(U) = R(A^k)$ *and* $N(U) = N(A^k)$.

Note that $\{T, S\}$ splitting reduces to index splitting if $m = n$, $T = R(U) = R(A^k)$ and $S = N(U) = N(A^k)$, where $k \geq ind(A)$. On the other hand, $\{T, S\}$ splitting reduces to proper splitting if $T = R(U^t) = R(A^t)$ and $S = N(U^t) = N(A^t)$.

The following result characterize the generalized inverse $A^{(2)}_{T,S}$ and it will be used in proving main results of this chapter.

Theorem 4.2.1. *(Theorem 2.1, [18]) Let $A \in \mathbb{R}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{R}^n$ of dimension $s \leq r$ and let S be a subspace of $\mathbb{R}^m$ of dimension $m - s$, such that $AT \oplus S = \mathbb{R}^m$. Assume that $A = U - V$ is a $\{T, S\}$ splitting of A and $dim(T) \leq rank(U)$. Then the generalized inverse $A^{(2)}_{T,S}$ satisfies the following conditions:*
(i) $U^{(2)}_{T,S} - A^{(2)}_{T,S} = -U^{(2)}_{T,S}VA^{(2)}_{T,S} = -A^{(2)}_{T,S}VU^{(2)}_{T,S}$,
(ii) $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S} = U^{(2)}_{T,S}(I - VU^{(2)}_{T,S})^{-1}$ *and*
(iii) $U^{(2)}_{T,S} = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + VA^{(2)}_{T,S})^{-1}$.

If $A = U - V$ is a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, then the following result gives equivalent conditions for $\rho(U^{(2)}_{T,S}V) < 1$.

Theorem 4.2.2. *(Theorem 2.2, [18]) Let $A = U - V$ is a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, such that the conditions of Theorem* 4.2.1 *are satisfied. Further, let $U^{(2)}_{T,S} \geq 0$ and $U^{(2)}_{T,S}V \geq 0$. Then the following statements are equivalent:*
(i) $A^{(2)}_{T,S} \geq 0$.
(ii) $A^{(2)}_{T,S}V \geq 0$.
(iii) $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1 + \rho(A^{(2)}_{T,S}V)} < 1$.

We conclude this section with a result that has motivated us to prove the main results of this chapter.

Theorem 4.2.3. *(Corollary 2.3, [18]) Let $A = U - V$ is a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, such that the conditions of Theorem* 4.2.1 *are satisfied. Suppose that $x \in T$, then:*
(i) The inverse $A^{(2)}_{T,S}b$ is the unique solution of the system $x = U^{(2)}_{T,S}Vx + U^{(2)}_{T,S}b$ for

any $b \in \mathbb{R}^n$.

(ii) The iteration $x^{i+1} = U^{(2)}_{T,S}Vx^i + U^{(2)}_{T,S}b$, $b \in \mathbb{R}^n$, *converges to* $A^{(2)}_{T,S}b$ *for every* $x^\circ \in \mathbb{R}^n$ *if and only if* $\rho(U^{(2)}_{T,S}V) < 1$.

As mentioned in Chapter 1, if the matrix A has two decompositions or splittings then the comparison of the spectral radius of the corresponding iteration matrices, is an important problem in analyzing the iterative scheme $x^{(k+1)} = Hx^{(k)} + c$ of the system $Ax = b$ (here $A = U - V$, where U is nonsingular and the matrix $H = U^{-1}V$ is called an iteration matrix [69] and $c = U^{-1}b$). The comparison of asymptotic rates of convergence of the iterative schemes induced by two splittings of a given matrix has been studied by many authors; for example Varga [69], Elsner and Song, to name a few. We obtain some convergence and comparison theorems for $\{T, S\}$ splittings in the next section.

4.3 Convergence results for $\{T, S\}$ splittings

We begin this section with a result that gives a relation between the eigenvalues of certain matrices induce by the $\{T, S\}$ splitting of a rectangular matrix.

Lemma 4.3.1. *Let* $A = U - V$ *be a* $\{T, S\}$ *splitting of* $A \in \mathbb{R}^{m\times n}$, *such that the conditions of Theorem* 4.2.1 *are satisfied. Let* μ_i, $1 \le i \le p$ *and* λ_j, $1 \le j \le p$ *be the eigenvalues of the matrices* $U^{(2)}_{T,S}V$ *(or* $VU^{(2)}_{T,S}$*) and* $A^{(2)}_{T,S}V$ *(or* $VA^{(2)}_{T,S}$*), respectively. Then for every* j, *we have* $1 + \lambda_j \neq 0$. *Also, for every* i, *there exists* j *such that* $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ *and for every* j, *there exists* i *such that* $\lambda_j = \frac{\mu_i}{1-\mu_i}$.

Proof. Let λ be an eigenvalue of the matrix $U^{(2)}_{T,S}V$ and x be the corresponding eigenvector. By Theorem 4.2.1, $U^{(2)}_{T,S} = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S}$. This implies that $(I + A^{(2)}_{T,S}V)U^{(2)}_{T,S}Vx = A^{(2)}_{T,S}Vx$. Then $(I + A^{(2)}_{T,S}V)\lambda x = A^{(2)}_{T,S}Vx$. Thus $A^{(2)}_{T,S}Vx = \frac{\lambda}{1-\lambda}x$ $(\lambda \neq 1)$. This shows that the matrices $U^{(2)}_{T,S}V$ and $A^{(2)}_{T,S}V$ have the same eigenvectors.

Now, let y be an eigenvector corresponding to the eigenvalue μ_i of the matrix $U^{(2)}_{T,S}V$. Then, y is also an eigenvector of the matrix $A^{(2)}_{T,S}V$ corresponding to some eigenvalue λ_j. By Theorem 4.2.1, $\mu_i y = U^{(2)}_{T,S}Vy = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S}Vy = \frac{\lambda_j}{1+\lambda_j}y$. This implies that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$. Similarly, one can show that for each j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$. □

Note that if $T = R(U^t) = R(A^t)$ and $S = N(U^t) = N(A^t)$ in Lemma 4.3.1 then it reduces to Lemma 2.6 in [44].

We now present a convergent result for $\{T, S\}$ splittings of a rectangular matrix. This result is a generalization of Lemma 3.4 in [43].

Theorem 4.3.1. *Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 4.2.1 are satisfied. Suppose that $A^{(2)}_{T,S}U \geq 0$ and $U^{(2)}_{T,S}V \geq 0$. Then* $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}U) - 1}{\rho(A^{(2)}_{T,S}U)} < 1.$

Proof. We have $U^{(2)}_{T,S}V \geq 0$. So, by the Perron-Frobenius theorem there exists a nonnegative vector $0 \neq x$ such that $U^{(2)}_{T,S}Vx = \rho(U^{(2)}_{T,S}V)x$. Hence $x \in R(U^{(2)}_{T,S}) = T = R(A^{(2)}_{T,S})$ so that $U^{(2)}_{T,S}Ux = x$. Also, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ by Theorem 4.2.1. So, $A^{(2)}_{T,S}U = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}U$. Then $A^{(2)}_{T,S}Ux = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}Ux = (I - U^{(2)}_{T,S}V)^{-1}x = \frac{1}{1-\rho(U^{(2)}_{T,S}V)}x$. Since $A^{(2)}_{T,S}U \geq 0$, it follows that $\frac{1}{1-\rho(U^{(2)}_{T,S}V)}$ is a nonnegative eigenvalue of $A^{(2)}_{T,S}U$. Hence $0 \leq \frac{1}{1-\rho(U^{(2)}_{T,S}V)} \leq \rho(A^{(2)}_{T,S}U)$. This implies that $\rho(U^{(2)}_{T,S}V) \leq \frac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)}$.

Again, the condition $A^{(2)}_{T,S}U \geq 0$ implies existence of a nonnegative vector $0 \neq y$ such that $A^{(2)}_{T,S}Uy = \rho(A^{(2)}_{T,S}U)y$. Then $y \in R(A^{(2)}_{T,S}) = R(U^{(2)}_{T,S})$. Therefore $\rho(A^{(2)}_{T,S}U)y = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}Uy = (I - U^{(2)}_{T,S}V)^{-1}y$. So, we have $(I - U^{(2)}_{T,S}V)^{-1}y = \rho(A^{(2)}_{T,S}U)y$. This implies that $\frac{1}{\rho(A^{(2)}_{T,S}U)}y = y - U^{(2)}_{T,S}Vy$ i.e., $U^{(2)}_{T,S}Vy = \frac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)}y$. Then $\rho(U^{(2)}_{T,S}V) \geq \frac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)}$. Now, from the earlier part of the proof it follows that $\rho(U^{(2)}_{T,S}V) =$

$\dfrac{\rho(A^{(2)}_{T,S}U)-1}{\rho(A^{(2)}_{T,S}U)} < 1.$ □

Next, we obtain another convergence result which generalizes Lemma 3.5 in [43].

Theorem 4.3.2. *Let $A = U - V$ is a $\{T,S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 4.2.1 are satisfied. Suppose that $U^{(2)}_{T,S}V \geq 0$. Then $A^{(2)}_{T,S}V \geq 0$ if and only if $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} < 1$.*

Proof. We first assume that $A^{(2)}_{T,S}V \geq 0$. Let λ and μ be the eigenvalues of $A^{(2)}_{T,S}V$ and $U^{(2)}_{T,S}V$, respectively. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, $\lambda \geq 0$. Then f is a strictly increasing function. We have $\mu = \frac{\lambda}{1+\lambda}$ by Lemma 4.3.1. So, μ attains its maximum when λ is maximum. However, λ is maximum when $\lambda = \rho(A^{(2)}_{T,S}V)$. As a result, the maximum value of μ is $\rho(U^{(2)}_{T,S}V)$. Hence $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} < 1$.
Conversly, assume that $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} < 1$. Since $A = U - V$ is a $\{T,S\}$ splitting, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ by Theorem 4.2.1. The condition $\rho(U^{(2)}_{T,S}V) < 1$ implies that $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k$. Therefore $A^{(2)}_{T,S}V = \sum_{k=1}^{\infty}(U^{(2)}_{T,S}V)^k \geq 0$. □

The following result provides some more properties of $\{T,S\}$ splitting, in addition to the properties discussed in Section 4.2.

Theorem 4.3.3. *Let $A = U - V$ be a $\{T,S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem 4.2.1 are satisfied. Suppose that $U^{(2)}_{T,S}V \geq 0$ and $A^{(2)}_{T,S}V \geq 0$. Then*

(i) $(I - U^{(2)}_{T,S}V)^{-1} \geq 0$.

(ii) $(I - U^{(2)}_{T,S}V)^{-1} \geq I$.

(iii) $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$.

Proof. (i) By Theorem 4.3.2, we have $\rho(U^{(2)}_{T,S}V) < 1$. Then by Theorem 1.1.3, $(I - U^{(2)}_{T,S}V)^{-1}$ exists and $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k \geq 0$ since $U^{(2)}_{T,S}V \geq 0$.
(ii) $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k = I + \sum_{k=1}^{\infty}(U^{(2)}_{T,S}V)^k \geq I$ since $U^{(2)}_{T,S}V \geq 0$.
(iii) From Theorem 4.2.1, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$. So $(I - U^{(2)}_{T,S}V)A^{(2)}_{T,S} = U^{(2)}_{T,S}$. Upon post multiplying by V, we have $A^{(2)}_{T,S}V - U^{(2)}_{T,S}V = U^{(2)}_{T,S}VA^{(2)}_{T,S}V \geq 0$ since $U^{(2)}_{T,S}V \geq 0$ and $A^{(2)}_{T,S}V \geq 0$. Hence $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$. □

Now we prove one more convergent result for $\{T, S\}$ splitting which is a generalization of Theorem 3.9 in [43].

Theorem 4.3.4. *Let $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$ such that the conditions of Theorem* 4.2.1 *are satisfied. Suppose that $U^{(2)}_{T,S}V \geq 0$, $\rho(U^{(2)}_{T,S}V) < 1$ and $A^{(2)}_{T,S} \geq 0$, then*
(i) $A^{(2)}_{T,S} \geq U^{(2)}_{T,S}$.
(ii) $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$.
(iii) $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1 + \rho(A^{(2)}_{T,S}V)}$.

Proof. (i) By Theorem 4.2.1, we have $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ so that $(I - U^{(2)}_{T,S}V)A^{(2)}_{T,S} = U^{(2)}_{T,S}$. Therefore $A^{(2)}_{T,S} - U^{(2)}_{T,S} = U^{(2)}_{T,S}VA^{(2)}_{T,S} \geq 0$. i.e., $A^{(2)}_{T,S} \geq U^{(2)}_{T,S}$.
(ii) $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}$ by Theorem 4.2.1. The condition $\rho(U^{(2)}_{T,S}V) < 1$ implies that $(I - U^{(2)}_{T,S}V)^{-1} = \sum_{k=0}^{\infty}(U^{(2)}_{T,S}V)^k$. Therefore $A^{(2)}_{T,S}V - U^{(2)}_{T,S}V = \sum_{k=2}^{\infty}(U^{(2)}_{T,S}V)^k \geq 0$. i.e., $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V$.
(iii) From (ii) we have $A^{(2)}_{T,S}V \geq U^{(2)}_{T,S}V \geq 0$. Hence $\rho(U^{(2)}_{T,S}V) = \dfrac{\rho(A^{(2)}_{T,S}V)}{1 + \rho(A^{(2)}_{T,S}V)}$ by Theorem 4.3.2. □

We next obtain comparison theorems for $\{T, S\}$ splittings.

4.4 Comparison results for $\{T, S\}$ splittings

Theorem 4.4.1. *Let $A \in \mathbb{R}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{R}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{R}^m$ of dimension $m - s$, such that $AT \oplus S = \mathbb{R}^m$. Let $A = U - V = P - Q$ be two $\{T, S\}$ splittings of A such that $dim(T) \le min\{rank(U), rank(P)\}$ and $U^{(2)}_{T,S} \ge 0$, $P^{(2)}_{T,S} \ge 0$, $V \ge 0$, $Q \ge 0$. If $A^{(2)}_{T,S} \ge 0$ and $Q \ge V$, then $1 > \rho(P^{(2)}_{T,S}Q) \ge \rho(U^{(2)}_{T,S}V)$.*

Proof. By Theorem 4.3.2, we have $\rho(U^{(2)}_{T,S}V) < 1$ and $\rho(P^{(2)}_{T,S}Q) < 1$. Also, $A^{(2)}_{T,S} \ge 0$ and $Q \ge V \ge 0$. Then $A^{(2)}_{T,S}Q \ge A^{(2)}_{T,S}V \ge 0$ and Theorem 1.1.4 yields $\rho(A^{(2)}_{T,S}Q) \ge \rho(A^{(2)}_{T,S}V)$. Let λ_1 and λ_2 be the eigenvalue of $A^{(2)}_{T,S}V$ and $A^{(2)}_{T,S}Q$, respectively. Then $1 > \rho(P^{(2)}_{T,S}Q) = \dfrac{\rho(A^{(2)}_{T,S}Q)}{1+\rho(A^{(2)}_{T,S}Q)} \ge \dfrac{\rho(A^{(2)}_{T,S}V)}{1+\rho(A^{(2)}_{T,S}V)} = \rho(U^{(2)}_{T,S}V)$ since $\frac{\lambda}{1+\lambda}$ is a strictly increasing function. □

Note that if $T = R(U^t) = R(P^t) = R(A^t)$ and $S = N(U^t) = N(P^t) = N(A^t)$ then the above result reduces to Theorem 3.2 in [28]. We illustrates Theorem 4.4.1 with The following example .

Example 4.4.1. *Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Set $U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Also, let $T = \{(x, x, 0)^t : x \in \mathbb{R}\}$ and $S = \{(0, y)^t : y \in \mathbb{R}\}$, then T is a subspace of $\mathbb{R}^3$ with dimension 1 and S is a subspace of $\mathbb{R}^2$ with dimension 1. Further $AT \oplus S = \mathbb{R}^2$, $UT \oplus S = \mathbb{R}^2$ and $PT \oplus S = \mathbb{R}^2$. So $A = U - V = P - Q$ are two $\{T, S\}$ splittings of A and $Q \ge V \ge 0$. Now $A^{(2)}_{T,S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \ge 0$, $U^{(2)}_{T,S} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \ge 0$, $P^{(2)}_{T,S} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \ge 0$, $U^{(2)}_{T,S}V =$*

$\frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ *and* $P^{(2)}_{T,S}Q = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$. *Thus,* $\rho(U^{(2)}_{T,S}V) = \frac{1}{2}$ *and* $\rho(P^{(2)}_{T,S}Q) = \frac{2}{3}$. *Hence* $1 > \rho(P^{(2)}_{T,S}Q) \geq \rho(U^{(2)}_{T,S}V)$.

Another comparison theorem for $\{T, S\}$ splitting is proved, next. This generalizes Theorem 3.3 in [28].

Theorem 4.4.2. *Let* $A = U - V = P - Q$ *be two* $\{T, S\}$ *splittings of* $A \in \mathbb{R}^{m\times n}$ *such that the conditions of Theorem* 4.4.1 *are satisfied. If* $A^{(2)}_{T,S} \geq 0$ *and* $U^{(2)}_{T,S} \geq P^{(2)}_{T,S}$, *then* $1 > \rho(P^{(2)}_{T,S}Q) \geq \rho(U^{(2)}_{T,S}V)$.

Proof. By Theorem 4.2.2, we have $\rho(U^{(2)}_{T,S}V) < 1$ and $\rho(P^{(2)}_{T,S}Q) < 1$. Also $\rho(U^{(2)}_{T,S}V)$ and $\rho(P^{(2)}_{T,S}Q)$ are strictly monotone increasing functions of $\rho(A^{(2)}_{T,S}V)$ and $\rho(A^{(2)}_{T,S}Q)$, respectively. Therefore, it is enough to show that $\rho(A^{(2)}_{T,S}Q) \geq \rho(A^{(2)}_{T,S}V)$. By the hypothesis, we have $A = U - V = P - Q$ two $\{T, S\}$ splittings satisfying the conditions of Theorem 4.4.1 and $A^{(2)}_{T,S} \geq 0$. So, $I + A^{(2)}_{T,S}V$ and $I + QA^{(2)}_{T,S}$ are both invertible and nonnegative. Now $U^{(2)}_{T,S} \geq P^{(2)}_{T,S}$ implies $A^{(2)}_{T,S}(I + QA^{(2)}_{T,S}) \geq (I + A^{(2)}_{T,S}V)A^{(2)}_{T,S}$ i.e., $A^{(2)}_{T,S}QA^{(2)}_{T,S} \geq A^{(2)}_{T,S}VA^{(2)}_{T,S}$. Then post multiplying by Q, and again by V, we have $(A^{(2)}_{T,S}Q)^2 \geq A^{(2)}_{T,S}VA^{(2)}_{T,S}Q$ and $A^{(2)}_{T,S}QA^{(2)}_{T,S}V \geq (A^{(2)}_{T,S}V)^2$. Therefore, by Theorem 1.1.4, we have

$$\rho^2(A^2_{T,S}Q) \geq \rho(A^{(2)}_{T,S}VA^{(2)}_{T,S}Q) = \rho(A^{(2)}_{T,S}QA^{(2)}_{T,S}V) \geq \rho^2(A^{(2)}_{T,S}V).$$

Hence $\rho(A^{(2)}_{T,S}Q) \geq \rho(A^{(2)}_{T,S}V)$. □

The following example illustrates Theorem 4.4.2.

Example 4.4.2. *Let* $A = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. *Set* $U = \begin{pmatrix}2 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}$, $V = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix} \geq 0$,

$P = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \geq 0$. *Also, let* $T = \{(x, 0, 0)^t : x \in \mathbb{R}\}$ *and* $S = \{(0, y)^t : y \in \mathbb{R}\}$, *then* T *is a subspace of* $\mathbb{R}^3$ *with dimension 1 and* S *is a subspace of* $\mathbb{R}^2$ *with dimension 1. Further* $AT \oplus S = \mathbb{R}^2$, $UT \oplus S = \mathbb{R}^2$ *and* $PT \oplus S = \mathbb{R}^2$. *So* $A = U - V = P - Q$ *are two* $\{T, S\}$ *splittings of* A.

Now $A^{(2)}_{T,S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$, $U^{(2)}_{T,S} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $P^{(2)}_{T,S} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $U^{(2)}_{T,S}V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $P^{(2)}_{T,S}Q = \frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *So* $U^{(2)}_{T,S} \geq P^{(2)}_{T,S} \geq 0$ *and* $\rho(U^{(2)}_{T,S}V) = \frac{1}{2}$, $\rho(P^{(2)}_{T,S}Q) = \frac{2}{3}$. *Hence* $1 > \rho(P^{(2)}_{T,S}Q) \geq \rho(U^{(2)}_{T,S}V)$.

CHAPTER 5

MOORE-PENROSE INVERSES OF GRAM MATRICES LEAVING A CONE INVARIANT IN AN INDEFINITE INNER PRODUCT SPACE

5.1 Introduction

In usual inner product setting, a matrix $A \in \mathbb{R}^{m\times n}$ is called cone nonnegative with respect to cones K_1 and K_2 if $AK_1 \subseteq K_2$. In this chapter, we characterize cone nonnegativity of Moore-Penrose inverses of Gram matrices in an indefinite inner product space with respect to indefinite matrix product. This will be done through the acuteness (or obtuseness) property of certain closed convex cones.

As we discussed in Chapter 1, monotonicity has been extended to characterize nonnegativity of generalized inverses. In particular, nonnegativity of the inverse of Gram operators has been studied in connection with certain optimization problems [13], where a characterization is proved. This characterization has been extended to operators between Hilbert spaces in [33] and [62]. In the later article, a completely new approach was proposed to prove this characterization. The following result is a finite dimensional version of this characterization.

Theorem 5.1.1. *Let $A \in \mathbb{R}^{m\times n}$ and K be a closed cone in $\mathbb{R}^n$ with $A^{\dagger}AK \subseteq K$. Then the following conditions are equivalent:*
(i) $(A^{\dagger})^{*}K^{*}$ is acute.
(ii) $(A^{*}A)^{\dagger}K^{*} \subseteq K + N(A)$.
(iii) AK is obtuse.

The above result is a motivation for results of this chapter. The sole aim of the present chapter is to extend this characterization of nonnegativity of the Moore-Penrose inverse of a Gram operator in an indefinite inner product space with the indefinite product of matrices, adopting the approach taken as in [62]. Again here, nonnegativity should be interpreted in terms of taking one cone into another. This result is proved in Theorem 5.3.2.

In Section 5.2, we collect certain preliminary results and fix the notation that will be used in the rest of this chapter. In Section 5.3, we prove a series of results which lead to the main result of this chapter. We conclude this chapter with certain remarks.

5.2 Definitions and preliminary results

We start this section with the definition of a Gram matrix in an indefinite inner product space with respect to an indefinite matrix product. This notion is used throughout this chapter.

Definition 5.2.1. *Let $A \in \mathbb{R}^{m\times n}$. Then $A^{[*]} \circ A$ is called Gram matrix of A with respect to the indefinite matrix product in an indefinite inner product space.*

Definition 5.2.2. *Let $A \in \mathbb{R}^{m\times n}$. The range space of A with respect to the indefinite matrix product, $\mathcal{R}(A)$ is defined by $\mathcal{R}(A) = \{y \in \mathbb{R}^m : y = A \circ x, x \in \mathbb{R}^n\}$ and the null space of A with respect to the indefinite matrix product, $\mathcal{N}(A)$ is defined by $\mathcal{N}(A) = \{x \in \mathbb{R}^n : A \circ x = 0\}$.*

Let $R(A)$ and $N(A)$ denote the range and null spaces of A with respect to the usual matrix product, respectively. Then it follows that $\mathcal{R}(A) = R(A)$ and $\mathcal{N}(A^{[*]}) = N(A^*)$.

We now move on to the definition of the Moore-Penrose inverse in an indefinite inner product space with respect to the indefinite matrix product.

Definition 5.2.3. *Let $A \in \mathbb{R}^{m\times n}$. Then the matrix $X \in \mathbb{R}^{n\times m}$ is called the Moore-Penrose inverse of A if it satisfies the following equations:*

$A \circ X \circ A = A$, $X \circ A \circ X = X$, $(A \circ X)^{[*]} = A \circ X$, $(X \circ A)^{[*]} = X \circ A$.

Such an X will be denoted by $A^{[\dagger]}$. It is shown in [55] that the Moore-Penrose inverse of any matrix exists over an indefinite inner product space with respect to the indefinite matrix product, whereas a similar result is false with the usual matrix product. It easily follows from the definition that for $A \in \mathbb{R}^{m\times n}$, $(A^{[\dagger]})^{[\dagger]} = A$ and $A^{[\dagger]} = NA^{\dagger}M$. If $N = M = I$ then $A^{[\dagger]} = A^{\dagger}$. We refer the reader to [5] (and the references cited there in) for a detailed study of $A^{\dagger}$.

In the next lemma, we collect some more properties of $A^{[\dagger]}$ that will be used in proving main results of this paper. These properties can be proved easily, by the direct verification of definitions.

Lemma 5.2.1. *Let $A \in \mathbb{R}^{m\times n}$. Then*
(i) $\left(A^{[*]}\right)^{[\dagger]} = \left(A^{[\dagger]}\right)^{[*]}$.
(ii) $(A^{[*]} \circ A)^{[\dagger]} = A^{[\dagger]} \circ (A^{[\dagger]})^{[*]}$.
(iii) $(A \circ I)^{[\dagger]} = I \circ A^{[\dagger]}$ *and* $(I \circ A)^{[\dagger]} = A^{[\dagger]} \circ I$.
(iv) $\mathcal{R}(A \circ A^{[\dagger]}) = \mathcal{R}(A)$ *and* $\mathcal{R}(A^{[\dagger]} \circ A) = \mathcal{R}(A^{[*]})$.
(v) $(A^{[*]} \circ A)^{[\dagger]} \circ (A^{[*]} \circ A) = A^{[\dagger]} \circ A$.

We now briefly discuss the notions of a cone and its dual.

Definition 5.2.4. *Let K be a subset of $\mathbb{R}^n$. Then K is called a cone if (i) $x, y \in K \Rightarrow x + y \in K$ and (ii) $x \in K$, and $\alpha \in \mathbb{R}$, $\alpha \geq 0 \Rightarrow \alpha x \in K$. The dual of a cone K in an indefinite inner product space is defined by $K^{[*]} = \{x \in \mathbb{R}^n : [x, t] \geq 0,$ for all $t \in K\}$. K is self dual if $K^{[*]} = K$.*

Let K be a cone, closed in $\mathbb{R}^n$ with usual topology and let K^* denote the dual of the cone K, in the Euclidean setting. Then

$$K^* = \{x \in \mathbb{R}^n : \langle x, t\rangle \geq 0, \text{ for all } t \in K\}$$

and $K^{**} = K$. Note that $K^{[*]} = NK^*$ and $K^{[*][*]} = (K^{[*]})^{[*]} = N^2K = K$. In particular, if $K = \mathbb{R}^n_+$ then $K^{[*]} = I \circ \mathbb{R}^n_+ = N\mathbb{R}^n_+$ and $K^{[*][*]} = \mathbb{R}^n_+$.

In the setting of an indefinite inner product space, a cone C is said to be acute if $[x, y] \geq 0$ for all $x, y \in C$ and C is said to be obtuse if $C^{[*]} \cap \{cl\ span\ C\}$ is acute. In particular, let $C = A \circ I \circ K$ then we say that $C = \{A \circ I \circ x : x \in K\}$ is obtuse if $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. According to Novikoff, the acuteness of a cone C in $\mathbb{R}^n$ is defined by the inclusion $C \subseteq C^*$. We can easily verify this condition in indefinite inner product spaces as $C \subseteq C^{[*]}$.

The next definition is an equivalent definition for cone nonnegativity of a matrix.

Definition 5.2.5. *Let K_1 and K_2 be cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A \in \mathbb{R}^{m\times n}$. Then A leaves a cone invariant (relative to K_1, K_2) with respect to the indefinite matrix product, if $A \circ K_1 \subseteq K_2$.*

Finally, we conclude this section with the following lemma which will be used frequently in this thesis.

Lemma 5.2.2. *(Lemma 2.2, [57]) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then, the linear equation $A \circ x = b$ has a solution iff $b \in \mathcal{R}(A)$. In this case, the general solution is given by $x = A^{[\dagger]} \circ b + z$ where $z \in \mathcal{N}(\mathcal{A})$.*

5.3 Main results

In the setting of an indefinite inner product space, for a given $A \in \mathbb{R}^{m\times n}$, Ramanathan and Sivakumar [57] derived a set of necessary and sufficient conditions for a cone to be invariant under $(A^{[*]} \circ A)^{[\dagger]}$. These conditions include pairwise acuteness (or pairwise obtuseness) of certain cones. In this chapter, we avoid using the notion of pairwise acuteness of cones and characterize the Moore-Penrose inverses of Gram matrices leaving a cone invariant in the approach of Sivakumar [62]. These results generalize the existing results of Sivakumar [62] in the finite dimensional setting from Euclidean spaces to indefinite inner product spaces.

In this section, we prove a series of results that lead up to the main theorem (Theorem 5.3.2). Some of these are interesting in their own right. First, we fix some notations. Throughout this section, we consider an $m \times n$ real matrix A satisfying the condition $A \circ I = I \circ A$ (i.e., $AN = MA$, where M, N are weight matrices) and K be a cone, closed in $\mathbb{R}^n$ with respect to the indefinite matrix (or vector) product. Also, we make a note that for any $A \in \mathbb{R}^{m \times n}$, if $A \circ I = I \circ A$ then $A^{[*]} = NA^*M = (MAN)^* = (ANN)^* = A^*$.

Lemma 5.3.1. *Let $A \in \mathbb{R}^{m\times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product. Then*

(i) $[A \circ x, y] = [x, A^{[*]} \circ y]$ *for all $x \in \mathbb{R}^n$ and for all $y \in \mathbb{R}^m$.*

(ii) $u \in (A \circ I \circ K)^{[*]} \Rightarrow (A \circ I)^{[*]} \circ u \in K^{[*]}$.

(iii) $A^{[\dagger]} \circ A \circ K \subseteq K \Leftrightarrow A^{[\dagger]} \circ A \circ K^{[*]} \subseteq K^{[*]}$.

Proof.
(i) $[A\circ x, y] = \langle A\circ x, My\rangle = \langle ANx, My\rangle = \langle x, NA^*My\rangle = [x, A^*My] = [x, A^{[*]}My] = [x, A^{[*]} \circ y]$, since $A^{[*]} = A^*$.
(ii) Let $u \in (A \circ I \circ K)^{[*]}$ and $r \in K$. Then $0 \leq [u, A \circ I \circ r] = [(A \circ I)^{[*]} \circ u, r]$, by part (i). Thus $(A \circ I)^{[*]} \circ u \in K^{[*]}$.
(iii) Let $A^{[\dagger]} \circ A \circ K \subseteq K$, $y = A^{[\dagger]} \circ A \circ x$ with $x \in K^{[*]}$, $u \in K$ and $u^1 = A^{[\dagger]} \circ A \circ u \in K$. Then $[y, u] = [A^{[\dagger]} \circ A \circ x, u] = [x, (A^{[\dagger]} \circ A)^{[*]} \circ u] = [x, A^{[\dagger]} \circ A \circ u] = [x, u^1] \geq 0$. This shows that $y \in K^{[*]}$. Hence $A^{[\dagger]} \circ A \circ K^{[*]} \subseteq K^{[*]}$. Similarly, one can easily prove the converse part. □

The condition (iii) in Lemma 5.3.1 is equivalent to "K is invariant under $A^{[\dagger]} \circ A$ if and only if $K^{[*]}$ is invariant under $A^{[\dagger]} \circ A$".

In the next result, we determine the set $((A^{[\dagger]})^{[*]} \circ I \circ K^{[*]})^{[*]}$ under the condition $A^{[\dagger]} \circ A \circ K \subseteq K$.

Theorem 5.3.1. *Let $A \in \mathbb{R}^{m\times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product satisfying the condition $A^{[\dagger]} \circ A \circ K \subseteq K$. Then $((A^{[\dagger]})^{[*]} \circ I \circ K^{[*]})^{[*]} = A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$.*

Proof. First, we prove that

$$(A \circ I \circ K)^{[*]} = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} + \mathcal{N}((A \circ I)^{[*]}). \tag{5.1}$$

For this, let $y \in (A \circ I \circ K)^{[*]}$. Then by part (ii) of Lemma 5.3.1, $z = (A \circ I)^{[*]} \circ y \in K^{[*]}$. So, by Lemma 5.2.2, $y = ((A \circ I)^{[*]})^{[\dagger]} \circ z + w$ for some $w \in \mathcal{N}((A \circ I)^{[*])}$. Then $y \in ((A \circ I)^{[*]})^{[\dagger]} \circ K^{[*]} + \mathcal{N}((A \circ I)^{[*]}) = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} + \mathcal{N}((A \circ I)^{[*]})$, by part (i) and (iii) of Lemma 5.2.1. This proves $(A \circ I \circ K)^{[*]} \subseteq (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} + \mathcal{N}((A \circ I)^{[*]})$.

Next, let $u = u^1 + u^2$, where $u^1 = (A^{[\dagger]})^{[*]} \circ I \circ l$ with $l \in K^{[*]}$ and $u^2 \in \mathcal{N}((A \circ I)^{[*]})$. Let $v = A \circ I \circ t$, $t \in K$ and set $t' = A^{[\dagger]} \circ A \circ t \in K$. Then $[u, v] = [u^1 + u^2, v] = [u^1, v] + [u^2, v] = [u^1, A \circ I \circ t] = [(A^{[\dagger]})^{[*]} \circ I \circ l, A \circ I \circ t] = [l, (A \circ I)^{[\dagger]} \circ A \circ I \circ t] = [l, A^{[\dagger]} \circ A \circ t] = [l, t'] \geq 0$, since $[u^2, v] = [u^2, A \circ I \circ t] = 0$ and by part (iii) of Lemma 5.2.1. Thus $u \in (A \circ I \circ K)^{[*]}$. This proves (1).

Now, we replace A by $((A^{[\dagger]})^{[*]}$ and K by $K^{[*]}$ in the equation (1), and use part (iii) of Lemma 5.3.1 to get the desired result. □

Remark 5.3.1. *The following example shows that Theorem 5.3.1 may not hold in the absence of the condition $A^{[\dagger]} \circ A \circ K \subseteq K$.*

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. *Then* $A^\dagger = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$ *and* $A^{[\dagger]} = NA^\dagger M = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$. *Let* $K = \mathbb{R}^3_+$ *then* $K^{[*]} = N\mathbb{R}^3_+$.

Suppose $x = (1,2,3)^t$. *Then* $A^{[\dagger]} \circ A \circ x = (1, \frac{-1}{2}, \frac{1}{2}) \notin K$. *Thus* $A^{[\dagger]} \circ A \circ K \nsubseteq K$. *Also* $A \circ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. *Therefore,* $\mathcal{N}((A \circ I)^{[*]})$ *contains only the zero vector. Let* $y = (2,5,8)^t \in K$ *and set* $y^1 = A \circ I \circ y = Ay = (2,3)^t \in A \circ I \circ K$. *Let* $v = N(1,2,0)^t = (1,-2,0)^t \in K^{[*]}$ *and* $z = (A^{[\dagger]})^{[*]} \circ I \circ v = (1,1)^t \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$. *Then* $[y^1, z] = \langle y^1, Mz \rangle = \langle (2,3)^t, (1,-1)^t \rangle < 0$, *so that* $y^1 \notin ((A^{[\dagger]})^{[*]} \circ I \circ K^{[*]})^{[*]}$.

5.3.1 Acuteness of cone

The following result is useful to prove the acuteness of certain cones.

Lemma 5.3.2. *Let* $A \in \mathbb{R}^{m \times n}$ *be such that* $A \circ I = I \circ A$ *and let* K *be a closed cone in* $\mathbb{R}^n$ *with respect to the indefinite matrix product satisfying the condition* $A^{[\dagger]} \circ A \circ K \subseteq K$. *Then* $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$.

Proof. Let $y = A \circ I \circ x \in (A \circ I \circ K)^{[*]}$. Then by part (*ii*) of Lemma 5.3.1, $(A \circ I)^{[*]} \circ y \in K^{[*]}$. Also, $y = (A \circ I) \circ (A \circ I)^{[\dagger]} \circ y = ((A \circ I) \circ (A \circ I)^{[\dagger]})^{[*]} \circ y = ((A \circ I)^{[\dagger]})^{[*]} \circ (A \circ I)^{[*]} \circ y = (A^{[\dagger]})^{[*]} \circ I \circ (A \circ I)^{[*]} \circ y \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$, proving that $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$.

Conversely, suppose that $x \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$. Then $x = ((A \circ I)^{[\dagger]})^{[*]} \circ u$ for some $u \in K^{[*]}$. This implies $x \in \mathcal{R}(A \circ I)$. Let $w \in K$, $v = A \circ I \circ w \in A \circ I \circ K$ and $w^1 = A^{[\dagger]} \circ A \circ w \in K$. Then we have $[x, v] = [(A^{[\dagger]})^{[*]} \circ I \circ u, A \circ I \circ w] = [u, A^{[\dagger]} \circ A \circ w] = [u, w^1] \geq 0$. Thus $x \in (A \circ I \circ K)^{[*]}$. □

In the next lemma, we obtain an equivalent condition for the acuteness of the cone $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$.

Lemma 5.3.3. *Let* $A \in \mathbb{R}^{m \times n}$ *be such that* $A \circ I = I \circ A$ *and let* K *be a closed cone in* $\mathbb{R}^n$ *with respect to the indefinite matrix product satisfying the condition* $A^{[\dagger]} \circ A \circ K \subseteq K$. *Then* $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ *is acute if and only if* $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq A \circ I \circ K$.

Proof. Suppose that $L = (A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. Then $L \subseteq L^{[*]}$. By Lemma 5.3.2 and Theorem 5.3.1, it follows that $L^{[*]} = ((A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I))^{[*]} = ((A^{[\dagger]})^{[*]} \circ I \circ K^{[*]})^{[*]} = A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$. So, $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$. However, we have to show that $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq A \circ I \circ K$. Let $x \in (A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$. Then $x = A \circ I \circ u + z$, with $u \in K$, $z \in \mathcal{N}((A \circ I)^{[*]})$. Since x and $A \circ I \circ u \in \mathcal{R}(A \circ I)$, it follows that $z \in \mathcal{R}(A \circ I) \cap \mathcal{N}((A \circ I)^{[*]}) = \{0\}$. Thus $x \in A \circ I \circ K$.

Conversely, let $x, y \in (A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq A \circ I \circ K$. Then $x = A \circ I \circ u$, $u \in K$. We also have $(A \circ I)^{[*]} \circ y \in K^{[*]}$. Now, $[x, y] = [A \circ I \circ u, y] = [u, (A \circ I)^{[*]} \circ y] \geq 0$. Thus $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. □

5.3.2 Cone nonnegativity of Moore-Penrose inverses of Gram matrices

In this section, we prove main results of this chapter. First, we obtain a necessary and sufficient conditions for a cone to be invariant under $(A^{[*]} \circ A)^{[\dagger]}$.

Lemma 5.3.4. *Let $A \in \mathbb{R}^{m \times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product satisfying the condition $A^{[\dagger]} \circ A \circ K \subseteq K$. Then the following are equivalent:*

(i) $(A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} \subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$.

(ii) $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.

(iii) $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K$.

Proof. $(i) \Rightarrow (ii)$:

For $x \in K^{[*]}$, let $y = (A^{[*]} \circ A)^{[\dagger]} \circ x = ((A \circ I)^{[*]} \circ (A \circ I))^{[\dagger]} \circ x = (A \circ I)^{[\dagger]} \circ$

$((A \circ I)^{[\dagger]})^{[*]} \circ x$. Then

$$\begin{aligned} A \circ I \circ y &= (A \circ I) \circ (A \circ I)^{[\dagger]} \circ ((A \circ I)^{[\dagger]})^{[*]} \circ x \\ &= ((A \circ I)^{[\dagger]})^{[*]} \circ x \\ &= (A^{[\dagger]})^{[*]} \circ I \circ x \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} \\ &\subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]}). \end{aligned}$$

Therefore $A \circ I \circ y = A \circ I \circ v + w$, $v \in K$, $w \in \mathcal{N}((A \circ I)^{[*]})$. So , $A \circ I \circ (y - v) \in \mathcal{R}(A \circ I) \cap \mathcal{N}((A \circ I)^{[*]}) = \{0\}$. Then $A \circ I \circ (y - v) = 0$. This implies, $y - v = u \in \mathcal{N}(A \circ I)$. Then $y = u + v$, $v \in K$, $u \in \mathcal{N}(A \circ I)$. This shows that $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.

$(ii) \Rightarrow (i)$:

Let $y = (A^{[\dagger]})^{[*]} \circ I \circ x$, $x \in K^{[*]}$. Then $y = ((A \circ I)^{[\dagger]})^{[*]} \circ x$ and $(A \circ I)^{[\dagger]} \circ y = (A \circ I)^{[\dagger]} \circ ((A \circ I)^{[\dagger]})^{[*]} \circ x = ((A \circ I)^{[*]} \circ (A \circ I))^{[\dagger]} \circ x = (A^{[*]} \circ A)^{[\dagger]} \circ x = u + v$, $u \in K, v \in \mathcal{N}(A \circ I)$. This implies that $y = ((A \circ I)^{[\dagger]})^{[\dagger]} \circ (u + v) + w$, $w \in \mathcal{N}((A \circ I)^{[\dagger]})$. Thus $y = A \circ I \circ u + w \in A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$.

$(ii) \Rightarrow (iii)$:

Let $x \in K^{[*]}$ and $y = (A^{[*]} \circ A)^{[\dagger]} \circ x$. Then $(A^{[*]} \circ A)^{[\dagger]} \circ x = u + v$ where $u \in K$, $v \in \mathcal{N}(A \circ I)$. This implies $x = (A^{[*]} \circ A) \circ (u + v) + w$, $w \in \mathcal{N}(A \circ I)$, so that $y = (A^{[*]} \circ A)^{[\dagger]} \circ (A^{[*]} \circ A) \circ u = A^{[\dagger]} \circ A \circ u \in K$, by part (v) of Lemma 5.2.1.

$(iii) \Rightarrow (ii)$:

This part is obvious. □

We also have a stronger one-way implication, given below.

Lemma 5.3.5. *Let $A \in \mathbb{R}^{m \times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product. If $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$ then $K^{[*]} \cap \mathcal{R}(A \circ I)^{[*]} \subseteq A^{[*]} \circ A \circ K + \mathcal{N}((A \circ I)$.*

Proof. Let $y = (A \circ I)^{[*]} \circ x \in K^{[*]}$. Then $(A^{[*]} \circ A)^{[\dagger]} \circ y = u + z$, $u \in K$, $z \in \mathcal{N}(A \circ I)$. From this $y = (A^{[*]} \circ A) \circ (u + z) + w$, $w \in \mathcal{N}(A^{[*]} \circ A)^{[\dagger]}$. Since $A^{[*]} \circ A = (A \circ I)^{[*]} \circ (A \circ I)$ and $z \in \mathcal{N}(A \circ I)$, we get $y = A^{[*]} \circ A \circ u + w \in A^{[*]} \circ A \circ K + \mathcal{N}(A \circ I)$. □

We are now in a position to prove the main result of this chapter.

Theorem 5.3.2. *(Main Result) Let $A \in \mathbb{R}^{m \times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product satisfying the condition $A^{[\dagger]} \circ A \circ K \subseteq K$. Let $C = A \circ I \circ K$ and $D = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$. Then the following conditions are equivalent:*
(i) D is acute.
(ii) $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.
(iii) C is obtuse.

Proof. $(i) \Rightarrow (ii)$:
Suppose D is acute then by definition, $D \subseteq D^{[*]}$. By Theorem 5.3.1, $D^{[*]} = A \circ I \circ K + \mathcal{N}(A \circ I)^{[*]}$. Thus $D \subseteq A \circ I \circ K + \mathcal{N}(A \circ I)^{[*]}$. Now, by Lemma 5.3.4, we obtain $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.
$(ii) \Rightarrow (i)$:
Suppose $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$. By Lemma 5.3.4, $D \subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$. Since $A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]}) = D^{[*]}$ by Theorem 5.3.1, we get $D \subseteq D^{[*]}$. Hence D is acute.
$(ii) \Rightarrow (iii)$ Suppose $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$. Note that $C = A \circ I \circ K$ is obtuse if $C^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. By Lemma 5.3.3, it is enough to show that $C^{[*]} \cap \mathcal{R}(A \circ I) \subseteq C$.

Let $y \in C^{[*]} \cap \mathcal{R}(A \circ I)$. Then $y = A \circ I \circ x$ and by part (ii) of Lemma 5.3.1, $(A \circ I)^{[*]} \circ y \in K^{[*]}$. So, $(A \circ I)^{[*]} \circ y \in K^{[*]} \cap \mathcal{R}(A \circ I)^{[*]}$. By Lemma 5.3.5, $(A \circ I)^{[*]} \circ y = A^{[*]} \circ A \circ u + z$ with $u \in K$, $z \in \mathcal{N}(A \circ I)$. Since $A^{[*]} \circ A = (A \circ I)^{[*]} \circ (A \circ I)$, it follows that $(A \circ I)^{[*]} \circ y, A^{[*]} \circ A \circ u \in \mathcal{R}(A \circ I)^{[*]}$. Thus $z \in \mathcal{R}(A \circ I)^{[*]} \cap \mathcal{N}(A \circ I) = \{0\}$. This implies $z = 0$. Then $(A \circ I)^{[*]} \circ y = A^{[*]} \circ A \circ u$. From this,

$$\begin{aligned} y &= ((A \circ I)^{[\dagger]})^{[*]} \circ ((A \circ I)^{[*]} \circ A \circ I \circ u) + w \\ &= ((A \circ I) \circ (A \circ I)^{[\dagger]})^{[*]} \circ (A \circ I) \circ u + w \\ &= (A \circ I) \circ (A \circ I)^{[\dagger]} \circ (A \circ I) \circ u + w \\ &= (A \circ I) \circ u + w, \end{aligned}$$

where $w \in \mathcal{N}((A \circ I)^{[*]})$.

Since $y \in \mathcal{R}(A \circ I)$, it follows that $w \in \mathcal{R}(A \circ I) \cap \mathcal{N}(A \circ I)^{[*]}) = \{0\}$. Thus $y \in A \circ I \circ K = C$.

$(iii) \Rightarrow (ii)$:

Let $C = A \circ I \circ K$ be obtuse. Then by definition, $C^{[*]} \cap \mathcal{R}(A \circ I) \subseteq C$. By Lemma 5.3.2, $(A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} \subseteq C$. Now by Lemma 5.3.4, $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$. $\square$

Corollary 5.3.1. *In addition to the conditions of Theorem 5.3.2, suppose that K is self dual (i.e., $K^{[*]} = K$). Then the conditions (i) and (iii) are equivalent to $(A^{[*]} \circ A)^{[\dagger]} \circ K \subseteq K + \mathcal{N}(A \circ I)$.*

The above corollary and Lemma 5.3.4 shows that $(A^{[*]} \circ A)^{[\dagger]}$ is cone invariant that justifies the title of the chapter.

5.3.3 Some remarks

(i) The inclusion $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$ does not appear to imply $(A^{[*]} \circ A)^{[\dagger]}$ leaves a cone invariant. However, due to Lemma 5.3.4 this inclusion is equivalent to $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K$ which clearly shows our requirement.

(ii) The following example illustrates Theorem 5.3.2. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $K = \mathbb{R}^3_+$. Then $A^\dagger = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$, $A^{[\dagger]} = NA^\dagger M = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $K^{[*]} = N\mathbb{R}^3_+$. Note that for $x^1 = (x, y, z)^t \in K$, $A^{[\dagger]} \circ A \circ x^1 = A^{[\dagger]}Ax^1 = \frac{1}{2}(x + z, 0, x + z)^t \in K$. Thus $A^{[\dagger]} \circ A \circ K \subseteq K$. And $(A^{[*]} \circ A)^\dagger = \frac{1}{16}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$. Therefore $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} = N(A^{[*]} \circ A)^\dagger NK^{[*]} \subseteq K$. Also one can easily verify that $C = A \circ I \circ K$ is obtuse and $D = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$ is acute.

(iii) Here, we show by an example that in the absence of the condition $A \circ I = I \circ A$, Theorem 5.3.2 may not hold. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = N$. Then clearly $A \circ I \neq I \circ A$. Let $K = \{(x, 0) : x \geq 0\}$ then $K^* = \{(x, y) : x \geq 0, y \in \mathbb{R}\}$ and $K^{[*]} = \{(y, x) : x \geq 0,\ y \in \mathbb{R}\}$. Also, $A^\dagger = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $A^{[\dagger]} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly $A^{[\dagger]} \circ A \circ K \subseteq K$ and $D = \left\{(\frac{x}{2}, \frac{x}{2}) : x \geq 0\right\}$ is acute but $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \nsubseteq K$ where $(A^{[*]} \circ A)^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

CHAPTER 6

CONCLUSIONS

Our thesis is primarily motivated by theoretical considerations. It studies generalizations of various characterizations of nonnegativity of group inverses and Moore-Penrose inverses. It also presents some convergence and comparison theorems. The highlights are the following.

- We obtain an extension of Fan's result (Lemma 2, [21]) from classical inverses to group inverses (Theorem 2.3.1).
- We generalize the result of Barker (Proposition 9, [3]) to the case of group inverse (Theorem 2.3.2).
- We derive a group inverse analogue of an important result of Elsener et al. (Theorem 3.4, [20]), in Theorem 2.4.1.
- We obtain certain comparison results as an application of Theorem 2.4.1.
- We extend the comparison results for double splittings of Shen and Huang (Theorem 3.1 and Theorem 3.2 in [60]) from ordinary inverses to Moore-Penrose inverses (Theorem 3.3.1 and Theorem 3.3.2).
- We obtain certain convergence and comparison results (Theorem 4.3.4, Theorem 4.4.1 and Theorem 4.4.2) similar to the results of ([43] and [28]) by using $\{T, S\}$ splittings which are introduced by Djordjević and Stanimirović.
- We characterize the Moore-Penrose inverses of Gram matrices leaving a cone invariant in terms of obtuseness or acuteness of certain cones (Theorem 5.3.2), generalizing the Sivakumar's result [62] from finite dimensional real Euclidean space to indefinite inner product space.

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