SOME STUDIES ON FUZZY SOLID TRANSPORTATION PROBLEMS WITH ROUGH INTERVALS, STOCHASTIC AND BUDGET CONSTRAINTS

A

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BY

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UNDER THE SUPERVISION OF

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Dedicated to

My Parents: Smt. Vijaya Laxmi &
Sri. Sadanandam

and

My Sister: Niveditha

DEPARTMENT OF MATHEMATICS
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CERTIFICATE

This is to certify that the thesis entitled **"SOME STUDIES ON FUZZY SOLID TRANSPORTATION PROBLEMS WITH ROUGH INTERVALS, STOCHASTIC AND BUDGET CONSTRAINTS"** submitted to the Department of Mathematics, National Institute of Technology, Warangal, is a record of bonafide research work carried out by **Mr. GOPAGANI NITHISH KUMAR**, Roll No. **701319**, for the award of Degree of Doctor of Philosophy in Mathematics under my supervision.

The contents of the thesis have not been submitted elsewhere for the award of any degree or diploma.

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DECLARATION

This is to certify that the work presented in the thesis entitled **"SOME STUDIES ON FUZZY SOLID TRANSPORTATION PROBLEMS WITH ROUGH INTERVALS, STOCHASTIC AND BUDGET CONSTRAINTS"** is a bonafide work done by me under the supervision of **Prof. Debashis Dutta** and was not submitted elsewhere for the award of any degree.

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-G. Nithish Kumar

About Re-Revision of Thesis (December 2017)

During the Re-Revision, drastic changes have been made.

1. From the suggestions of the reviewers, some theoretical contents also **updated**.
2. **Algorithms** are given in Chapter 2 to Chapter 8 respectively.
3. Chapter wise conclusions are **removed**. The **overall** conclusions are made in Chapter 9.
4. More possible comparative studies have been **incorporated**.
5. The style of presentation is **updated**.
6. A separate section on the literature on Expected value is compiled.

-G. Nithish Kumar

Contents

Chapter 1

Introduction

In this Chapter, the entire thesis writing began. Section 1.1 provides the abstract of the thesis. Section 1.2 gives some key words. Section 1.3 presents a list of abbreviations used in the thesis. Section 1.4 introduces some basic concepts and definitions concerning the thesis. Section 1.5 presents the literature review in detail.

1.1 Motivation and Abstract of the Thesis

The transportation problem (TP) is a well-known optimization problem in operational research, in which two kinds of constraints are taken into consideration, i.e., source constraint and destination constraint. But in the real system, we always deal with other constraints besides the source constraint and destination constraint, such as product type constraint or transportation mode constraint.

If more than one objective is to be considered and optimized at the same time in a STP, then the problem is called multi-objective solid transportation problem (MOSTP). Besides the source, destination and conveyance capacity in an STP, there may exist some other constraints. For example, budget constraints may arise due to limited budget, space constraints may arise due to limited space in warehouses, stores, etc.

Due to insufficient information, lack of evidence, fluctuating financial market, the available data of a transportation system such as transportation costs, resources, demands and conveyance capacities are not always crisp or precise. For example the transportation cost depends upon fuel price, tax charges, labour charges, etc., each of which are fluctuated from time to time. It will be more realistic to express those parameters by fuzzy numbers.

Fuzzy set theory is a generalization of the conventional set theory to represent vagueness or imprecision in everyday life. Thus, fuzzy sets have found applications in various fields such as Pattern Recognition, Optimization Techniques, etc. In fuzzy set theory, an element x in a fuzzy set A has a degree of membership $\mu_A(x)$. The range of the membership function is [0,1]. A fuzzy number is a convex normalized fuzzy set of the real line R, with a piecewise continuous membership function.

In order to deal with the uncertain optimization problems, fuzzy and stochastic approaches are commonly used to describe the imprecise characteristics. In stochastic programming the uncertain coefficients were regarded as random variables and their probability distributions are assumed to be known. In fuzzy programming the constraints and objective function are viewed as fuzzy sets and their membership functions also need to be known. In these two kinds of approaches, the membership functions and probability distributions play an important roles. However, it is sometimes difficult to specify an appropriate membership function or accurate probability distribution in an uncertain environment.

The fixed charge solid transportation problem is an extension of classical transportation problem in which a fixed profit is incurred, independent of the amount transported, along with a variable profit that is proportional to the amount shipped. The fixed charge solid transportation has two kinds of profits: direct profit and fixed charge profit.

The entire thesis divided into nine chapters.

Chapter one is introductory in nature. Chapters two, three and four explored the fuzzy solid transportation, fractional solid transportation and stochastic solid transportation models with interval budget constraints respectively. Chapters five and six study the multi-objective solid transportation problems through goal programming approach in the presence of a defuzzification model. Chapters seven and eight deal with Rough interval approach. Algorithms are given by Chapter two to eight and study of comparisons with other methods are included in Chapter five and eight. Chapter nine gives the key findings and scope for future study.

The brief description of each chapter is given below.

The aim of the present thesis is to present numerical solutions for solid transportation models with various conditions. In the real world, the parameters in the models are seldom known exactly and have to be estimated.

Chapter - 1 is introductory in nature and gave motivation to the investigations carried out in the thesis. A brief survey of relevant literature, notations, abstract and keywords were drawn to exhibit the importance of the problems considered.

Chapter - 2, deals with fuzzy solid transportation problem with interval budget constraint. The objective is to determine the optimal total cost of the solid transportation problem with the supply, the demand and the conveyance satisfying the transportation requirement. A method is proposed for the fuzzy objective value of the fuzzy solid transportation problem. Based on the extension principle, the fuzzy solid transportation problem is transformed into a pair of mathematical programs that is employed to calculate the lower and upper bounds of the fuzzy total transportation cost at possibility level α. From different values of α, the membership function of the objective value is constructed. To illustrate the results of the proposed model, we have given numerical example and presented the computational result.

Chapter - 3, deals with the solid fractional transportation problem (SFTP). Based on the α -cut representation of fuzzy sets and the extension principle, a fractional program is formulated to find the fuzzy objective value of fuzzy SFTP when, the cost coefficients, supply, demand quantities and conveyance capacities are fuzzy numbers and the additional constraints on the total budget at each destination which is an interval type. Under Zadeh's extension principle, a pair of two level mathematical programs is formulated to calculate the fuzzy objective value of SFTP with fuzzy parameters. By applying the dual formulation and variable substitution techniques, the two-level mathematical programs are transformed into one level linear program. Taking different values of α, the membership function of objective value is constructed and a numerical example was given to illustrate the proposed model.

In **Chapter - 4**, a fuzzy stochastic solid transportation problem (FSSTP) was formulated with random demand and capacities of conveyances with budget constraints. Goal programming (GP) approach was applied to solve the said solid transportation problem (STP). This chapter also presents fuzzy goal programming models (FGP) for the stochastic aspect in STP. It was considered that the demand, conveyance capacities are random and expressed as fuzzy-stochastic constraints. Two dimensional representation of a proposed FSSTP was derived and solved numerically. The optimum results of this model are compared with the solid transportation model with different budgets.

Chapter - 5 considers the multi-objective solid transportation problem with fuzzy coefficients for the objectives and constraints is modeled and then solved. Fuzzy goal programming was used to the multi-objective solid transportation problem, and an optimal compromise solution obtained. Meanwhile, expected values of the fuzzy objective functions are considered to derive crisp values. In this method, a defuzzification model, which is an applications of fuzzy linear programming and conditions for a solid transportation problem are imposed. Three numerical examples were presented using the above mentioned methodology and the appropriate comparative study is also included.

In **Chapter - 6**, an attempt has been made to study the multi objective solid transportation problem with L-R coefficients. As for the object with L-R coefficient in the linear programming, the Researcher integrated the method, which could also be changed into to a fuzzy optimal solution to multi-object linear programming. Meanwhile, determination of this model may cause the constraint field of linear programming to be empty sets after subjectively the flexible indexes p_1, p_2, p_3 were given. Using this method and the classical algorithms solution could be obtained to the problem. At the end, A numerical example is given to illustrate the proposed model.

In **Chapter - 7**, the solid transportation problem with fuzzy coefficients for the objectives and constraints with rough interval was modeled and solved. Here, the researcher formulated two solid transportation problems with interval coefficients considering the lower approximation and the upper approximation of the rough intervals. From these two solid transportation problems four different classical solid transportation problems were constituted. The concept of the completely satisfactory solution, rather satisfactory solutions, surely optimal range, possibly optimal range and rough optimal range are discussed. The proposed procedure is validated with the help of a numerical example.

In **Chapter - 8**, the researcher present a study on rough interval approach to determine the preferred compromise solution for fuzzy fixed charge solid transportation problem. The proposed model is formulated as fuzzy coefficients for the direct profit, fixed charge and constraints with rough interval. Here, researcher constructed two solid transportation problems with interval coefficients considering the lower and upper approximation of the rough intervals. Moreover, from these two solid transportation problems four different classical solid transportation problems were constituted and solved. Expected values of the fuzzy objective functions (with direct profit and fixed charge profit) were considered to derive the crisp values. Numerical example was given in order to show applicability of the proposed model. Sensitivity analysis and comparative study was included.

Chapter - 9 presents the main conclusions of all the Chapters of the thesis along with some directions for future research work.

1.2 Keywords

Single-objective and Multi-objective, Fixed Charge, Solid Transportation Problem, Rough Interval, Stochastic, L-R coefficients, Membership Function, Fuzzy Number, Fractional Programming, Fuzzy Goal Programming.

1.3 Abbreviations

TP	:	Transportation Problem
STP	:	Solid Transportation Problem
TFN	:	Triangular Fuzzy Number
TrFN	:	Trapezoidal Fuzzy Number
FLP	:	Fuzzy Linear Programming
LFP	:	Linear Fractional Programming
SFTP	:	Solid Fractional Transportation Problem
MOSTP	:	Multi-Objective Solid Transportation Problem
FCSTP	:	Fixed Charge Solid Transportation Problem
GP	:	Goal Programming
FGP	:	Fuzzy Goal Programming
SSTP	:	Stochastic Solid Transportation Problem
FSSTP	:	Fuzzy Stochastic Solid Transportation Problem
RI	:	Rough Interval
FSFTP	:	Fuzzy Solid Fractional Transportation Problem
MOFSTP	:	Multi-Objective Fuzzy Solid Transportation Problem
STPIC	:	Solid Transportation Problem with Interval Coefficient
FCSTPIC	:	Fixed Charge Solid Transportation Problem with Interval Coefficient
FCSTPRIC	:	Fixed Charge Solid Transportation Problem with Rough Interval Coefficient

1.4 Basic Definitions

The basic definitions are given as follows.

Definition 1.1. Fuzzy Set

Fuzzy sets introduced by Zadeh [108], as a mathematical tool to represent ambiguity and vagueness are a generalization of the classical (crisp) set and it is a class of objects with membership grades defined by a membership function.
Let U be a universal set. A fuzzy set $\tilde{A}$ of U is defined by a membership function

$$\mu_{\tilde{A}}(x) : U \longrightarrow [0,1], \tag{1.1}$$

where $\mu_{\tilde{A}}(x)$ denotes the membership grade (or degree) of x in $\tilde{A}$, and is called the membership function.

In a classical set, an element of the universe either belongs to or does not belong to the set while in a fuzzy set, the degree of membership of each element ranges over the unit interval. A fuzzy set $\tilde{A}$ on the given universal set U is a set of ordered pairs:

$$\tilde{A} = \{x, \mu_{\tilde{A}}(x) : x \in U\}, \tag{1.2}$$

where the first element of which denotes the element and the second the degree of membership.

Example: $\tilde{A}$ = "Integers close to 10". Then fuzzy set $\tilde{A}$ can be written as
$\tilde{A} = \{x, \mu_{\tilde{A}}(x) : x$ is an integer close to $10\}$
=(6, 0.2), (7, 0.4), (8, 0.7), (9, 0.8), (10, 1), (11, 0.6), (12, 0.2).

Definition 1.2. α-cut of a Fuzzy Set

An α-cut of $\tilde{A}$ is defined by $A_\alpha = \{x : \mu_{\tilde{A}}(x) = \alpha, \alpha \geq 0\}$.

Definition 1.3. Fuzzy Number

A fuzzy set $\tilde{A}$ on a universal set U is a fuzzy number iff $\tilde{A}$ satisfies two conditions:

1. $\tilde{A}$ is normal, i.e.,

$$sup_{x \in U} \mu_{\tilde{A}}(x) = 1, \tag{1.3}$$

2. $\tilde{A}$ is convex, i.e.,

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)), \forall x, y \in U, \forall \in [0,1]. \tag{1.4}$$

Definition 1.4. Triangular Fuzzy Number (TFN)

A fuzzy number $\tilde{A} = (a_1, a_2, a_3)$, is called the triangular fuzzy number where $a_1 < a_2 < a_3$, if the membership function of $\tilde{A}$ is defined by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{when} \quad a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & \text{when} \quad a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases}$$

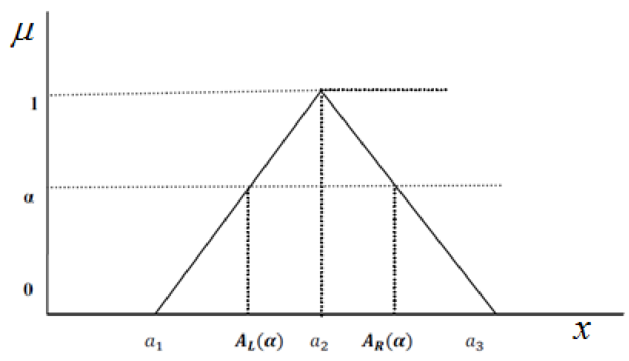

Figure 1.1: Triangular fuzzy number

Definition 1.5. Trapezoidal Fuzzy Number (TrFN)

A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, is called the trapezoidal fuzzy number where $a_1 < a_2 < a_3 < a_4$, if the membership function of $\tilde{A}$ is defined by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{when} \quad a_1 \leq x \leq a_2 \\ 1, & \text{when} \quad a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & \text{when} \quad a_3 \leq x \leq a_4 \\ 0, & \text{otherwise} \end{cases}$$

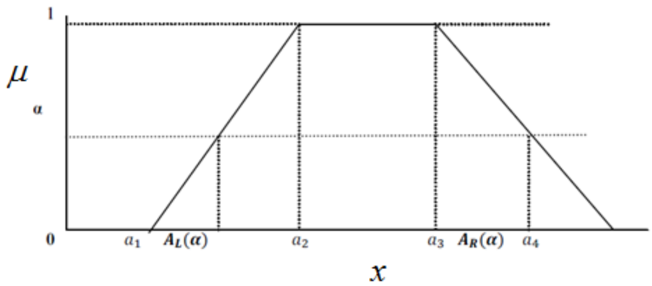

Figure 1.2: Trapezoidal Fuzzy Number

Definition 1.6. Fuzzy Linear Programming (FLP) Problem

A Fuzzy Linear Programming Problem with m fuzzy equality constraints and n fuzzy variables may be formulated as follows:

max (or min) $(\tilde{C}^T \otimes \tilde{X})$,

subject to $\tilde{A} \otimes \tilde{X} (\lesssim, \approxeq\cong \; or \; \gtrsim) \tilde{b}$

$\tilde{X}$ is a non-negative fuzzy number,

where $\tilde{C}^T = [\tilde{c}_j]_{1\times n}, \tilde{X} = [\tilde{x}_j]_{n\times 1}, \tilde{A} = [\tilde{a}_{ij}]_{m\times n}, \tilde{b} = [\tilde{b}_i]_{m\times 1}$,

and the parameters $\tilde{a}_{ij}, \tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in F(R)$

where $F(R)$ is the set of fuzzy numbers.

Definition 1.7. Linear Fractional Programming (LFP) Problem

A Linear-fractional programming problem is one whose objective function is a ratio of two linear functions satisfying some linear constraints.

Example:

max $\quad Z = \frac{f_1(x)}{f_2(x)}$,

subject to $g_1(x) \leq 0, g_2(x) \leq 0, x \geq 0$,

f_1, f_2, g_1, g_2 are linear functions,

$f_2(x) \neq 0$ for any x.

Definition 1.8. Solid Transportation Problem (STP)

Let us consider m sources, n destinations and k conveyance in a solid transportation problem. At each source, let s_i be the amount of a homogeneous product we want to transport to n destinations to satisfy the demand for d_j units of the product. Here e_k called conveyance that denotes the units of this product that can be carried by k different modes of transportation, such as the land transportation by car or train, and ocean shipping. A penalty value of the unit shipping cost represented by c_{ijk} of a product from origin to destination by means of the conveyance. We need to determine a feasible way of shipping the available amounts to satisfy the demand so that the total transportation cost is minimized.

Let x_{ijk} denote the number of units to be transported from source i to destination j through conveyance capacities k. The mathematical form of the solid transportation problem with transportation costs, availabilities and conveyance capacities is given below:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} \\
\text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i \qquad i = 1, 2, \dots m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j \qquad j = 1, 2, \dots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k \qquad k = 1, 2, \dots K \\
& x_{ijk} \ge 0, \quad \forall i, j, k.
\end{aligned} \tag{1.5}$$

Definition 1.9. Multi-Objective Solid Transportation Problem (MOSTP)

A multi-objective fuzzy solid transportation problem is formulated as follows:

$$\begin{aligned}
Z_r = \min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}^{r}x_{ijk}, \qquad r = 1, 2, \dots R \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, \qquad i = 1, 2, \dots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, \qquad j = 1, 2, \dots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \qquad k = 1, 2, \dots K, \\
& x_{ijk} \ge 0, \forall i, j, k.
\end{aligned} \tag{1.6}$$

Here, e_k is called conveyance, that denotes units of this product which is carried by k different modes of transportation and also the objectives of $Z_r(r = 1, 2, \dots R)$ need to be minimized.

Remark 1.1: In a multi-objective optimization problem two or more objectives are simultaneously optimized under some constraints. These objectives may be in conflict with each other, or may not be. Further, there is no single optimum solution, but there is a solution set which would bring Pareto optimal solutions. Pareto optimal solutions are a set of trade-offs between different objectives and are nondominated solutions.

Definition 1.10. Goal Programming (GP) (Charnes and Cooper [13])
Goal programming (GP) Models were originally introduced by Charnes and Cooper in 1961 for a linear programming (LP) model. Goal programming is a multi-criteria decision making (MCDM) approach that allows the simultaneous solution of a system of multiple and conflicting objectives.

In the simplest version of goal programming, the decision maker (DM) sets the goal for each objective that he/she wishes to attain. The optimum solution is then defined as the one that minimizes the total deviations from the set goals.
Thus, the goal programming formulation of multi-objective optimization problem leads to

$$\begin{aligned} \min \quad & [\sum_{j=1}^{n}(d_j^+ + d_j^-)^p]^{\frac{1}{p}} \\ \text{subject to} \quad & g(j)(X) \leq 0, \\ & W_j(X) - d_j^+ + d_j^- = b_j, \\ & d_j^+ d_j^- = 0, \\ & d_j^+ \geq 0, d_j^- \geq 0. \end{aligned} \tag{1.7}$$

Here, $g(j)(X) = j^{th}$ constraint function $(j = 1, 2, \dots n)$,
$W_j = j^{th}$ objective function,
b_j goal set by the decision maker for W_j,
d_j^-= under-deviation from the j^{th} goal,
d_j^+= over-deviation from the j^{th} goal,
p = distance parameter $1 \leq p \leq \infty$.

Goal programming is used to perform three types of analysis:

1. To determine the required resources to achieve a desired set of objectives.

2. To determine the degree of attainment of the goals with the available resources.

3. To provide the best satisfying solution under a varying amount of resources and priorities of the goals.

Definition 1.11. Rough Interval (Hamzeheea *et al.*[35])

The qualitative value A is called a rough interval when one can assign two closed intervals A_* and A^* on R to it where $A_* \subseteq A^*$. Moreover,

1. If $x \in A_*$ then A surely takes x (denoted by $x \in A$).

2. If $x \in A^*$ then A possibly takes x.

3. If $x \notin A^*$ then A surely does not take x (denoted by $x \notin A$).

A_* and A^* are called the lower approximation interval (LAI) and the upper approximation interval (UAI) of A , respectively. Further, A is denoted by $A = (A_*, A^*)$. For a rough variable A, the lower approximation interval means that the variable A takes the values in that interval in normal cases but for a special case the variable takes the value from the upper approximation interval, which imply, the variable is bounded by the upper approximation interval as it cant take a value outside this. Note that the intervals A_* and A^* are not complement to each other.

Definition 1.12. Rough Interval Arithmetic

Rough Interval, a special case of a rough set proposed by Rebolledo [90], satisfy all the rough sets properties and basic concepts, including the upper and lower approximation definitions.

According to Rebolledo some of these arithmetic operations of rough intervals are given below:

Let $A = ([\underline{a}^l, \underline{a}^u], [\bar{a}^l, \bar{a}^u])$ and $B = ([\underline{b}^l, \underline{b}^u], [\bar{b}^l, \bar{b}^u])$ are two rough intervals.

1.Addition : $A + B = ([\underline{a}^l + \underline{b}^l, \underline{a}^u + \underline{b}^u], [\bar{a}^l + \bar{b}^l, \bar{a}^u + \bar{b}^u])$.

2.Subtraction : $A - B = ([\underline{a}^l - \underline{b}^u, \underline{a}^u - \underline{b}^l], [\bar{a}^l - \bar{b}^u, \bar{a}^u - \bar{b}^l])$.

3.Negation : $-A = ([-\underline{a}^u, -\underline{a}^l], [-\bar{a}^u, -\bar{a}^l])$.

4.Union : $A \cup B = ([\min\{\underline{a}^l, \underline{b}^l\}, \max\{\underline{a}^u, \underline{b}^u\}], [\min\{\bar{a}^l, \bar{b}^l\}, \max\{\bar{a}^u, \bar{b}^u\}])$.

5.Intersection : $A \cap B = ([\max\{\underline{a}^l, \underline{b}^l\}, \min\{\underline{a}^u, \underline{b}^u\}], [\max\{\bar{a}^l, \bar{b}^l\}, \min\{\bar{a}^u, \bar{b}^u\}])$.

1.5 Literature Survey

There were many production systems with transportation operations where raw materials of the company are transported from source (origin) to demand (destination) by different types of conveyances like trucks, ships, goods trains, cargo flights, etc. This type of transportation problem, known as solid transportation problem (STP), is the generalization of traditional transportation problem. The transportation problem (TP) is first defined by Hitchcock [38]. The solid transportation problem, first stated by Schell [92] and Haley [34].

The fuzzy set theory, developed by Zadeh [108], is a generalization of the conventional set theory to represent vagueness or imprecision in a strict mathematical framework. The philosophy of fuzzy sets is very close to human thinking. Hence, fuzzy sets have found applications in diverse fields. Bellman and Zadeh [6] applied the notion of fuzzy sets for decision-making theory, considering conflicts between constraint equation and objective equation of the general programming, and proposed the max-min operation method in order to determine the optimal decision of the two solutions.

The technique of fuzzy linear programming enlarges the range of applications of the linear programming problem. It allows the decision maker to consider tolerances for values of decision model parameters in a natural and direct way. It is of a great importance when it is not possible to determine the decision model parameters exactly.

1.5.1 Literature on Transportation Problem (TP)

Since the transportation problem is essentially a linear program, one uniformly apply the existing fuzzy linear programming techniques (Buckly [8], Chanas *et al.*[12] and Hadi Basirzadeh [5]) to the fuzzy transportation problem.

$$\begin{aligned} Z &= \min \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij}, \\ \text{subject to} \sum_{j=1}^{n} x_{ij} &= \tilde{a}_i, \qquad i = 1, 2, \ldots m, \\ \sum_{i=1}^{m} x_{ijk} &= \tilde{b}_j, \qquad j = 1, 2, \ldots n, \\ x_{ij} &\geq 0, \forall i, j. \end{aligned} \tag{1.8}$$

in which the transportation costs $\tilde{c}_{ij}$, supply $\tilde{a}_i$ and demand $\tilde{b}_j$ quantities are fuzzy quantities.

The method of Julien [47] and Parra *et al.*[84] is able to find the possibility distribution of the objective value, provided all the inequality constraints are of " $\geq$" type or " $\leq$" type. Mondal [74] had studied transportation problem with a budgetary constraint in the deterministic case. Dinagar and Palanivel [22] investigated fuzzy transportation problem, with the aid of trapezoidal fuzzy numbers.

$$
\begin{aligned}
Z = \min & \sum_{i=1}^{m}\sum_{j=1}^{n}[\tilde{c}_{ij}^1, \tilde{c}_{ij}^2, \tilde{c}_{ij}^3, \tilde{c}_{ij}^4][x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4] \\
\text{subject to} & \sum_{j=1}^{n} x_{ij} = [\tilde{a}_i^1, \tilde{a}_i^2, \tilde{a}_i^3, \tilde{a}_i^4], \qquad i = 1, 2, \ldots m, \\
& \sum_{i=1}^{m} x_{ijk} = [\tilde{b}_j^1, \tilde{b}_j^2, \tilde{b}_j^3, \tilde{b}_j^4], \qquad j = 1, 2, \ldots n, \\
& [x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4] \geq 0, \forall i, j.
\end{aligned}
\tag{1.9}
$$

1.5.2 Literature on Solid Transportation Problem (STP)

However, in many realistic transportation situations the demands at various demand points are random variables. the study on STP in uncertain environment was started during last two decades. An STP with one or more random or fuzzy parameter is defined as a stochastic solid transportation problem (SSTP). A bicriteria STP in stochastic environment was solved by Yang and Feng [106]. Ojha *et al.*[79] using the analytic hierarchy process for a stochastic discounted multi-objective STP for breakable items. Bit *et al.*[7] developed a fuzzy programming model for a multi-objective STP. Jimenez and Verdegay [45] presented an evolutionary algorithm based on parametric approach to solve fuzzy STP. Li *et al.*[63] considered an improved genetic algorithm to solve multi-objective STP in fuzzy environment, where total fuzzy transportation cost is optimized.

The fixed charge solid transportation problem is an extension of classical transportation problem. Kennington and Unger [51] have developed a new branch-and-bound procedure specialized for the fixed-charge transportation problem.

1.5.3 Literature on Multi-Objective Solid Transportation Problem (MOSTP)

If optimized objective is more than one in the STP, then the problem is called multi-objective solid transportation problem (MOSTP). Zimmermann [110] first introduced fuzzy set theory into the conventional linear programming problem, and combined the fuzzy linear programming (FLP) model with multi-objective programming (MOP) into fuzzy multi-objective linear programming. The MOSTP was solved by several investigators using various methods. Bit *et al.*[7] used fuzzy programming approach, Ida *et al.*[41] presented a neural network method. Gao and Liu [30] developed two-phase fuzzy algorithms to solve MOSTP. Kundu *et al.*[57] presented a multi-objective, multi-item solid transportation problem. Pramanik *et al.*[88] have developed a multi-objective STP in a fuzzy random environment. Li *et al.*[63] presented a genetic algorithm for solving the MOSTP with coefficients of the objective function as fuzzy numbers. Jimnez and Verdegay [46] applied an evolutionary algorithm based on parametric approach to solve fuzzy STP. In addition, Li *et al.*[63] designed a neural network approach for multi-criteria STP and they also presented an improved genetic algorithm to solve MOSTP with fuzzy numbers. Gao and Liu [30] developed a two-phase fuzzy goal programming technique for multi-objective transportation problem. Gen *et al.*[31] gave a genetic algorithm for solving bicriteria fuzzy STP.

Ida *et al.*[41] presented a neural network method to solve a MOSTP. Gao and Liu [30] developed two-phase fuzzy algorithms to solve multi-objective STP. Yang and Liu [105] presented an algorithm for fuzzy fixed charge solid transportation problem in minimization type, Tao and Xu [95] developed a class of rough multiple objective programming and its application to a solid transportation problem. Li *et al.*[63] designed a neural network approach for multicriteria STP. Hussein [40] introduced the complete solutions of multi objective transportation problems with possibilistic coefficients. Chakraborty *et al.*[9] used a fuzzy programming approach to solve a MOSTP. Ojha *et al.*[78] formulated a STP with discounted costs, fixed charges and vehicles costs as a linear programming problem. Ammar and Youness [1] introduced the solution of MOTP with fuzzy objectives, fuzzy sources, and fuzzy destinations. Recently, Pramanik *et al.*[88] have developed a multi-objective STP in a fuzzy random environment.

The operations on fuzzy numbers could be founded in Dubois and Prade [25]. A detailed introduction to the theory of fuzzy sets could be found in Kauffman [48] and in Klir and Bo [53]. Over the past few decades, some researchers started to apply fuzzy set theory in inventory management problems. Hannan [36, 37] formulated the linear programming with multiple fuzzy goals.

Decision makers sometimes set such goals, even when they are unattainable within the available resources. Such problems were tackled with the help of the techniques of goal programming. Whether the goals are attainable or not, the objective function is stated in such a way that it's optimization means "as close as possible" to the indicated goals. The concept of Goal Programming was introduced by Charnes and Cooper [13]. Some extensions of linear goal programming for multi-objective analysis could be found in Ignizio [42, 43], Lee [61], Ijiri [44]. The goal programming was applied in a fuzzy environment by Narasimhan [75, 76]. Charnes and Cooper [14, 15] introduced the concept of goal programming for multiple linear fractional optimization problems which was extended by Kornbluth and Steuer [54, 55]. Tiwari *et al.*[96] proposed fuzzy goal programming as an additive model. The GP approach to fuzzy programming problems introduced by Mohamed [72] is extended to solve fuzzy multi-objective fractional linear programming problems. Using trapezoidal membership function the objectives are transformed into fuzzy goals by means of assigning an aspiration level to each of them. Several researchers considered transportation problem in stochastic environments (Chalam [11], Cooper [17, 18]) and fuzzy environments (Kaur and Kumar [49], [50]).

1.5.4 Literature on Fractional Programming

The linear and nonlinear models of fractional programming problems had been initially studied by Charnes *et al.*[14] and Dinkelbach [23]. The fractional programming problems have been studied extensively by many researchers. Mjelde [71] maximized the ratio of the return and the cost in resource allocation problems, Kydland [60] on the other hand maximized the profit per unit time in a cargo-loading problem. Arora *et al.*[3] discussed a fractional bulk transportation problem in which the numerator is quadratic in nature and the denominator is linear.

Lin [64] proposed iterative labeling algorithms to determine the sensitivity ranges of the fractional assignment problem. Xu *et al.*[103] utilize a new algorithm to deal with the linear fractional minimal cost flow problem on network. Wang *et al.*[98] solve the bi-level linear fractional programming problem by means of an optimization algorithm based on the duality gap of the lower level problem.

Dutta *et al.*[26, 28] developed a fuzzy set theoretic approach for the multiple objective linear fractional programming, and then presented the comments over it. The effect of tolerance in fuzzy fractional programming was presented by Dutta *et al.*[27]. A restricted class of multi-objective linear fractional programming problems were developed by Dutta *et al.*[29]. Mohamed [72] had studied the relationship between goal programming and fuzzy programming. Chen and Tzeng [16] applied the fuzzy multi-objective approach to the supply chain model. Kumar *et al.*[56] applied fuzzy goal programming approach for vendor selection problem in a supply chain.

Mohanty *et al.*[73] proposed a fuzzy approach for multi-objective programming problem and its equivalent goal programming problem with appropriate priorities and aspiration levels. Pal and Basu [82] presented a goal programming method for solving fractional programming problems via dynamic programming. Pal *et al.*[83] formulated a goal programming procedure for fuzzy multi-objective linear fractional programming problem. Pramanik and Roy [89] used the fuzzy goal programming approach to multilevel programming problems. Chakraborty and Chatterjee [10] used multi-criteria decision-making methods for the selection of materials with minimum data.

1.5.5 Literature on Expected value operator

Liu and Liu [69] presented the expected value model for fuzzy programming. Yang and Liu [105] applied expected value model, chance-constrained programming model and dependent-chance programming in fixed charge solid transportation problem in fuzzy environment. Li and Wang [62] presented generalized expected value model for stochastic programming in transportation problems. Cui and Sheng [20] presented the expected-constrained programming for an uncertain solid transportation problem was given based on uncertainty theory. Baidya and Maiti [4] studied a STP with safety factor under different uncertain environments. Halder and Maiti [33] investigated some special fixed charge multi-item solid transportation problems in crisp and fuzzy environments.

In recent years, the interval analysis method was developed to model the uncertainty in uncertain inventory optimization problems, in which the bounds of the uncertain coefficients are only required, not necessarily knowing the probability distributions or membership functions. Charnes *et al.*[15] proposed an idea for solving the linear programming problems in which the constraints were assumed as closed intervals.

Tong [97] studied interval number and fuzzy number linear programming where the coefficients of the objective function and constraints were all interval numbers, and the possible interval of the solution was obtained by taking the maximum value range and minimum value range inequalities as constraint conditions. The KarushKuhnTucker (KKT) optimality conditions play an important role in the area of optimization theory and has been studied for over a century.

Many approaches to interval-valued optimization problems have been explored in considerable details, while few papers studied the KKT optimality conditions for interval-valued optimization problems. Wu [100, 101, 102] studied the KKT optimality conditions in an optimization problem with interval-valued objective function. Oliveira *et al.*[80] presented an overview of multiple objective linear programming models with interval coefficients. Liu [70] applied geometric programming to profit maximization with interval coefficients and quantity discount.

1.5.6 Literature on Rough Intervals (RI)

In literature there were so many works in which TP/STP minimizes the corresponding transportation cost. But the profit maximizing transportation problems have been investigated by only few researchers. In many real life situations it was found that the main objective of the problem is to maximize under some conditions.

Pawlak [85] was the first person to present the Rough set theory. The relationships of rough set theory to many other theories have been extensively investigated by Pawlak and Skowron [86, 87]. In particular, its relationships with fuzzy set theory, the theory of evidence, Boolean reasoning methods and statistical methods. Rough set theory has a great application in different fields, for example presented Greco *et al.*[32] multi-criteria decision analysis, Nasiri and Mashinchi [77] presented decision analysis, machine learning, knowledge acquisition, and knowledge discovery from a database. Weigou [99] developed decision algorithms. Arabani and Nashaei [2] presented civil engineering problems (New approach to simplify dams location) and other areas. In the recent era, some interesting work has been developed in theoretical aspect and applied into many practical fields such as Data Envelopment Analysis (DEA) (Shafiee and Shams-e-alam [93], Xu *et al.*[104]), Lin [65] has presented data mining, some other researchers have presented multi-criteria decision analysis, signal processing etc. Liu [67] proposed the concept of rough variable which was a measurable function from rough space to the set of real numbers. Liu [68] discussed some inequalities of rough variables and convergence concept of sequence of rough variables. Robolledo [90] has presented the rough interval (RI). Youness [107] proposed a type of rough programming, in which two optimal solutions (surely optimal solution and possibly optimal solution) were defined and solution are well discussed by Osman *et al.*[81]. Kundu *et al.*[58] developed a solid transportation model considering crisp and rough cost. Tao *et al.*[95] presented an application of rough multiple objective programming in solid transportation problem.

Chapter 2

Fuzzy Solid Transportation Problem based on Extension Principle with Interval Budget Constraints

2.1 Introduction

Transportation models have wide applications in logistics and supply chain management for improving service and reduce the cost.

In this chapter, the researcher investigated a solution for the fuzzy solid transportation problem with interval valued budget at each destination. An assessment of different results of the model was presented. The researcher considered solution procedure that was able to calculate the fuzzy objective value of the fuzzy solid transportation problem. The idea is to apply Zadeh's extension principle [109]. A pair of two-level mathematical programs was formulated to calculate the lower and upper bounds of the α-level cut of the objective value. Researcher introduced the crisp conversion of the constraints of the respective model and made use of Hu and Wangs [39] approach based on interval ranking. Based on the extension principle, the fuzzy solid transportation problem was transformed into a pair of mathematical programs that was employed to calculate the lower and upper bounds of the fuzzy total transportation cost at possibility level α. From different values of α, the membership function of the objective value is constructed. Since the objective value was fuzzy, the values of the decision variables derived in this paper are fuzzy as well. An example was illustrated for this model.

Section 2.2 gives the formulation of the problem in fuzzy solid transportation problem. Section 2.3 deals with the solution procedure of the problem deriving the membership function. An algorithm is given in section 2.4. Section 2.5 presents a numerical example to explain the proposed method.

2.2 Formulation of the Problem

STP is a problem of transporting goods from some sources to some destinations through some conveyances (modes of transportation), and the main objective is to find the optimal transportation plan so that the total transportation cost is minimum. Let us consider the m sources, n destinations and k conveyance in a solid transportation problem. At each source, let s_i be the amount of a homogeneous product we want to transport to n destinations to satisfy the demand for d_j units of the product. Here e_k called conveyance denotes the units of this product that could be carried by k different modes of transportation, interval budget at the destination, such as the land transportation by car or train, and sea voyage. A penalty value of the unit shipping cost represents by c_{ijk} of a product from origin to destination by means of the conveyance. We need to determine a feasible way of shipping the available amounts to satisfy the demand so that the total transportation cost is minimized.

The mathematical form of the solid transportation problem with interval budget constraints, transportation costs, availabilities and conveyance capacities is given below:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} & & \\
\text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & & i = 1, 2, \dots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & & j = 1, 2, \dots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & & k = 1, 2, \dots K, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} c_{ijk}x_{ijk} \le [b_j^L, b_j^R], & & j = 1, 2, \dots n,
\end{aligned} \tag{2.1}$$

$$x_{ijk} \geq 0, \quad \forall\, i, j, k.$$

Intuitively, if any of the parameters x_{ijk},s_i, d_j or e_k is fuzzy, the total transportation cost becomes fuzzy. Then the Model (2.1) turns into the fuzzy solid transportation problem with interval budget constraints.

Suppose the unit shipping cost c_{ijk}, supply s_i, demand d_j, conveyance capacity e_k and budget intervals were approximately known. They can be represented by the convex fuzzy numbers C_{ijk}, S_i, D_j and E_k respectively, with membership functions $\mu_{\tilde{C}_{ijk}}$,$\mu_{\tilde{S}_i}$, $\mu_{\tilde{D}_j}$ and $\mu_{\tilde{E}_k}$:

$$\begin{aligned} \tilde{C}_{ijk} &= \{(c_{ijk}, \mu_{\tilde{C}_{ijk}}(c_{ijk}))|c_{ijk} \in S(\tilde{C}_{ijk})\} \\ \tilde{S}_i &= \{(s_i, \mu_{\tilde{S}_i}(s_i))|s_i \in S(\tilde{S}_i)\} \\ \tilde{D}_j &= \{(d_i, \mu_{\tilde{D}_j}(d_j))|d_j \in S(\tilde{D}_j)\} \\ \tilde{E}_k &= \{(e_k, \mu_{\tilde{E}_k}(e_k))|e_k \in S(\tilde{E}_k)\} \end{aligned} \tag{2.2}$$

where $S(\tilde{C}_{ijk})$, $S(\tilde{S}_i)$, $S(\tilde{D}_j)$ and $S(\tilde{E}_k)$ were the supports of $\tilde{C}_{ijk}$, $\tilde{S}_i$, $\tilde{D}_j$ and $\tilde{E}_k$ which denote the universe sets of the unit shipping cost, the quantity supplied by the origin, the quantity required by the destination, and the capacity carried by the conveyance, respectively.
The fuzzy objective function $\tilde{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\tilde{C}_{ijk}x_{ijk}$, which is to be minimized, together with the following constraints, constitutes the fuzzy solid transportation problem:
Using Hu and Wang's approach [39] on budget constraint crisp conversion was followed.

$$\sum_{i=1}^{m}\sum_{k=1}^{K} C_{ijk}x_{ijk} \leq \frac{b_j^L + b_j^R}{2}, j = 1, 2, \ldots n \tag{2.3}$$

Without the loss of generality, all the supply and demand quantities and conveyance capacities were assumed to be convex fuzzy numbers as the crisp values could be represented by degenerated membership functions which have only one value in their domains. In the next section, the solution procedure for fuzzy solid transportation problem with fuzzy supply, requirement and conveyance capacity was developed.

2.3 Solution Methodology

The focus had been laid on deriving the membership function of the total transportation cost $\tilde{Z}$. Since $\tilde{Z}$ is a fuzzy number, instead of a crisp number and it can't be minimized directly. To tackle this problem, one can transform the fuzzy solid transportation problem, which is based on Zadeh's extension principle to a family of mathematical programs to be solved.

Based on the extension principle, the membership function $\mu_{\tilde{z}}$ could be defined as:

$$\mu_{\tilde{z}}(z) = supmin\{\mu_{\tilde{C}_{ijk}}(c_{ijk}), \mu_{\tilde{S}_i}(s_i), \mu_{\tilde{D}_j}(d_j), \mu_{\tilde{E}_k}(e_k) \forall i, j, k | z = Z(c, s, d, e)\} \quad (2.4)$$

viewed as the application of this extension principle to the α -cuts of $\tilde{Z}$. Let us denote the α -cuts of $\tilde{C}_{ijk}$, $\tilde{S}_i$, $\tilde{D}_j$ and $\tilde{E}_k$ as

$$(\tilde{C}_{ijk})_\alpha = \{c_{ijk} \in S(\tilde{C}_{ijk}) | \mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha\} = [(\tilde{C}_{ijk})^L_\alpha, (\tilde{C}_{ijk})^U_\alpha], \quad (2.5.1)$$

$$(\tilde{S}_i)_\alpha = \{s_i \in S(\tilde{S}_i) | \mu_{\tilde{S}_i}(s_i) \geq \alpha\} = [(\tilde{S}_i)^L_\alpha, (\tilde{S}_i)^U_\alpha], \quad (2.5.2)$$

$$(\tilde{D}_j)_\alpha = \{d_j \in S(\tilde{D}_j) | \mu_{\tilde{D}_j}(d_j) \geq \alpha\} = [(\tilde{D}_j)^L_\alpha, (\tilde{D}_j)^U_\alpha], \quad (2.5.3)$$

$$(\tilde{E}_k)_\alpha = \{e_k \in S(\tilde{E}_k) | \mu_{\tilde{E}_i}(e_k) \geq \alpha\} = [(\tilde{E}_k)^L_\alpha, (\tilde{E}_k)^U_\alpha], \quad (2.5.4)$$

These intervals indicate where the unit shipping cost, supply, demand, and conveyance lie at possibility level α. In (2.4), several membership functions were involved. To derive $\mu_{\tilde{z}}$ in closed form is hardly possible. According to (2.4), $\mu_{\tilde{z}}$ is the minimum of $\mu_{\tilde{C}_{ijk}}$, $\mu_{\tilde{S}_i}$, $\mu_{\tilde{D}_j}$ and $\mu_{\tilde{E}_k}$, $\forall i,j,k$. We need $\mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha, \mu_{\tilde{S}_i}(s_i) \geq \alpha, \mu_{\tilde{D}_j}(d_j) \geq \alpha$ or $\mu_{\tilde{E}_k}(e_k) \geq \alpha$ and at least one $\mu_{\tilde{C}_{ijk}}(c_{ijk}), \mu_{\tilde{S}_i}(s_i), \mu_{\tilde{D}_j}(d_j), \mu_{\tilde{E}_k}(e_k) \forall i,j,k$ equal to α such that $z = Z(c,s,d,e)$ to satisfy $\mu_{\tilde{z}} = \alpha$. To find the membership function $\mu_{\tilde{z}}$, it suffices to find the left shape function and right shape function of $\mu_{\tilde{z}}$, which is equivalent to finding the lower bound Z_α^L and upper bound Z_α^U of the α-cuts of $\tilde{z}$. Since Z_α^L is the minimum of $z = Z(c,s,d,e)$ and Z_α^U is the maximum of $z = Z(c,s,d,e)$, they can be expressed as:

$Z_\alpha^L = \min\{Z(c,s,d,e) | (\tilde{C}_{ijk})_\alpha^L \leq c_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (\tilde{S}_i)_\alpha^U, (\tilde{D}_j)_\alpha^L \leq d_j \leq (\tilde{D}_j)_\alpha^U, (\tilde{E}_k)_\alpha^L \leq e_k \leq (\tilde{E}_k)_\alpha^U \quad \forall\ i,j,k\}$,

$Z_\alpha^U = \max\{Z(c,s,d,e) | (\tilde{C}_{ijk})_\alpha^L \leq c_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (\tilde{S}_i)_\alpha^U, (\tilde{D}_j)_\alpha^L \leq d_j \leq (\tilde{D}_j)_\alpha^U, (\tilde{E}_k)_\alpha^L \leq e_k \leq (\tilde{E}_k)_\alpha^U \quad \forall\ i,j,k\}$,

This could be reformulated as the following pair of two-level mathematical programs:

$$
\begin{array}{r}
Z_\alpha^L = \min \\
(\tilde{C}_{ijk})_\alpha^L \leq c_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, \\
(\tilde{S}_i)_\alpha^L \leq s_i \leq (\tilde{S}_i)_\alpha^U, \\
(\tilde{D}_j)_\alpha^L \leq d_j \leq (\tilde{D}_j)_\alpha^U, \\
(\tilde{E}_k)_\alpha^L \leq e_k \leq (\tilde{E}_k)_\alpha^U, \\
\forall\ i,j,k.
\end{array}
\left\{
\begin{array}{rll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} c_{ijk} x_{ijk} & \\
\text{subject to} & \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq s_i, & i = 1,2,\ldots m \\
& \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq d_j, & j = 1,2,\ldots n \\
& \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq e_k, & k = 1,2,\ldots K \\
& \sum_{i=1}^{m} \sum_{k=1}^{K} c_{ijk}^L x_{ijk} \leq \frac{b_j^L + b_j^R}{2}, & j = 1,2,\ldots n \\
& x_{ijk} \geq 0, \forall i,j,k. &
\end{array}
\right.
\tag{2.6.1}
$$

$$Z_\alpha^U = \max_{\substack{(\tilde{C}_{ijk})_\alpha^L \le c_{ijk} \le (\tilde{C}_{ijk})_\alpha^U, \\ (\tilde{S}_i)_\alpha^L \le s_i \le (\tilde{S}_i)_\alpha^U, \\ (\tilde{D}_j)_\alpha^L \le d_j \le (\tilde{D}_j)_\alpha^U, \\ (\tilde{E}_k)_\alpha^L \le e_k \le (\tilde{E}_k)_\alpha^U, \\ \forall\, i,j,k.}} \left\{ \begin{array}{lll} \min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} & \\ \text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1,2,\ldots m, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1,2,\ldots n, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & k = 1,2,\ldots K, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} c_{ijk}^U x_{ijk} \le \frac{b_j^L + b_j^R}{2}, & j = 1,2,\ldots n, \\ & x_{ijk} \ge 0, \forall i,j,k. & \end{array} \right. \tag{2.6.2}$$

In Model (2.6.1), the inner program calculates the objective value for each c_{ijk}, s_i, d_j and e_k specified by the outer program, while the outer program determines the values of c_{ijk}, s_i, d_jande_k that generate the smallest objective value Z_L. The objective value is the lower bound of the objective value for Model (2.3).

By the same token, the inner program of Model (2.6.2) calculated the objective value for each given value of $c_{ijk}, s_i, d_j$and e_k, while the outer program determines the values of c_{ijk}, s_i, d_j and e_k that produce the largest objective value. The objective value is the upper bound of the objective value for Model (2.3). Since the value of α varies in Model (2.6.1 and 2.6.2), it can also be regarded as a pair of parametric programming model.

A necessary and sufficient condition for Model (2.6.1 and 2.6.2) to have feasible solutions is $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} d_j$. In the first level of Model (2.6.1 and 2.6.2) s_i, d_j and e_k are allowed to vary in the range of $[(\tilde{S}_i)_\alpha^L, (\tilde{S}_i)_\alpha^U]$, $[(\tilde{D}_j)_\alpha^L, (\tilde{D}_j)_\alpha^U]$ and $[(E_k)_\alpha^L, (E_k)_\alpha^U]$, respectively. However, to ensure the transportation problem of the second level to be feasible, it felt necessary that the constraint $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} d_j$ was imposed in the outer program.

Here $B_j = \frac{b_j^L + b_j^R}{2}, \qquad j = 1, 2, \dots n.$

Hence, Models (2.6.1 and 2.6.2) becomes:

$$
\begin{array}{r}
Z_\alpha^L = \min \\
(\tilde{C}_{ijk})_\alpha^L \le c_{ijk} \le (\tilde{C}_{ijk})_\alpha^U, \\
(\tilde{S}_i)_\alpha^L \le s_i \le (\tilde{S}_i)_\alpha^U, \\
(\tilde{D}_j)_\alpha^L \le d_j \le (\tilde{D}_j)_\alpha^U, \\
(\tilde{E}_k)_\alpha^L \le e_k \le (\tilde{E}_k)_\alpha^U, \\
\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \\
\sum_{k=1}^{K} e_k \ge \sum_{j=1}^{n} d_j, \\
\forall i, j, k.
\end{array}
\left\{
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} & \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1, 2, \dots m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1, 2, \dots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & k = 1, 2, \dots K \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} c_{ijk}x_{ijk} \le B_j, & j = 1, 2, \dots n \\
& x_{ijk} \ge 0, \forall i, j, k. &
\end{array}
\right.
\tag{2.7.1}
$$

$$
\begin{array}{r}
Z_\alpha^U = \max \\
(\tilde{C}_{ijk})_\alpha^L \le c_{ijk} \le (\tilde{C}_{ijk})_\alpha^U, \\
(\tilde{S}_i)_\alpha^L \le s_i \le (\tilde{S}_i)_\alpha^U, \\
(\tilde{D}_j)_\alpha^L \le d_j \le (\tilde{D}_j)_\alpha^U, \\
(\tilde{E}_k)_\alpha^L \le e_k \le (\tilde{E}_k)_\alpha^U, \\
\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \\
\sum_{k=1}^{K} e_k \ge \sum_{j=1}^{n} d_j, \\
\forall i, j, k.
\end{array}
\left\{
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} & \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1, 2, \dots m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1, 2, \dots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & k = 1, 2, \dots K \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} c_{ijk}x_{ijk} \le B_j, & j = 1, 2, \dots n \\
& x_{ijk} \ge 0, \forall i, j, k. &
\end{array}
\right.
\tag{2.7.2}
$$

In above Models (2.7.1 and 2.7.2) are infeasible when $\sum_{i=1}^{m} S_{\alpha=0}^{U} \le \sum_{j=1}^{n} D_{\alpha=0}^{L}$ for any α level. In other words, a fuzzy transportation problem is feasible when upper bound of the total fuzzy supply is greater than or equal to the lower bound of the total fuzzy demand. To derive the lower bound of the objective value in Model (2.7.1), we can directly set c_{ijk} to its lower bound $(C_{ijk})^L_\alpha, \forall i, j, k$ to find the minimum objective value.

Hence, Model (2.7.1) could be reformulated as:

$$
\begin{array}{r}
Z^L_\alpha = \min \\
(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, \\
(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, \\
(\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, \\
(\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha, \\
\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \\
\sum_{k=1}^{K} e_k \ge \sum_{j=1}^{n} d_j, \\
\forall i, j, k.
\end{array}
\left\{
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} & \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1, 2, \dots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1, 2, \dots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & k = 1, 2, \dots K, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} c_{ijk}x_{ijk} \le B_j, & j = 1, 2, \dots n, \\
& x_{ijk} \ge 0, \forall i, j, k. &
\end{array}
\right.
\tag{2.8}
$$

Since Model (2.8) is to find the minimum of all the minimum objective values, one can combine the constraints of inner program and outer program together and simplify the two-level mathematical program to the conventional one-level program as follows:

$$
\begin{array}{lll}
Z^L_\alpha = \min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} & \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1, 2, \dots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1, 2, \dots n,
\end{array}
\tag{2.9}
$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1, 2, \ldots K,$$

$$\sum_{i=1}^{m}\sum_{k=1}^{K} \tilde{C}_{ijk}^{L} x_{ijk} \leq B_j, \qquad j = 1, 2, \ldots n,$$

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \sum_{k=1}^{K} e_k \geq \sum_{j=1}^{n} d_j,$$

$$(\tilde{C}_{ijk})_{\alpha}^{L} \leq c_{ijk} \leq (\tilde{C}_{ijk})_{\alpha}^{U},$$

$$(\tilde{S}_i)_{\alpha}^{L} \leq s_i \leq (\tilde{S}_i)_{\alpha}^{U},$$

$$(\tilde{D}_j)_{\alpha}^{L} \leq d_j \leq (\tilde{D}_j)_{\alpha}^{U},$$

$$(\tilde{E}_k)_{\alpha}^{L} \leq e_k \leq (\tilde{E}_k)_{\alpha}^{U},$$

$$x_{ijk} \geq 0, \quad \forall\ i, j, k.$$

This model is a linear program which could be solved easily. In this model, since all c_{ijk} have been set to the lower bounds of their α-cuts, that is,$\mu_{\tilde{C}_{ijk}}(c_{ijk}) = \alpha$ this assures $\mu_{\tilde{z}}(z) = \alpha$ as required by (2.4).

To solve Model (2.7.2), the outer program and inner program have different directions for optimization, one for maximization and another for minimization. A transformation is required to make a solution obtainable. The dual of inner program is formulated to become a maximization problem to be consistent with the maximization operation of outer program. It is well known from the duality theorem of linear programming that the primal model and the dual model have the same objective value. Thus, Model (2.7.2) becomes:

$$
\begin{aligned}
&Z^U_\alpha = \max \\
&(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, \\
&(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, \\
&(\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, \\
&(\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha, \\
&\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \\
&\sum_{k=1}^{K} e_k \ge \sum_{j=1}^{n} d_j, \quad \forall\ i,j,k.
\end{aligned}
\left\{
\begin{aligned}
\max \quad & -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{K} e_k w_k - \sum_{j=1}^{n} B_j y_j \\
\text{subject to} \quad & -u_i + v_j - w_k - y_j \le c_{ijk}, \\
& i = 1,2,\ldots m, j = 1,2,\ldots n, k = 1,2,\ldots K, \\
& u_i, v_j, w_k, y_j \ge 0,\ \forall\ i,j,k.
\end{aligned}
\right.
\tag{2.10}
$$

Since $(C_{ijk})^L_\alpha \le c_{ijk} \le (C_{ijk})^U_\alpha$, in Model (2.10), $\forall i,j,k$ one can derive the upper bound of the objective value by setting c_{ijk} to its upper bound because this gives the largest feasible region. Thus, we can reformulate Model (2.10) as:

$$
\begin{aligned}
&Z^U_\alpha = \max \\
&(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, \\
&(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, \\
&(\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, \\
&(\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha, \\
&\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \\
&\sum_{k=1}^{K} e_k \ge \sum_{j=1}^{n} d_j, \forall\ i,j,k.
\end{aligned}
\left\{
\begin{aligned}
\max \quad & -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{K} e_k w_k - \sum_{j=1}^{n} B_j y_j \\
\text{subject to} \quad & -u_i + v_j - w_k \le (\tilde{C}_{ijk})^L, \\
& i = 1,2,\ldots m, j = 1,2,\ldots n, k = 1,2,\ldots K, \\
& u_i, v_j, w_k \ge 0, \forall i,j,k.
\end{aligned}
\right.
\tag{2.11}
$$

Now, since both outer program and inner program perform the same maximization operation, their constraints could be combined to form the following one-level mathematical program:

$$
\begin{aligned}
Z_\alpha^U = \max \quad & -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{K} e_k w_k - \sum_{j=1}^{n} B_j y_j \\
\text{subject to} \quad & -u_i + v_j - w_k \le (\tilde{C}_{ijk})^L, \\
& i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots K \\
& \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \\
& \sum_{k=1}^{K} e_k \ge \sum_{j=1}^{n} d_j, \\
& i = 1, 2, \ldots m \\
& (\tilde{S}_i)_\alpha^L \le s_i \le (\tilde{S}_i)_\alpha^U, \qquad j = 1, 2, \ldots n \\
& (\tilde{D}_j)_\alpha^L \le d_j \le (\tilde{D}_j)_\alpha^U, \qquad j = 1, 2, \ldots n \\
& (\tilde{E}_k)_\alpha^L \le e_k \le (\tilde{E}_k)_\alpha^U, \qquad k = 1, 2, \ldots K \\
& x_{ijk} \ge 0, \forall\ i, j, k.
\end{aligned}
\tag{2.12}
$$

This model is a linearly constrained nonlinear program. There were several effective and efficient methods for solving this Model (2.12). Similar to Model (2.9), since all c_{ijk} have been set to the upper bounds of their α-cuts, that is, $\mu_{\tilde{C}_{ijk}}(\tilde{C}_{ijk}) = \alpha$ this assures $\mu_{\tilde{z}}(z) = \alpha$ as required by (2.4).

If the total supply and the total conveyance capacity were greater than the total demand at all α values, respectively, i.e., $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$, $\sum_{i=1}^{m} (\tilde{S}_i)_{\alpha=0}^L \ge \sum_{j=1}^{n} (\tilde{D}_j)_{\alpha=0}^U$ and $\sum_{k=1}^{K} (\tilde{E}_k)_{\alpha=0}^L \ge \sum_{j=1}^{n} (\tilde{D}_j)_{\alpha=0}^U$ then the constraints $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$ could be deleted from Model (2.12). Multiplying constraints (2.12) by u_i, v_j and w_k respectively, and substituting $s_i u_i$ by p_i, $d_j v_j$ by q_j, and $e_k w_k$ by r_k, Model (2.12) was transformed into the following linear program:

$$
\begin{aligned}
Z_\alpha^U = \max \quad & -\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j - \sum_{k=1}^{K} r_k \\
\text{subject to} \quad & -u_i + v_j - w_k \leq (\tilde{C}_{ijk})^L, \\
& i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2 \ldots K \\
& (\tilde{S}_i)_\alpha^L u_i \leq p_i \leq (\tilde{S}_i)_\alpha^U u_i, \qquad i = 1, 2, \ldots m \\
& (\tilde{D}_j)_\alpha^L v_j \leq q_j \leq (\tilde{D}_j)_\alpha^U v_j, \qquad j = 1, 2, \ldots n \\
& (\tilde{E}_k)_\alpha^L w_k \leq r_k \leq (\tilde{E}_k)_\alpha^U w_k, \qquad k = 1, 2, \ldots K \\
& p_i, q_j, r_k \geq 0, \quad \forall\, i, j, k.
\end{aligned} \tag{2.13}
$$

In this case, the upper bound of the total transportation cost Z_α^L at α level could be found more easily. Problems (2.7.1) and (2.7.2) were assured to be feasible if the lower bound of the total fuzzy demand is smaller than both of the upper bound of the total fuzzy supply and the upper bound of the total conveyance capacity, i.e., $\sum_{j=1}^{n} (\tilde{D}_j)_{\alpha=0}^{L} \leq \sum_{i=1}^{m} (\tilde{S}_i)_{\alpha=0}^{U}$ and $\sum_{j=1}^{n} (\tilde{D}_j)_{\alpha=0}^{L} \leq \sum_{k=1}^{K} (\tilde{E}_k)_{\alpha=0}^{U}$.

2.4 Algorithm

Step 1 Consider a STP model as given in (2.1).

Step 2 Formulate the pair of two level mathematical problems as shown in (2.6.1) and (2.6.2).

Step 3 Transform (2.6.1) as (2.9) and solve for different α values ranging from 0 to 1 with step length 0.1.

Step 4 Transform (2.6.2) as (2.13) and solve for different α values ranging from 0 to 1 with step length 0.1.

Step 5 For different α values, analyze the optimal solutions with lower and upper bound.

2.5 Numerical Example

As an illustration of the proposed approach, consider a fuzzy solid transportation problem with two fuzzy supplies, three fuzzy demands, two conveyance capacities and three budget intervals in nature. The notations used in this example is (a, b, c, d) for a trapezoidal fuzzy number with $a, b, c$and d as the coordinates of the four vertices of the trapezoid and (x, y, z) for the triangular fuzzy number with x, y, z as the coordinates of the three vertices of the triangle. The problem has the following mathematical form:

$$
\begin{aligned}
\min \quad & (20,30,40)x_{111} + 70x_{112} + 60x_{121} + 60x_{122} + 50x_{131} + 30x_{132} \\
& + (10,20,30)x_{211} + 40x_{212} + 20x_{221} + 50x_{222} + 40x_{231} + 50x_{232} \\
\text{subject to} \quad & x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} \leq (70,80,100,120), \\
& x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} \leq (60,70,90), \\
& x_{111} + x_{112} + x_{211} + x_{212} \geq (10,30,40,50), \\
& x_{121} + x_{122} + x_{221} + x_{222} \geq (40,50,60), \\
& x_{131} + x_{132} + x_{231} + x_{232} \geq (30,40,60,70), \\
& x_{111} + x_{121} + x_{131} + x_{211} + x_{231} \leq (70,80,100), \\
& x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} \leq (60,70,90), \\
& 20x_{111} + 70x_{112} + 10x_{211} + 40x_{212} \leq [3450,3750], \\
& 60x_{121} + 20x_{122} + 30x_{221} + 50x_{222} \leq [2585,2615], \\
& 50x_{131} + 30x_{132} + 40x_{231} + 50x_{232} \leq [2860,2940], \\
& x_{ijk} \geq 0, i = 1,2, j = 1,2,3, k = 1,2.
\end{aligned}
\tag{2.14}
$$

The total Supply $S = \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 = (130, 150, 170, 210)$ the total demand $D = \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 = (80, 120, 150, 180)$ and the total conveyance capacity $E = \tilde{E}_1 + \tilde{E}_2 +$

$\tilde{E}_3 = (130, 150, 190)$ and the intervals of budgets are $[3450, 3750]$, $[2585, 2615]$ and $[2860, 2940]$. Since $S \bigcap D \bigcap E \neq \emptyset$ Problem has feasible solutions.

$$
\begin{array}{l}
Z^L_\alpha = \min \\
70 + 10\alpha \leq s_1 \leq 120 - 20\alpha, \\
60 + 10\alpha \leq s_2 \leq 90 - 20\alpha, \\
10 + 10\alpha \leq d_1 \leq 50 - 10\alpha, \\
40 + 10\alpha \leq d_2 \leq 60 - 10\alpha, \\
30 + 10\alpha \leq d_3 \leq 70 - 10\alpha, \\
70 + 10\alpha \leq e_1 \leq 100 - 20\alpha, \\
60 + 10\alpha \leq e_2 \leq 90 - 20\alpha, \\
s_1 + s_2 \geq d_1 + d_2 + d_3, \\
e_1 + e_2 \geq d_1 + d_2 + d_3, \\
x_{ijk} \geq 0, i = 1, 2, j = 1, 2, 3, \\
k = 1, 2.
\end{array}
\left\{
\begin{array}{ll}
\min & (20, 30, 40)x_{111} + 70x_{112} + 60x_{121} + 60x_{122} + 50x_{131} + 30x_{132} \\
 & +(10, 20, 30)x_{211} + 40x_{212} + 20x_{221} + 50x_{222} + 40x_{231} + 50x_{232} \\
 & \text{subject to} \\
 & x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} \leq (70, 80, 100, 120), \\
 & x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} \leq (60, 70, 90), \\
 & x_{111} + x_{112} + x_{211} + x_{212} \geq (10, 30, 40, 50), \\
 & x_{121} + x_{122} + x_{221} + x_{222} \geq (40, 50, 60), \\
 & x_{131} + x_{132} + x_{231} + x_{232} \geq (30, 40, 60, 70), \\
 & x_{111} + x_{121} + x_{131} + x_{211} + x_{231} \leq (70, 80, 100), \\
 & x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} \leq (60, 70, 90), \\
 & 20x_{111} + 70x_{112} + 10x_{211} + 40x_{212} \leq [3450, 3750], \\
 & 60x_{121} + 20x_{122} + 30x_{221} + 50x_{222} \leq [2585, 2615], \\
 & 50x_{131} + 30x_{132} + 40x_{231} + 50x_{232} \leq [2860, 2940], \\
 & x_{ijk} \geq 0, i = 1, 2, j = 1, 2, 3, k = 1, 2.
\end{array}
\right.
\tag{2.15.1}
$$

$$Z_\alpha^U = \max_{\substack{70+10\alpha \le s_1 \le 120-20\alpha,\\ 60+10\alpha \le s_2 \le 90-20\alpha,\\ 10+10\alpha \le d_1 \le 50-10\alpha,\\ 40+10\alpha \le d_2 \le 60-10\alpha,\\ 30+10\alpha \le d_3 \le 70-10\alpha,\\ 70+10\alpha \le e_1 \le 100-20\alpha,\\ 60+10\alpha \le e_2 \le 90-20\alpha,\\ s_1+s_2 \ge d_1+d_2+d_3,\\ e_1+e_2 \ge d_1+d_2+d_3,\\ x_{ijk}\ge 0, i=1,2, j=1,2,3,\\ k=1,2.}} \left\{ \begin{array}{l} \min \quad (20,30,40)x_{111}+70x_{112}+60x_{121}+60x_{122}+50x_{131}+30x_{132} \\ +(10,20,30)x_{211}+40x_{212}+20x_{221}+50x_{222}+40x_{231}+50x_{232} \\ \text{subject to} \\ x_{111}+x_{112}+x_{121}+x_{122}+x_{131}+x_{132} \le (70,80,100,120), \\ x_{211}+x_{212}+x_{221}+x_{222}+x_{231}+x_{232} \le (60,70,90), \\ x_{111}+x_{112}+x_{211}+x_{212} \ge (10,30,40,50), \\ x_{121}+x_{122}+x_{221}+x_{222} \ge (40,50,60), \\ x_{131}+x_{132}+x_{231}+x_{232} \ge (30,40,60,70), \\ x_{111}+x_{121}+x_{131}+x_{211}+x_{231} \le (70,80,100), \\ x_{112}+x_{122}+x_{132}+x_{212}+x_{222}+x_{232} \le (60,70,90), \\ 20x_{111}+70x_{112}+10x_{211}+40x_{212} \le [3450,3750], \\ 60x_{121}+20x_{122}+30x_{221}+50x_{222} \le [2585,2615], \\ 50x_{131}+30x_{132}+40x_{231}+50x_{232} \le [2860,2940], \\ x_{ijk} \ge 0, i=1,2, j=1,2,3, k=1,2. \end{array} \right. \tag{2.15.2}$$

According to Models (2.9) and (2.13), the lower and upper bounds of $\tilde{Z}$ at possibility level α could be formulated as:

$$\begin{aligned} Z_\alpha^L = \min \quad & (20,30,40)x_{111}+70x_{112}+60x_{121}+60x_{122}+50x_{131}+30x_{132} \\ & +(10,20,30)x_{211}+40x_{212}+20x_{221}+50x_{222}+40x_{231}+50x_{232} \\ \text{subject to} \quad & x_{111}+x_{112}+x_{121}+x_{122}+x_{131}+x_{132} \le s_1, \\ & x_{211}+x_{212}+x_{221}+x_{222}+x_{231}+x_{232} \le s_2, \\ & x_{111}+x_{112}+x_{211}+x_{212} \ge d_1, \\ & x_{121}+x_{122}+x_{221}+x_{222} \ge d_2, \\ & x_{131}+x_{132}+x_{231}+x_{232} \ge d_3, \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
& x_{111} + x_{121} + x_{131} + x_{211} + x_{231} \leq e_1, \\
& x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} \leq e_2, \\
& 20x_{111} + 70x_{112} + 10x_{211} + 40x_{212} \leq 3600, \\
& 60x_{121} + 20x_{122} + 30x_{221} + 50x_{222} \leq 2600, \\
& 50x_{131} + 30x_{132} + 40x_{231} + 50x_{232} \leq 2900, \\
& s_1 + s_2 \geq d_1 + d_2 + d_3, \\
& e_1 + e_2 \geq d_1 + d_2 + d_3, \\
& 70 + 10\alpha \leq s_1 \leq 120 - 20\alpha, \\
& 60 + 10\alpha \leq s_2 \leq 90 - 20\alpha, \\
& 10 + 10\alpha \leq d_1 \leq 50 - 10\alpha, \\
& 40 + 10\alpha \leq d_2 \leq 60 - 10\alpha, \\
& 30 + 10\alpha \leq d_3 \leq 70 - 10\alpha, \\
& 70 + 10\alpha \leq e_1 \leq 100 - 20\alpha, \\
& 60 + 10\alpha \leq e_2 \leq 90 - 20\alpha, \\
& x_{ijk} \geq 0, i = 1,2, j = 1,2,3, k = 1,2.
\end{aligned}
$$

$$
\begin{aligned}
Z_\alpha^U = \max \quad & - s_1u_1 - s_2u_2 + d_1v_1 + d_2v_2 + d_3v_3 - e_1w_1 - e_2w_2 \\
& - 3600y_1 - 2600y_2 - 2900y_3 \\
\text{subject to} \quad & - u_1 + v_1 - w_1 - 20y_1 \leq 40 - 10\alpha, \\
& - u_1 + v_1 - w_2 - 70y_1 \leq 70, \\
& - u_1 + v_2 - w_1 - 60y_2 \leq 60, \\
& - u_1 + v_2 - w_2 - 20y_2 \leq 20, \\
& - u_1 + v_3 - w_1 - 50y_3 \leq 50, \\
& - u_1 + v_3 - w_2 - 30y_3 \leq 30,
\end{aligned}
$$

$$
\begin{aligned}
&- u_2 + v_1 - w_1 - (30 - 10\alpha)y_1 \leq (30 - 10\alpha),\\
&- u_2 + v_1 - w_2 - 40y_1 \leq 40,\\
&- u_2 + v_2 - w_1 - 30y_2 \leq 30,\\
&- u_2 + v_2 - w_2 - 50y_2 \leq 50,\\
&- u_2 + v_3 - w_1 - 40y_3 \leq 40,\\
&- u_2 + v_3 - w_2 - 50y_3 \leq 50,\\
&s_1 + s_2 \geq d_1 + d_2 + d_3,\\
&e_1 + e_2 \geq d_1 + d_2 + d_3,\\
&70 + 10\alpha \leq s_1 \leq 120 - 20\alpha,\\
&60 + 10\alpha \leq s_2 \leq 90 - 20\alpha,\\
&10 + 10\alpha \leq d_1 \leq 50 - 10\alpha,\\
&40 + 10\alpha \leq d_2 \leq 60 - 10\alpha,\\
&30 + 10\alpha \leq d_3 \leq 70 - 10\alpha,\\
&70 + 10\alpha \leq e_1 \leq 100 - 20\alpha,\\
&60 + 10\alpha \leq e_2 \leq 90 - 20\alpha,\\
&u_1, u_2, v_1, v_2, v_3, w_1, w_2 \geq 0.
\end{aligned}
$$

The researcher solved the above two problems by using Lingo, Table-2.1 lists the α-cuts of the total transportation cost at 11 distinct α values: $0, 0.1, 0.2, 0.3, \ldots 1.0$ and Fig.2.1 depict the membership function of the total transportation cost of this example.

The α value indicates level of possibility and degree of uncertainty for the obtained information. The greater the α value, the greater the level of possibility and the lower the degree of uncertainty is.

Since the fuzzy total transportation cost lied in a range, its most likely value falls

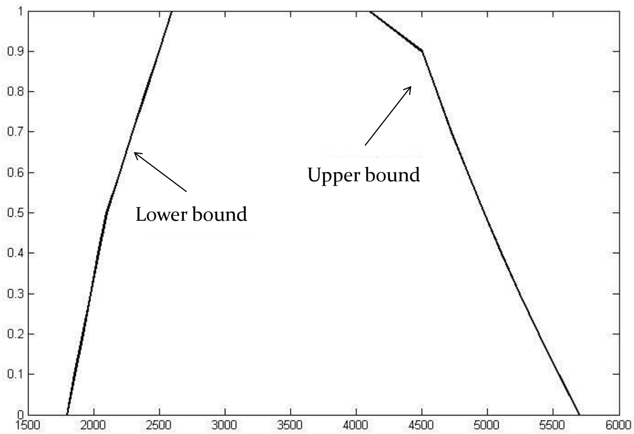

Figure 2.1: The membership function of the total transportation cost

Table 2.1: The α -cuts of the total transportation cost

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z_{α}^{L}	1800	1860	1920	1980	2040	2100	2200	2300	2400	2500	2600
Z_{α}^{U}	5700	5543	5392	5247	5108	4975	4848	4727	4612	4503	4100

between 2600 and 4100, and its value impossible to falls outside the range of 1800 and 5700. For α =0, the lower bound of Z^* =1800 occurs at x_{122}^*=40, x_{132}^* =30, x_{211}^* =10 with s_1= 120, s_2 =90, d_1 =10, d_2 =40, d_3 =30, e_1 =100, e_2 =90, and the other decision variables are 0. The upper bound of Z^* =5700 occurs at x_{111}^* =40, x_{122}^* =10, x_{132}^* =70, x_{211}^* =10, x_{221}^* =50 with s_1 = 120, s_2 =60, d_1 =50, d_2 =60, d_3 =70, e_1 =100, e_2 =80, and the other decision variables are zero. At other extreme end of α =1, the lower bound of Z^* =2600 occurs at x_{122}^* =30, x_{132}^* =40, x_{211}^* =20, x_{221}^* =20 with s_1 = 80, s_2 =70, d_1 =20, d_2 =50, d_3 =40, e_1 =80, e_2 =70, and the other decision variables are 0. The upper bound of =4100 occurs at x_{111}^* =10, x_{122}^* =10, x_{132}^* =60, x_{211}^* =30, x_{221}^* =40 with s_1 = 80, s_2 =70, d_1 =40, d_2 =50, d_3 =60, e_1 =80, e_2 =70, and the other decision variables are zero.

Chapter 3

Fuzzy Solid Fractional Transportation Problem with Interval Budget Constraint

3.1 Introduction

In real life the trasportation cost and labor cost plays an impotent role in transportation network. **In this Chapter**, the researcher worked on the Solid fractional transportation problem (SFTP) with fuzzy parameters. Based on Zadeh's extension principle, utilized a pair of two-level mathematical programs to find the α-cuts of the fuzzy objective value of the SFTP. With the application of the dual formulation and variable substitution techniques, the pair of two-level mathematical programs were transformed into a pair of ordinary one-level linear programs to solve. At specific α-cut, solving the pair of linear programs produces the bounds of the objective value of the fuzzy SFTP. By enumerating various values of α, the membership function of $\tilde{Z}$ was approximated numerically.

Section 3.2 gives the nature of SFTP, followed with a two-level mathematical programming formulation for deriving the bounds of the fuzzy objective values. Section 3.3 describes how to transform the two-level mathematical program into the conventional one-level linear programming problem. An algorithm is given in section 3.4. Section 3.5 presents a numerical example.

3.2 Formulation of the Problem

It is assumed that a company has m sources and n destinations in a solid transportation problem. Here e_k called conveyance denotes the units of this product that could be carried by k different modes of transportation, interval budget at the j^{th} destination, such as the land transportation by car or train, and the ocean shipping. A penalty value of the unit shipping cost represented by c_{ijk}. If a product from i^{th} origin to j^{th} destination by means of the k^{th} conveyance, and the values of θ and β are given in fixed costs. We need to determine a feasible way of shipping the available amounts to satisfy the demand such that the total transportation cost to total labor cost is minimized.

The given x_{ijk} denotes the number of units to be transported from source i to destination j through conveyance capacities k. The mathematical form of the solid fractional transportation problem with interval valued budget constraint, transportation costs, availabilities and conveyance capacities given below:

$$
\begin{aligned}
Z = \min & \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} + \theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}x_{ijk} + \beta} \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_i, \qquad i = 1,2,\ldots,m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_j, \qquad j = 1,2,\ldots,n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1,2,\ldots,K \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} f_{ijk}x_{ijk} \leq [b_j^L, b_j^R], \quad j = 1,2,\ldots,n \\
& x_{ijk} \geq 0, \quad \forall i,j,k.
\end{aligned}
\tag{3.1}
$$

Intuitively, if any of the given parameters x_{ijk}, s_i, d_j or e_k is fuzzy, the total transportation cost becomes fuzzy as well and budget constraint is taken with interval. Then the Model (3.1) turns into the fuzzy solid transportation problem with interval valued budget constraint.

Suppose the unit shipping cost c_{ijk} & a_{ijk}, supply s_i, demand d_j, conveyance capacity e_k and budget intervals are approximately known. They can be represented by the convex fuzzy numbers $\tilde{C}_{ijk}$, $\tilde{S}_i$, $\tilde{D}_j$ and $\tilde{E}_k$ respectively, with membership functions $\mu_{\tilde{C}_{ijk}}$, $\mu_{\tilde{A}_{ijk}}$, $\mu_{\tilde{S}_i}$, $\mu_{\tilde{D}_j}$ and $\mu_{\tilde{E}_k}$:

$$\begin{aligned}
\tilde{C}_{ijk} &= \{(c_{ijk}, \mu_{\tilde{C}_{ijk}}(c_{ijk}))|c_{ijk} \in S(\tilde{C}_{ijk})\} \\
\tilde{A}_{ijk} &= \{(a_{ijk}, \mu_{\tilde{A}_{ijk}}(a_{ijk}))|a_{ijk} \in S(\tilde{A}_{ijk})\} \\
\tilde{S}_i &= \{(s_i, \mu_{\tilde{S}_i}(s_i))|s_i \in S(\tilde{S}_i)\} \\
\tilde{D}_j &= \{(d_i, \mu_{\tilde{D}_j}(d_j))|d_j \in S(\tilde{D}_j)\} \\
\tilde{E}_k &= \{(e_k, \mu_{\tilde{E}_k}(e_k))|e_k \in S(\tilde{E}_k)\}
\end{aligned} \tag{3.2}$$

where $S(\tilde{C}_{ijk})$, $S(\tilde{A}_{ijk})$, $S(\tilde{S}_i)$,$S(\tilde{D}_j)$ and $S(\tilde{E}_k)$ are the supports of $\tilde{C}_{ijk}$,$\tilde{S}_i$,$\tilde{D}_j$ and $\tilde{E}_k$ which denote the universe sets of the unit shipping cost, the quantity supplied by the origin, the quantity required by the destination, and the capacity carried by the conveyance, respectively.

Using Hu and Wang's approach [39] on budget constraint the following crisp conversion was executed.

$$\sum_{i=1}^{m}\sum_{k=1}^{K} c_{ijk}x_{ijk} \leq \frac{b_j^L + b_j^R}{2}, j = 1, 2, \ldots, n \tag{3.3}$$

Without the loss of generality, all the supply and demand quantities and conveyance capacities are assumed to be convex fuzzy numbers as the crisp values could be represented by degenerated membership functions which have only one value in their domains. In the next section, the solution procedure for fuzzy solid transportation problem with fuzzy supply, requirement and conveyance capacity was developed.

3.3 Solution Methodology

Researcher endeavored in deriving the membership function of the total transportation cost $\tilde{Z}$. Since $\tilde{Z}$ was a fuzzy number, instead of a crisp number, it couldn't be minimized directly. To tackle this problem, one could transform the fuzzy solid transportation problem, which was based on Zadeh's extension principle to a family of mathematical programs to be solved.

Based on the extension principle, the membership function $\mu_{\tilde{z}}$ can be defined as:

$$\mu_{\tilde{z}}(z) = sup \min\{\mu_{\tilde{C}_{ijk}}(c_{ijk}), \mu_{\tilde{S}_i}(s_i), \mu_{\tilde{D}_j}(d_j), \mu_{\tilde{E}_k}(e_k) \forall i, j, k | z = Z(c, s, d, e)\} \quad (3.4)$$

viewed as the application of this extension principle to the α -cuts of $\tilde{Z}$. Following Dong and Wong [24] let us denote the α -cuts of $\tilde{C}_{ijk}$, $\tilde{S}_i$, $\tilde{D}_j$ and $\tilde{E}_k$ as:

$$(\tilde{C}_{ijk})_\alpha = \{c_{ijk} \in S(\tilde{C}_{ijk}) | \mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha\} = [(\tilde{C}_{ijk})^L_\alpha, (\tilde{C}_{ijk})^U_\alpha], \quad (3.5.1)$$

$$(\tilde{A}_{ijk})_\alpha = \{a_{ijk} \in S(\tilde{A}_{ijk}) | \mu_{\tilde{A}_{ijk}}(a_{ijk}) \geq \alpha\} = [(\tilde{A}_{ijk})^L_\alpha, (\tilde{A}_{ijk})^U_\alpha], \quad (3.5.2)$$

$$(\tilde{S}_i)_\alpha = \{s_i \in S(\tilde{S}_i) | \mu_{\tilde{S}_i}(s_i) \geq \alpha\} = [(\tilde{S}_i)^L_\alpha, (\tilde{S}_i)^U_\alpha], \quad (3.5.3)$$

$$(\tilde{D}_j)_\alpha = \{d_j \in S(\tilde{D}_j) | \mu_{\tilde{D}_j}(d_j) \geq \alpha\} = [(\tilde{D}_j)^L_\alpha, (\tilde{D}_j)^U_\alpha], \quad (3.5.4)$$

$$(\tilde{E}_k)_\alpha = \{e_k \in S(\tilde{E}_k) | \mu_{\tilde{E}_i}(e_k) \geq \alpha\} = [(\tilde{E}_k)^L_\alpha, (\tilde{E}_k)^U_\alpha], \quad (3.5.5)$$

These intervals indicate where the unit shipping cost, supply, demand, and conveyance lie at possibility level α. In (3.4), several membership functions were involved. To derive $\mu_{\tilde{z}}$ in closed form is hardly possible. According to (3.4), $\mu_{\tilde{z}}$ is the supmin of $\mu_{\tilde{C}_{ijk}}$, $\mu_{\tilde{S}_i}$, $\mu_{\tilde{D}_j}$ and $\mu_{\tilde{E}_k}$, $\forall i, j, k$. It required $\mu_{\tilde{C}_{ijk}}(c_{ijk}) \geq \alpha$, $\mu_{\tilde{A}_{ijk}}(a_{ijk}) \geq \alpha$, $\mu_{\tilde{S}_i}(s_i) \geq \alpha$, $\mu_{\tilde{D}_j}(d_j) \geq \alpha$ or $\mu_{\tilde{E}_k}(e_k) \geq \alpha$ and at least one $\mu_{\tilde{C}_{ijk}}(c_{ijk})$, $\mu_{\tilde{A}_{ijk}}(a_{ijk})$, $\mu_{\tilde{S}_i}(s_i)$, $\mu_{\tilde{D}_j}(d_j)$, $\mu_{\tilde{E}_k}(e_k), \forall i, j, k$ equal to α such that $z = Z(c, a, s, d, e)$ to satisfy $\mu_{\tilde{z}} = \alpha$. To find the membership function $\mu_{\tilde{z}}$, it suffices to find the left shape function and right shape function of $\mu_{\tilde{z}}$, which is equivalent to finding the lower bound Z^L_α and upper bound Z^U_α of the α-cuts of $\tilde{Z}$. Since Z^L_α is the minimum of $z = Z(c, a, s, d, e)$ and Z^U_α is the maximum of $z = Z(c, a, s, d, e)$, they could be expressed as:

$$Z^L_\alpha = \min\{Z(c,a,s,d,e)|(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, (A_{ijk})^L_\alpha \le a_{ijk} \le (A_{ijk})^U_\alpha,$$

$$(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, (\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, (\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha \ \ \forall\, i,j,k\} \qquad (3.6)$$

$$Z^U_\alpha = \max\{Z(c,a,s,d,e)|(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, (A_{ijk})^L_\alpha \le a_{ijk} \le (A_{ijk})^U_\alpha,$$

$$(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, (\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, (\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha \ \ \forall\, i,j,k\} \qquad (3.7)$$

This could be reformulated as the following pair of two-level mathematical programs:

Where $z = Z(c,a,s,d,e)$ is defined in (3.1) note that $z = Z(c,a,s,d,e)$ is a mathematical program with minimization as the objective function. Therefore, models (3.6) and (3.7) are two level programs, with $z = Z(c,a,s,d,e)$ as the linear program. For each set of cost c_{ijk}, supply s_i, demand d_j, conveyance capacity e_k values defined by respective α -cuts in the outer program (first level), the objective value was calculated in the inner program (second level). The set of c_{ijk}, s_i, d_j, and e_k Values, which produced the largest and smallest objective values were determined at the first level by models (3.6) and (3.7), respectively. By enumerating various values of α, the membership function of $\tilde{Z}$ was approximated numerically. This could be reformulated as the following pair of two-level mathematical programs:

$$Z^L_\alpha = \min_{\substack{(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, \\ (\tilde{A}_{ijk})^L_\alpha \le a_{ijk} \le (\tilde{A}_{ijk})^U_\alpha, \\ (\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, \\ (\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, \\ (\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha, \\ \forall\, i,j,k.}} \left\{ \begin{array}{lll} & Z = \min \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} + \theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}x_{ijk} + \beta} & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1,2,\ldots m \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1,2,\ldots n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & k = 1,2,\ldots K \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} \tilde{C}^L_{ijk}x_{ijk} \le \frac{b^L_j + b^R_j}{2}, & j = 1,2,\ldots n \\ & x_{ijk} \ge 0, \quad \forall\, i,j,k. & \end{array} \right. \tag{3.8}$$

$$Z^U_\alpha = \max_{\substack{(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, \\ (\tilde{A}_{ijk})^L_\alpha \le a_{ijk} \le (\tilde{A}_{ijk})^U_\alpha, \\ (\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, \\ (\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, \\ (\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha, \\ \forall\, i,j,k.}} \left\{ \begin{array}{lll} & Z = \min \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} + \theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}x_{ijk} + \beta} & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, & i = 1,2,\ldots m \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j, & j = 1,2,\ldots n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, & k = 1,2,\ldots K \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} \tilde{C}^U_{ijk}x_{ijk} \le \frac{b^L_j + b^R_j}{2}, & j = 1,2,\ldots n \\ & x_{ijk} \ge 0, \quad \forall\, i,j,k. & \end{array} \right. \tag{3.9}$$

In, Model (3.8), the inner program calculates the objective value of each c_{ijk}, s_i, d_j and e_k specified by the outer program, while the outer program determines the values of c_{ijk}, s_i, d_j and e_k that generated the smallest objective value Z^L. The objective value was the lower bound of the objective value for Model (3.3).

By the same token, the inner program of Model (3.9) calculated the objective value for each given value of c_{ijk}, s_i, d_j and e_k, while the outer program determined the values of c_{ijk}, s_i, d_j and e_k that produce the largest objective value. The objective value Z^U was the upper bound of the objective value for Model (3.3). Since the value of α varies in, Models (3.8 and 3.9), it could also be regarded as a pair of parametric programming model.

A necessary and sufficient condition was provided for Model (3.8 and 3.9) to have feasible solutions $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{K} e_k \geq \sum_{j=1}^{n} d_j$. In the first level of the model (3.8 and 3.9), d_j and e_k were allowed to vary in the range of $[(S_i)_\alpha^L, (S_i)_\alpha^U], [(D_j)_\alpha^L, (D_j)_\alpha^U]$, and $[(E_k)_\alpha^L, (E_k)_\alpha^U]$ respectively. However, to ensure the transportation problem of the second level to be feasible, it felt necessary that the constraint $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j$ be imposed in the outer program.

The interval $[Z_\alpha^L, Z_\alpha^U]$ in the α- cut of $\tilde{Z}$, and $\{Z_\alpha^L, 0 \leq \alpha \leq 1\}$ and $\{Z_\alpha^U, 0 \leq \alpha \leq 1\}$ are the left and right-shape functions of$\mu_{\tilde{Z}}$, respectively, from which fuzzy number $\tilde{Z}$ is formed.

3.3.1 Construction of Upper Bound

In the previous section it was shown that to find the upper bound of the objective value of fuzzy FTP described in the Model (3.1), it was required to solve the two-level mathematical program of Model (3.9). Since the outer program and inner program have different directions for optimization, that is, one for maximization and one for minimization, solving Model (3.8) were not so straightforward. We could solve more easily if we transform the inner program of (3.6).

Since $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}x_{_{ijk}} + \beta > 0$ for every feasible $x_{_{ijk}}$in (3.8), according to Charnes and Cooper [14], let $t = 1/(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}x_{_{ijk}} + \beta)$, and $y_{ijk} = x_{ijk}t$, it made possible transform the inner program of (3.8) into the following linear program:

$$\begin{aligned}
&\min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}y_{_{ijk}} + \theta t \\
\text{subject to} &- \sum_{j=1}^{n}\sum_{k=1}^{K} y_{ijk} + s_i t \geq 0, && i = 1, 2, \ldots, m, \\
&\sum_{i=1}^{m}\sum_{k=1}^{K} y_{ijk} - d_j t \geq 0, && j = 1, 2, \ldots, n, \\
&- \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ijk} + e_k t \geq 0, && k = 1, 2, \ldots, K, \\
&- \sum_{i=1}^{m}\sum_{k=1}^{K} f_{ijk}y_{ijk} + B_j t \geq 0, && j = 1, 2, \ldots, n, \\
&\textstyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}y_{_{ijk}} + \beta t = 1, \\
&t > 0,\ y_{ijk} \geq 0, \quad \forall\ i, j, k.
\end{aligned} \tag{3.10}$$

Here $B_j = \frac{b_j^L + b_j^R}{2}$, $j = 1, 2, \ldots n$.

By the duality theorem, if both primal and dual problems are feasible, then they both have optimal solutions owning the same objective value. Craven and Mond [19], Schaile [91] and Sherali [94] have discussed the duality relationship of the linear fractional program. The dual formulation of (3.10) is given by

$$\begin{aligned}
&\max \lambda \\
&\text{subject to} - u_i + v_j - w_k - B_j + a_{ijk}\lambda \leq c_{ijk}, \\
&\qquad i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots K.
\end{aligned} \tag{3.11}$$

$$\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j + \sum_{k=1}^{K} e_k w_k + \sum_{j=1}^{n} B_j g_j + \beta\lambda \leq \theta,$$

$$u_i, v_j, w_k \geq 0, \quad \forall\ i, j, k, \quad \lambda \text{ unrestricted in sign.}$$

After this replacement, both the linear and outer programs have the same direction for optimization:

$$\begin{array}{r} Z_\alpha^U = \max \\ (\tilde{C}_{ijk})_\alpha^L \leq c_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, \\ (\tilde{A}_{ijk})_\alpha^L \leq a_{ijk} \leq (\tilde{A}_{ijk})_\alpha^U, \\ (\tilde{S}_i)_\alpha^L \leq s_i \leq (\tilde{S}_i)_\alpha^U, \\ (\tilde{D}_j)_\alpha^L \leq d_j \leq (\tilde{D}_j)_\alpha^U, \\ (\tilde{E}_k)_\alpha^L \leq e_k \leq (\tilde{E}_k)_\alpha^U, \\ \forall\ i, j, k \end{array} \left\{ \begin{array}{l} \max \quad \lambda \\ \qquad -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{K} e_k w_k - \sum_{j=1}^{n} B_j y_j \\ \text{subject to} - u_i + v_j - w_k - y_j \leq c_{ijk} \\ \qquad i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots K \\ \qquad \sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j + \sum_{k=1}^{K} e_k w_k + \sum_{j=1}^{n} B_j g_j + \beta\lambda \leq \theta, \\ \qquad u_i, v_j, w_k \geq 0, \quad \forall\ i, j, k, \lambda \text{ unrestricted in sign.} \end{array} \right. \tag{3.12}$$

Because the linear program and outer program have the same maximization operation, they could be merged into a one-level program with the constraints at the two levels considered simultaneously.

$$Z_\alpha^U = \max \ \lambda \tag{3.13.1}$$

$$\text{subject to} - u_i + v_j - w_k - B_j + a_{ijk}\lambda \leq c_{ijk},$$

$$i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots K. \tag{3.13.2}$$

$$\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j + \sum_{k=1}^{K} e_k w_k + \sum_{j=1}^{n} B_j g_j + \beta\lambda \leq \theta, \tag{3.13.3}$$

$$(\tilde{C}_{ijk})_\alpha^L \leq c_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, \tag{3.13.4}$$

$$(\tilde{A}_{ijk})^L_\alpha \le a_{ijk} \le (\tilde{A}_{ijk})^U_\alpha, \tag{3.13.5}$$

$$(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha, \tag{3.13.6}$$

$$(\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha, \tag{3.13.7}$$

$$(\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha, \tag{3.13.8}$$

$$u_i, v_j, w_k \ge 0,\ \forall\ i, j, k,\ \lambda \text{ unrestricted in sign.}$$

In (3.13.4) directed $(\tilde{C}_{ijk})^L_\alpha \le c_{ijk} \le (\tilde{C}_{ijk})^U_\alpha, i = 1, 2, \dots m, j = 1, 2, \dots n, k = 1, 2, \dots K$ from which it was possible to obtain the upper bound of the objective value by setting c_{ijk} its bounds $(\tilde{C}_{ijk})^U_\alpha, \forall i, j$, in (3.13.2) Because this gave the largest feasible region, now Model (3.13.1) could be reformulated as

$$Z^U_\alpha = \max\ \lambda \tag{3.14.1}$$

$$\text{subject to} - u_i + v_j - w_k - B_j + a_{ijk}\lambda \le (\tilde{C}_{ijk})^U_\alpha,$$

$$i = 1, 2, \dots m, j = 1, 2, \dots .n, k = 1, 2, \dots K. \tag{3.14.2}$$

$$\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j + \sum_{k=1}^{K} e_k w_k + \sum_{j=1}^{n} B_j g_j + \beta\lambda \le \theta, \tag{3.14.3}$$

$$(\tilde{A}_{ijk})^L_\alpha \le a_{ijk} \le (\tilde{A}_{ijk})^U_\alpha, \tag{3.14.4}$$

$$(\tilde{S}_i)^L_\alpha \le s_i \le (\tilde{S}_i)^U_\alpha \qquad i = 1, 2, \dots m, \tag{3.14.5}$$

$$(\tilde{D}_j)^L_\alpha \le d_j \le (\tilde{D}_j)^U_\alpha \qquad j = 1, 2, \dots n, \tag{3.14.6}$$

$$(\tilde{E}_k)^L_\alpha \le e_k \le (\tilde{E}_k)^U_\alpha \qquad k = 1, 2, \dots K, \tag{3.14.7}$$

$$u_i, v_j, w_k \ge 0, \forall i, j, k, \lambda \text{ unrestricted in sign.}$$

Because c_{ijk}, a_{ijk}, θ and β were shipping costs and fixed costs, this implies that the objective value in Model (3.8) is positive one. Based on the dual relationship, the objective value of Model (3.14.1), the dual of Model (3.8), needed to be a positive one as well. It implies that the variable $\lambda > 0$. Model (3.14.1) is a nonlinear program due to the $a_{ijk}\lambda, s_i u_i, v_j d_j$ and $e_k w_k$. However, the variable substitutions of $\delta_{ijk} = a_{ijk}\lambda, p_i = s_i u_i, q_j = v_j d_j, r_k = e_k w_k$ and $o_j = B_j g_j$, could be applied to transform the nonlinear program into a linear one since $\lambda > 0, u_i \geq 0, v_j \geq 0$, and $w_k \geq 0$. It made possible to multiply constraints (3.14.4), (3.14.5), (3.14.6) and (3.14.7) by λ, u_i, v_j and w_k respectively, for $i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots K$. The resulted linear program is

$$
\begin{aligned}
Z_\alpha^U &= \max \ \lambda \\
&\text{subject to} - u_i + v_j - w_k - B_j + \delta_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, \\
&\qquad i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots K. \\
&\qquad \sum_{i=1}^{m} p_i - \sum_{j=1}^{n} q_j + \sum_{k=1}^{K} r_k + \sum_{j=1}^{n} o_j + \beta\lambda \leq \theta, \\
&\qquad (\tilde{A}_{ijk})_\alpha^L \leq \delta_{ijk} \leq (\tilde{A}_{ijk})_\alpha^U, \\
&\qquad (\tilde{S}_i)_\alpha^L u_i \leq p_i \leq (\tilde{S}_i)_\alpha^U u_i \qquad i = 1, 2, \ldots m, \\
&\qquad (\tilde{D}_j)_\alpha^L v_j \leq q_j \leq (\tilde{D}_j)_\alpha^U v_j \qquad j = 1, 2, \ldots n, \\
&\qquad (\tilde{E}_k)_\alpha^L w_k \leq r_k \leq (\tilde{E}_k)_\alpha^U w_k \qquad k = 1, 2, \ldots K, \\
&\qquad u_i, v_j, w_k \geq 0, \forall i, j, k, \quad \lambda \text{ unrestricted in sign.}
\end{aligned}
\tag{3.15}
$$

The upper bound of the objective value, Z_α^U of the fuzzy solid fractional transportation problem could be obtained by Solving Model (3.15).

3.3.2 Construction of Lower Bound

Since both the inner program and the outer program of (3.9) have the same direction, minimization, for optimization, they could be merged into a conventional one-level mathematical program with the constraints of the two programs considered at the same time.

$$
\begin{aligned}
Z = \min & \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} + \theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} a_{ijk}x_{ijk} + \beta} \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i \qquad i = 1, 2, \ldots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j \qquad j = 1, 2, \ldots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k \qquad k = 1, 2, \ldots K, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} f_{ijk}x_{ijk} \le \frac{(b_j^L + b_j^R)}{2}, \quad j = 1, 2, \ldots n, \\
& (\tilde{C}_{ijk})_\alpha^L \le c_{ijk} \le (\tilde{C}_{ijk})_\alpha^U, \\
& (\tilde{A}_{ijk})_\alpha^L \le a_{ijk} \le (\tilde{A}_{ijk})_\alpha^U, \\
& (\tilde{S}_i)_\alpha^L \le s_i \le (\tilde{S}_i)_\alpha^U, \qquad i = 1, 2, \ldots m, \\
& (\tilde{D}_j)_\alpha^L \le d_j \le (\tilde{D}_j)_\alpha^U, \qquad j = 1, 2, \ldots n, \\
& (\tilde{E}_k)_\alpha^L \le e_k \le (\tilde{E}_k)_\alpha^U, \qquad k = 1, 2, \ldots K, \\
& x_{ijk} \ge 0, \quad \forall\, i, j, k.
\end{aligned}
\tag{3.16}
$$

Similar to (3.9), based on the rule of Charnes and Cooper [14], one can transform (3.16) into the following mathematical program

$$Z_\alpha^L = \min \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} c_{ijk} y_{_{ijk}} + \theta t \tag{3.17.1}$$

$$\text{subject to } -\sum_{j=1}^{n} \sum_{k=1}^{K} y_{ijk} + s_i t \geq 0, \qquad i = 1, 2, \dots m, \tag{3.17.2}$$

$$\sum_{i=1}^{m} \sum_{k=1}^{K} y_{ijk} - d_j t \geq 0, \qquad j = 1, 2, \dots n, \tag{3.17.3}$$

$$-\sum_{i=1}^{m} \sum_{j=1}^{n} y_{ijk} + e_k t \geq 0, \qquad k = 1, 2, \dots K, \tag{3.17.4}$$

$$-\sum_{i=1}^{m} \sum_{k=1}^{K} f_{ijk} y_{ijk} + B_j t \geq 0, \qquad j = 1, 2, \dots n, \tag{3.17.5}$$

$$\sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{n} \sum\nolimits_{k=1}^{K} a_{ijk} y_{_{ijk}} + \beta t = 1, \tag{3.17.6}$$

$$(\tilde{C}_{ijk})_\alpha^L \leq c_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U, \tag{3.17.7}$$

$$(\tilde{A}_{ijk})_\alpha^L \leq a_{ijk} \leq (\tilde{A}_{ijk})_\alpha^U, \tag{3.17.8}$$

$$(\tilde{S}_i)_\alpha^L \leq s_i \leq (\tilde{S}_i)_\alpha^U, \qquad i = 1, 2, \dots m, \tag{3.17.9}$$

$$(\tilde{D}_j)_\alpha^L \leq d_j \leq (\tilde{D}_j)_\alpha^U, \qquad j = 1, 2, \dots n, \tag{3.17.10}$$

$$(\tilde{E}_k)_\alpha^L \leq e_k \leq (\tilde{E}_k)_\alpha^U, \qquad k = 1, 2, \dots K, \tag{3.17.11}$$

$$x_{ijk} \geq 0, \quad \forall i, j, k.$$

For nonnegative y_{ijk}, they had $(\tilde{C}_{ijk})_\alpha^L y_{ijk} \leq c_{ijk} y_{ijk} \leq (\tilde{C}_{ijk})_\alpha^U y_{ijk}$. In searching for the minimal value of the objective function, the fuzzy parameterc_{ijk}, must reach its lower bound. Consequently, we have$Z_\alpha^L = \min \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} C_{ijk}^L y_{_{ijk}} + \theta t$. The variable transformation technique was utilized to the nonlinear terms$s_i t,\ d_j t, w_k t,\ g_j t$ and $a_{ijk} y_{ijk}$. The constraints (3.17.9), (3.17.10) and (3.17.11) could be multiplied by t and constraint (3.17.7) could be multiplied by y_{ijk}. We substituted P_i, Q_j, R_k, G_j

and ξ_{ijk} for $s_i t, d_j t, e_k t$ and $A_{ijk}y_{ijk}$ respectively. Now, it could be rewrite (3.1) as the following mathematical formulations

$$
\begin{aligned}
Z_\alpha^L = \min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} (\tilde{C}_{ijk})^L y_{ijk} + \theta t \\
\text{subject to } & -\sum_{j=1}^{n}\sum_{k=1}^{K} y_{ijk} + P_i \geq 0, && i = 1, 2, \ldots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} y_{ijk} - Q_j \geq 0, && j = 1, 2, \ldots n, \\
& -\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ijk} + R_k \geq 0, && k = 1, 2, \ldots K, \\
& -\sum_{i=1}^{m}\sum_{k=1}^{K} y_{ijk} + G_j \geq 0, && j = 1, 2, \ldots n, \\
& \sum\nolimits_{i=1}^{m}\sum\nolimits_{j=1}^{n}\sum\nolimits_{k=1}^{K} \xi_{ijk} + \beta t = 1, \\
& (\tilde{A}_{ijk})_\alpha^L y_{ijk} \leq \xi_{ijk} \leq (\tilde{A}_{ijk})_\alpha^U y_{ijk}, \\
& (\tilde{S}_i)_\alpha^L t \leq s_i \leq (\tilde{S}_i)_\alpha^U t && i = 1, 2, \ldots m, \\
& (\tilde{D}_j)_\alpha^L t \leq d_j \leq (\tilde{D}_j)_\alpha^U t && j = 1, 2, \ldots n, \\
& (\tilde{E}_k)_\alpha^L t \leq e_k \leq (\tilde{E}_k)_\alpha^U t && k = 1, 2, \ldots K, \\
& t > 0, y_{ijk} \geq 0, \quad \forall i, j, k.
\end{aligned}
\tag{3.18}
$$

This model is a linear programming problem, and it is possible to obtain lower bound of the objective value Z_α^L, by solving (3.18). Together with Z_α^L was solved from (3.5), [Z_α^L,Z_α^U] constitutes the interval that the objective value of the Fuzzy SFTP.

For two possibility levels α_1 and α_2 such that $0 < \alpha_2 < \alpha_1 \leq 1$, the feasible regions defined by α_1 in Model (3.8 and 3.9) were smaller to those defined by α_2, respectively, as a result $Z_{\alpha_1}^U \leq Z_{\alpha_2}^U$ and $Z_{\alpha_1}^L \geq Z_{\alpha_2}^L$; in other words, the right shape function was non increasing and the left shape function is non decreasing. This property, based on the definition of "convex fuzzy set" (Zimmermann [111]), assures the convexity of $\tilde{Z}$. If both Z_α^L, Z_α^U were invertible with respect to α, then a right shape function of $R(z) = (Z_\alpha^U)^{-1}$ and left shape function of $L(z) = (Z_\alpha^L)^{-1}$ could be obtained. From $R(z)$ and $L(z)$ the membership function $\mu_{\tilde{Z}}$ was constructed as:

$$\mu_{\tilde{z}} = \begin{cases} L(z), & Z_{\alpha=0}^L \leq z \leq Z_{\alpha=1}^L \\ 1, & Z_{\alpha=1}^L \leq z \leq Z_{\alpha=1}^U \\ R(z), & Z_{\alpha=1}^U \leq z \leq Z_{\alpha=0}^L \end{cases}$$

In most cases, the values of Z_α^L and Z_α^U may not be solved analytically. However, numerical solutions for Z_α^U and Z_α^L at different possibility level α could be collected to depict the shapes of $R(z)$ and $L(z)$, respectively.

3.4 Algorithm

Step 1 Consider a SFTP model as given in (3.1).

Step 2 Formulate the pair of two level mathematical problems as shown in (3.8) and (3.9).

Step 3 Transform (3.8) as (3.15) and solve for different α values ranging from 0 to 1 with step length 0.1.

Step 4 Transform (3.9) as (3.18) and and solve for different α values ranging from 0 to 1 with step length 0.1.

Step 5 For different α values, analyze the optimal solutions with lower and upper bound.

3.5 Numerical Example

As an illustration of the proposed approach, consider a fuzzy solid fractional transportation problem with two fuzzy supplies, three fuzzy demands, two conveyance capacities and three budget intervals in nature. The notations used in this example are trapezoidal fuzzy number and triangular fuzzy numbers. Let

$$\tilde{Z} = \min \frac{\begin{bmatrix} (15,16,17)x_{111} + 18x_{112} + 12x_{121}(20,21,22)x_{122} + 10x_{131} + (4,5,6)x_{132} \\ + 17x_{211} + 20x_{212} + (21,22,23)x_{221} + 20x_{222} + 21x_{231} + (19,20,21)x_{232} + 650 \end{bmatrix}}{\begin{bmatrix} 6x_{111} + (7,8,9,10)x_{112} + 10x_{121}6x_{122} + (11,12,13)x_{131} + 3x_{132} \\ + 13x_{211} + (8,9,10)x_{212} + 12x_{221} + (2,3,4)x_{222} + 20x_{231} + 15x_{232} + 700 \end{bmatrix}}$$

$$\begin{aligned} \text{subject to } & x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} \leq (60,70,80), \\ & x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} \leq (20,40,60), \\ & x_{111} + x_{112} + x_{211} + x_{212} \geq (10,30,40), \\ & x_{121} + x_{122} + x_{221} + x_{222} \geq (20,30,40), \\ & x_{131} + x_{132} + x_{231} + x_{232} \geq (40,50,60,70), \\ & x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} \leq (20,30,40), \\ & x_{112} + x_{212} + x_{132} + x_{212} + x_{222} + x_{232} \leq (30,40,60), \\ & (15,16,17)x_{111} + 18x_{112} + 17x_{211} + 20x_{212} \geq [15,25], \\ & 12x_{121} + (20,21,22)x_{122} + (21,22,23)x_{221} + 20x_{222} \geq [17,21], \\ & 10x_{131} + (4,5,6)x_{132} + 21x_{231} + (19,20,21)x_{232} \geq [11,31], \\ & x_{ijk} \geq 0,\ i = 1,2,\ j = 1,2,3,\ k = 1,2. \end{aligned} \tag{3.19}$$

The total Supply $S = \tilde{S}_1 + \tilde{S}_2 = (80,110,140)$,the total demand$D = \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 = (70,110,120,150)$, and the total conveyance capacity $E = \tilde{E}_1 + \tilde{E}_2 = (50,70,100)$ and the intervals of budgets are [15, 25], [17, 21] and [11, 31]. Since $S \bigcap D \bigcap E \neq \varnothing$, problem has feasible solutions.

$$Z_\alpha^U = \max_{\substack{(60+10\alpha)\le s_1\le(80-10\alpha),\\ (20+20\alpha)\le s_2\le(60-20\alpha),\\ (10+20\alpha)\le d_1\le(40-10\alpha),\\ (20+10\alpha)\le d_2\le(40-10\alpha),\\ (40+10\alpha)\le d_3\le(70-10\alpha),\\ (20+10\alpha)\le e_1\le(40-10\alpha),\\ (30+10\alpha)\le e_2\le(60-10\alpha),\\ \forall\, i,j,k.}} \left\{ \begin{array}{l} \min \dfrac{\begin{bmatrix} (15,16,17)x_{111}+18x_{112}+12x_{121}(20,21,22)x_{122}+10x_{131} \\ +(4,5,6)x_{132}+17x_{211}+20x_{212}+(21,22,23)x_{221} \\ +20x_{222}+21x_{231}+(19,20,21)x_{232}+650 \end{bmatrix}}{\begin{bmatrix} 6x_{111}+(7,8,9,10)x_{112}+10x_{121}6x_{122}+(11,12,13)x_{131} \\ +3x_{132}+13x_{211}+(8,9,10)x_{212} \\ +12x_{221}+(2,3,4)x_{222}+20x_{231}+15x_{232}+700 \end{bmatrix}} \\ \text{subject to} \\ x_{111}+x_{112}+x_{121}+x_{122}+x_{131}+x_{132}\le(60,70,80), \\ x_{211}+x_{212}+x_{221}+x_{222}+x_{231}+x_{232}\le(20,40,60), \\ x_{111}+x_{112}+x_{211}+x_{212}\ge(10,30,40), \\ x_{121}+x_{122}+x_{221}+x_{222}\ge(20,30,40), \\ x_{131}+x_{132}+x_{231}+x_{232}\ge(40,50,60,70), \\ x_{111}+x_{121}+x_{131}+x_{211}+x_{221}+x_{231}\le(20,30,40), \\ x_{112}+x_{212}+x_{132}+x_{212}+x_{222}+x_{232}\le(30,40,60), \\ (15,16,17)x_{111}+18x_{112}+17x_{211}+20x_{212}\ge[15,25], \\ 12x_{121}+(20,21,22)x_{122}+(21,22,23)x_{221}+20x_{222}\ge[17,21], \\ 10x_{131}+(4,5,6)x_{132}+21x_{231}+(19,20,21)x_{232}\ge[11,31], \\ x_{ijk}\ge 0,\ i=1,2,\ j=1,2,3,\ k=1,2. \end{array} \right. \tag{3.20}$$

$$Z^L_\alpha = \min \left\{ \begin{array}{l} \min \dfrac{\left[\begin{array}{l}(15,16,17)x_{111} + 18x_{112} + 12x_{121}(20,21,22)x_{122} + 10x_{131} \\ + (4,5,6)x_{132} + 17x_{211} + 20x_{212} + (21,22,23)x_{221} \\ + 20x_{222} + 21x_{231} + (19,20,21)x_{232} + 650\end{array}\right]}{\left[\begin{array}{l}6x_{111} + (7,8,9,10)x_{112} + 10x_{121}6x_{122} + (11,12,13)x_{131} \\ + 3x_{132} + 13x_{211} + (8,9,10)x_{212} \\ + 12x_{221} + (2,3,4)x_{222} + 20x_{231} + 15x_{232} + 700\end{array}\right]} \\ \text{subject to} \\ x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} \leq (60,70,80), \\ x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} \leq (20,40,60), \\ x_{111} + x_{112} + x_{211} + x_{212} \geq (10,30,40), \\ x_{121} + x_{122} + x_{221} + x_{222} \geq (20,30,40), \\ x_{131} + x_{132} + x_{231} + x_{232} \geq (40,50,60,70), \\ x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} \leq (20,30,40), \\ x_{112} + x_{212} + x_{132} + x_{212} + x_{222} + x_{232} \leq (30,40,60), \\ (15,16,17)x_{111} + 18x_{112} + 17x_{211} + 20x_{212} \geq [15,25], \\ 12x_{121} + (20,21,22)x_{122} + (21,22,23)x_{221} + 20x_{222} \geq [17,21], \\ 10x_{131} + (4,5,6)x_{132} + 21x_{231} + (19,20,21)x_{232} \geq [11,31], \\ x_{ijk} \geq 0,\ i = 1,2,\ j = 1,2,3,\ k = 1,2. \end{array} \right.$$

$$\begin{array}{l} (60+10\alpha) \leq s_1 \leq (80-10\alpha), \\ (20+20\alpha) \leq s_2 \leq (60-20\alpha), \\ (10+20\alpha) \leq d_1 \leq (40-10\alpha), \\ (20+10\alpha) \leq d_2 \leq (40-10\alpha), \\ (40+10\alpha) \leq d_3 \leq (70-10\alpha), \\ (20+10\alpha) \leq e_1 \leq (40-10\alpha), \\ (30+10\alpha) \leq e_2 \leq (60-10\alpha), \\ \forall\ i,j,k. \end{array} \tag{3.21}$$

According to models (3.15) and (3.18), the lower and upper bounds of $\tilde{Z}$at the possibility level α could be formulated as:

$$
\begin{aligned}
Z_\alpha^U = \max \quad & \lambda \\
\text{subject to} \quad & -u_1 + v_1 - w_1 - (15+\alpha)g_1 + \delta_{111} \leq (17-\alpha), \\
& -u_1 + v_1 - w_2 - 18g_1 + \delta_{112} \leq 18, \\
& -u_1 + v_2 - w_1 - 12g_2 + \delta_{121} \leq 12, \\
& -u_1 + v_2 - w_2 - (20+\alpha)g_2 + \delta_{122} \leq (22-\alpha), \\
& -u_1 + v_3 - w_1 - 21g_3 + \delta_{131} \leq 10, \\
& -u_1 + v_3 - w_2 - (4+\alpha)g_3 + \delta_{132} \leq (6-\alpha), \\
& -u_2 + v_1 - w_1 - 17g_1 + \delta_{211} \leq 17, \\
& -u_2 + v_1 - w_2 - 20g_1 + \delta_{212} \leq 20, \\
& -u_2 + v_2 - w_1 - (21+\alpha)g_2 + \delta_{221} \leq (23-\alpha), \\
& -u_2 + v_2 - w_2 - 20g_2 + \delta_{222} \leq 20, \\
& -u_2 + v_3 - w_1 - 21g_3 + \delta_{231} \leq 21, \\
& -u_2 + v_3 - w_2 - (19+\alpha)g_3 + \delta_{232} \leq (21-\alpha), \\
& p_1 + p_2 - q_1 - q_2 - q_3 + r_1 + r_2 + 700\lambda \leq 650, \\
& \delta_{111} = 6\lambda, (7+\alpha)\lambda \leq \delta_{112} \leq (10-\alpha)\lambda, \\
& \delta_{121} = 10\lambda, \delta_{122} = 6\lambda, \\
& (10+\alpha)\lambda \leq \delta_{131} \leq (13-\alpha)\lambda, \\
& \delta_{132} = 3\lambda, \delta_{211} = 13\lambda, \\
& (8+\alpha)\lambda \leq \delta_{212} \leq (10-\alpha)\lambda, \\
& \delta_{221} = 12\lambda, (2+\alpha)\lambda \leq \delta_{222} \leq (4-\alpha)\lambda, \\
& \delta_{231} = 20\lambda, \delta_{232} = 15\lambda, \\
& (60+10\alpha)u_1 \leq p_1 \leq (80-10\alpha)u_1, (20+20\alpha)u_2 \leq p_2 \leq (60-20\alpha)u_2, \\
& (10+20\alpha)v_1 \leq q_1 \leq (40-10\alpha)v_1, (20+10\alpha)v_2 \leq q_2 \leq (40-10\alpha)v_2, \\
& (40+10\alpha)v_3 \leq q_3 \leq (70-10\alpha)v_3, \\
& (20+10\alpha)w_1 \leq r_1 \leq (40-10\alpha)w_1, (30+10\alpha)w_2 \leq r_2 \leq (60-10\alpha)w_2, \\
& \lambda > 0, \\
& u_1, u_2, v_1, v_2, v_3, w_1, w_2 \geq 0.
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
Z_\alpha^L = & \min(15+\alpha)y_{111} + 18y_{112} + 12x_{121}(20+\alpha)y_{122} + 10y_{131} + (4+\alpha)y_{132} \\
& + 17y_{211} + 20y_{212} + (21+\alpha)y_{221} + 20y_{222} + 21y_{231} + (19+\alpha)y_{232} + 650t \\
\text{subject to } & - y_{111} - y_{112} - y_{121} - y_{122} - y_{131} - y_{132} + s_1 \geq 0, \\
& - y_{211} - y_{212} - y_{221} - y_{222} - y_{231} - y_{232} + s_2 \geq 0, \\
& y_{111} + y_{112} + y_{211} + y_{212} - d_1 \geq 0, \\
& y_{121} + y_{122} + y_{221} + y_{222} - d_2 \geq 0, \\
& y_{131} + y_{132} + y_{231} + y_{232} - d_3 \geq 0, \\
& - y_{111} - y_{121} - y_{131} - y_{211} - y_{221} - y_{231} + e_1 \geq 0, \\
& - y_{112} - y_{212} - y_{132} - y_{212} - y_{222} - y_{232} + e_2 \geq 0, \\
& - (15+\alpha)y_{111} - 18y_{112} - 17y_{211} - 20y_{212} + b_1 \geq 0, \\
& - 12y_{121} - (20+\alpha)y_{122} - (21+\alpha)y_{221} - 20y_{222} + b_2 \geq 0, \\
& - 10y_{131} - (4+\alpha)y_{132} - 21y_{231} - (19+\alpha)y_{232} + b_3 \geq 0, \\
& \xi_{111} + \xi_{112} + \xi_{121} + \xi_{122} + \xi_{131} + \xi_{132} + \xi_{211} + \xi_{212} + \xi_{221} + \xi_{222} + \xi_{231} + \xi_{232} + 700t = 1, \\
& \xi_{111} = 6y_{111}, (7+\alpha)y_{112} \leq \xi_{112} \leq (10-\alpha)y_{112}, \\
& \xi_{121} = 10y_{121}, \xi_{122} = 6y_{122}, \\
& (10+\alpha)y_{131} \leq \xi_{131} \leq (13-\alpha)y_{131}, \\
& \xi_{132} = 3y_{132}, \xi_{211} = 13y_{211}, \\
& (8+\alpha)y_{212} \leq \xi_{212} \leq (10-\alpha)y_{212}, \\
& \xi_{221} = 12y_{221}, (2+\alpha)y_{222} \leq \xi_{222} \leq (4-\alpha)y_{222}, \\
& \xi_{231} = 20y_{231}, \xi_{232} = 15y_{232}, \\
& (60+10\alpha)t \leq s_1 \leq (80-10\alpha)t, (20+20\alpha)t \leq s_2 \leq (60-20\alpha)t, \\
& (10+20\alpha)t \leq d_1 \leq (40-10\alpha)t, (20+10\alpha)t \leq d_2 \leq (40-10\alpha)t, \\
& (40+10\alpha)t \leq d_3 \leq (70-10\alpha)t, \\
& (20+10\alpha)t \leq e_1 \leq (40-10\alpha)t, (30+10\alpha)t \leq e_2 \leq (60-10\alpha)t, \\
& \lambda > 0, \\
& y_{ijk} \geq 0,\ i = 1,2.,\ j = 1,2,3.,\ k = 1,2.
\end{aligned}
\tag{3.23}
$$

Table 3.1 lists the α-cuts if the value of total transportation cost of 11 distinct α values: $0, 0.1, 0.2, 0.3, \ldots 1.0$ and Figure 3.1 depict the membership function of the total transportation cost of this example. The α-cuts of Z represents the possibility that the value of the game would appear in the associated range. Since the fuzzy objective value lied in the range, different α-cuts show different interval and the uncertainty level of yet lowest possibility, indicating that the objective value would never fall outside of this range. Most likely the value fell between 0.5930 and 0.9890. The membership function of $\tilde{Z}$, as constructed from 100 α-cuts, is depicted in the Figure 3.1.

At α-cuts=0, the lower bound $Z^L_{\alpha=0}$ was solved as 0.5930, occurring at $x^*_{121} = 30$, $x^*_{132} = 10$, x^*_{211}=40, $x^*_{131} = 60, x^*_{111} = x^*_{112} = x^*_{122} = x^*_{212} = x^*_{221} = x^*_{222} = x^*_{231} = x^*_{232} = 0$ and the total supply is 140. The upper bound $Z^U_{\alpha=0}$ was solved as 0.9890, which occursed at $x^*_{121} = 10$, $x^*_{132} = 30$, x^*_{211}=20, $x^*_{131} = 20, x^*_{111} = x^*_{112} = x^*_{122} = x^*_{212} = x^*_{221} = x^*_{222} = x^*_{231} = x^*_{232} = 0$. In this case, the total amount being shipped was 80. At α-cut=1, the lower bound $Z^L_{\alpha=1}$=0.9645 occurs at $x^*_{121} = 10$, $x^*_{132} = 30$, x^*_{211}=20, $x^*_{131} = 20, x^*_{111} = x^*_{112} = x^*_{122} = x^*_{212} = x^*_{221} = x^*_{222} = x^*_{231} = x^*_{232} = 0$, and the total transportation quantity was 150. Notably, the objective value associated with the largest total quantity being shipped need not be the highest. In this example, the largest possible amount to be shipped is 140, which is the largest total supply. The objective value for this amount is Z^*=0.5930. However, the highest objective value is Z^*=0.9890, occurring at the total amount of 80.

Table 3.1: The α -cuts of the total transportation cost

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Z^U_α	0.9890	0.9858	0.9825	0.9791	0.9755	0.9719	0.9681	0.9643	0.9604	0.9564	0.9524
Z^L_α	0.5930	0.6134	0.6310	0.6515	0.6753	0.7045	0.7474	0.8058	0.8614	0.9141	0.9645

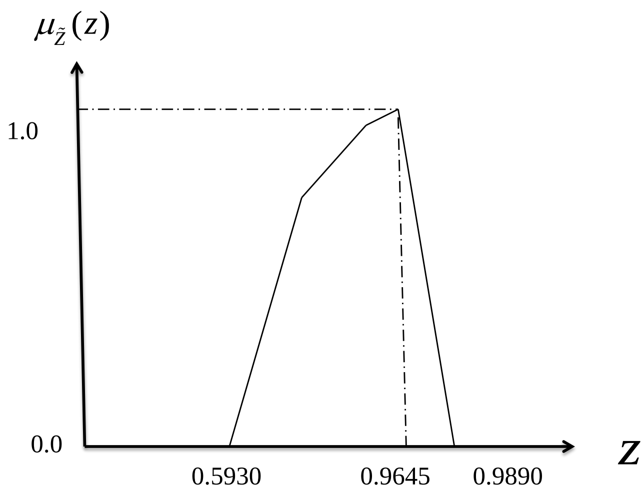

Figure 3.1: Membership function of total transportation cost

Chapter 4

Goal Programming Approach to Stochastic Solid Transportation Problem under Budget Constraint

4.1 Introduction

In real life the demand of an item may not be crisp. So **in this chapter**, a fuzzy stochastic solid transportation problem (FSSTP) is formulated with random demand and capacities of conveyances with budget constraints. Goal programming (GP) approach was applied to solve the said solid transportation problem (STP) under several constraints. This chapter also presents fuzzy goal programming models (FGP) for the stochastic aspect in STP. The researcher considered demand, conveyance capacities are random and expressed as fuzzy-stochastic constraints. Moreover, as a particular case three dimensional representation of an existing model Chalam [11] is also presented. The optimum results of this model were compared with the solid transportation model with different budgets.

Section 4.1 gives the introduction. Section 4.2 gives the formulation of the problem in the goal programming (GP) formulation. Section 4.3 deals with stochastic aspects of transportation problem. Section 4.4 deals with fuzzy goal programming (FGP) formulation of the problem with random demands. The algorithm is given in section 4.5. Section 4.6 presents a numerical example.

4.2 Goal Programming (GP) Formulation

Let us consider a transportation situation where s_i, d_j and e_k be the quantities of a commodity available, demand and conveyance at the i^{th} origin $(i = 1, 2, \ldots, m)$, j^{th} destination through the k^{th} conveyances $(k = 1, 2, \ldots, K)$, respectively. Let c_{ijk} be the per unit transportation cost for transporting one unit from i^{th} origin to j^{th} destination by k^{th} conveyance. Further B is the budget allotted for the transportation which is less than the minimum total cost of transportation. Hence, the problem is to find the optimum values of x_{ijk} (where x_{ijk} is the number of units of the commodity to be transported from i^{th} origin to j^{th} destination by k^{th} conveyance).

$$\begin{aligned} Z = \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} \\ \text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_i, \qquad i = 1,2,\ldots,m \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_j, \qquad j = 1,2,\ldots,n \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1,2,\ldots,K \\ & x_{ijk} \geq 0, \quad \forall i,j,k. \end{aligned} \tag{4.1}$$

This model is applicable only when all the data is deterministic.

$Z = \min \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} \leq B$

When the budgetary constraint is operating in the system, the demand needs at all the demand points cannot be fulfilled simultaneously and each demand point tries to realize its demand requirement competing with the others. Thus the problem is to be viewed as a multiple objective goal programming problem.

The GP version of problem (4.1) is expressed as

$$\begin{aligned} \min \quad & Z(g) = \sum_{j=1}^{n} g_j^- \\ \text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_i, \qquad i = 1,2,\ldots m, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} - g_j^+ + g_j^- = d_j, \quad j = 1,2,\ldots,\ n, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1,2,\ldots K, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} \leq B \end{aligned} \tag{4.2}$$

$$g_j^+ . g_j^- = 0, \qquad j = 1, 2, \ldots n.$$

$$x_{ijk}, g_j^+, g_j^- \geq 0, \quad \forall\, i, j, k.$$

Here, g_j^+, g_j^- are deviational variables from the j^{th} goal.

4.3 Stochastic aspect of Solid Transportation Problem

We know that if Y be a normal variate with mean m and standard deviation σ, then it satisfies the following relations.

$$P[m - \sigma < Y < m + \sigma] = 0.6826$$

$$P[m - 2\sigma < Y < m + 2\sigma] = 0.9544$$

$$P[m - 3\sigma < Y < m + 3\sigma] = 0.9973$$

The above probabilities furnished that if there was a normal distribution of 10,000 cases, then 6826, 9544 and 9973 cases were expected to be between the ranges$(m - \sigma, m + \sigma)$, $(m - 2\sigma, m + 2\sigma)$ and $(m - 3\sigma, m + 3\sigma)$ respectively. The general rule is that in a normal distribution, $m \pm 3\sigma$ will give us the range within which a normal mean (m) is expected to vary. These are the limitation within which the actual mean would be vary. Using these properties of a normal variate, the above problem could be reduced to equivalence crisp problems as discussed below.

4.4 Fuzzy Goal Programming Formulation of Stochastic Solid Transportation Problem (SSTP)

The FGP formulation of SSTP with Budgetary constraint was given in the following.

Find x_{ijk}

$$
\begin{aligned}
\text{such that } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, && i = 1, 2, \dots m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \gtrsim d_j, && j = 1, 2, \dots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, && k = 1, 2, \dots K \\
& \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk} x_{ijk} \le B, \\
& x_{ijk} \ge 0, \quad \forall\, i, j, k.
\end{aligned}
\tag{4.3}
$$

Where the deterministic equivalent of the fuzzy stochastic constraint symbol "$\gtrsim$" stood approximately or fuzzily greater than or equal to. As such, problems cannot be solved in this form, for that crisp equivalents models were required.

Here it has been discussed the three minimum levels of fulfillments by all the demand points simultaneously viz., lower bound σ_j, lower bound $2\sigma_j$, lower bound $3\sigma_j$. Let A_j be the fuzzy set corresponding to the j^{th} demand constraint $\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \ge d_j$, $j = 1, 2, \dots, n$.

Following Zimmermann [110] it could be defined the membership function of A_j as

$$\mu_{A_j} = \begin{cases} 1, & \text{if} \quad \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_j, \\ \dfrac{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K} x_{ijk} - d_j^*}{d_j - d_j^*}, & \text{if} \quad d_j^* < \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} < d_j \\ 0 & \text{if} \quad \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \leq d_j^*, \end{cases} \tag{4.4}$$

Where d_j^* is the lower tolerance limit of $\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K} x_{ijk} \geq d_j$.
Initially we took d_j^* as $(d_j - \sigma_j)$ with this membership function, if the solution is infeasible relax its value by taking $d_j^* = (d_j - 2\sigma_j)$ and if the corresponding solution is found still infeasible then further relax it by $d_j^* = (d_j - 3\sigma_j)$.

The situation here is one of competitiveness since all the demand points compete with each other to realize the minimum level of fulfillment prescribed. Thus we define the overall decision function D by the intersection operator as $D = \bigcap_{j=1}^{n} A_j$. The membership function of D becomes

$$\mu_D = \bigwedge_{j=1}^{n} \mu_{A_j} = Min\{\mu_{A_j}\}$$

The crisp equivalent of (4.3) can now be defined as

max μ_D

Subject to $\mu_D \leq \mu_{A_j}$

$$\lambda = \min_j \left[\frac{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K} x_{ijk} - d_j^*}{d_j - d_j^*} \right]$$

It is formulated the above expression as equivalent to

$$
\begin{aligned}
\max \quad & \lambda \\
\text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \le s_i, && i = 1, 2, \ldots m \\
& \lambda \le \left[\frac{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K} x_{ijk} - d_j^*}{d_j - d_j^*} \right], && j = 1, 2, \ldots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, && k = 1, 2, \ldots, K \\
& \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} \le B, \\
& \lambda, x_{ijk} \ge 0, \quad \forall\ i, j, k.
\end{aligned}
\tag{4.5}
$$

4.5 Algorithm

Step 1 Consider the solid transportation problem with budget constraint as given in (4.1).

Step 2 Solve (4.1) using the given data and fix the goal in (4.3).

Step 3 For fuzzy goal programming of stochastic STP formulate the membership function as given in (4.4).

Step 4 By using simplex method or by any standard solve the problem (4.5)

4.6 Numerical Example

Ganga Distributor, a well known sugar brand in South India (Telangana State) dispatches its product from two sources to three destinations through two specific conveyances. It has some main stocking depots and distribution centers in different cities of Telangana. Here two sources (i.e., m=2), three destinations (i.e., n=3), two conveyances (i.e., K=2) are considered from its two stocking deports; three distribution centers cost against two types of conveyances by road (small and big size truck) for 12 months were collected. Source (s_1, s_2), demands (d_1, d_2, d_3), capacities of conveyances (e_1, e_2) and goals and total transportation cost presented in Table-4.1. We consider the normal distribution curves of the raw data approximately 68%, 95% and 99% data between $(m-\sigma, m+\sigma)$, $(m-2\sigma, m+2\sigma)$ and $(m-3\sigma, m+3\sigma)$ respectively.

Table 4.1: Cost matrix (in \$)

c_{ij1}		
10	8	12
13	10	15

c_{ij2}		
14	8	10
17	12	15

i.e.,

$$\begin{aligned}
\min \quad & 10x_{111} + 14x_{112} + 8x_{121} + 8x_{122} + 12x_{131} + 10x_{132} \\
& + 13x_{211} + 17x_{212} + 10x_{221} + 12x_{222} + 15x_{231} + 15x_{232} \\
\text{subject to } & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 24, \\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 32, \\
& x_{111} + x_{211} + x_{112} + x_{212} \geq 18, \\
& x_{121} + x_{221} + x_{122} + x_{222} \geq 21, \\
& x_{131} + x_{132} + x_{231} + x_{232} \geq 17, \\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \leq 46, \\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \leq 52, \\
& x_{ijk} \geq 0, \quad \forall\, i, j, k.
\end{aligned} \tag{4.6}$$

Using the raw data from the above Table-4.1, optimum results were found as x_{111}= 7, x_{132}= 17, x_{211}=11 and x_{221}= 21 with minimum transportation cost of \$593.

4.6.1 Goal Programming model

When the budget is assuming \$500, the goal programming model follow as

$$
\begin{aligned}
\min \ Z = \sum_{j=1}^{3} g_j^- \\
\text{subject to } & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 24, \\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 32, \\
& x_{111} + x_{211} + x_{112} + x_{212} + g_1^- - g_1^+ = 18, \\
& x_{121} + x_{221} + x_{122} + x_{222} + g_2^- - g_2^+ = 21, \\
& x_{131} + x_{132} + x_{231} + x_{232} + g_3^- - g_3^+ = 17, \\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \leq 46, \\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \leq 52, \\
& 10x_{111} + 8x_{121} + 12x_{131} + 14x_{112} + 8x_{122} + 10x_{132} \\
& + 13x_{211} + 10x_{221} + 15x_{231} + 17x_{212} + 12x_{222} + 15x_{232} \leq 500, \\
& x_{ijk} \geq 0, \quad \forall \ i, j, k, \\
& g_j^+, g_j^- \geq 0, \\
& g_j^+ . g_j^- = 0.
\end{aligned} \tag{4.7}
$$

Since all goals have an equal priority, solving the above problem using Lingo software, it made possible to get the deficit achievement function as 7.1538 (shown in Table-4.2).

Table 4.2: Result of model 4.7

Conveyance 1	Demand			
Supply	1	0	0	
	3.84615	21	0	
Conveyance 2	Demand			
Supply	0	0	17	
	0	0	0	
Existing Demand	18	21	17	56
Fulfillment demand	10.846154	21	17	48.846154
Deficit	7.153846	0	0	7.153846

4.6.2 Fuzzy Goal programming formulation of SSTP model

For the present numerical problem, among the four parameters-transportation goal cost, supply, demand and conveyance capacities, some may be picked random and others are crisp. So different models could be formulated with several combinations of these parameters as random and crisp.

Here it was considered demand as random and expressed as fuzzy-stochastic constraints. $Y_1 \sim N(18,3)$, $Y_2 \sim N(21,3)\, and\, Y_3 \sim N(17,2)$ The left parts of the $1\sigma_j$ intervals, i.e., (d_j^*, d_j), for Y_1, Y_2 and Y_3 are (15, 18), (18, 21), (15, 17) respectively. However, with these (d_j^*, d_j) values, Model-(4.8) has no feasible solution.

Therefore, relax the d_j^* values by taking the left part $2\sigma_j$ intervals. Now Y_1, Y_2 and Y_3 values are (12, 18), (15, 21), (13, 17) respectively. Even with these (d_j^*, d_j) values, Model (4.8) has no feasible solution.

Hence, further relax the d_j^* values by taking the left part of $3\sigma_j$ intervals. Now Y_1, Y_2 and Y_3 values are (9, 18), (12, 21), (11, 17) respectively.

Similarly, it was considered demand, conveyance capacities as random and expressed as fuzzy-stochastic constraints.

$$\text{Let}\, Y_1 \sim N(18,3),\, Y_2 \sim N(21,3)\, and\, Y_3 \sim N(17,2)$$
$$E_1 \sim N(46,3),\; E_2 \sim N(52,2)$$

If only demand parameters were random, Model (4.7) becomes:

$$\begin{aligned}
\max \quad & \lambda \\
\text{subject to } & x_{111}+x_{121}+x_{131}+x_{112}+x_{122}+x_{132} \leq 24, \\
& x_{211}+x_{221}+x_{231}+x_{212}+x_{222}+x_{232} \leq 32, \\
& x_{111}+x_{211}+x_{112}+x_{212}-9\lambda \geq 9, \\
& x_{121}+x_{221}+x_{122}+x_{222}-9\lambda \geq 12, \\
& x_{131}+x_{132}+x_{231}+x_{232}-6\lambda \geq 11, \\
& x_{111}+x_{211}+x_{131}+x_{121}+x_{221}+x_{231} \leq 46, \\
& x_{112}+x_{212}+x_{132}+x_{122}+x_{222}+x_{232} \leq 52, \\
& 10x_{111}+8x_{121}+12x_{131}+14x_{112}+8x_{122}+10x_{132} \\
& +13x_{211}+10x_{221}+15x_{231}+17x_{212}+12x_{222}+15x_{232} \leq 500, \\
& \lambda, x_{ijk} \geq 0, \quad \forall\, i,j,k.
\end{aligned} \tag{4.8}$$

Solving the above model the optimum results were obtained and presented in Table-4.3. It was also observed that the distributor has total demand of sugar 56 tons fulfilling the total mean demand was 48.168428 tons for this model. Thus the pattern of distribution controlled and the total deficit of 7.831572 tons were distributed among all the demand points.

Now let it be consider that the demand and conveyance parameters are random, Model (4.6) becomes

$$
\begin{aligned}
\max \quad & \lambda \\
\text{subject to}\ & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 24, \\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 32, \\
& x_{111} + x_{211} + x_{112} + x_{212} - 9\lambda \geq 9, \\
& x_{121} + x_{221} + x_{122} + x_{222} - 9\lambda \geq 12, \\
& x_{131} + x_{132} + x_{231} + x_{232} - 6\lambda \geq 11, \\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \leq 37, \\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \leq 46, \\
& 10x_{111} + 8x_{121} + 12x_{131} + 14x_{112} + 8x_{122} + 10x_{132} \\
& + 13x_{211} + 10x_{221} + 15x_{231} + 17x_{212} + 12x_{222} + 15x_{232} \leq 500, \\
& \lambda, x_{ijk} \geq 0, \quad \forall\, i, j, k.
\end{aligned} \tag{4.9}
$$

Table 4.3: Result of model 4.8

Conveyance 1 (CC1)	Demand			(CC1)	Existing	Fulfillment	Deficit
Supply	8.957895 6.105263	0 18.06316	0 0		46	33.126	12.874
Conveyance 2 (CC2)	Demand			(CC2)			
Supply	0 0	0 0	15.04211 0		52	15.042	36.958
Existing demand	18	21	17	56	98	48.168	49.832
Fulfillment demand	15.063158	18.06316	15.04211	48.168428			
Deficit	2.936842	2.93684	1.95789	7.831572			

By applying the GP approach to the numerical example, it has been observed that the total deficit of 7.153846 tons of sugar borne alone (shown in Table-4.2). This is due to the fact that GP is based on the principle of minimizing the sum of the deviations from goal values, as the deviations are unrestricted. However, in FGP approach, the deviations were restricted to lie in a pre-specified interval at all goals.

Thus the pattern of distribution was controlled and the total deficit of 7.831572 tons which was distributed among all the demand points shown in Table-4.3. The fuzzy intersection operator ensured fulfillment of a certain amount of demand by all the demand points.

The results of Model 4.8 in Table- 4.4 were represented in a budget values 530 and 550 by applying (4.2) and (4.5) models, i.e., when two random conveyances were used for Model 4.8. It was interesting to note that through total demand deficit in GP model (4.846154 tons) was much less than that (5.305263 tons) FGP model. Similarly, we can observe that demand deficit when budget is 550 in GP and FGP models (shown in Table-4.4). This was because when two random conveyances were available, in the minimization of total cost.

Table 4.4: Results of the budget (with values 530 and 550 respectively), random demand, conveyances, general SSTP using GP and FGP.

Budget		Optimal solutions of x_{ijk}			Existing demand $(d_j, m\sigma_j)$	Fulfillment demand	Demand deficit	Existing capacities $(e_k, m\sigma_k)$	Fulfillment capacities	Capacities deficit
530	GP model	7 6.153846	0 21	0 0	((18,9), (21,9), (17,6))	(13.153846, 21, 17)	4.846154	(46,9)	34.153846	51.153846
		0 0	0 0	17 0				(52, 6)	17	
	FGP model	8.326316 7.684211	0 19.01053	0 0	((18,9), (21,9), (17,6))	(16.010527, 19.01053, 15.67368)	5.305263	(46,9)	35.021057	50.694737
		0 0	0 0	15.67368 0				(52, 6)	15.67368	
550	GP model	7 7.692308	0 21	0 0	((18,9), (21,9), (17,6))	(14.692308, 21, 17)	3.307692	(46,9)	35.692308	52.692308
		0 0	0 0	17 0				(52, 6)	17	
	FGP model	7.905263 8.736842	0 19.64211	0 0	((18,9), (21,9), (17,6))	(16.643105, 19.64211, 16.09474)	3.620045	(46,9)	36.284215	52.378955
		0 0	0 0	16.09474 0				(52, 6)	16.09474	

Chapter 5

Multi-Objective Fuzzy Solid Transportation Problem based on Expected Value and the Goal Programming Approach

5.1 Introduction

To the best of the author's knowledge, it might be noticed that previous studies did not include the goal programming approach for solving multi-objective fuzzy solid transportation problem based on expected value models. It was shown that the optimal solution of the (MOFSTP) could be found simply by solving an equivalent crisp LP problem.

So in this **chapter**, a fuzzy goal programming approach for solving multi-objective fuzzy solid transportation problem (MOFSTP) with fuzzy constraints is presented. The objective is to determine the crisp model with corresponding defuzzified values under the conditions and the expected value models in objective functions for triangular and trapezoidal membership functions. Then multi-objective problems are solved by the fuzzy goal programming approach and three numerical examples were given to illustrate the proposed model.

Section 5.2 introduces the preliminaries and notations. Section 5.3 gives the formulation of the problem. Section 5.4 provides a defuzzification process. Section 5.5, deals with the solution methodology. Section 5.6 presents three numerical examples with analysis of results. Section 5.7 gives the comparative study.

5.2 Preliminaries

5.2.1 Expected value operator on fuzzy number

Liu *et al.*[67] presented a novel definition of expected value of fuzzy variable and proposed a new class of fuzzy programming called fuzzy expected value models. Yang and Liu [105] applied expected value model, chance-constrained programming model and dependent-chance programming in fixed charge solid transportation problem in fuzzy environment. kundu *et al.*[59] presented MOSTP under various uncertain environments. In this chapter, we develop a goal programming approach and consider the MOFSTP based on expected value model. Also, we consider sources, demands, and conveyance capacities as fuzzy.

Definition 5.1. (Liu *et al.*[67]) Let $\tilde{\xi}$ be a fuzzy variable. Then the expected value of $\tilde{\xi}$ is defined as

$$E\left[\xi\right] = \int_0^{\infty} Cr\left\{\xi \geq r\right\}dr - \int_{-\infty}^{0} Cr\left\{\xi \leq r\right\}dr,$$

(where Cr stands for credibility measure)
provided that at least one of the two integral is finite. If $\tilde{\xi}$ is a triangular fuzzy variable (r_1, r_2, r_3), then the expected value of $\tilde{\xi}$ is $(1/4)(r_1 + 2r_2 + r_3)$. If $\tilde{\xi}$ is a trapezoidal fuzzy variable (r_1, r_2, r_3, r_4), then the expected value of $\tilde{\xi}$ is $(1/4)(r_1 + r_2 + r_3 + r_4)$.

5.2.2 Defuzzification

Kikuchi [52] proposed a defuzzification method to find the most appropriate set of crisp numbers. For each of many possible sets of values that satisfy the relationships the lowest membership grade was checked and the set whose lowest membership grade was the highest chosen as the best set of values for the problem. This process is performed using the fuzzy linear programming method.

Let the membership function for each value as $\mu_A\left(x\right)$, $\mu_B\left(y\right)$ and $\mu_C\left(z\right)$ where A, B, C are the fuzzy set of approximate numbers, if there was corresponding crisp values x, y, z. But each of them satisfy the relationship $R_j\left(x\right)$, $j \in N$ among them.

Then the following defuzzified model is formulated:

$$
\begin{aligned}
\max \quad & \lambda, \\
\text{subject to} \quad & \mu_A(x) \geq \lambda, \qquad \mu_B(y) \geq \lambda, \qquad \mu_C(z) \geq \lambda, \\
& \text{and the relationship } R_j(x), j \in N, \\
& \lambda \geq 0, \quad A, B, C \geq 0.
\end{aligned}
$$

where λ is the minimum degree of membership that one of the values A, B, C takes, i.e., $\lambda^* = \text{Max } \lambda = \text{Max Min } [\mu_A(x), \mu_B(y), \mu_C(z)]$.

Kikuchi [52] applied this method to a traffic volume consistency problem taking all observed values as triangular fuzzy numbers. Dey and Yadav [21] modified this method with trapezoidal fuzzy numbers.

5.3 Problem Formulation

A multi-objective fuzzy solid transportation problem (MOFSTP) is formulated, where c_{ijk}^r, s_i, d_j and e_k are all fuzzy numbers:

$$
\begin{aligned}
Z_r = \min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \tilde{c}_{ijk}^r x_{ijk}, \quad && r = 1, 2, \ldots R \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \tilde{s}_i, && i = 1, 2, \ldots m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \tilde{d}_j, && j = 1, 2, \ldots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \tilde{e}_k, && k = 1, 2, \ldots K \\
& x_{ijk} \geq 0, \forall i, j, k
\end{aligned}
\tag{5.1}
$$

Consider a product to be transported from m sources to n destinations in a STP. At each source, let s_i be the amount of a homogeneous product we wanted to be transported to n destinations to satisfy the demand for d_j units of the product. Here e_k, called conveyance, denotes the units of this product that can be carried by k different modes of transportation and also the objectives $Z_r (r = 1, 2, \ldots R)$ are to be minimized.

5.4 Defuzzification

We considered s_i, d_j and e_k $(\forall i, j, k)$ are any triangular or trapezoidal fuzzy numbers (here triangular fuzzy numbers were denoted by $s_i = (s_i^1, s_i^2, s_i^3), d_j = (d_j^1, d_j^2, d_j^3)$ and $e_k = (e_k^1, e_k^2, e_k^3)$, whereas trapezoidal fuzzy numbers were denoted by $s_i = (s_i^1, s_i^2, s_i^3, s_i^4), d_j = (d_j^1, d_j^2, d_j^3, d_j^4)$ and $e_k = (e_k^1, e_k^2, e_k^3, e_k^4)$) with their membership functions as μ_{s_i}, μ_{d_j}andμ_{e_k} respectively. Now to solve the above problem, first find the corresponding crisp numbers, say, s_{ic}, d_{jc} and $e_{kc}(\forall i, j, k)$ so that for each item, total available resources greater than or equal to the total demands and also total conveyance capacities greater than or equal to the total demands for all items, i.e.

$$\sum_{i=1}^{m} s_{ic} \geq \sum_{j=1}^{n} d_{jc}, \qquad \sum_{k=1}^{K} e_{kc} \geq \sum_{j=1}^{n} d_{jc} \tag{5.2}$$

For this purpose the defuzzification method based on fuzzy linear programming was applied. The method is to introduce an auxiliary variable and formulate the following linear programming: where λ is the minimum degree of membership that one of the values of the variables s_{ic}, d_{jc}, e_{kc} takes, i.e.

$\max \lambda = \lambda^* = MaxMin[\mu_{s_i}(s_{ic}), \mu_{d_j}(d_{jc}), \mu_{e_k}(e_{kc})]$, where for triangular fuzzy numbers:

$$\mu_{s_i}(s_{ic}) = \begin{cases} \dfrac{s_i - s_i^1}{s_i^2 - s_i^1}, & \text{when} \quad s_i^1 \le s_{ic} \le s_i^2; \\ \dfrac{s_i^1 - s_i}{s_i^3 - s_i^2}, & \text{when} \quad s_i^2 \le s_{ic} \le s_i^3; \\ 0, & \text{otherwise.} \end{cases}$$

and for trapezoidal fuzzy numbers:

$$\mu_{s_i}(s_{ic}) = \begin{cases} \dfrac{s_i - s_i^1}{s_i^2 - s_i^1}, & \text{when} \quad s_i^1 \le s_{ic} \le s_i^2; \\ 1, & \text{when} \quad s_i^2 \le s_{ic} \le s_i^3; \\ \dfrac{s_i^4 - s_i}{s_i^4 - s_i^3}, & \text{when} \quad s_i^3 \le s_{ic} \le s_i^4; \\ 0, & \text{otherwise.} \end{cases}$$

and similarly for $\mu_{d_j}(d_{jc})$, $\mu_{e_k}(e_{kc})$.

Now if it was denoted the right and the left sides of membership functions $\mu_{s_i}(s_{ic})$ by $\mu_{s_i}^l(s_{ic})$ and $\mu_{s_i}^r(s_{ic})$ respectively and so on for $\mu_{d_j}(d_{jc})$, $\mu_{e_k}(e_{kc})$. Then the following auxiliary model is formulated

$$\begin{aligned} \max \quad & \lambda, \\ \text{subject to } & \mu_{s_i}^l(s_{ic}) \ge \lambda, \qquad \mu_{s_i}^r(s_{ic}) \ge \lambda, \\ & \mu_{d_j}^l(d_{jc}) \ge \lambda, \qquad \mu_{d_j}^r(d_{jc}) \ge \lambda, \\ & \mu_{e_k}^l(e_{kc}) \ge \lambda, \qquad \mu_{e_k}^r(e_{kc}) \ge \lambda, \\ & \sum_{i=1}^{m} s_{ic} \ge \sum_{j=1}^{n} d_{jc}, \qquad \sum_{k=1}^{K} e_{kc} \ge \sum_{j=1}^{n} d_{jc}, \\ & \lambda \ge 0, \quad \forall i, j, k. \end{aligned} \tag{5.3}$$

5.5 Solution Methodology

After obtaining the defuzzified values s_{ic}, d_{jc} and $e_{kc}(\forall i, j, k)$ through the above procedure, problem 5.1 became as,

$$\begin{aligned}
\min Z_r = & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \tilde{c}_{ijk}^{r} x_{ijk}, \quad r = 1, 2, 3 \ldots, R \\
\text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_{ic}, \qquad i = 1, 2, \ldots m \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_{jc}, \qquad j = 1, 2, \ldots n \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_{kc}, \qquad k = 1, 2, \ldots K \\
& x_{ijk} \geq 0, \quad \forall i, j, k.
\end{aligned} \tag{5.4}$$

Now, we use the following method to solve the problem.

5.5.1 Using Expected Value

Here we minimize the expected value of the objective functions and then the above problem becomes

$$\min E\left[Z_r\right] = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}^{r} x_{ijk}\right], \qquad r = 1, 2, 3 \ldots, R \tag{5.5.1}$$

$$\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_{ic}, \qquad i = 1, 2, \ldots m \tag{5.5.2}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_{jc}, \qquad j = 1, 2, \ldots, n \tag{5.5.3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_{kc}, \qquad k = 1, 2, \ldots, K \tag{5.5.4}$$

$x_{ijk} \geq 0, \forall i, j, k.$

which is equivalently written as (using Liu *et al.*[67])

$$\min E\left[Z_r\right] = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}^{r}\right] x_{ijk}, \qquad r = 1, 2, 3 \ldots, R \tag{5.6}$$

subject to the constraints (5.5.2) - (5.5.4), $x_{ijk} \geq 0, \quad \forall\ i, j, k.$

The expected value model could be formulated for the model (5.1) by using expected value to both the objective functions and the constraints. But here the crisp equivalence form (the deterministic values of supplies $E\left[\tilde{s}_i\right]$, demands $E[\tilde{d}_j]$ and conveyance capacities $E\left[\tilde{e}_k\right]$) might not satisfy the required conditions (5.2). So this method gave a feasible solution only when the fuzzy supplies, demands and conveyance capacities are so that their respective expected values automatically satisfy those conditions.

5.5.2 Algorithm

The procedure to solve MOFSTP based on fuzzy goal programming techniques is given below:

Step 1 Solve (5.3) to obtain the defuzzified values.

Step 2 Formulate the model (5.6).

Step 3 Solve multi-objective problem as a single objective problem each time using only one objective ($r = 1, 2, \ldots R$) ignore all other objectives, to obtain the optimal solution X^{r*} of R different single objective problems.

Step 4 Calculate the values of all the R objective functions at all these R optimal solutions $X^{r*}(r = 1, 2, \ldots R)$ and find the lower bound and upper bound for each objective function given by $L_t = \bar{Z}_r(X^{t*}), t = 1, 2, ...R$ and $U_t = Max\{\bar{Z}_r(X^{1*}), \bar{Z}_r(X^{2*}), \ldots \bar{Z}_r(X^{R*})\}$, respectively.

Step 5 Define a membership function μ_t for the R_{th} objective function as follows:

$$\mu_t\left(\bar{Z}_t(x)\right) = \begin{cases} 1, & \text{if} \quad \bar{Z}_t \leq L_t \\ \dfrac{U_t - \bar{Z}_t}{U_t - L_t}, \quad \text{if} & L_t \leq \bar{Z}_r(x) \leq U_t \\ 0, & \text{if} \quad \bar{Z}_t \geq L_t \end{cases}$$

Then the linear goal programming model for MOFSTP can be formulated as:

$$\begin{aligned} \max \quad & \lambda, \\ \text{subject to } & \frac{U_t - \bar{Z}_t}{U_t - L_t} + d_r^- - d_r^+ = 1, \\ & \lambda \geq d_r^-, \qquad\qquad r = 1, 2, \ldots, R, \\ & d_r^- d_r^+ = 0, \\ & \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq s_{ic}, \qquad i = 1, 2, \ldots, m \\ & \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq d_{jc}, \qquad j = 1, 2, \ldots, n \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq e_{kc}, \qquad k = 1, 2, \ldots, K \\ & d_r^-, d_r^+ \geq 0, \\ & \lambda \leq 1, \lambda \geq 0, \\ & x_{ijk} \geq 0, \quad \forall i, j, k \end{aligned}$$

Step 6 Solve this crisp model and the obtained solution will be the optimal compromise solution of MOFSTP.

5.6 Numerical Example

Example 1

It was considered two objective functions with triangular and trapezoidal fuzzy numbers given in Table-5.1 and Table-5.2 to illustrate the proposed method.

The supplies, demands and conveyance capacities are given as $\tilde{s}_1$=(21, 23, 25),

Table 5.1: Penalties/costs $\tilde{c}^1_{ijk}$

i	k=1 j			k=2 j		
	1	2	3	1	2	3
1	(8, 9, 11)	(4, 6, 9, 11)	(10, 12, 14, 16)	(9, 11, 13, 15)	(6, 8, 10)	(7, 9, 12, 14)
2	(8, 10, 13, 15)	(6, 7, 8, 9)	(11, 13, 15, 17)	(10, 11, 13,15)	(6, 8, 10, 12)	(14, 16, 18, 20)

Table 5.2: Penalties/costs $\tilde{c}^2_{ijk}$

i	k=1 j			k=2 j		
	1	2	3	1	2	3
1	(9, 10, 12)	(5, 8, 10, 12)	(10, 11, 12, 13)	(11, 13, 14, 15)	(6, 7, 9, 11)	(8, 10, 11, 13)
2	(11, 13, 14, 16)	(7, 9, 12, 14)	(12,14, 16, 18)	(14, 16, 20)	(9, 11, 13, 14)	(13, 14, 15, 16)

$\tilde{s}_2$=(28, 32, 35, 37), $\tilde{d}_1$= (14, 16, 19), $\tilde{d}_2$= (17, 20, 22, 25), $\tilde{d}_3$ = (12, 15, 18, 21), $\tilde{e}_1$= (21, 24, 26), $\tilde{e}_2$= (24, 26, 27, 30). Then apply fuzzy programming in (5.3) and obtained defuzzified values are s_{1c}= 22.27, s_{2c} = 30.54, d_{1c} = 17.09, d_{2c} = 21.82, d_{3c} = 13.90, e_{1c} = 24.73, e_{2c} = 28.09. In the following the proposed steps of the previous section (i.e., step 3 to 6) was applied and the results are: λ = 0.80985, x^1_{111}= 17.09, x^1_{122}= 4.4951, x^1_{132}= 0.6849, x^1_{221}= 7.64, x^2_{111}= 22.7, x^2_{222}= 9.6849, x^2_{232}= 13.2151, d^-_1= 0.80985, d^-_2= 0.80985, $\bar{Z}_1$= 574.1754 and $\bar{Z}_2$= 605.085.

5.7 Comparative Study

Example 1

The following numerical example presented by Kundu *et al.*[57] was considered to explain the efficiency of the proposed Method. The data was given in Table 5.3–5.6.

Table 5.3: Penalties/costs $\tilde{c}^1_{ijk}$

i	k=1 j 1	2	3	k=2 j 1	2	3
1	(5, 8, 9, 11)	(4, 6, 9, 11)	(10, 12, 14, 16)	(9, 11, 13, 15)	(6, 8, 10)	(7, 9, 12, 14)
2	(8, 10, 13, 15)	(6, 7, 8, 9)	(11, 13, 15, 17)	(10, 11, 13,15)	(6, 8, 10, 12)	(14, 16, 18, 20)

Table 5.4: Penalties/costs $\tilde{c}^2_{ijk}$

i	k=1 j 1	2	3	k=2 j 1	2	3
1	(9, 10, 12)	(5, 8, 10, 12)	(10, 11, 12, 13)	(11, 13, 14, 15)	(6, 7, 9, 11)	(8, 10, 11, 13)
2	(11, 13, 14, 16)	(7, 9, 12, 14)	(12,14, 16, 18)	(14, 16, 20)	(9, 11, 13, 14)	(13, 14, 15, 16)

Table 5.5: Penalties/costs $\tilde{c}^3_{ijk}$

i	k=1 j 1	2	3	k=2 j 1	2	3
1	(4, 5, 7, 8)	(3, 5, 6, 8)	(7, 9, 10, 12)	(6, 7, 8, 9)	(4, 6, 7, 9)	(5, 7, 9, 11)
2	(6, 8, 9, 11)	(5, 6, 7, 8)	(6, 7, 9, 10)	(4, 6, 8, 10)	(7, 9,11, 13)	(9,10, 11, 12)

Table 5.6: Penalties/costs $\tilde{c}^4_{ijk}$

i	k=1 j 1	2	3	k=2 j 1	2	3
1	(5, 7, 9, 10)	(4, 6, 7, 9)	(9, 11, 12, 13)	(7, 8, 9, 10)	(4, 5, 7, 8)	(8,10, 11, 12)
2	(10, 11, 13, 14)	(6, 7, 8, 9)	(7, 9, 11, 12)	(6, 8, 10, 12)	(5, 7, 9, 11)	(9, 10, 12, 14)

$\tilde{s}^1_1$=(21, 24, 26, 28), $\tilde{s}^1_2$=(28, 32, 35, 37), $\tilde{d}^1_1$= (14, 16, 19, 22), $\tilde{d}^1_2$= (17, 20, 22, 25), $\tilde{d}^1_3$=(12, 15, 18, 21), $\tilde{s}^2_1$=(32, 34, 37, 39), $\tilde{s}^2_2$=(25, 28, 30, 33), $\tilde{d}^2_1$=(20, 23, 25, 28), $\tilde{d}_2^2$=(16,18, 19, 22), $\tilde{d}_3^2$=(15, 17, 19, 21), $\tilde{e}_1$=(46, 49, 51, 53), $\tilde{e}_2$=(51, 53, 56, 59). Then applying the proposed method, we get the following result $\lambda = 0.1133810$, x_{111}^1= 15.8, x_{221}^1= 19.7, x_{231}^1= 6.8, x_{132}^1= 7.90, x_{111}^2= 22.7, x_{122}^2= 2.41, x_{132}^2= 8.685, x_{221}^2= 15.385, x_{231}^2= 8.1145, d_1^-= 0.1133, d_2^-= 0.1133, $\bar{Z}_1$= 1110.183 and $\bar{Z}_2$= 814.396. The other variables that are not in the above have a zero value. Table-5.7 shows

Table 5.7: Comparisons of optimal solutions

Kundu *et al.* [57]		**Proposed method**
Fuzzy linear programming	Global criterion method	Goal programming approach
1139.536	1144.894	1110.183
837.4808	832.1250	814.396

the comparison of the results of the objective values of $\bar{Z}_1$ and $\bar{Z}_2$ of the present example with the results obtained by Kundu *et al.*[57]. It was shown that the optimal solution of the proposed problem gave better results by using fuzzy Goal programming approach when compared to Fuzzy linear programming and Global criterion method.

Example 2 (Bit, A. K. *et al.*[7])

Let us consider a multi-objective solid transportation problem with mixed constraints.

$$\min \quad Z_r = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} c_{ijk}^r x_{ijk}, \qquad r = 1, 2, 3.$$

$$\text{subject to} \quad \sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk} = 8, \quad \sum_{j=1}^{3}\sum_{k=1}^{3} x_{2jk} \geq 9, \quad \sum_{j=1}^{3}\sum_{k=1}^{3} x_{3jk} \leq 5,$$

$$\sum_{i=1}^{3}\sum_{k=1}^{3} x_{i1k} = 7, \quad \sum_{i=1}^{3}\sum_{k=1}^{3} x_{i2k} \geq 6, \quad \sum_{i=1}^{3}\sum_{k=1}^{3} x_{i3k} \leq 5,$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij1} = 10, \quad \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij2} \geq 5, \quad \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij3} \leq 6,$$

$$x_{ijk} \geq 0, \quad \forall i, j, k.$$

Table 5.8: Data for Three objective functions

	j	1			2			3		
	i. k	1	2	3	1	2	3	1	2	3
c^1	1	(8, 9, 10)	(10,12,14)	(7, 9, 11)	(3, 6, 9)	(8, 9, 10)	(5, 7, 9)	(2, 3, 4)	(6, 7, 8)	(5, 7, 9)
	2	(4, 5, 6)	(5, 6, 7)	(3, 5, 7)	(7,9, 11)	(8, 11, 14)	(1, 3, 5)	(5, 6, 7)	(6, 8, 10)	(5, 6, 7)
	3	(1, 2, 3)	(1, 2, 3)	(1, 1, 1)	(1, 2, 3)	(6, 7, 8)	(6, 7, 8)	(1, 1, 1)	(8, 9, 10)	(1, 3, 5)
c^2	1	(1, 2, 3)	(8, 9, 10)	(6, 8, 10)	(1, 1, 1)	(2, 4, 6)	(1, 1, 1)	(7, 9, 11)	(7, 9, 11)	(4, 5, 6)
	2	(1, 2, 3)	(7, 8, 9)	(1, 1, 1)	(3, 4, 5)	(3, 5, 7)	(1, 2, 3)	(6, 8, 10)	(5, 6, 7)	(8, 9, 10)
	3	(4, 5, 6)	(1, 2, 3)	(5, 7, 9)	(6, 8, 10)	(8, 9, 10)	(6, 7, 8)	(3, 5, 7)	(1, 2, 3)	(3, 5, 7)
c^3	1	(1, 2, 3)	(2, 4, 6)	(5, 6, 7)	(2, 3, 4)	(4, 6, 8)	(3, 4, 5)	(6, 8, 10)	(2, 4, 6)	(7, 9, 11)
	2	(1, 2, 3)	(3, 5, 7)	(1, 3, 5)	(3, 5, 7)	(4, 6, 8)	(4, 6, 8)	(7, 9, 11)	(4, 6, 8)	(1, 3, 5)
	3	(1, 1, 1)	(8, 9, 10)	(1, 1, 1)	(7, 8, 9)	(2, 3, 4)	(7, 9, 11)	(3, 5, 7)	(5, 7, 9)	(10, 11, 12)

By using the proposed method, we get the following optimal compromise solution as $\lambda = 0.3322039$, x_{121}= 7.17, x_{122}= 0.8295, x_{211}= 2.82, x_{212}= 2.77, x_{223}= 3.39, x_{312}= 1.39136, d_1^-= 0.3322, d_2^-= 0.3322, d_3^-= 0.3322 $\bar{Z}_1$= 94.2678, $\bar{Z}_2$= 47.9457 and $\bar{Z}_3$= 78.91. The other variables that are not in the above have a zero value.

It was observed from the Table-5.9 the objective values of $\bar{Z}_1$, $\bar{Z}_2$ and $\bar{Z}_3$ of the

Table 5.9: **Comparisons of optimal solutions**

Bit, A. K. *et al.* [7]	**Li, Y. *et al.*** [63]	**Proposed method**
Fuzzy programming	improved GA	Goal programming approach
$Z_1 = 94.271$	$Z_1 = 94.5$	$Z_1 = 94.2678$
$Z_2 = 47.952$	$Z_2 = 57.5$	$Z_2 = 47.9457$
$Z_3 = 78.94$	$Z_3 = 67.0$	$Z_3 = 78.91$

above example were in good agreement with the results obtained by Bit, A. K. *et al.*[7] and Li, Y. *et al.*[63].

Chapter 6

Multi-Objective Fuzzy Solid Transportation Problem with L-R coefficients

6.1 Introduction

In this chapter, a solution procedure has been given for the multi objective solid transportation problem with L-R coefficients in the objective function. When the objective function with L-R coefficient in the linear programming is integrated the method, that can be changed into a fuzzy optimal solution to multi-object linear programming. Meanwhile, determination of this model might cause the constraint field of linear programming to be empty sets after the flexible indexes p_1, p_2, p_3 are given subjectively. The new method is a systematic procedure, easy to apply and could be utilized for all types of solid transportation problems either maximize or minimize objective function.

The rest of this chapter is organized as follows. In Section 6.2, preliminary knowledge about fuzzy linear programming is provided. Section 6.3 deals with formulation of STP problem. Section 6.4 describes the model formulation of fuzzy LP in MOSTP with LR-coefficients. The algorithm is given in 6.5. Section 6.6 presents a numerical example.

6.2 Fuzzy Linear Programming and its Algorithm

Suppose that $x = (x_1, x_2, \dots x_n)^T$ is an n-dimensional decision vector, $c = (c_1, c_2, \dots c_n)$ is an n-dimensional objective coefficient vector, $A = a_{ij}(1 \leq i \leq m, 1 \leq j \leq n)$is an m $\times$ n-dimensional constraint coefficient matrix, $b = (b_1, b_2, \dots b_m)^T$is an m-dimensional constant vector, and fuzzified objective and constraint function in the ordinary linear programming, then

$$\widetilde{\max}(\text{ or } \widetilde{\min})z = cx$$
$$\text{subject to } Ax \underset{\sim}{\leq} b \tag{6.1}$$
$$\forall x \geq 0$$

It could be as fuzzy linear programming. Let the rank $(A) = m$. "$\underset{\sim}{\leq}$" denotes the fuzzy version of "$\leq$" and has the linguistic interaction "essentially smaller than or equal to" (Zimmermann [110]). $\widetilde{max}$ represents fuzzy maximizing, where $cx = \sum_{j=1}^{n} c_j x_j Ax = (\sum_{j=1}^{n} a_{ij} x_j)$ $(i = 1, 2, \ldots m)$. The membership function of fuzzy objective $\tilde{g}(x)$ is

$$\mu_{\tilde{G}}(x) = \tilde{g}(\sum_{j=1}^{n} c_j x_j)$$

$$= \begin{cases} 0, & \text{when } \sum_{j=1}^{n} c_j x_j \leq z_0, \\ \frac{1}{p_0}(\sum_{j=1}^{n} c_j x_j - z_0), & \text{when } z_0 < \sum_{j=1}^{n} c_j x_j \leq z_0 + p_0, \\ 1, & \text{when } \sum_{j=1}^{n} c_j x_j > z_0 + p_0, \end{cases}$$

Let $t_0 = \sum_{j=1}^{n} c_j x_j$ then the membership function of $\tilde{g}(t_0)$ is shown as figure 6.1.

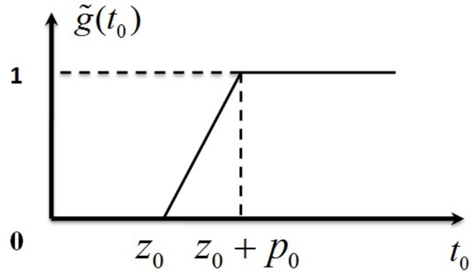

Figure 6.1: Membership function of $\tilde{g}(t_0)$.

The membership function of fuzzy constraints is written as below:

$$\mu_{\tilde{S}}(x) = \tilde{f}(\sum_{j=1}^{n} c_j x_j)$$

$$= \begin{cases} 1, & \text{when } \sum_{j=1}^{n} a_{ij}x_j \leq b_i, \\ 1 - \frac{1}{p_i}(\sum_{j=1}^{n} a_{ij}x_j - b_j), & \text{when } b_i < \sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i, \\ 0, & \text{when } \sum_{j=1}^{n} a_{ij}x_j > b_i + p_i \end{cases}$$

Let $t_i = \sum_{j=1}^{n} a_{ij}x_j$ then $f(t_i)$ membership function shown in the figure 6.2 where as

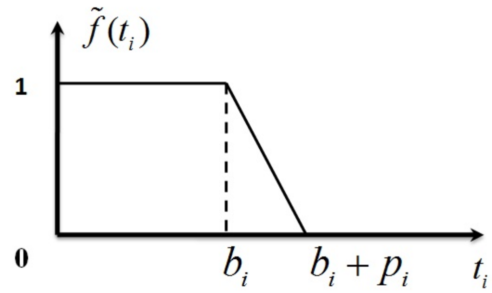

Figure 6.2: Membership function of $f(t_i)$.

$p_i \geq 0\,(1 \leq i \leq m)$ is a flexible index by an appropriate choice.

Consider a symmetric form fuzzy linear programming (6.1), written as $\mu_{\tilde{S}} = \tilde{S}_f$ and $\mu_{\tilde{G}} = \tilde{M}_f$, and we call it condition and unconditional fuzzy superiority set of f concerning constraint $\tilde{S}$, respectively.

Theorem 6.1 If $\bar{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an optimal solution in (6.1), then $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an optimal solution in ((6.1)) and they have constrained optimal level of α.

Zimmermann [110] initiated arithmetic to Problem (6.1).

Here we introduced its solution procedure in the following. First to find an ordinary linear programming:

$$\begin{aligned} \max \quad & z = cx \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \max z &= cx \\ \text{subject to} \quad & Ax \leq b + p, \\ & x \geq 0. \end{aligned}$$

It is obtained a maximum value z_0 and z_0+p_0, where $b+p = (b_1 + p_1, b_2 + p_2, \ldots b_n + p_n)^T$. Here, z_0 is a function maximum under the constraint condition $Ax \leq b$ obeyed strictly (the membership degree is $\mu_{\tilde{S}}(x) = 1$ at this time). $z_0 + p_0$ is a function maximum when the constraint condition to be relaxed as $Ax \leq b + p$ (the membership degree is $\mu_{\tilde{S}}(x) = 0$ at this time). z_0 and $z_0 + p_0$ corresponds to two extreme cases $\mu_{\tilde{S}}(x) = 1$ and $\mu_{\tilde{S}}(x) = 0$, which could adequate lower membership degree $\mu_{\tilde{S}}(x)$, so that the optimal value improved, lying between z_0 and $z_0 + p_0$.

$$\begin{aligned} \max \text{ G } &= \alpha \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij}x_j + p_i\alpha \leq b_i + p_i (1 \leq i \leq m), \\ & \sum_{j=1}^{n} c_j x_j - p_0\alpha \geq z_0, \\ & 0 \leq \alpha \leq 1, \forall x_j \geq 0, j = 1, 2, \ldots n. \end{aligned} \tag{6.3}$$

The researcher found its optimal solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ by using simplex method, thus optimal point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ in (6.1) was obtained by Theorem 6.1 and the objective function value is $\mu_{\tilde{D}}(x^*) = \alpha^*$.

6.3 Problem Formulation

For instance when considered a product is to be transported from m sources to n destinations in a STP, at each source, let s_i be the amount of a product expected to be transported to n destinations to satisfy the demand for d_j units of the product. Here e_k, called conveyance, denotes the units of this product that could be carried by k different modes of transportation.

A multi-objective fuzzy solid transportation problem is formulated in the following example.

$$\begin{aligned}
\max \quad & \tilde{Z}^r = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \tilde{c}_{ijk}^r x_{ijk}, r = 1, 2, \ldots R \\
\text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_i, i = 1, 2, \ldots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_j, j = 1, 2, \ldots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, k = 1, 2, \ldots K, \\
& x_{ijk} \geq 0, \forall i, j, k.
\end{aligned} \tag{6.4}$$

6.4 Fuzzy Linear Programming in MOSTP with L-R Coefficients

Consider

max $\quad \tilde{Z} = \tilde{c}x$

subject to $Ax \leq b$

$x \geq 0$.

Where $\tilde{c} = (c_{ijk}, \underline{c}_{ijk}, \bar{c}_{ijk})_{LR}$ and

$$\tilde{Z} = (Z, \underline{Z}, \bar{Z})_{LR} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk}, \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \underline{c}_{ijk}x_{ijk}, \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \bar{c}_{ijk}x_{ijk} \right)_{LR}$$

are all L-R numbers in (6.4) is approximately equivalent to a linear programming with three objectives

$$\begin{aligned}
\max &\left(Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} c_{ijk}x_{ijk} \right), \\
\min &\left(\bar{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \underline{c}_{ijk}x_{ijk} \right), \\
\max &\left(\bar{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \bar{c}_{ijk}x_{ijk} \right), \\
\text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq s_i, \qquad i = 1, 2, \ldots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq d_j, \qquad j = 1, 2, \ldots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1, 2, \ldots K, \\
& x_{ijk} \geq 0, \quad \forall i, j, k
\end{aligned} \tag{6.5}$$

Find an optimal solution to each objective individually with respect to the constraints.

Here, researcher considered a three-objective, and the membership functions, are given by

$$\mu_{\tilde{G}}(x) = f_1(Z)$$

$$= \begin{cases} 0, & \text{when } z \leq z_0, \\ 1 - \frac{1}{p_0}(z_1 - Z), & \text{when } z_0 < z \leq z_0 + p_0, \\ 1, & \text{when } z > z_0 + p_0 \end{cases} \tag{6.6}$$

Where $z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} c_{ijk}x_{ijk}$, $z_1 = z_0 + p_0$ and p_0 is flexible index value.
Now, using the Theorem 6.1, presented the equivalent crisp linear programming of Model (6.4) as follows:

$$\begin{aligned} \text{Z } &= \max \quad \alpha \\ \text{subject to } & \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} c_{ijk}x_{ijk} - p_1\alpha \geq z_1, \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \underline{c}_{ijk}x_{ijk} - p_2\alpha \geq z_2, \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} \bar{c}_{ijk}x_{ijk} - p_3\alpha \geq z_3 \\ & \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq s_i, \qquad i = 1, 2, \ldots m, \\ & \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq d_j, \qquad j = 1, 2, \ldots n, \end{aligned} \tag{6.7}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1, 2, \ldots K,$$

$$0 \leq \alpha \leq 1, \forall x_{ijk} \geq 0.$$

6.5 Algorithm

Step 1 Input the data in (6.4), assume as level (0).

Step 2 Find the optimal solutions for each objective individually with given constraints as given in (6.5).

Step 3 For fuzzy goal programming of stochastic STP formulate the membership function as given in (6.6).

Step 4 Formulate the model (6.7).

Step 5 Solve (6.7) which will give solution for (6.4) with LR-coefficients at level (0).

Step 6 For level (-1), take -1 value at each point in L-R numbers then goto Step 2.

Step 7 For level (+1), add +1 value at each point in L-R numbers then goto Step 2.

6.6 Numerical Example

To demonstrate the potentiality of the proposed model, it felt necessary to present an example of coal transportation problem. Coal is a kind of crucial energy source in the development of economy and society. Accordingly, economically coal transportation is also an important issue in the coal transport from mines to the different areas. For the convenience of description, it was summarized the problem as follows. Suppose that there were two coal mines to supply the coal for three cities. During the process of transportation, two kinds of conveyances were available to be selected, i.e., train and cargo ship. In the beginning of this task, the decision maker needs to obtain the basic data, since the transportation plan is made in advance, it is not possible get this data exactly.

Table 6.1: Profit matrix

c_{ij1}			c_{ij2}		
(11, 6, 4)	(10, 3, 4)	(11, 5, 2)	(18, 5, 3)	(11, 4, 2)	(12, 6, 1)
(13, 7, 3)	(11, 4, 6)	(10, 4, 1)	(12, 3, 4)	(13, 5, 2)	(15, 7, 4)

$$
\begin{aligned}
\text{subject to } & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 35, \\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 30, \\
& x_{111} + x_{211} + x_{112} + x_{212} \geq 9, \\
& x_{121} + x_{221} + x_{122} + x_{222} \geq 7, \\
& x_{131} + x_{132} + x_{231} + x_{232} \geq 8, \\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \leq 25, \\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \leq 26, \\
& x_{ijk} \geq 0, \forall i, j, k.
\end{aligned}
$$

Level (0): This problem is equivalent to

$$
\begin{aligned}
\max Z \; = & \; 11x_{111} + 18x_{112} + 10x_{121} + 11x_{122} + 11x_{131} + 12x_{132} \\
& + 13x_{211} + 12x_{212} + 11x_{221} + 13x_{222} + 10x_{231} + 15x_{232}
\end{aligned}
$$

$$\begin{aligned}
\min \underline{Z} &= 6x_{111} + 5x_{112} + 3x_{121} + 4x_{122} + 5x_{131} + 6x_{132}\\
&\quad + 7x_{211} + 3x_{212} + 4x_{221} + 5x_{222} + 4x_{231} + 7x_{232}\\
\max \bar{Z} &= 4x_{111} + 3x_{112} + 4x_{121} + 2x_{122} + 2x_{131} + 1x_{132}\\
&\quad + 3x_{211} + 4x_{212} + 6x_{221} + 2x_{222} + 1x_{231} + 4x_{232}\\
\text{subject to}\quad & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 35,\\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 30,\\
& x_{111} + x_{211} + x_{112} + x_{212} \geq 9,\\
& x_{121} + x_{221} + x_{122} + x_{222} \geq 7,\\
& x_{131} + x_{132} + x_{231} + x_{232} \geq 8,\\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \leq 25,\\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \leq 26,\\
& x_{ijk} \geq 0, \quad \forall\ i, j, k.
\end{aligned}$$

Now solving each objective individually with respect to the given constraints, we obtain $x^{(1)}_{112} = 26$, $x^{(1)}_{131} = 8$, $x^{(1)}_{211} = 10, x^{(1)}_{221} = 7, Z^{(1)} = 763$ when $\underline{Z}^{(1)} = 268$, $\bar{Z}^{(1)} = 166$. Similarly, we obtain the optimum solution for second and third objective as follows.

The optimum solution of second objective is: $x^{(2)}_{121} = 7$, $x^{(2)}_{212} = 9$, $x^{(2)}_{231} = 8, Z^{(2)} = 258$ when $\underline{Z}^{(2)} = 80$, $\bar{Z}^{(2)} = 72$.

The optimum solution of third objective is: $x^{(3)}_{111} = 3, x^{(3)}_{112} = 18, x^{(3)}_{221} = 22$, $x^{(3)}_{232} = 8, Z^{(3)} = 719$ when $\underline{Z}^{(3)} = 252, \bar{Z}^{(3)} = 230$ and all other variables are zero.

Here, given flexible index for $p_1 = 3$, $p_2 = 8$ and $p_3 = 9$ for fuzzy objective sets, let us construct the membership functions as given below:

$$\mu_{m_1}(x) = f_1(Z) = \begin{cases} 0, & z < 760 \\ 1 - \frac{1}{3}(763 - Z), & 760 \leq z < 763 \\ 1, & z \geq 763 \end{cases}$$

$$\mu_{m_2}(x) = f_2(Z) = \begin{cases} 1, & z \leq 258 \\ 1 - \frac{1}{8}(Z - 250), & 258 < z < 266 \\ 0, & z \geq 266 \end{cases}$$

$$\mu_{m_3}(x) = f_3(Z) = \begin{cases} 0, & z < 710 \\ 1 - \frac{1}{9}(719 - Z), & 710 \leq z < 719 \\ 1, & z \geq 719 \end{cases}$$

Then the problem is changed into an ordinarily linear programming problem.

$$\max \alpha$$

$$\text{subject to } 1 - \frac{1}{3}(763 - \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} c_{ijk}x_{ijk}) \geq \alpha,$$

$$1 - \frac{1}{8}(\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} \underline{c}_{ijk}x_{ijk} - 258) \geq \alpha,$$

$$1 - \frac{1}{9}(719 - \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} \bar{c}_{ijk}x_{ijk}) \geq \alpha,$$

$$x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 35,$$

$$x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 30,$$

$$x_{111} + x_{211} + x_{112} + x_{212} \geq 9,$$

$$x_{121} + x_{221} + x_{122} + x_{222} \geq 7,$$

$$
\begin{aligned}
& x_{131}+x_{132}+x_{231}+x_{232} \geq 8, \\
& x_{111}+x_{211}+x_{131}+x_{121}+x_{221}+x_{231} \leq 25, \\
& x_{112}+x_{212}+x_{132}+x_{122}+x_{222}+x_{232} \leq 26, \\
& x_{ijk} \geq 0, \quad \forall\, i,j,k.
\end{aligned}
\tag{6.8}
$$

i.e.,

$$
\begin{aligned}
\max \quad & \alpha \\
\text{subject to } & 11x_{111}+18x_{112}+10x_{121}+11x_{122}+11x_{131}+12x_{132} \\
& +13x_{211}+12x_{212}+11x_{221}+13x_{222}+10x_{231}+15x_{232}-3\alpha \geq 760, \\
& 6x_{111}+5x_{112}+3x_{121}+4x_{122}+5x_{131}+6x_{132} \\
& +7x_{211}+3x_{212}+4x_{221}+5x_{222}+4x_{231}+7x_{232}+8\alpha \leq 266, \\
& 4x_{111}+3x_{112}+4x_{121}+2x_{122}+2x_{131}+1x_{132} \\
& +3x_{211}+4x_{212}+6x_{221}+2x_{222}+1x_{231}+4x_{232}-9\alpha \geq 710, \\
& x_{111}+x_{121}+x_{131}+x_{112}+x_{122}+x_{132} \leq 35, \\
& x_{211}+x_{221}+x_{231}+x_{212}+x_{222}+x_{232} \leq 30, \\
& x_{111}+x_{211}+x_{112}+x_{212} \geq 9, \\
& x_{121}+x_{221}+x_{122}+x_{222} \geq 7, \\
& x_{131}+x_{132}+x_{231}+x_{232} \geq 8, \\
& x_{111}+x_{211}+x_{131}+x_{121}+x_{221}+x_{231} \leq 25, \\
& x_{112}+x_{212}+x_{132}+x_{122}+x_{222}+x_{232} \leq 26, \\
& x_{ijk} \geq 0, \forall i,j,k.
\end{aligned}
$$

The optimal solution is obtained as $\alpha^* = 0.2$, $x^*_{112} = 26$, $x^*_{131} = 8$, $x^*_{211} = 8.8$, $x^*_{221} = 8.2$ correspondingly $Z = 760.6$, $\underline{Z} = 264.4$, $\bar{Z} = 169.6$. Then the approximately fuzzy optimal value is $\tilde{Z}^* = (760.6, 264.4, 169.6)_{LR}$.

Level (-1): To solve the above considered problem, convert the problem as in profit matrix and take -1 value at each point (left level) in L-R numbers with flexible index values $p_1 = 3$, $p_2 = 8$, $p_3 = 9$ (as we concentrate more in maximizing and minimizing).

$$\begin{aligned}
\max Z &= 11x_{111} + 18x_{112} + 10x_{121} + 11x_{122} + 11x_{131} + 12x_{132} \\
&\quad + 13x_{211} + 12x_{212} + 11x_{221} + 13x_{222} + 10x_{231} + 15x_{232} \\
\min \underline{Z} &= 5x_{111} + 4x_{112} + 2x_{121} + 3x_{122} + 4x_{131} + 5x_{132} \\
&\quad + 6x_{211} + 2x_{212} + 3x_{221} + 4x_{222} + 3x_{231} + 6x_{232} \\
\max \bar{Z} &= 3x_{111} + 2x_{112} + 3x_{121} + 1x_{122} + 1x_{131} + 1x_{132} \\
&\quad + 2x_{211} + 3x_{212} + 5x_{221} + 1x_{222} + 1x_{231} + 3x_{232} \\
\text{subject to } & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \le 35, \\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \le 30, \\
& x_{111} + x_{211} + x_{112} + x_{212} \ge 9, \\
& x_{121} + x_{221} + x_{122} + x_{222} \ge 7, \\
& x_{131} + x_{132} + x_{231} + x_{232} \ge 8, \\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \le 25, \\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \le 26, \\
& x_{ijk} \ge 0, \quad \forall\, i, j, k.
\end{aligned}$$

Now solving each objective individually with respect to the given constraints, we obtain $x^{(1)}_{112} = 26$, $x^{(1)}_{131} = 8$, $x^{(1)}_{211} = 10$, $x^{(1)}_{221} = 7$,$Z^{(1)} = 763$ when $\underline{Z}^{(1)} = 217$, $\bar{Z}^{(1)} = 115$. $x^{(2)}_{121} = 7$, $x^{(2)}_{212} = 9$, $x^{(2)}_{231} = 8$,$Z^{(2)} = 258$ when $\underline{Z}^{(2)} = 56$, $\bar{Z}^{(2)} = 56$. $x^{(3)}_{111} = 3$, $x^{(3)}_{112} = 18$, $x^{(3)}_{221} = 22$, $x^{(3)}_{232} = 8$, $Z^{(3)} = 719$ when $\underline{Z}^{(3)} = 192$, $\bar{Z}^{(3)} = 179$ and all other variables are zero.

Here, flexible index values $p_1 = 3$, $p_2 = 8$, $p_3 = 9$ are given for fuzzy objective sets and applying above procedure. Meanwhile, it could not be found find any feasible solution for this numerical example. However for any other example with suitable data set, this method could provide feasible solution.

Level (+1): Similarly, reseracher considered problem as in the cost matrix and took +1 value at each point (right level) of L-R coefficients with flexible index values are $p_1 = 16$, $p_2 = 24$, $p_3 = 22$. The profit matrix becomes

$$
\begin{aligned}
\max\ Z &= 11x_{111} + 18x_{112} + 10x_{121} + 11x_{122} + 11x_{131} + 12x_{132} \\
&\quad + 13x_{211} + 12x_{212} + 11x_{221} + 13x_{222} + 10x_{231} + 15x_{232} \\
\min\ \underline{Z} &= 7x_{111} + 6x_{112} + 4x_{121} + 5x_{122} + 6x_{131} + 7x_{132} \\
&\quad + 8x_{211} + 4x_{212} + 5x_{221} + 6x_{222} + 5x_{231} + 8x_{232} \\
\max\ \bar{Z} &= 5x_{111} + 4x_{112} + 5x_{121} + 3x_{122} + 3x_{131} + 3x_{132} \\
&\quad + 4x_{211} + 5x_{212} + 7x_{221} + 2x_{222} + 3x_{231} + 5x_{232} \\
\text{subject to}\ & x_{111} + x_{121} + x_{131} + x_{112} + x_{122} + x_{132} \leq 35, \\
& x_{211} + x_{221} + x_{231} + x_{212} + x_{222} + x_{232} \leq 30, \\
& x_{111} + x_{211} + x_{112} + x_{212} \geq 9, \\
& x_{121} + x_{221} + x_{122} + x_{222} \geq 7, \\
& x_{131} + x_{132} + x_{231} + x_{232} \geq 8, \\
& x_{111} + x_{211} + x_{131} + x_{121} + x_{221} + x_{231} \leq 25, \\
& x_{112} + x_{212} + x_{132} + x_{122} + x_{222} + x_{232} \leq 26, \\
& x_{ijk} \geq 0, \quad \forall\ i, j, k.
\end{aligned}
$$

Now solving each objective individually with respect to the system constraints with flexible index values $p_1 = 16$, $p_2 = 24$, $p_3 = 22$ the values of objective functions are calculated as follows:

The optimal solution is obtained as $\alpha^* = 0.7$, $x^*_{112} = 26$, $x^*_{131} = 8$, $x^*_{211} = 8.167$,$x^*_{221} =$ 8.834 correspondingly $Z = 759.35$, $\underline{Z} = 313.5$, $\bar{Z} = 222.5$. The corresponding results are listed in the Table 6.2 with different flexible index values. Suppose that when the optimal solution of membership function, the decision maker is satisfied, then the interactive process is stopped, so we set flexible index values (16, 24, 22), obtain that the satisfied solutions were detailed as $\alpha^* = 0.7$, $x^*_{112} = 26$, $x^*_{131} = 8$, $x^*_{211} = 8.167$, $x^*_{221} = 8.834$. The corresponding optimal values are $\tilde{Z}^* = (759.35, 313.5, 222.5)_{LR}$.

Table 6.2: Results for different flexible index values

Level	**Each objective individually results**	**Flexible index value**	**Optimal solutions**
0	(763, 268, 116)	(3, 8, 9)	(760.6, 315.4, 220.6)
-1	(763, 217, 115)	(3, 8, 9)	Infeasible solution
-1	(763, 217, 115)	(16, 24, 22)	Infeasible solution
+1	(763, 319, 217)	(16, 24, 22)	(759.35, 313.5, 222.5)
+1	(763, 319, 217)	(33, 78, 89)	(739.35, 313.89, 255.5)
+1	(763, 319, 217)	(33, 85, 81)	(748.03, 296.55, 239.45)

Chapter 7

A Rough Interval Approach to solve the Expected cost value of Fuzzy Solid Transportation Problem

7.1 Introduction

In this chapter, the researcher presents a new fuzzy solid transportation problem (i.e. Sources, demands and conveyance capacities are fuzzy) with expected cost value and rough interval constraints that maximized the profit. Through research, it showed that the optimal solution of the fuzzy solid transportation problem could be found simply by solving proposed models and our model discussed some concepts like as surely optimal range, possibly optimal range, completely satisfactory solutions, rather satisfactory solutions, rough interval range. In order to solve the model conveniently, researcher had discussed the crisp model with corresponding expected value models in objective functions for triangular and trapezoidal membership functions, here it was showed that the problem could be converted into two different solid transportation problems with interval coefficients, and then further each of these two solid transportation problem can be regenerated into four classical solid transportation problem. A numerical example was given to illustrate the proposed model.

Section 7.2 introduces some preliminaries and notations. Section 7.3 gives the problem formulation of fuzzy STP with rough interval coefficients. Section 7.4 provides description of the proposed model. Section 7.5 gives solution procedure (algorithm). Section 7.6 presents a numerical example.

7.2 Preliminaries

In this section, the basic knowledge of rough set theory is recalled. As Rough intervals, rough interval arithmetic and ordering of rough interval were introduced briefly in Chapter 1 (definition 1.11 and 1.12). The expected value operator is discussed below.

Expected value of triangular and trapezoidal fuzzy numbers:

Definition 7.1. (Liu et al. [67]) Let $\tilde{\xi}$ be a fuzzy variable. Then the expected value of $\tilde{\xi}$ is defined as

$$E[\xi] = \int_0^{\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr \tag{7.1}$$

provided that at least one of the two integral is finite. If $\tilde{\xi}$ is a triangular fuzzy variable (r_1, r_2, r_3), then the expected value of $\tilde{\xi}$is $(1/4)(r_1 + 2r_2 + r_3)$. If $\tilde{\xi}$ is a trapezoidal fuzzy variable (r_1, r_2, r_3, r_4), , then the expected value of $\tilde{\xi}$is $(1/4)(r_1 + r_2 + r_3 + r_4)$. Here Cr is called credibility measure.

7.3 Problem Formulation

It is assumed that a product is to be transported from m sources to n destinations in a STP, at each source, let s_i be the amount of a homogeneous product we want to transport to n destinations to satisfy the demand for d_j units of the product. Here e_k, called conveyance, denotes the units of this product that could be carried by k different modes of transportation and also the objectives Z is to be maximized.

A solid transportation problem is formulated with fuzzy objective functions and different types of constraints with rough interval coefficients could be written as.

$$\begin{aligned}
\text{Z} \ &= \max \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \tilde{c}_{ijk} x_{ijk} \\
\text{subject to } &\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u], [\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots .m, \\
&\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u], [\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots .n, \\
&\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u], [\bar{e}_k^l, \bar{e}_k^u]), k = 1, 2, \ldots K, \\
\text{where } & x_{ijk} \geq 0, \forall\, i, j, k.
\end{aligned} \tag{7.2}$$

A value of the unit shipping cost $\tilde{c}_{ijk}$ is associated with fuzzy transportation from i^{th} origin to j^{th} destination by the k^{th} conveyance. We need to determine a feasible way of shipping the available amounts to satisfy the demand such that the profit is maximized.

Remark 7.1: Using the rough interval properties given in section 7.2, we have the followings proposals:

$$[\underline{s}_i^l, \underline{s}_i^u] \subseteq [\bar{s}_i^l, \bar{s}_i^u] \Rightarrow \bar{s}_i^l \leq \underline{s}_i^l \leq \underline{s}_i^u \leq \bar{s}_i^u$$

$$[\underline{d}_j^l, \underline{d}_j^u] \subseteq [\bar{d}_j^l, \bar{d}_j^u] \Rightarrow \bar{d}_j^l \leq \underline{d}_j^l \leq \underline{d}_j^u \leq \bar{d}_j^u$$

$$[\underline{e}_k^l, \underline{e}_k^u] \subseteq [\bar{e}_k^l, \bar{e}_k^u] \Rightarrow \bar{e}_k^l \leq \underline{e}_k^l \leq \underline{e}_k^u \leq \bar{e}_k^u$$

where i=1,2,. . . m, j=1,2,. . . n and k=1,2,. . . K.

Here we define some sets which will help to develop the methodology and theoretical background that leads us to solve the proposed solid transportation problem under rough interval approximation.

Definition 7.2. In Problem (7.2), we define the following sets:

1. $\bar{U}^l = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^l \right., i = 1, 2, \ldots, m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^u, j = 1, 2, \ldots, n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^l, k = 1, 2, \ldots, K \text{and} x_{ijk} \geq 0, \forall i, j, k \end{array} \right\}$

2. $\underline{U}^l = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^l \right., i = 1, 2, \ldots, m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^u, j = 1, 2, \ldots, n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^l, k = 1, 2, \ldots, K \text{and} x_{ijk} \geq 0, \forall i, j, k \end{array} \right\}$

3. $\bar{U}^u = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^u \right., i = 1, 2, \ldots, m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^l, j = 1, 2, \ldots, n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^u, k = 1, 2, \ldots, K \text{and} x_{ijk} \geq 0, \forall i, j, k \end{array} \right\}$

4. $\underline{U}^u = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^u \right., i = 1, 2, \ldots, m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^l, j = 1, 2, \ldots, n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^u, k = 1, 2, \ldots, K \text{and} x_{ijk} \geq 0, \forall i, j, k \end{array} \right\}$

Where $p = m \times n \times K$.

Proposition 7.2.1:

Here we propose that for the sets $\bar{U}^l, \bar{U}^u, \underline{U}^l$ and $\underline{U}^u$ that we define by definitions are (according to Hamzeheea *et al.*[35]) have the following property:

$$\bar{U}^l \subseteq \underline{U}^l \subseteq \underline{U}^u \subseteq \bar{U}^u$$

Proof: Using remark 7.1 we have, Let $x \in \bar{U}^l$ which implies,

$$\left(x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^l \right., \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^u, \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^l \right)$$

$$\left(\Rightarrow x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^l \right., \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^u, \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^l \right) \Rightarrow x \in \underline{U}^l,$$

$$\left(\Rightarrow x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^u \right., \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^l, \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^u \right) \Rightarrow x \in \underline{U}^u,$$

$$\left(\Rightarrow x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^u \right., \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^l, \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^u \right) \Rightarrow x \in \bar{U}^u,$$

for all $i = 1, 2, \ldots m, j = 1, 2, \ldots n,$ and $k = 1, 2, \ldots K$ where $p = m \times n \times K$.

So it is proved that for any $x \in \bar{U}^l$ we find that $x \in \underline{U}^l$, $x \in \underline{U}^u$, $x \in \bar{U}^u$.

Therefore,

A. $\bar{U}^l \subseteq \underline{U}^l$

B. $\bar{U}^l \subseteq \underline{U}^u$

C. $\underline{U}^u \subseteq \bar{U}^u$

D. $\underline{U}^l \subseteq \bar{U}^u$

E. $\underline{U}^l \subseteq \bar{U}^u$

Now combining the above we can say that,$\bar{U}^l \subseteq \underline{U}^l \subseteq \underline{U}^u \subseteq \bar{U}^u$. Hence proved.

7.4 Solution Methodology

Now, we use the following method to solve the problem given in (7.2).

Using Expected Value:

Here we minimize the expected value of the objective functions and then the above problem become as:

$$\begin{aligned}
\max \ \ \text{E[Z]= E}&\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\tilde{c}_{ijk}x_{ijk}\right]\\
\text{subject to } &\sum_{j=1}^{n}\sum_{k=1}^{K}x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u], [\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots m,\\
&\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u], [\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots n,\\
&\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u], [\bar{e}_k^l, \bar{e}_k^u]), k = 1, 2, \ldots K,\\
&\text{where } x_{ijk} \geq 0, \forall i, j, k.
\end{aligned}$$

This is equivalently written as:

$$
\begin{aligned}
\max \quad \mathrm{E[Z]} &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\
\text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u], [\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u], [\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u], [\bar{e}_k^l, \bar{e}_k^u]), k = 1, 2, \ldots K, \\
& \text{where } x_{ijk} \geq 0, \forall i, j, k.
\end{aligned}
\tag{7.3}
$$

The expected value model (Liu *et al.*[67]) could be formulated for the model (7.3) by using expected value to both (the objective functions). But here the crisp equivalence may not satisfy the required conditions (total available resources greater than or equal to the total demands and also total conveyance capacities greater than or equal to the total demands for all items). So this method gave a feasible solution only when the rough interval coefficients supplies, demands and conveyance capacities were rough interval coefficients so that their respective expected values automatically satisfy those conditions.

Solid transportation problem with interval coefficient-7.1 (STPIC-7.1):

In order to solve the Model (7.3), here we have a solid transportation problem with interval coefficient which we denote as STPIC-7.1.

$$
\begin{aligned}
\max \ \mathrm{E[Z]} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\
\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u]), i = 1, 2, \ldots m, \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u]), j = 1, 2, \ldots n, \\
\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u]), k = 1, 2, \ldots K, \\
\text{where } x_{ijk} \geq 0, \forall i, j, k.
\end{aligned}
\tag{7.4}
$$

Solid transportation problem with interval coefficient-7.2 (STPIC-7.2):

In order to solve the Model (7.3), here we have another solid transportation problem with interval coefficient which we denote as STPIC-2.

$$
\begin{aligned}
\max \ \mathrm{E[Z]} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\
\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq ([\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots m, \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq ([\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots n, \\
\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq ([\bar{e}_k^l, \bar{e}_k^u]), k = 1, 2, \ldots K,
\end{aligned}
\tag{7.5}
$$

where $x_{ijk} \geq 0, \forall i, j, k$.

Theorem 7.1[Hamzeheea *et al.*[35]]

Suppose the optimal range of STPIC-7.1 exists, then it is equal to the surely optimal range of the Model (7.3).

The optimal range of the STPIC-1 could be obtained by solving two classical transportation problem given by STP-7.1.1 and STP-7.1.2.

Solid Transportation Problem-7.1.1 (STP-7.1.1):

$$\begin{aligned} \max \quad & \underline{Z}^l = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\ \text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^l, i = 1, 2, \ldots m, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^u, j = 1, 2, \ldots n, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^l, k = 1, 2, \ldots K, \\ & \text{where } x_{ijk} \geq 0, \forall i, j, k. \end{aligned} \tag{7.6}$$

Solid Transportation Problem-7.1.2 (STP-7.1.2):

$$\begin{aligned} \max \quad & \underline{Z}^l = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\ \text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^u, i = 1, 2, \ldots m, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^l, j = 1, 2, \ldots n, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^u, k = 1, 2, \ldots K, \\ & \text{where } x_{ijk} \geq 0, \forall i, j, k. \end{aligned} \tag{7.7}$$

Now from definition 7.1, the feasible set of STP-7.1.1 and STP-7.1.2 is equal to $\underline{U}^l$ and $\underline{U}^u$ respectively. Hence the optimal range of STPIC-7.1 by the interval $[\underline{z}^l, \underline{z}^u]$.

Theorem 7.2 [Hamzeheea *et al.*[35]]

Suppose the optimal range of STPIC-7.2 exists, then it is equal to the surely optimal range of the Model (7.3).

The optimal range of the STPIC-7.2 could be obtained by solving two classical solid transportation problems given by STP-7.2.1 and STP-7.2.2.

Solid Transportation Problem-7.2.1 (STP-7.2.1):

$$\begin{aligned} \max \quad \bar{Z}^l &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\ \text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^l, i = 1, 2, \dots m, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^u, j = 1, 2, \dots n, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^l, k = 1, 2, \dots K, \\ & \text{where } x_{ijk} \geq 0, \forall i, j, k. \end{aligned} \tag{7.8}$$

Solid Transportation Problem-7.2.2 (STP-7.2.2):

$$\begin{aligned} \max \quad \bar{Z}^u &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} E\left[\tilde{c}_{ijk}\right] x_{ijk} \\ \text{subject to} \quad & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^u, i = 1, 2, \dots m, \\ & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^l, j = 1, 2, \dots n, \\ & \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^u, k = 1, 2, \dots K, \\ & \text{where } x_{ijk} \geq 0, \forall i, j, k. \end{aligned} \tag{7.9}$$

Now from definition 7.1 the feasible sets of STP-7.2.1 and STP-7.2.2 are equal to $\bar{U}^l$ and $\bar{U}^u$ respectively. Hence the optimal range of STPIC-7.2 was given by the interval $[\bar{z}^l, \bar{z}^u]$.

Now we can claim that the interval $[\bar{z}^l, \bar{z}^u]$ and possibly optimal range of the Model

(7.3) both are equal and we have to show it. We need to prove $[\underline{z}^l, \underline{z}^u] \subseteq [\bar{z}^l, \bar{z}^u]$ or equivalently $\bar{z}^l \leq \underline{z}^l \leq \underline{z}^u \leq \bar{z}^u$. In order to show it, we need to prove $\bar{U}^l \subseteq \underline{U}^l$ and $\underline{U}^u \subseteq \bar{U}^u$. The remaining proof is given in proposition 7.1.

7.5 Algorithm

Input As an input is considered one solid transportation problem with rough interval coefficient (STPRIC) given by the Model (7.3).

Step 1 Break down the Model (7.3) into two transportation problem with interval coefficients given by STPIC-7.1 and STPIC-7.2.

Step 2 Find out the surely optimal range $[\underline{z}^{sl}, \underline{z}^{su}]$ by solving STPIC-7.1 by breaking down them to two classical transportation problem given by STP-7.1.1 and STP-7.1.2.

Step 3 In a similar process with step 2, we have to find the possibly optimal range $[\bar{z}^{pl}, \bar{z}^{pu}]$ by solving the STPIC-7.2.

Step 4 In this step we may have three possible outcomes depending on the set of decision variables. The three possible outcomes are as follows.

1. If STPIC-7.1 and STPIC-7.2 have their optimal ranges, then the main problem i.e. STPRIC has a rough ranges and it is given by ($[\underline{z}^{sl}, \underline{z}^{su}][\bar{z}^{pl}, \bar{z}^{pu}]$).
2. If STPIC-7.1 and STPIC-7.2 has unbounded range then the STPRIC has unbounded range.
3. The infeasibility of STPIC-1 and STPIC-2 directly implies the infeasibility of STPRIC.

7.6 Numerical Example

We considered objective functions with triangular and trapezoidal fuzzy numbers to illustrate the proposed method to show the effectiveness and efficiency. Supply, demand and conveyance capacities are taken as rough interval form.

$$\begin{aligned}\max\ \text{Profit} =&(1,3,7)x_{111} + (2,5,8,9)x_{121} + (4,7,9,11)x_{112} + (3,5,6,8)x_{211}\\ &+ (3,4,5,6)x_{221} + (5,7,8,9)x_{122} + (1,2,3)x_{222} + (4,5,6,7)x_{212}\end{aligned}$$

$$\begin{aligned}\text{subject to } & x_{111} + x_{121} + x_{112} + x_{122} \leq ([5,10],[4,18]),\\ & x_{211} + x_{221} + x_{212} + x_{222} \leq ([5.5,8],[3,9]),\\ & x_{111} + x_{211} + x_{112} + x_{212} \geq ([3,3.5],[2,3.5]),\\ & x_{121} + x_{221} + x_{122} + x_{222} \geq ([2,2.5],[1,2.5]),\\ & x_{111} + x_{211} + x_{121} + x_{221} \leq ([2.5,6.5],[2,8]),\\ & x_{112} + x_{212} + x_{122} + x_{222} \leq ([4,7.5],[4,10]),\end{aligned}$$

where $x_{ijk} \geq 0, \forall\, i,j,k.$

Now following the proposed methodology discussed in theorems 7.1 and 7.2. We have four classical solid transportation problem like as STP-7.1.1, STP-7.1.2, STP-7.2.1 and STP-7.2.2, given in section 7.3. We solve these STPs of the model using LINGO software.

After solving the problems we got optimal results and they were given as follows.

STP-7.1.1 Optimal value $\underline{z}^l$=44.5, optimal solution x_{121}=1, x_{112}=2.5, x_{211}=1.5, x_{122}=1.5 and all other variables are zero.

STP-7.1.2 Optimal value $\underline{z}^u$=95.125, optimal solution x_{121}=2.5, x_{112}=7.5, x_{211}=4 and all other variables are zero.

STP-7.2.1 Optimal value $\bar{z}^l$=40.75, optimal solution x_{112}=1.5, x_{211}=2, x_{122}=2.5

and all other variables are zero.

STP-7.2.2 Optimal value $\bar{z}^u$=125.5, optimal solution x_{121}=8, x_{112}=10 and all other variables are zero.

With these optimal values we have the following solution for the problem as follows, $[\underline{z}^l, \underline{z}^u]$=[44.5, 95.125] is the surly optimal range and these optimal solutions are two rather satisfactory solutions, $[\bar{z}^l, \bar{z}^u]$=[40.75, 125.5] is the possibly optimal range and these optimal solutions are completely satisfactory solutions and [44.5, 95.125], [40.75, 125.5] is the rough optimal range. Here the obtained results are involved in the transportation system regarding quantity of production of respective goods. Using the expected value operator and rough interval. In all the cases we managed to reach the optimal solution. As we know that using the rough interval tool, we were make the solution space or feasible region of the problem more flexible.

Chapter 8

A Fuzzy Fixed Charge Solid Transportation Problem with Rough Interval Approach with Sensitivity Analysis

8.1 Introduction

In this chapter, the researcher presents a new fuzzy fixed charge solid transportation problem with rough interval constraints. The fixed charge solid transportation problem is an extension of classical transportation problem in which a fixed profit is incurred, independent of the amount transported, along with a variable profit that is proportionate to the amount shipped. The fixed charge solid transportation has two kinds of profits: direct profit and fixed charge profit. Here, we could construct two solid transportation problems with interval coefficients considering the lower approximation and the upper approximation of the rough intervals. The researcher finds that the optimal solution of the fuzzy solid transportation problem could be found simply by solving proposed models and our model discussed some concepts like as surely optimal range, possibly optimal range, completely satisfactory solutions, rather satisfactory solutions, rough interval range. A numerical example is given to illustrate the proposed model. Our results are compared with those obtained by Pardip Kundu *et al.*[58], using the proposed models for the same numerical example.

Section 8.2 introduces some preliminaries. Section 8.3 gives the problem formulation of fuzzy STP with rough interval coefficients. Section 8.4 provides description of the proposed model. Section 8.5 gives solution procedure (algorithm). Section 8.6 presents a numerical example. Section 8.7 presents two numerical examples and sensitivity analysis. Section 8.8 gives the comparative study.

8.2 Preliminaries

The Rough intervals, rough interval arithmetic and ordering of rough interval have given in Chapter 1 (definition 1.11 and 1.12). The expected value of triangular and trapezoidal fuzzy numbers have given in Chapter 5 (definition 5.1).

8.3 Problem Formulation

It is assumed that a product is to be transported from m sources to n destinations in a STP. At each source, let s_i be the amount of a homogeneous product we want to transport to n destinations to satisfy the demand for d_j units of the product. Here e_k called conveyance, denotes the units of this product that can be carried by k different modes of transportation and also the objectives Z should be maximized.

A solid transportation problem is formulated with fuzzy objective functions, fuzzy fixed charge and different types of constraints with rough interval coefficients could be written as.

$$
\begin{aligned}
Z = \max \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} & (\tilde{c}_{ijk}x_{ijk} + \tilde{\eta}_{ijk}y_{ijk}) \\
\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} & x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u], [\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots m, \\
\sum_{i=1}^{m}\sum_{k=1}^{K} & x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u], [\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots n, \\
\sum_{i=1}^{m}\sum_{j=1}^{n} & x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u], [\bar{e}_k^l \bar{e}_k^u]), k = 1, 2, \ldots K, \\
\text{where } & x_{ijk} \geq 0, \quad \forall\ i, j, k, \\
& y_{ijk} = 0 \text{ if } \ x_{ijk} = 0, \\
& y_{ijk} = 1 \text{ if } \ x_{ijk} \geq 0.
\end{aligned} \tag{8.1}
$$

A value of the unit shipping cost $\tilde{c}_{ijk}$ is associated with fuzzy transportation from i^{th} origin to j^{th} destination by the k^{th} conveyance. We need to determine a feasible way of shipping the available amounts to satisfy the demand such that the profit is maximized.

Remark 8.1: Using the rough interval properties as given in section 8.2, the following proposals are made:

$$[\underline{s}_i^l, \underline{s}_i^u] \subseteq [\bar{s}_i^l, \bar{s}_i^u] \Rightarrow \bar{s}_i^l \leq \underline{s}_i^l \leq \underline{s}_i^u \leq \bar{s}_i^u$$

$$[\underline{d}_j^l, \underline{d}_j^u] \subseteq [\bar{d}_j^l, \bar{d}_j^u] \Rightarrow \bar{d}_j^l \leq \underline{d}_j^l \leq \underline{d}_j^u \leq \bar{d}_j^u$$

$$[\underline{e}_k^l, \underline{e}_k^u] \subseteq [\bar{e}_k^l, \bar{e}_k^u] \Rightarrow \bar{e}_k^l \leq \underline{e}_k^l \leq \underline{e}_k^u \leq \bar{e}_k^u$$

where $i = 1, 2, \ldots m, j = 1, 2, \ldots n$ and $k = 1, 2, \ldots K$.

Here, researcher tried to define some sets which would help to developing the methodology and theoretical background that would lead to solve the proposed solid transportation problem under rough interval approximation.

Definition 8.1. In problem (8.1), we define the following sets:

$$1.\bar{U}^l = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^l \right., i = 1, 2, \ldots m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^u, j = 1, 2, \ldots n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^l, k = 1, 2, \ldots K \text{ and } x_{ijk} \geq 0, \forall i, j, k \end{array} \right\}$$

$$2.\underline{U}^l = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^l \right., i = 1, 2, \ldots m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^u, j = 1, 2, \ldots n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^l, k = 1, 2, \ldots K \text{ and } x_{ijk} \geq 0, \forall i, j, k. \end{array} \right\}$$

$$3.\bar{U}^u = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^u \right., i = 1, 2, \ldots m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^l, j = 1, 2, \ldots n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^u, k = 1, 2, \ldots K \text{ and } x_{ijk} \geq 0, \forall i, j, k. \end{array} \right\}$$

$$4.\underline{U}^u = \left\{ \begin{array}{l} x \in R^p \left| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^u \right., i = 1, 2, \ldots m; \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^l, j = 1, 2, \ldots n; \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^u, k = 1, 2, \ldots K \text{ and } x_{ijk} \geq 0, \forall i, j, k. \end{array} \right\}$$

where $p = m \times n \times K$.

Proposition 8.1

Here, researcher proposed that for the sets $\bar{U}^l, \bar{U}^u, \underline{U}^l$ and $\underline{U}^u$ that we define by definitions are (according to Hamzeheea *et al.*[35]) have the following property: $\bar{U}^l \subseteq \underline{U}^l \subseteq \underline{U}^u \subseteq \bar{U}^u$

Proof: Using remark 8.1 we have, Let $x \in \bar{U}^l$ which implies,

$$\left(x \in R^p \middle| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^l, \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^u, \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^l \right)$$

$$\begin{pmatrix} \Rightarrow x \in R^p \middle| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^l, \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^u, \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^l \end{pmatrix} \Rightarrow x \in \underline{U}^l,$$

$$\begin{pmatrix} \Rightarrow x \in R^p \middle| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^u, \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^l, \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^u \end{pmatrix} \Rightarrow x \in \underline{U}^u,$$

$$\begin{pmatrix} \Rightarrow x \in R^p \middle| \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^u, \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^l, \\ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^u \end{pmatrix} \Rightarrow x \in \bar{U}^u,$$

for all

$$i = 1, 2, \ldots m, \ j = 1, 2, \ldots n \ \text{ and } \ k = 1, 2, \ldots K,$$

where $p = m \times n \times K$.

So we prove that for any $x \in \bar{U}^l$ we find that $x \in \underline{U}^l, x \in \underline{U}^u, x \in \bar{U}^u$. Therefore,

A. $\bar{U}^l \subseteq \underline{U}^l$

B. $\bar{U}^l \subseteq \underline{U}^u$

C. $\underline{U}^u \subseteq \bar{U}^u$

D. $\underline{U}^l \subseteq \bar{U}^u$

E. $\underline{U}^l \subseteq \bar{U}^u$

Now combining the above we can say that $\bar{U}^l \subseteq \underline{U}^l \subseteq \underline{U}^u \subseteq \bar{U}^u$. Hence proved.

8.4 Solution Methodology

Now, The following method is used to solve the problem given in (8.1).

Using Expected Value

Here when the researcher minimizes the expected value of the objective functions and fixed charges, then the above problem becomes as:

$$
\begin{aligned}
\max\ E[Z] = E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}(\tilde{c}_{ijk}x_{ijk} + \tilde{\eta}_{ijk}y_{ijk})\right] \\
\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u], [\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots m, \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u], [\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots n, \\
\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u], [\bar{e}_k^l, \bar{e}_k^u])\ , k = 1, 2, \ldots K, \\
x_{ijk} \geq 0,\ \forall\ i, j, k \\
y_{ijk} = 0 \text{ if } x_{ijk} = 0, \\
y_{ijk} = 1 \text{ if } x_{ijk} > 0.
\end{aligned}
$$

This is equivalently written as:

$$
\begin{aligned}
\max \mathrm{E[Z]} &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}(E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to } &\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq ([\underline{s}_i^l, \underline{s}_i^u], [\bar{s}_i^l, \bar{s}_i^u]), i = 1, 2, \ldots m, \\
&\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq ([\underline{d}_j^l, \underline{d}_j^u], [\bar{d}_j^l, \bar{d}_j^u]), j = 1, 2, \ldots n,
\end{aligned} \tag{8.2}
$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq ([\underline{e}_k^l, \underline{e}_k^u], [\bar{e}_k^l, \bar{e}_k^u]), k = 1, 2, \ldots K,$$

$$x_{ijk} \geq 0, \quad \forall\ i, j, k$$

$$y_{ijk} = 0 \text{ if } x_{ijk} = 0,$$

$$y_{ijk} = 1 \text{ if } x_{ijk} > 0.$$

The expected value model (Liu *et al.*[67]) could be formulated for the model (8.2) by using expected value to both the objective functions. But here the crisp equivalence might not satisfy the required conditions (total available resources greater than or equal to the total demands and also total conveyance capacities greater than or equal to the total demands for all items). So this method provides a feasible solution only when the rough interval coefficients supplies, demands and conveyance capacities are rough interval coefficients. So that their respective expected values automatically satisfy those conditions.

Fixed charge solid transportation problem with interval coefficient-8.1 (FCSTPIC-8.1):

In order to solve the problem (8.2), here we have a fixed charge solid transportation problem with interval coefficient which we denote as FCSTPIC-8.1.

$$\begin{aligned}
\max \quad \text{E[Z]} \ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} (E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to } &\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq [\underline{s}_i^l, \underline{s}_i^u], i = 1, 2, \ldots m. \\
&\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq [\underline{d}_j^l, \underline{d}_j^u], j = 1, 2, \ldots n.
\end{aligned} \tag{8.3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq [\underline{e}_k^l, \underline{e}_k^u], k = 1, 2, \ldots K.$$

$$x_{ijk} \geq 0, \forall\ i, j, k,$$

$$y_{ijk} = 0 \text{ if } \ x_{ijk} = 0,$$

$$y_{ijk} = 1 \text{ if } \ x_{ijk} > 0.$$

Fixed charge solid transportation problem with interval coefficient-8.2 (FCSTPIC-8.2):

In order to solve the problem (8.2), here we have another fixed charge solid transportation problem with interval coefficient which we denote as FCSTPIC-8.2.

$$\begin{aligned}
\max \quad \text{E[Z]} &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} (E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to} & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq [\bar{s}_i^l, \bar{s}_i^u], \quad i = 1, 2, \ldots m. \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq [\bar{d}_j^l, \bar{d}_j^u], \quad j = 1, 2, \ldots n. \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq [\bar{e}_k^l, \bar{e}_k^u], \quad k = 1, 2, \ldots K. \\
& x_{ijk} \geq 0, \forall\ i, j, k \\
& y_{ijk} = 0, \quad \text{if} \quad x_{ijk} = 0, \\
& y_{ijk} = 1, \quad \text{if} \quad x_{ijk} > 0.
\end{aligned} \tag{8.4}$$

Remark 8.2(Using Hamzeheea *et al.*[35])

Suppose that the optimal range of FCSTPIC-8.1 exists. Then it is equal to the surely optimal range of the Model (8.2). The optimal range of the FCSTPIC-8.1 could be obtained by solving two classical transportation problems given by FCSTP-8.1.1

and FCSTP-8.1.2.

Fixed Charge Solid Transportation Problem-8.1.1

(FCSTP-8.1.1):

$$\begin{aligned}
\underline{z}^l = \max \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} (E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^l, \quad i = 1, 2, \ldots m. \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^u, \quad j = 1, 2, \ldots n. \\
\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^l, \quad k = 1, 2, \ldots K. \\
x_{ijk} \geq 0, \forall\ i, j, k, \\
y_{ijk} = 0, \quad \text{if} \quad x_{ijk} = 0, \\
y_{ijk} = 1, \quad \text{if} \quad x_{ijk} > 0.
\end{aligned} \tag{8.5}$$

Fixed Charge Solid Transportation Problem-8.1.2

(FCSTP-8.1.2):

$$\begin{aligned}
\underline{z}^u = \max \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} (E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \underline{s}_i^u, \quad i = 1, 2, \ldots m. \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \underline{d}_j^l, \quad j = 1, 2, \ldots n. \\
\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \underline{e}_k^u, \quad k = 1, 2, \ldots K. \\
x_{ijk} \geq 0, \forall i, j, k,
\end{aligned} \tag{8.6}$$

$$y_{ijk} = 0, \quad \text{if} \quad x_{ijk} = 0,$$

$$y_{ijk} = 1, \quad \text{if} \quad x_{ijk} > 0.$$

Now from the definition 8.1, the feasible set of FCSTP-8.1.1 and FCSTP-8.1.2 is equal to $\underline{U}^l$ and $\underline{U}^u$ respectively. Hence the optimal range of FCSTPIC-8.1 was given by the interval $[\underline{z}^l, \underline{z}^u]$. Now we could claim that the interval was surly optimal range of the problem (8.2).

Remark 8.3(Using Hamzeheea *et al.*[35])
Suppose that the optimal range of FCSTPIC-8.2 exists. Then it is equal to the surely optimal range of the Model (8.2). The optimal range of the FCSTPIC-8.2 could be obtained by solving two classical solid transportation problems given by FCSTP-8.2.1 and FCSTP-8.2.2.

Fixed Charge Solid Transportation Problem-8.2.1 (FCSTP-8.2.1):

$$\begin{aligned}
\bar{z}^l &= \max \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} (E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to } & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^l, \quad i = 1, 2, \ldots m. \\
& \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^u, \quad j = 1, 2, \ldots n. \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^l, \quad k = 1, 2, \ldots K. \\
& x_{ijk} \geq 0, \ \forall \ i, j, k, \\
& y_{ijk} = 0, \quad \text{if} \quad x_{ijk} = 0, \\
& y_{ijk} = 1, \quad \text{if} \quad x_{ijk} > 0.
\end{aligned} \tag{8.7}$$

Fixed Charge Solid Transportation Problem-8.2.2 (FCSTP-8.2.2):

$$
\begin{aligned}
\bar{z}^u \quad &= \max \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} (E[c_{ijk}]x_{ijk} + E[\eta_{ijk}]y_{ijk}) \\
\text{subject to} \quad & \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk} \leq \bar{s}_i^u, \quad i = 1, 2, \ldots m. \\
& \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk} \geq \bar{d}_j^l, \quad j = 1, 2, \ldots n. \\
& \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq \bar{e}_k^u, \quad k = 1, 2, \ldots K. \\
& x_{ijk} \geq 0, \ \forall i, j, k, \\
& y_{ijk} = 0 \text{ if } x_{ijk} = 0, \\
& y_{ijk} = 1 \text{ if } x_{ijk} > 0.
\end{aligned} \tag{8.8}
$$

Now from the definition (8.1) the feasible sets of FCSTP-8.2.1 and FCSTP 8.2.2 are equal to $\bar{U}^l$ and $\bar{U}^u$ respectively. Hence the optimal range of FCSTPIC-8.2 was given by the interval$[\bar{z}^l, \bar{z}^u]$. Now we could claim that the interval $[\bar{z}^l, \bar{z}^u]$ was possibly optimal range of the problem (8.2) both are equal and we have to show it.

We need to prove $[\underline{z}^l, \underline{z}^u] \subseteq [\bar{z}^l, \bar{z}^u]$ or equivalently

$$\bar{z}^l \leq \underline{z}^l \leq \underline{z}^u \leq \bar{z}^u.$$

In order to show it, one need to prove $\bar{U}^l \subseteq \underline{U}^l$ and$\underline{U}^u \subseteq \bar{U}^u$. The remaining proof is given in proposition (8.1).

Algorithm 8.5 solves a more general form of problem (8.1). The process is summarized as follows:

8.5 Algorithm

Input As an input consider one fixed charge solid transportation problem with rough interval coefficient (FCSTPRIC) given by the Model (8.2).

Step 1 Break down the given Model (8.2) into two transportation problems with interval coefficients given as FCSTPIC-8.1 and FCSTPIC-8.2.

Step 2 Find out the surely optimal range $[\underline{z}^{sl}, \underline{z}^{su}]$ by solving FCSTPIC-8.1 by breaking down them into two classical transportation problems given as FCSTP-8.1.1 and FCSTP-8.1.2.

Step 3 In a similar process with step 2, we have to find the possibly optimal range $[\bar{z}^{pl}, \bar{z}^{pu}]$ by solving the FCSTPIC-8.2.

Step 4 In this step we might have three possible outcomes depending on the set of decision variables.

The two possible outcomes are as follows:

A. If FCSTPIC-8.1 and FCSTPIC-8.2 have their optimal ranges, then the main problem i.e, FCSTPRIC has a rough ranges and it is given by $([\underline{z}^{sl}, \underline{z}^{su}][\bar{z}^{pl}, \bar{z}^{pu}])$.

B. The infeasibility of FCSTPIC-8.1 and FCSTPIC-8.2 directly implies the infeasibility of FCSTPRIC.

The above outcomes are validated in the next section.

8.6 Numerical Example

We considered objective functions with triangular and trapezoidal fuzzy numbers to illustrate the proposed method to show the effectiveness and efficiency. Supply, demand and conveyance capacities were taken as rough interval form.

Table 8.1: Profit matrix $\tilde{c}_{ijk}$ for numerical example

$\tilde{c}_{ij1}$		$\tilde{c}_{ij2}$	
(1,3,7)	(2,5,8,9)	(4,7,9,11)	(5,7,8,9)
(3,5,6,8)	(3,4,5,6)	(4,5,6,7)	(1,2,3)

Table 8.2: Fixed charge matrix $\tilde{c}_{ijk}$ for numerical example

$\tilde{\eta}_{ij1}$		$\tilde{\eta}_{ij2}$	
(15,20,30)	(20,24,30)	(20,26,32)	(25,30,35,40)
(10,16,20,24)	(20,27,30,35)	(15,20,24,28)	(22,26,30,34)

$$\begin{aligned} \text{subject to} \quad & x_{111} + x_{121} + x_{112} + x_{122} \leq ([5, 10], [4, 18]), \\ & x_{211} + x_{221} + x_{212} + x_{222} \leq ([5.5, 8], [3, 9]), \\ & x_{111} + x_{211} + x_{112} + x_{212} \geq ([3, 3.5], [2, 3.5]), \\ & x_{121} + x_{221} + x_{122} + x_{222} \geq ([2, 2.5], [1, 2.5]), \end{aligned} \tag{8.9}$$

$$x_{111} + x_{211} + x_{121} + x_{221} \leq ([2.5, 6.5], [2, 8]),$$

$$x_{112} + x_{212} + x_{122} + x_{222} \leq ([4, 7.5], [4, 10]),$$

$$\text{where } \ x_{ijk} \geq 0, \ \forall\, i, j, k,$$

$$y_{ijk} = 0 \ \text{ if } x_{ijk} = 0,$$

$$y_{ijk} = 1 \ \text{ if } x_{ijk} > 0.$$

Now following the proposed methodology discussed in remarks 8.1 and 8.2, we have four classical fixed charge solid transportation problem like as

FCSTP-8.1.1, FCSTP-8.1.2, FCSTP-8.2.1 and FCSTP-8.2.2, given in section 8.3 and solve these FCSTP model. After solving the problems we got optimal results and they were given below.

FCSTP-8.1.1 Optimal value $\underline{z}^l$=145.1, optimal solution x_{121}=1, x_{112}=2.5, x_{211}=1.5, x_{122}=1.5 and all other variables are zero.

FCSTP-8.1.2 Optimal value $\underline{z}^u$=163.225, optimal solution x_{121}=2.5, x_{112}=7.5, x_{211}=4 and all other variables are zero.

FCSTP-8.2.1 Optimal value $\bar{z}^l$=116.75, optimal solution x_{112}=1.5, x_{211}=2, x_{122}=2.5 and all other variables are zero.

FCSTP-8.2.2 Optimal value $\bar{z}^u$=176.1, optimal solution x_{121}=8, x_{112}=10 and all other variables are zero.

With these optimal values we derived the following solution for the problem they are, $[\underline{z}^l, \underline{z}^u]$ = [145.1, 163.225] the surly optimal range and these optimal solutions are two rather satisfactory solutions, $[\bar{z}^l, \bar{z}^u]$=[116.75, 176.1] is the possible optimal range and these optimal solutions are two completely satisfactory solutions [145.1, 163.225], [116.75, 176.1] is the rough optimal range, which validates the outcome “A” of the previous section. Here the obtained results are involved in the transportation system regarding quantity of production of respective goods. Using the expected value operator and rough interval, in all the case we reached the optimal solution. As it was clear that using the rough interval tool we make the solution space or feasible region of the problem more flexible.

8.7 Sensitivity Analysis

Here, the right side constraints of numerical example was changed (-1 level)(given in above section 8.6 (0 level)) to ([4, 9], [3, 17]), ([4.5, 7], [2, 8]), ([2, 2.5], [1, 1.5]), ([1, 1.5], [0, 1.5]), ([1.5, 5.5], [1, 7]), ([3, 6.5], [3, 9]) in the proposed methodology.

Table 8.3: Profit matrix $\tilde{c}_{ijk}$ for numerical example

$\tilde{c}_{ij1}$		$\tilde{c}_{ij2}$	
(1,3,7)	(2,5,8,9)	(4,7,9,11)	(5,7,8,9)
(3,5,6,8)	(3,4,5,6)	(4,5,6,7)	(1,2,3)

Table 8.4: Fixed charge matrix $\tilde{c}_{ijk}$ for numerical example

$\tilde{\eta}_{ij1}$		$\tilde{\eta}_{ij2}$	
(15,20,30)	(20,24,30)	(20,26,32)	(25,30,35,40)
(10,16,20,24)	(20,27,30,35)	(15,20,24,28)	(22,26,30,34)

$$\begin{aligned} \text{subject to} \quad & x_{111} + x_{121} + x_{112} + x_{122} \leq ([4, 9], [3, 17]), \\ & x_{211} + x_{221} + x_{212} + x_{222} \leq ([4.5, 7], [2, 8]), \\ & x_{111} + x_{211} + x_{112} + x_{212} \geq ([2, 2.5], [1, 1.5]), \\ & x_{121} + x_{221} + x_{122} + x_{222} \geq ([1, 1.5], [0, 1.5]), \\ & x_{111} + x_{211} + x_{121} + x_{221} \leq ([1.5, 5.5], [1, 7]), \\ & x_{112} + x_{212} + x_{122} + x_{222} \leq ([3, 6.5], [3, 9]), \end{aligned} \tag{8.10}$$

$$\begin{aligned} & \text{where}\, x_{ijk} \geq 0, \forall\, i, j, k, \\ & y_{ijk} = 0 \;\text{ if } x_{ijk} = 0, \\ & y_{ijk} = 1 \;\text{ if } x_{ijk} > 0. \end{aligned}$$

Now following the proposed methodology discussed in remarks 8.1 and 8.2, we have four classical fixed charge solid transportation problem like, FCSTP-8.1.1, FCSTP-8.1.2, FCSTP-8.2.1 and FCSTP-8.2.2, given in section 8.3. After solving the prob-

lems we get optimal results and they are two completely satisfactory solutions [114.6, 149.975], [104, 162.35] is the rough optimal range.
Similarly here, take the right side constraints for numerical example (+2 level) to ([7, 12], [6, 20]), ([7.5, 10], [5, 11]), ([5, 5.5], [4, 5.5]), ([4, 4.5], [3, 4.5]), ([4.5, 8.5], [4, 10]), ([6, 9.5], [6, 12]) and these optimal solutions are two completely satisfactory solutions [170.6, 188.975], [142.5, 220.1] is the rough optimal range, considered to explain the proposed Methods' efficiency.

Here, it is observed that the results are involved in the transportation system regarding quantity of production of respective goods. Using the expected value operator and rough interval, in all the cases the optimal solutions are reached. As it was hypothesised that using the rough interval tool we make the solution space or feasible region of the problem more flexible.
Table-8.5 shows that the comparison of the results of the present examples. It is shown that the optimal solution of the proposed problem gives better results.

Table 8.5: **Sensitivity analysis of (8.10)**

Level	Surely optimal range	Possibly optimal range	Rough interval ranges
-1	[114.6, 149.975]	[104, 162.35]	([114.6, 149.975], [104, 162.35])
0	[145.1, 163.225]	[116.75, 176.1]	([145.1, 163.225], [116.75, 176.1])
+2	[170.6, 188.975]	[142.5, 220.1]	([170.6, 188.975], [142.5, 220.1])

It is also to be noted that Model (8.10) does not have optimal range at level +1 due to infeasibility. Thus it validates outcome "B" of the previous section.

8.8 Comparative Study

In this section, let us consider a numerical example presented by Pradeep Kundu *et al.*[58]. Table 8.6 and Figure 8.1 presents the comparison analysis between the results obtained using rough interval approximation and the results of Pradip Kundu *et al.*[58] using chance-constrained programming. Figure 8.1 represents the surly optimal ranges and possibly optimal ranges. Here, both the ranges obtained by Pradeep Kundu *et al.*[58] are subset of proposed method.

Table 8.6: **Comparisons of optimal solutions**

Pradip Kundu *et al.* [58] Chance-constrained programming	**Proposed method**
([572.54,579.536],[471.427,630.2688])	([474, 581.80], [408.60, 670.2])

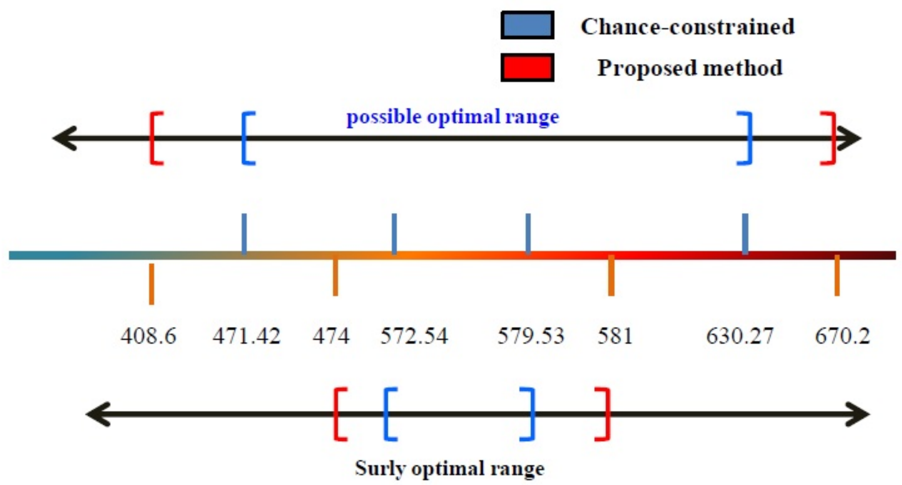

Figure 8.1: Optimal values represented as a Rough Interval.

Chapter 9

Conclusions and Scope for Future Work

This Chapter presents the main conclusions of all the chapters in the thesis (section 9.1) and some directions for future research work (section 9.2).

9.1 Conclusions

Thesis is devoted to study on fuzzy solid transportation problems with rough intervals, stochastic and budget constraints. Chapter-wise conclusions of the thesis are as follows:

Chapter 1 is introductory in nature. This chapter includes the abstract of the thesis, key words, some definitions, literature survey of 111 references and list of notations used throughout the work.

In Chapter 2, a solution procedure has been developed for a fuzzy solid transportation problem with fuzzy supply, requirement, conveyance capacity and budget interval using Hu and Wang's approach and fuzzy programming approach. In a frame work at genuine field problem, the technique could be used in an effective way.

In Chapter 3, developed a method to find the fuzzy objective value of fuzzy solid fractional transportation problem (SFTP). A two level mathematical programming technique to find the α-cuts of the fuzzy SFTP problem. At a specific α-cut, solving the pair of linear program produced the bounds of the objective value of the SFTP. The objective value of SFTP is expressed by approximating membership function rather than by a point value, more information was provided for making better decisions. The α-cuts of different possibility levels could be used to approximate the membership function. An example was given to illustrates the proposed model. With the ability of calculating the objective value of the SFTP under the fuzzy environment, it might help initiate wider applications.

In Chapter 4, FSSTP Has been formulated with the fuzzy-random environment and solved by GP and FGP approaches. Here, fuzzy constraints with random demand and conveyance parameters have been transformed into crisp forms using a new technique assuming normal distribution to the demand variants with different limits. We considered two different budgetary constraints and observed the effects on demand deficits and capacities deficits. The present model also could be extended to include some restrictions on the conveyances and on the unit transportation cost with respect to the amount to be transported.

In Chapter 5, presented a fuzzy Goal programming approach for solving MOSTP with fuzzy constraints (i.e., sources, demands and conveyance capacities are fuzzy). In order to solve the model conveniently, we have discussed the crisp model with corresponding defuzzified values under the conditions and the expected value models in objective functions for triangular and trapezoidal membership functions. Then multi-objective problems were solved using the fuzzy goal programming approach and three numerical examples are given to illustrate the proposed model. The optimal solution of the proposed problem gave better results by using fuzzy Goal programming approach when compared to Fuzzy linear programming and Global criterion method.

In Chapter 6, a solution procedure for the fuzzy linear programming MOSTP model with L-R coefficients has developed. The flexible indexes were given in objective function. In order to sustain the existence of an optimal solution. The model was illustrated with a numerical example and results were compared at level of -1, 0 and +1. In this regard the decision maker might choose the best among those results according to the requirement. So this helps the decision maker.

Chapter 7 presented a new fuzzy solid transportation problem with expected profit value and rough interval constraints (i.e. Sources, demands and conveyance capacities are fuzzy) that maximized the profit. In order to solve the model conveniently, we have discussed the crisp model with corresponding expected value models with objective functions in triangular and trapezoidal membership functions. An algorithm is proposed. A numerical example was given to illustrate the proposed model. As part of solution of the problems some concepts on optimal solutions were discussed.

In Chapter 8, a new fuzzy fixed charge solid transportation problem with expected profit value and rough interval constraints with sensitivity analysis was presented. We have discussed the crisp model with corresponding expected value models in objective functions for triangular and trapezoidal membership functions. We have shown the problem that could be converted into two different solid transportation problems with interval coefficients, and then further each of these two solid transportation problems could be regenerated into four classical solid transportation problems. A numerical example was given to illustrate the proposed model, sensitivity and comparative study was included.

9.2 Future Directions

This thesis mainly focuses on solid transportation problems with rough intervals, stochastic and budget constraints models in fuzzy environment. There are several interesting opportunities for future researches in this area. The extension of fuzzy solid transportation problem models could be done for stochastic with different parameters like demand rate, varies time were concerning to the real life problem. For future researches, the following areas are recommended:

- Extending some models to multi-item case is another direction for further work.
- The models presented in the thesis could be extended or applied to other similar uncertain models in other areas such as inventory control, ecology, sustainable form management, etc.
- Some Evolutionary Techniques like Genetic Algorithm (GA) & Particle Swarm Optimization (PSO) may be applied.
- Demand function may be considered as Ramp Type function.
- Case studies with real life problems.

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LIST OF PUBLICATIONS

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4. "A Fuzzy Fixed Charge Solid Transportation Problem with Rough Interval Approach", *Global Science and Technology Forum 2016*,**Article ID:** 20,(2016), DOI: 10.5176/2251-1938_ORS16.20.

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