RESONANCE TYPE FLOWS DUE TO OSCILLATIONS OF SYMMETRIC BODIES IN FLUIDS WITH COUPLE-STRESSES

A THESIS

SUBMITTED TO

NATIONAL INSTITUTE OF TECHNOLOGY, WARANGAL, INDIA

FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

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JULY - 2018

Dedicated

To

My Beloved Parents

Sri Tangudu Narayana Rao

&

Smt. Tangudu Kalavathi

For their infinite love and support

DECLARATION

This is to certify that the work presented in the thesis entitled **" RESONANCE TYPE FLOWS DUE TO OSCILLATIONS OF SYMMETRIC BODIES IN FLUIDS WITH COUPLE-STRESSES "***is a bonafide work done by me under the supervision of* ***PROF. J.V. RAMANA MURTHY*** *and co-supervision of* ***PROF. G.S. BHASKARA RAO*** *was not submitted elsewhere for the award of any degree.*

I declare that this written submission represents my ideas in my own words. Where-ever others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea / data / fact /source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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CERTIFICATE

This is to certify that the thesis entitled **“ RESONANCE TYPE FLOWS DUE TO OSCILLATIONS OF SYMMETRIC BODIES IN FLUIDS WITH COUPLE-STRESSES”** *submitted to National Institute of Technology, Warangal for the award of the degree of* ***DOCTOR OF PHILOSOPHY****, is the bonafide research work done by* ***Mr. GOVINDARAO TANGUDU*** *under our supervision. The contents of this thesis have not been submitted elsewhere for the award of any degree.*

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ACKNOWLEDGEMENT

I gratefully acknowledge my Supervisor, **Prof. J. V. Ramana Murthy**, Department of Mathematics, National Institute of Technology Warangal -506004, India, for his guidance, time, patience, invaluable discussions, continuous support and constant encouragement throughout the preparation of the thesis. With great patience and his insight, he has mentored me in developing my research abilities, critical thinking, and research writing skills at every stage of my research work. He inspired and motivated me, first for pursuing research and then encouraged me to work on this particular field. Without him, it would not have been possible to achieve this work. I also thank his wife **Smt. J. Prasanna Ramana Murthy** for her hospitality and her patience during our elongated discussions.

I express my sincere gratitude to my Co-Supervisor, **Dr. G. S. Bhaskara Rao**, Professor and HOD, Basic Sciences and Humanities, Gudlavalleru Engineering College, Gudlavalleru-521356, Krishna for his constant support and encouragement throughout the research.

I wish to thank **Prof. D. Srinivasacharya**, Head of the Department, Department of Mathematics, for his constant moral support and encouragement at every stage.

I extended my sincere thanks to the members of the Doctoral Scrutiny Committee, **Prof.L. Ram Gopal Reddy,** Department of Physics and Dean (Student Welfare), **Dr. J. Pranitha,** Assistant Professor, Department of Mathematics for spending their valuable time in providing suggestions, feedback, and support while my reviewing the progress of my work.

I take this opportunity to express my thanks from the bottom of my heart to the dynamic personality, (Late) **Prof. T. K. V. Iyengar**, Retired Professor, Department of Mathematics, **Prof. Y. N. Reddy**, Department of Mathematics for giving me the opportunity to do research. And their affectionate support, encouragement and valuable suggestions.

I would like to thank **Prof. G. Radhakrishnamacharya,** Retired Professor, Department of Mathematics, for his support, encouragement and valuable suggestions during my progress presentations.

I am also grateful to thank **Prof. K. N. S. Kasi Viswanadham,** former Head, Department of Mathematics, **Prof. D. Dutta,** former Head, Department of Mathematics for their constant support and encouragement during my research work.

I extended my thanks to **Dr. P. Muthu, Dr. T. Kurmayya, Dr. Ch. Ram Reddy** and **Dr. D. Bhargavi** for their moral support and encouragement throughout my stay in Warangal.

I also thank **Dr. H. P. Rani, Dr. A. Benerji Babu,** and **Dr. R. S. Selvaraj**, for their concern and encouragement during my research.

I would like to thank the office staff of the department **Mr. S. Narayana, Mr.T.Kiran Kumar, Mr. J. Narasaiah** and **Mr. B. Sadanandam** for their support and cooperation.

I thank Prof. **N. V. Ramana Rao**, Director, National Institute of Technology, Warangal.

I also wish to thank all co-research scholars of the Department of Mathematics and friends for their moral support and enthusiastic help throughout my stay at NIT Warangal. Special thanks to **Dr. M. Phani Kumar, Dr. J. Srinivas, Dr. B. Srinivasulu, Mr. M. Pavan Kumar Reddy, Dr. Ch. Venkata Rao, Dr. T. Ramesh, Dr. Krishnaih, Dr. J. A. Ranga Babu, Mr. Ravi Chandra, Mr. Sudheer**.

I gratefully acknowledge the management members of **Gudlavalleru Engineering College**, Gudlavalleru, for their extended support and encouragement for carrying out my Ph.D. work.

I also convey my heartful thanks to **Prof. P. Ravindra Babu**, Principal, **Prof.G.V.S.N.R.V Prasad**, Vice-Principal (Academics), **Prof. B. Karuna Kumar**, Vice-Principal (Administration), **Prof. P. Nageswara Reddy**, Director(AS&A), for their wholehearted support and cooperation.

I express my sincere thanks to some of my colleagues and friends in Gudlavalleru Engineering College **Mr. D. Surendra Babu, Dr. S. Suresh, Mr.S.Sivaram, Mr. D. K. Pavan Kumar, Mr. T. Babu Rao, Mr. V. Naveen Kumar, Mr. B. Akkayya, Mr. Sk. Abdul Mathin, Mr. Sk. Bajivali, Dr.P.S.V.Srinu Babu** and all other faculty members of the department for their moral support and encouragement.

I also convey my sincere thanks to all the teaching and non-teaching staff of Gudlavalleru Engineering College who are support directly or indirectly in various aspects like classwork adjustment while going on leave for research work, and technical aspects in documentation.

I thank my gurus **Mr. Ramesh, Mr. Suresh, Mr. Haribabu, Mr. K. Siva Prasad** for their inspiring teaching, moral support and well-wishing in various stages of my studies.

I wish to express my sincere gratitude to my friends **Mr. M. Satya Krishna, Mr. B. Rakesh, Mr. S. Ratna Kishore, Mr. A. Narayana, Mr. K. Suhash Chand, Mr. K. Chakradhar** and all my friends for their memories. Especially their support for the completion of my studies.

I owe special thanks to my parents **Sri. Narayana Rao** and **Smt. Kalavathi**, my brother **Sri. Narasimha Raju** and **Smt. Sukanya**, their children **Chi. Abhishek**, **Chi. Karthik**, my father in law **Sri. Sadanandam** and **Smt. Bharathi** for their moral support and love.

Last but not least, there are no words to express thanks to my better half **Smt. Sridevi** and my children **Chi. Sathwik, Chi. Kavya Sree**, who have always been an endless source of love and support to me. And for her cooperation and understanding.

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ABSTRACT

In the classical flow problems, for the case of non-Newtonian fluids, the oscillation problems are very important to study, since many of the elastic properties of the dilute polymer solutions can be determined by the oscillation processes. In non-Newtonian fluids, fluids with Couple stresses are having special status, since the oscillations generated in these fluids exhibit effect of Couple stresses on the Drag or Couple.

The problems of the different oscillations of symmetric bodies (like circular cylinder or sphere) along/about its axis of symmetry in an incompressible Micro-polar fluid/Couple-stress fluid and the flow generated due to these oscillations in the fluid is considered. The Stokes flow is considered by neglecting nonlinear convective terms in the equations of motion on the assumption that the flow is so slow that oscillations Reynolds number is less than unity. The solution of this case cannot be obtained as limiting case of non-resonance problem. The velocity and micro-rotation components of the flow for the case of *resonance* and *non-resonance* are obtained. The Drag / Couple / Skin friction are derived analytically and the effect of physical parameters like Micro-polarity and Couple stress parameter on the Drag / Couple / Skin friction due to oscillations is shown through graphs.

The thesis consists of twelve chapters and Four parts. Part - I and Chapter one is introductory in nature. Part – II is devoted to flows generated in Micro-polar fluids and contains Five chapters (Chapters two to six). Part – III is devoted to flows in the Couple stress fluids and contains Five chapters (Chapters seven to eleven). Part - IV and Chapter twelve gives concluding remarks of the thesis and possible directions in which further work can be carried out.

In all these chapters, the expressions for the velocity, micro-rotation for Micro-polar fluids and velocity field for Couple-stress fluids are obtained. The Drag/Couple/Skin friction is derived analytically and the effect of physical parameters like Reynolds number and Couple stress parameter on the Drag/Couple /Skin friction are studied graphically.

NOMENCLATURE

(R, θ, Z)	Cylindrical co-ordinate system
$(\bar{e}_R, \bar{e}_\theta, \bar{e}_Z)$	Base vector in cylindrical co-ordinate system
(R, θ, ϕ)	Spherical co-ordinate system
$(\bar{e}_R, \bar{e}_\theta, \bar{e}_\phi)$	Base vector in spherical co-ordinate system
∇_1, ∇	Dimensional and Non-dimensional gradient operator
R, r	Dimensional and Non-dimensional Distance from origin (L)
$\bar{Q}, \bar{q}$	Dimensional and Non-dimensional Fluid velocity vector (LT^{-1})
$\bar{l}, \bar{\upsilon}$	Dimensional and Non-dimensional Micro-rotation vector
U, V, W	Dimensional Velocity components (LT^{-1})
u, v, w	Non-dimensional Velocity components
$\mathcal{A}, \mathcal{B}, \mathcal{C}$	Dimensional Micro-rotation components
A, B, C	Non-dimensional Micro-rotation components
P, p	Dimensional and Non-dimensional Fluid pressure at any point ($ML^{-1}T^{-2}$)
Ψ, ψ	Dimensional and Non-dimensional Stream function
$\tau, t = \frac{U_0}{a}$	Dimensional and Non-dimensional Time (T)
$\sigma, \varpi = \frac{a\sigma}{U_0}$	Dimensional and Non-dimensional Frequency parameter (T^{-1})
$\mathcal{J}, J$	Dimensional and Non-dimensional Gyration coefficient (MLT^{-1})
D^*, D	Dimensional and Non-dimensional Drag
C^*, C	Dimensional and Non-dimensional Couple
$\boldsymbol{c_f}$	Skin friction
ζ	Non-dimensional Swirl
ρ	Density of the fluid (ML^{-3})
μ	Viscosity coefficient ($ML^{-1}T^{-1}$)

k	Micro-viscosity coefficient ($ML^{-1}T^{-1}$)
α, β, γ	Couple-stress viscosity coefficients ($ML^{-1}T^{-1}$)
T_{ij}	Cauchy's Stress components
M_{ij}	Couple-stress components
e_{ij}	Strain rate tensor
$s = \frac{ka^2}{\gamma}$	Couple-stress parameter for Micro-polar fluid
S	Couple-stress parameter for Couple-stress fluid
$c = \frac{k}{\mu + k}$	Cross viscosity coefficient or Micro-polarity parameter
$Re = \frac{\rho U_0 a}{\mu}$	Reynolds number and $R_0 = Re(1 - c)$
$\epsilon = \frac{\alpha + \beta + \gamma}{\gamma}$	Another Couple stress parameter
a	Radius of the Sphere/Circular cylinder
L	Length of the Circular cylinder
a_0, a_1, a_2	Constants (for Micro-polar fluids)
A_0, A_1, A_2	Constants (for Couple-stress fluids)
∇^2, D_c^2	Operators which are formed in cylinder problems
E^2, D_s^2	Operators which are formed in sphere problems
$K_0, K_1, K_{\frac{3}{2}}$	Modified Bessel functions of second kind of orders 0, 1, 3/2

CONTENTS

Part – I

Introduction

Chapter 1

Introduction

In the classical study of steady flow of fluids past bodies or flow due to rotation of bodies, there exists vast literature to find the Drag or Couple acting on the body. In nature many problems involve unsteady transient flows and oscillatory flows. Oscillatory flows are very important to determine the properties of the fluid. Stokes (1851**)** was one of the first scientists who provided an analytical solution for motion pendulum and an unsteady, one-dimensional flow problem, namely the solution for the fluid motion induced by the sudden movement of a plate. These one dimensional problems on plate are referred to as Stokes first and second problems. Latter many types of unsteady flow problems were investigated by several researchers. The following works are a few to quote on different flow problems: Rayleigh (1916) on revolving fluids, Benjamin (1957) on formation waves on inclined plane, Benny (1966) on waves in liquid films, Schlichting (1968) on boundary layer theory, Batchelor (1970) on slender body theory, Kim and Troesch (1989) on oscillating cylinders. Many unsteady and oscillation problems can be found in famous treatises by Happel and Brenner(1973) on low Reynold number flows, Van Dyke (1975) on perturbation methods, Fung (1984) on Bio-dynamics, Pozrikidis (2009) on singularity methods and numerical computations.

Many elastic properties of dilute polymers can be detected and measured conveniently by a suitable choice of oscillatory flows. The problems that are concerned with the effects of free stream oscillations are of physical significance. The problems of unsteady flows were initiated by Lighthill (1954) by giving analytical solution for stream function due to oscillating streaming flow past a cylinder and proved that the amplitudes of heat transfer fluctuations are much reduced if the frequency on coming stream flow is above a critical frequency. Thomas et al. (1966) examined the flow due to the unsteady motion of a sphere with convective terms present in a elastic viscous liquid using Laplace Transform technique. Later the paper presented by Frater (1967) has got well recognition in which he has discussed the problems of oscillating sphere in an elastic viscous fluid and discussed the effect of relaxation time parameter on the Drag. Latter many authors have studied the phenomena of oscillations of external flow over a non- zero mean velocity. Lai et al.

(1978) have considered the flow due to oscillating sphere in an elastic-viscous fluid by neglecting the nonlinear terms. In the same paper they considered the flow due to a sphere accelerating with a periodic and arbitrary motion in the visco-elastic fluid using Fourier Transform technique and obtained expressions for Drag experienced by the sphere. Variable Viscosity and Inclined Magnetic Field on the Peristaltic Motion of a Non-Newtonian fluid in an inclined asymmetric channel was studied by Afsar Khan et al. (2016). Flow generated by slow steady rotation of a permeable sphere in a Micro-polar fluid was analyzed by Aparna et al. (2012).

1.1 Literature Survey

In non-Newtonian fluids, several Stokes flow problems concerning Micro-polar fluids have been studied by researchers over the past five decades ever since Eringen (1966) introduced the Micro-polar fluid theory. Lakshmana Rao et al. (1971, 1972, 1983, 1987) studied the oscillatory flows of sphere, circular cylinder, spheroid and elliptic cylinder in incompressible Micro-polar fluids, the main thrust of the investigation being the determination of the Drag or Couple as the case may be on the oscillating body.

Oscillatory flow problems were first analysed by the analytical solution to find the effect of elastic parameters on the Drag by Frater (1968) when the oscillations of circular cylinder and sphere were examined in a visco-elastic fluid. In non-Newtonian fluids, Micro-polar fluids and Couples-stress fluids which support body Couples and exhibit Couple stresses are of a special type. As in the case of other non-Newtonian fluids, the properties of Micro-polar and Couple-stress fluids can be determined by generating the flows due to oscillations. Similarly, Stokesian flows in the case of Couple-stress fluids were studied by Stokes (1966, 1968, 1971) and Jain et al. (1972). Ariman (1967) and Liu (1971) studied Micro-polar fluid flows in annular region and their instabilities respectively. Latter very good treatises on fluids exhibiting Couple stresses were written by Stokes (1984) and Lukaszewicz (1999). In the pioneering works of Lakshmana Rao et al. (1980), flows generated due to oscillations of circular cylinder, sphere, spheroid and elliptic cylinder in Micro-polar fluids were analyzed.

The flow problems in Couple-stress fluids have been attracting many researchers due their Mathematical simplicity and elegance and importance in many applications. Ramkissoon et al. (1990, 1991) and Rajagopal (1983) considered a flow generated due to longitudinal and torsional oscillations of a uniform cylindrical rod in polar fluids and non-Newtonian fluids. In these papers, the authors derived a formula for Drag on the object using a limit on stream function. The flows due to longitudinal and torsional oscillations of a cylinder in various fluids were investigated by different authors namely Bandelli et al. (1994), Pontrelli (1997), Calmelet-Eluhu et al. (1998). Ramana Murthy et al. (2009, 2010) studied the flow of Micro-polar fluid under transverse magnetic field with suction.

However in all these problems, as far as the authors know, special cases, which are branded as oscillatory flows of "*Resonance*" type that arise when the material parameters of the fluids are constrained in a particular form (to be stated later) have not been investigated until recently. The rare but distinct possibility of resonance flows has been noticed in the works of Lakshmana Rao et al. (1983, 1987), Ramana Murthy et al. (2011), Aparna et al. (2012), Nagaraju et al. (2014) and the investigation in this case is mathematically more complicated than in the usual non-resonance type flows. But in these papers, the case of resonance was not studied by the authors. This type of flows arise whenever oscillations of a body take place in any non-Newtonian fluids.

Many natural fluids like blood, oils and paints are non-Newtonian fluids. The fluids which exhibit Couple stresses and body Couples are called polar fluids. In these type of fluids, the well-known fluids are 1) Micro-polar fluids 2) Couple-stress fluids and 3) Polar fluids. These fluids exhibit length elongation property. In this thesis, we consider two fluids viz, Micro-polar fluids and Couple-stress fluids.

1.2 Micro-polar Fluid Theory

The Micro-polar fluids introduced by Eringen (1966). It is well known fact that in many of the real fluids, the shear behavior cannot be characterized by Newtonian relationships and hence researchers have proposed diverse non-Newtonian fluid theories to explain the deviation in the behavior of real fluids with that of Newtonian fluids. One such theory is that of Micro-polar fluids. These fluids are isotropic polar fluids in which deformation of molecules is neglected and these contribute a subclass of the simple Micro-polar fluids. Physically, a Micro-polar fluid model can represent fluids whose molecules can rotate independently of the fluid stream function and its local vorticity.

Micro-polar fluid contributes a medium whose behavior during its flow is affected by micro-rotation vector at any point, which represents the local rotational motion of the fluid molecules contained in a given fluid volume element. The fluid medium sustains Couple stress and micro-rotation. For Micro-polar fluids, stress tensor is not symmetric. This fluid model constitutes a substantial generalization of the Navier-Stokes model and can be used to analyze the behavior of lubricants, liquid crystals and animal blood.

The field equations of the Micro-polar fluids are representable in terms of the velocity vector $\bar{Q}$ and the micro-ratation vector $\bar{l}$ associated with each particle in the fluid medium. The micro-ratation vector $\bar{l}$ represents the rotation in an average sense of the rigid particles centered in a small volume element about the centroid of the element.

The field equations for velocity and micro-rotation of an incompressible Micro-polar fluid as derived by Eringen (1966) are given by:

$$\frac{\partial\rho}{\partial\tau} + div(\rho\bar{Q}) = 0 \quad (1.1)$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \quad (1.2)$$

$$\rho J(\frac{\partial\bar{l}}{\partial\tau} + \bar{Q}.\nabla_1\bar{l}) = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1.\bar{l}) \quad (1.3)$$

where τ is time, ρ is density of the fluid, μ is coefficient of viscosity, k is coefficient microviscosity, J is micro-gyration coefficient and α,β, γ are coefficients of Couple stress viscosities. These confirm to the inequality

$$k \geq 0, 2\mu + k \geq 0, 3\lambda + 2\mu + k \geq 0, \gamma \geq 0, |\beta| \leq \gamma, 3\alpha + \beta + \gamma \geq 0 \quad (1.4)$$

The constitutive equations for the stress components T_{ij} and Couple stress components M_{ij} for Micro-polar fluids are given by

$$T_{ij} = -P\delta_{ij} + \frac{1}{2}(2\mu + k)(u_{i,j} + u_{j,i}) + ke_{ijr}(w_r - l_r) \quad (1.5)$$

$$M_{ij} = \alpha l_{i,i}\delta_{i,j} + \beta l_{i,j} + \gamma l_{j,i} \quad (1.6)$$

where the permutation tensor $e_{ijk} = \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ 1 & \text{if } i,j,k \text{ are cyclic} \\ -1 & \text{if } i,j,k \text{ are anti} - \text{cyclic} \end{cases}$ (1.7)

and w_r = r^{th} component of $\frac{1}{2}(\,curl\,\boldsymbol{Q}\,)$.

1.3 Couple-stress Fluid Theory

Another theory which appeared almost simultaneously in 1966 along with Micro-polar fluid theory to explain the deviation in the behavior of real fluids with that of Newtonian fluids in the theory of Couple-stress fluids. This theory initiated by Stokes (1966), is a simple generalization of the classical theory of viscous fluids. This theory allows for the presence of Couple stresses and body coupes in the fluid medium. The concept of Couple stresses arises due to the way in which the mechanical interactions in the fluid medium are modeled. In this theory, the rotational field is defined in terms of the velocity field itself and the rotation vector equals to half of the curl of the velocity vector. Here again, stress vector is not symmetric. This theory also has several industrial and scientific applications which comprise pumping fluids such as synthetic fluids, liquid crystals, animal blood etc.

Couple-stress fluids introduced by Stokes (1966), are fluids consisting of rigid randomly oriented particles suspended in a viscous medium. The characterizing features that distinguish the Couple-stress fluid theory from the Newtonian fluid theory are the presence of the Couple-stresses and body Couples in the fluid medium and the non symmetry of the stress tensor. In Micro-polar fluid theory, the micro structure of the fluid is taken into account, and this accounts for the polar effects that arise in the fluid. In Couple-stress fluid theory, the micro structure is not taken into account. The polar effects are a consequence of assuming that the mechanical interaction of one part of a body on another across a surface is equivalent to a force together with a moment distribution. Here the rotation is associated with each particle is the vorticity vector equals to half of the curl of the velocity vector at any point in the fluid medium.

The Couple-stress fluid theory constitutes the simplest generalization of the classical Newtonian viscous fluid theory that shows all the important features and effects of the Couple stresses and results in equations that are similar to the Navier-Stokes equations.

The basic equations of an incompressible Couple stress fluid introduced by Stokes (1966) are given by:

$$div\bar{Q} = 0 \tag{1.8}$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{1.9}$$

where $\bar{Q}$ is fluid velocity vector, ρ is density, τ is time, μ is viscosity coefficient.

For Couple stress fluids, the stress components T_{ij} and Couple stress tensor M satisfy the following constitutive equations.

$$T = -PI + \lambda(\nabla_1 . Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2} I \times (\nabla_1 . M) \tag{1.10}$$

$$M = mI + 2\eta\nabla_1(\nabla_1 \times Q) + 2\eta'[\nabla_1(\nabla_1 \times Q)]^T \tag{1.11}$$

The problems related to cylinder are to be solved in cylindrical polar coordinate system and problems related to sphere are solved in spherical coordinate system. Hence the expressions for strain rate tensors and stress tensors are given below. Cowin et al. (1970) proposed boundary conditions suitable to polar fluids.

1.4 Cylindrical Co-ordinate System

A cylindrical coordinate system is a three-dimensional coordinate system. Which specifies a point position by the distance from a chosen reference axis, the direction from the axis relative to a chosen reference direction, and the distance from a chosen reference plane perpendicular to the axis. Generally, cylindrical co-ordinate system is taken as (R, θ, Z).

1.4.1 Strain Rate Tensor in Cylindrical Co-ordinates

$$\text{Strain rate tensor} = E = [e_{ij}] = \frac{1}{2}\,[\nabla\bar{Q} + \nabla\bar{Q}^T] \tag{1.12}$$

$$\nabla\bar{Q} = \left(\bar{e}_r \frac{\partial}{\partial R} + \frac{\bar{e}_\theta}{R}\frac{\partial}{\partial \theta} + \bar{e}_z \frac{\partial}{\partial Z}\right)(U\bar{e_r} + V\overline{e_\theta} + W\bar{e_z})$$

$$\nabla\bar{Q} = \bar{e_r}\bar{e_r}\frac{\partial U}{\partial R} + \bar{e_r}U\frac{\partial \bar{e_r}}{\partial R} + \bar{e_r}\overline{e_\theta}\frac{\partial V}{\partial R} + V\bar{e_r}\frac{\partial \overline{e_\theta}}{\partial R} + \bar{e_r}\overline{e_\emptyset}\frac{\partial W}{\partial R} + W\bar{e_r}\frac{\partial \bar{e_z}}{\partial R}$$

$$+\frac{\overline{e_\theta}\bar{e_r}}{R}\frac{\partial U}{\partial \theta} + \frac{U\overline{e_\theta}}{R}\frac{\partial \bar{e_r}}{\partial \theta} + \frac{\overline{e_\theta}\,\overline{e_\theta}}{R}\frac{\partial V}{\partial \theta} + \frac{V\overline{e_\theta}}{R}\frac{\partial \overline{e_\theta}}{\partial \theta} + \frac{\overline{e_\theta}\bar{e_z}}{R}\frac{\partial W}{\partial \theta} + \frac{\overline{e_\theta}W}{R}\frac{\partial \bar{e_z}}{\partial \theta}$$

$$+\bar{e_z}\bar{e_r}\frac{\partial U}{\partial Z} + U\bar{e_z}\frac{\partial \bar{e_r}}{\partial Z} + \bar{e_z}\overline{e_\theta}\frac{\partial V}{\partial Z} + V\bar{e_z}\frac{\partial \overline{e_\theta}}{\partial Z} + \bar{e_z}\bar{e_z}\frac{\partial W}{\partial Z} + W\bar{e_z}\frac{\partial \bar{e_z}}{\partial Z} \tag{1.13}$$

Partial derivatives of basic unit vectors are given by

$$\left.\begin{array}{lll}\frac{\partial \overline{e_r}}{\partial R}=0, & \frac{\partial \overline{e_r}}{\partial \theta}=\overline{e_\theta}, & \frac{\partial \overline{e_r}}{\partial Z}=0 \\ \frac{\partial \overline{e_\theta}}{\partial R}=0, & \frac{\partial \overline{e_\theta}}{\partial \theta}=-\overline{e_r}, & \frac{\partial \overline{e_\theta}}{\partial Z}=0 \\ \frac{\partial \overline{e_z}}{\partial R}=0, & \frac{\partial \overline{e_z}}{\partial \theta}=0, & \frac{\partial \overline{e_z}}{\partial Z}=0\end{array}\right\} \tag{1.14}$$

Substituting (1.14) in (1.13), we get

$$\nabla\bar{Q}=\overline{e_r}\,\overline{e_r}\frac{\partial U}{\partial R}+\overline{e_r}\,\overline{e_\theta}\frac{\partial V}{\partial R}+\overline{e_r}\,\overline{e_z}\frac{\partial W}{\partial R}+\frac{\overline{e_\theta}\,\overline{e_r}}{R}\frac{\partial U}{\partial \theta}+\overline{e_\theta}\,\overline{e_\theta}\left[\frac{U}{R}+\frac{1}{R}\frac{\partial V}{\partial R}\right]-\overline{e_\theta}\,\overline{e_r}\frac{V}{R}+\frac{\overline{e_\theta}\,\overline{e_z}}{R}\frac{\partial W}{\partial \theta}+\overline{e_z}\,\overline{e_r}\frac{\partial U}{\partial Z}+\overline{e_z}\,\overline{e_\theta}\frac{\partial V}{\partial Z}+\overline{e_z}\,\overline{e_z}\frac{\partial W}{\partial Z}$$

The same in matrix form can be written as

$$\nabla\bar{Q}=\begin{bmatrix}\frac{\partial U}{\partial R} & \frac{\partial V}{\partial R} & \frac{\partial W}{\partial R}\\ \frac{1}{R}\left[\frac{\partial U}{\partial \theta}-V\right] & \frac{1}{R}\left[\frac{\partial V}{\partial \theta}+U\right] & \frac{1}{R}\frac{\partial W}{\partial \theta}\\ \frac{\partial U}{\partial Z} & \frac{\partial V}{\partial Z} & \frac{\partial W}{\partial Z}\end{bmatrix} \tag{1.15}$$

The strain rate tensor $E=\frac{1}{2}\ [\nabla\bar{Q}+\nabla\bar{Q}^T]$

$$E=\frac{1}{2}\begin{bmatrix}2\frac{\partial U}{\partial R} & \left[\frac{\partial V}{\partial R}+\frac{1}{R}\frac{\partial U}{\partial \theta}-\frac{V}{R}\right] & \left[\frac{\partial W}{\partial R}+\frac{\partial U}{\partial Z}\right]\\ \left[\frac{\partial V}{\partial R}+\frac{1}{R}\frac{\partial U}{\partial \theta}-\frac{V}{R}\right] & \frac{2}{R}\left[\frac{\partial V}{\partial \theta}+U\right] & \left[\frac{1}{R}\frac{\partial W}{\partial \theta}+\frac{\partial V}{\partial Z}\right]\\ \left[\frac{\partial W}{\partial R}+\frac{\partial U}{\partial Z}\right] & \left[\frac{1}{R}\frac{\partial W}{\partial \theta}+\frac{\partial V}{\partial Z}\right] & 2\frac{\partial W}{\partial Z}\end{bmatrix} \tag{1.16}$$

$$\nabla.M=\left[\frac{1}{R}\frac{\partial}{\partial R}(RM_{RR})+\frac{1}{R}\frac{\partial}{\partial \theta}M_{\theta R}+\frac{\partial}{\partial Z}M_{ZR}-\frac{M_{R\theta}}{R}\right]\bar{e}_R+\left[\frac{1}{R^2}\frac{\partial}{\partial R}(R^2\,M_{R\theta})+\frac{1}{R}\frac{\partial}{\partial \theta}M_{\theta\theta}+\frac{\partial}{\partial Z}M_{Z\theta}+\frac{M_{\theta Z}-M_{R\theta}}{R}\right]\bar{e}_\theta+\left[\frac{1}{R}\frac{\partial}{\partial R}(RM_{RZ})+\frac{1}{R}\frac{\partial}{\partial \theta}M_{\theta Z}+\frac{\partial}{\partial Z}M_{ZZ}\right]\bar{e}_Z \tag{1.17}$$

1.5 Spherical Co-ordinate System

A spherical coordinate system is a three-dimensional coordinate system where the position of a point is specified by three numbers: the *radial distance* of that point from a fixed origin, its *polar angle* measured from a fixed zenith direction, and the *azimuth angle* of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. Generally cylindrical co-ordinate system taken as (R,θ,ϕ).

1.5.1 Strain Rate Tensor in Spherical Co-ordinates

Strain rate tensor $= E = \left[e_{ij}\right] = \frac{1}{2}\ [\nabla\bar{Q} + \nabla\bar{Q}^T]$ (1.18)

$$\nabla\bar{Q} = \left(\bar{e_r}\frac{\partial}{\partial R} + \frac{\bar{e_\theta}}{R}\frac{\partial}{\partial\theta} + \frac{\bar{e_\emptyset}}{RSin\theta}\frac{\partial}{\partial\emptyset}\right)(U\bar{e_r} + V\overline{e_\theta} + W\overline{e_\emptyset})$$

$$= \bar{e_r}\bar{e_r}\frac{\partial U}{\partial R} + \bar{e_r}U\frac{\partial\bar{e_r}}{\partial R} + \bar{e_r}\overline{e_\theta}\frac{\partial V}{\partial R} + V\bar{e_r}\frac{\partial\overline{e_\theta}}{\partial R} + \bar{e_r}\overline{e_\emptyset}\frac{\partial W}{\partial R} + W\bar{e_r}\frac{\partial\overline{e_\emptyset}}{\partial R}$$

$$+ \frac{\overline{e_\theta}\bar{e_r}}{R}\frac{\partial U}{\partial\theta} + \frac{U\overline{e_\theta}}{R}\frac{\partial\bar{e_r}}{\partial\theta} + \frac{\overline{e_\theta}\overline{e_\theta}}{R}\frac{\partial V}{\partial\theta} + \frac{V\overline{e_\theta}}{R}\frac{\partial\overline{e_\theta}}{\partial\theta} + \frac{\overline{e_\theta}\overline{e_\emptyset}}{R}\frac{\partial W}{\partial\theta} + \frac{\overline{e_\theta}W}{R}\frac{\partial\overline{e_\emptyset}}{\partial\theta}$$

$$+ \frac{\overline{e_\emptyset}\bar{e_r}}{RSin\theta}\frac{\partial U}{\partial\emptyset} + \frac{U\overline{e_\emptyset}}{RSin\theta}\frac{\partial\bar{e_r}}{\partial\emptyset} + \frac{\overline{e_\emptyset}\overline{e_\theta}}{RSin\theta}\frac{\partial V}{\partial\emptyset} + \frac{\overline{Ve_\emptyset}}{RSin\theta}\frac{\partial\overline{e_\theta}}{\partial\emptyset} + \frac{\overline{e_\emptyset}\overline{e_\emptyset}}{RSin\theta}\frac{\partial W}{\partial\emptyset} + \frac{W\overline{e_\emptyset}}{RSin\theta}\frac{\partial\overline{e_\emptyset}}{\partial\emptyset}$$

Partial derivatives of basic unit vectors are given by

$\frac{\partial\bar{e_r}}{\partial R} = 0$ $\quad\frac{\partial\bar{e_r}}{\partial\theta} = \overline{e_\theta}$ $\quad\frac{\partial\bar{e_r}}{\partial\emptyset} = \overline{e_\emptyset}sin\theta$

$\frac{\partial\overline{e_\theta}}{\partial R} = 0$ $\quad\frac{\partial\overline{e_\theta}}{\partial\theta} = -\bar{e_r}$ $\quad\frac{\partial\overline{e_\theta}}{\partial\emptyset} = \overline{e_\emptyset}cos\theta$

$\frac{\partial\overline{e_\emptyset}}{\partial R} = 0$ $\quad\frac{\partial\overline{e_\emptyset}}{\partial\theta} = 0$ $\quad\frac{\partial\overline{e_\emptyset}}{\partial\emptyset} = -\bar{e_r}\ sin\theta - \overline{e_\theta}cos\theta$

Substituting these in (1.18), we get

$$\nabla\bar{Q} = \bar{e_r}\bar{e_r}\frac{\partial U}{\partial R} + \bar{e_r}\overline{e_\theta}\frac{\partial V}{\partial R} + \bar{e_r}\overline{e_\emptyset}\frac{\partial W}{\partial R} + \frac{\overline{e_\theta}\bar{e_r}}{R}\frac{\partial U}{\partial\theta} + \frac{U\overline{e_\theta}\overline{e_\theta}}{R} + \frac{\overline{e_\theta}\overline{e_\theta}}{R}\frac{\partial V}{\partial\theta} - \frac{V\overline{e_\theta}}{R}\bar{e_r} +$$

$$+ \frac{\overline{e_\emptyset}\bar{e_r}}{RSin\theta}\frac{\partial U}{\partial\emptyset} + \frac{U\overline{e_\emptyset}\overline{e_\emptyset}}{R} + \frac{\overline{e_\emptyset}\overline{e_\theta}}{RSin\theta}\frac{\partial V}{\partial\emptyset} + \frac{Vcot\theta}{R}\overline{e_\emptyset e_\emptyset} + \frac{\overline{e_\emptyset}\overline{e_\emptyset}}{RSin\theta}\frac{\partial W}{\partial\emptyset} - \frac{W}{R}\overline{e_\emptyset}\bar{e_r}$$
$$- \frac{Wcot\theta}{R}\overline{e_\emptyset e_\theta} + \frac{\overline{e_\theta}\overline{e_\emptyset}}{R}\frac{\partial W}{\partial\theta}$$

In the matrix, it is given by

$$\nabla\bar{Q} = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{\partial V}{\partial R} & \frac{\partial W}{\partial R} \\ \frac{1}{R}\left[\frac{\partial U}{\partial\theta} - V\right] & \frac{1}{R}\left[\frac{\partial V}{\partial\theta} + U\right] & \frac{1}{R}\frac{\partial W}{\partial\theta} \\ \frac{1}{R}\left[\frac{1}{Sin\theta}\frac{\partial U}{\partial\emptyset} - W\right] & \frac{1}{R}\left[\frac{1}{Sin\theta}\frac{\partial V}{\partial\emptyset} - WCot\theta\right] & \frac{1}{R}\left[\frac{1}{Sin\theta}\frac{\partial W}{\partial\emptyset} + VCot\theta + U\right] \end{bmatrix} \quad (1.19)$$

$$E = \frac{1}{2}\ [\nabla\bar{Q} + \nabla\bar{Q}^T]$$

$$E = \frac{1}{2R}\begin{bmatrix} 2R\frac{\partial U}{\partial R} & \frac{\partial U}{\partial \theta} + R\frac{\partial V}{\partial R} - V & \frac{1}{Sin\theta}\frac{\partial U}{\partial \emptyset} + R\frac{\partial W}{\partial R} - W \\ \frac{\partial U}{\partial \theta} + R\frac{\partial V}{\partial R} - V & 2\left[\frac{\partial V}{\partial \theta} + U\right] & \frac{1}{Sin\theta}\frac{\partial V}{\partial \emptyset} + \frac{\partial W}{\partial \theta} - WCot\theta \\ \frac{1}{Sin\theta}\frac{\partial U}{\partial \emptyset} + R\frac{\partial W}{\partial R} - W & \frac{1}{Sin\theta}\frac{\partial V}{\partial \emptyset} + \frac{\partial W}{\partial \theta} - WCot\theta & 2\left[\frac{1}{Sin\theta}\frac{\partial W}{\partial \emptyset} + VCot\theta + U\right] \end{bmatrix} \quad (1.20)$$

$$\nabla . M = \left[\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 M_{RR}) + \frac{1}{Rsin\theta}\frac{\partial}{\partial \theta}(M_{\theta R} sin\theta) + \frac{1}{Rsin\theta}\frac{\partial}{\partial \emptyset}M_{\emptyset R} - \frac{M_{\theta\theta} + M_{\emptyset\emptyset}}{R}\right]\bar{e}_R +$$
$$\left[\frac{1}{R^3}\frac{\partial}{\partial R}(R^3 M_{R\theta}) + \frac{1}{Rsin\theta}\frac{\partial}{\partial \theta}(M_{\theta\theta} sin\theta) + \frac{1}{Rsin\theta}\frac{\partial}{\partial \emptyset}M_{\emptyset\emptyset} + \frac{M_{\theta R} - M_{R\theta} - M_{\emptyset\emptyset}cot\theta}{R}\right]\bar{e}_\theta +$$
$$\left[\frac{1}{R^3}\frac{\partial}{\partial R}(R^3 M_{R\emptyset}) + \frac{1}{Rsin\theta}\frac{\partial}{\partial \theta}(M_{\theta\emptyset} sin\theta) + \frac{1}{Rsin\theta}\frac{\partial}{\partial \emptyset}M_{\emptyset\emptyset} + \frac{M_{\emptyset R} - M_{R\emptyset} + M_{\emptyset\theta}cot\theta}{R}\right]\bar{e}_\emptyset \quad (1.21)$$

1.6 Modified Bessel equation and functions

The recurrence relations and other equations for Bessel functions are taken from Andrei D. Polyanin and Valentin F Zaitsev (2003) and Sneddon (1956).
The Bessel differential equation is defined as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0 \quad (1.22)$$

The solution of this equation is $y = c_1 J_n(kx) + c_2 Y_n(kx)$

The differential equation for modified Bessel functions is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(k^2 + \frac{n^2}{x^2}\right)y = 0 \text{ or } x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (k^2x^2 + n^2)y = 0 \quad (1.23)$$

The solution of the above equation is $y = c_1 I_n(kx) + c_2 K_n(kx)$

$K_n(x)$ satisfies the following recurrence relations

$$-2K'_n(x) = K_{n-1}(x) + K_{n+1}(x) \quad (1.24a)$$

$$\frac{2n}{x}K_n(x) = K_{n+1}(x) - K_{n-1}(x) \quad (1.24b)$$

$$xK'_n(x) = -xK_{n-1}(x) - nK_n(x) \quad (1.24c)$$

$$xK'_n(x) = nK_n(x) - xK_{n+1}(x) \quad (1.24d)$$

$$K'_0(x) = -K_1(x) \quad (1.24e)$$

The differential equation

$$\frac{d^2y}{dx^2}+\frac{1-2\alpha}{x}\frac{dy}{dx}-\left(1+\frac{n^2-\alpha^2}{x^2}\right)y=0$$

has a solution as $y=x^{\alpha}K_n(x)$

If $\alpha=\frac{1}{2}\ and\ n=\frac{3}{2}$ the above differential equation reduces to

$$x^2\frac{d^2y}{dx^2}-(x^2+2)y=0 \qquad (1.25)$$

This equation has a solution as $y=x^{\frac{1}{2}}K_{\frac{3}{2}}(x)$ (1.26)

If $\alpha=0\ and\ n=1$ the differential equation and the corresponding solution are given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+1)y=0 \qquad (1.27)$$

And $y=K_1(x)$ (1.28)

The following formulae on Bessel functions are very much useful for simplications

$$\frac{d}{dr}\left[\sqrt{r}K_{\frac{3}{2}}(\lambda r)\right]=-\frac{K_{\frac{3}{2}}(\lambda r)}{\sqrt{r}}\Delta_1(\lambda r) \qquad (1.29)$$

$$\frac{d}{dr}\left[\frac{K_{\frac{3}{2}}(\lambda r)}{\sqrt{r}}\right]=-\frac{K_{\frac{3}{2}}(\lambda r)}{r^{\frac{3}{2}}}\left(1+\Delta_1(\lambda r)\right) \qquad (1.30)$$

Where $\Delta_1(x)=1+\frac{x\mathrm{K}_{\frac{1}{2}}(x)}{K_{\frac{3}{2}}(x)}=\frac{1+x+x^2}{1+x}$ (1.31)

$$\frac{d}{dr}\left[r^{\frac{3}{2}}K'_{\frac{3}{2}}(\lambda r)\right]=\frac{1}{\lambda\sqrt{r}}\left[\left(\lambda^2r^2+\frac{3}{2}\right)K_{\frac{3}{2}}(\lambda r)-\frac{\lambda r}{2}K_{\frac{1}{2}}(\lambda r)\right] \qquad (1.32)$$

1.7 Problems on Oscillations

The work on the oscillating flows in different geometries is listed below.

1.7.1 Oscillating flows in infinite plate geometry

Many authors studied several problems related to infinite plate geometry. Liu (1966, 1967) studied flows in dusty gas generated by oscillation of an infinite flat plate and impulsive motion of an infinite flat plate in a dusty gas. Baral (1967) studied parallel plate problem of unsteady flow of conducting liquid between two parallel plates. Soundalgekar et al. (1974) studied oscillatory flow past an infinite plate with constant suction and investigated effects of Couple stresses on the flow. Jyotirmoy Sinha Roy et al. (1981) investigated visco-elastic flow between two infinite parallel porous plates where one plate oscillating and the other one is in uniform motion.

Unsteady flow between two oscillating plates was studied by Evelyn et al. (1982). Ramamurthy et al. (1987) studied the steady streaming generated between two infinite parallel plates where one is vibrating plate and another one is a fixed plate in a dusty fluid. Rashmi et al. (2007) also studied unsteady flow of a dusty fluid generated between two oscillating plates under varying constant pressure gradient. Yanqing Wang et al. (2017) deliberated on analytical study for vibration of longitudinally moving plate submerged in infinite liquid domain.

1.7.2 Oscillating flows through tubes

There is a vast literature for problems through tubes. Vijay Kumar Stokes (1968) analysed effects of Couple stresses in fluids on hydromagnetic channel flows. Owen et al. (2006) studied steady flow of Micro-polar fluid through a circular pipe, in this he considered a transverse magnetic field with constant suction / injection. Owen et al. (2006) studied an Oldroyd-B liquid flow generated due to performing longitudinal and torsional oscillations of a straight circular tube with different frequencies. Ramana Murthy et al. (2009, 2010) studied Steady and unsteady flow of Micro-polar fluid through a circular pipe under a transverse magnetic field with constant suction / injection. Ramana Murthy et al. (2011) studied steady flow of Micro-polar fluid in a rectangular channel, in this transverse magnetic field with suction considered.

1.7.3 Oscillating flows in spherical geometry

Stimson et al. (1926) studied viscous fluid flow due to the motion of two spheres. Frater (1967, 1968) studied oscillatory flows in an elastico-viscous fluid, and evaluated Drag on sphere, damping force on a body. Verma et al. (1971) studied oscillating flow past a fixed porous sphere. Stokes (1971) analysed effects of Couple stresses in fluids on the creeping flow past a sphere. Lai et al. (1978) studied elastic-viscous fluid flow generated due to rectilinear oscillations of a sphere and evaluated Drag on a sphere. Lakshmana Rao et al. (1970, 1971, 1981, 1983) studied slow stationary flow past a sphere and the oscillatory flows of sphere and spheroid in incompressible Micro-polar fluids, the main thrust of the investigation being the determination of the Drag or Couple as the case may be on the oscillating body. Iyengar et al. (2001) studied rectilinear oscillations, rotary oscillations of approximate sphere in an incompressible viscous fluid and Micro-polar fluid respectively. Iyengar et al. (1993, 2004) studied Stokes flow of an incompressible Micro-polar fluid past an approximate sphere and oscillatory flow of a Micro-polar fluid generated due to rotary oscillations of two concentric spheres. Aparna et al. (2012) studied incompressible Micro-polar fluid flow of permeable sphere performing rotary oscillations. Recently oscillatory flows of composite sphere and spherical particle were studied by Ashmawy (2015, 2016).

1.7.4 Oscillating flows in cylindrical geometry

Flow of Micro-polar Fluid between two concentric Cylinders was studied by Ariman et al. (1967). Frater (1968) studied oscillatory flow circular cylinder in an elastico-viscous fluid and evaluated Drag on a circular cylinder. Lakshmana Rao et al. (1972, 1987) studied the oscillatory flows of circular cylinder and elliptic cylinder in an incompressible Micro-polar fluid. Ramkissoon et al. (1990) studied oscillatory flow due to cylinder performing longitudinal and torsional oscillations. Rao et al. (1992) computationally studied unsteady viscous fluid flow of circular cylinder oscillating transversely and longitudinally in a uniform flow at high Reynolds number. Calmelet-Eluhu et al. (1998), Fetecau et al. (2006), Nagaraju et al. (2014) studied oscillatory flows of circular cylinder subject to longitudinal and torsional Oscillations. Anwar et al. (2004), Mehrdad Massoudi et al. (2008), Ramana Murthy et al. (2010) studied oscillatory flows mainly due to longitudinal and torsional Oscillations of circular cylinder numerically.

1.8 Drag/Couple/Skin friction

Drag is a force acting on the entire body in the direction of fluid flow. In fluid dynamics, Drag (may be due to air resistance or fluid resistance) refers to forces which act on a solid object in the direction of the fluid velocity. Unlike other resistive forces, such as dry friction, which is nearly independent of velocity, drag forces depend on velocity. Drag forces always decrease fluid velocity relative to free velocity when there is no body. If $\boldsymbol{k}$ is the direction of flow, then the expression for Drag is

$$D = \int \boldsymbol{k}.\boldsymbol{T_n} ds = \int \boldsymbol{k}.\boldsymbol{T}.\boldsymbol{n} ds \qquad (1.33)$$

Skin friction is a force per unit area. It is friction between a moving fluid and surface of the body. The Drag is a force acting on the entire body where as Skin friction is force per unit area. When the body rotates or perform rotary oscillations, moment of Couple is considered. This moment of force in the direction of axis symmetry is taken as Couple. If ***k*** is axis of symmetry and ***f*** is force then Couple is taken as

$$C=\int \boldsymbol{k.(r{\times}f)}\, ds =\int \boldsymbol{k.(r{\times}T_n)}\, ds= \int \boldsymbol{k.(r{\times}T.\, n)}\, ds \tag{1.34}$$

1.9 Operators

In this work, we come across some operators like E^2 and ∇^2 as follows. When a circular cylinder performs rectilinear oscillations in incompressible Micro-polar fluid or Couple-stress fluid, we get the following operators

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \text{ and } D_c^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2} \tag{1.35}$$

When a sphere performs rectilinear oscillations in incompressible Micro-polar fluid or Couple-stress fluid, we get the following operators

$$E^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{cot\theta}{r^2}\frac{\partial}{\partial\theta} \text{ and } D_s^2=\frac{d^2}{dr^2}-\frac{2}{r^2} \tag{1.36}$$

Whenever a flow is generated in a Micro-polar fluid or Couple-stress fluid due to rectilinear oscillations, the stream function ψ of the flow satisfies the following equation

$$E^2(E^2-\lambda_1^2)\,(E^2-\lambda_2^2)\,\psi=0 \tag{1.37}$$

By taking ψ = f(r) sinθ for cylinder and ψ = f(r) $\sin^2\theta$ for sphere, the above expression reduces to

$$D_c^2(D_c^2-\lambda_1^2)\,(D_c^2-\lambda_2^2)\,f=0 \tag{1.38a}$$

Or

$$D_s^2(D_s^2-\lambda_1^2)\,(D_s^2-\lambda_2^2)\,f=0 \tag{1.38b}$$

as the case may be, where E^2 is stoke stream function operator and λ_1, λ_2 contain material constants of the fluid. There arises a case $\lambda_1 = \lambda_2 = \lambda$ and stream function in this case follows the equation

$$E^2\left(E^2-\lambda^2\right)^2\psi=0 \tag{1.39}$$

This reduces to

$$D_c^2\left(D_c^2-\lambda^2\right)^2 f=0 \text{ or to } D_s^2\left(D_s^2-\lambda^2\right)^2 f=0 \tag{1.40}$$

The solution for in this case cannot be obtained by taking the limit $\lambda_1\rightarrow\lambda_2=\lambda$. Hence special attention is to be taken in this case. This case is referred to as " *Resonance*".

Similarly, whenever a flow is generated in a Micro-polar fluid or Couple-stress fluid due to rotary oscillations, the toroidal velocity V of the flow satisfies the following equation

$$(D_c^2 - \lambda_1^2)\,(D_c^2 - \lambda_2^2)\,V = 0 \qquad (1.41)$$

where D_c^2 is stokes operator and λ_1, λ_2 contain material constants of the fluid. There arises a case $\lambda_1 = \lambda_2 = \lambda$ and velocity V, in this case, follows the equation

$$\left(D_c^2 - \lambda^2\right)^2 V = 0 \qquad (1.42)$$

The solution for in this case cannot be obtained by taking the limit $\lambda_1 \rightarrow \lambda_2 = \lambda$. Hence special attention is to be taken in this case. This case is referred to as " *Resonance*".

1.10 Determination of Parameters

1.10.1 Micro-polar fluids

The material parameters in the Micro-polar fluids are related by the relation:

$$\lambda_1^2 + \lambda_2^2 = (2-c)s + i(J + \varpi R_0) = B_0 \text{ and } \lambda_1^2\lambda_2^2 = i\varpi R_0(2s + iJ) = C_0 \qquad (1.43)$$

When resonance occurs the parametres c, s, $R_0\varpi$ and J are related in the following way

$$(2-c)s = J - \varpi R_0 \text{ and } (2-c)J = \varpi R_0(2+c) \qquad (1.44)$$

Hence for resonance by fixing two parameters, the other two can be determined by the above equations. Then solving quadratic equation

$$x^2 - B_0 x + C_0 = 0$$

for x we get the value of λ^2

When there is non-resonance, all the parameters c, s, $R_0\varpi$ and J can be chosen independently. Then solving above equation for x, we get λ_1^2, λ_2^2.

If we fix $|\lambda^2|$, then in the case of resonance we have

$$\lambda^2 = \frac{(2-c)}{2c} s(c + 2i)$$

Hence $s = \sqrt{\frac{4c^2|\lambda^2|}{(2-c)^2(4+c^2)}}$

If we fix, c and s, we get J and $R_0\varpi$ from the relations

$$2c\,J = s(4 - c^2) \;\; and \;\; cR_0\varpi = s(2-c)^2$$

If c and J are fixed , then $R_0\varpi = \frac{J(2-c)}{2+c}$ *and* $s = \frac{(J - R_0\varpi)}{(2-c)}$ (1.45)

When there is non-resonance, if $|\lambda^2|$ is fixed, s, c, J, $R_0\varpi$ are related by the following Way (let $a = 2 - c, b = J + R_0\varpi,\ \ \alpha = a^2s^2 - b^2 + 4JR_0\varpi,$
$\beta = (2ab - 8R_0\varpi)s$ and $p = \alpha^2 + \beta^2$)

$$2\lambda^2 = as + ib + \sqrt{(as+ib)^2 + 4(J - i2s)R0\varpi}$$

⇨ $2\lambda^2 = as + \sqrt{\frac{\sqrt{p}+\alpha}{2}} + i(b + \sqrt{\frac{\sqrt{p}-\alpha}{2}})$

Taking modulus we get,

$$4|\lambda^4| - a^2s^2 - b^2 = \sqrt{p} + \sqrt{2}(as\sqrt{\sqrt{p}+\alpha} + b\sqrt{\sqrt{p}-\alpha})$$

Re-arranging and simplifying we get;(let $R_1 = R_0\varpi$)

$$4R_1^2(4R_1^2 - a^2|\lambda^4|)s^4$$
$$+\left\{-a^2|\lambda^{12}| + \left(8R_1^2 + \frac{a^2b^2}{2}\right)|\lambda^8| - R_1^2(a^2J^2 + 4b^2)|\lambda^4| + 8J^2R_1^4\right\}s^2$$
$$+|\lambda^{16}| - b^2|\lambda^{12}| + 2JR_1(J^2 + JR_1 + R_1^2)|\lambda^8| - b^2J^2R_1^2|\lambda^4| + J^4R_1^4 = 0$$

Solving this we get s.

1.10.2 Couple-stress fluids

The parameters are related by ($R_1 = R_0\varpi$, $q = S^2 + 16R_1^2$)

$$2\lambda^2 = s + \sqrt{s^2 - 4iR_1s}$$

⇨ $2\lambda^2 = S + \sqrt{\frac{S^2+\sqrt{S^2q}}{2}} + i\sqrt{\frac{-S^2+\sqrt{S^2q}}{2}}$

⇨ This can be rewritten as: $4|\lambda^4| = S^2 + \sqrt{S^2q} + \sqrt{2}S\sqrt{S^2 + \sqrt{S^2q}}$

Rearranging and simplifying we get: ($L = |\lambda^4|$)

$$R_1^2(R_1^2 - L)\,S^4 - L^2(L + 2R_1^2)S^2 + L^4 = 0 \tag{1.46}$$

For resonance we have

$$S = 4iR_1 = 2\lambda^2 \tag{1.47}$$

1.11 Planning of Thesis

In this thesis, we propose to investigate this case of resonance type flows in Micro-polar fluids generated due to rectilinear/rotary/longitudinal oscillations of circular cylinder/sphere. The velocity and micro-rotation (for Micro-polar fluid) and Drag / Couple / Skin friction acting on the body is obtained. The effect of physical parameters like Reynolds number, Micro-polarity and Couple stress parameter on the Drag / Couple / Skin friction due to oscillations is shown through graphs. The problems attempted in this thesis for Micro-polar and Couple-stress fluids are listed below:

i) Rectilinear Oscillations of a Circular Cylinder

ii) Rotary oscillations of a Circular Cylinder

iii) Longitudinal oscillations of a Circular Cylinder

iv) Rectilinear Oscillations of a Sphere

v) Rotary Oscillations of a Sphere

The thesis consists of twelve chapters and Four parts. Part - I and Chapter one is introductory in nature. Part – II is devoted to flows generated in Micro-polar fluids and contains Five chapters (Chapters two to six). Part – III is devoted to flows in the Couple stress fluids and contains Five chapters (Chapters seven to eleven). Part - IV and Chapter twelve gives concluding remarks of the thesis and possible directions in which further work can be carried out.

Chapter I : Introduction.

In this chapter, we introduce the two non-Newtonian fluid theories, Micro-polar theory introduced by Eringen and Couple-stress fluid theory introduced by Stokes and present a brief review of the problems related to the thesis available in the existing literature.

Chapter II : Rectilinear oscillations of a circular cylinder in a Micro-polar fluid.

The flow is assumed to be governed by Eringen's Micro-polar fluid flow equations. The flow of an incompressible Micro-polar fluid generated due to rectilinear oscillations of a circular cylinder about a diameter of the cylinder is considered. The flow is so slow that Oscillations Reynolds number is less than unity and hence nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered. The stream function and Drag acting on cylinder are obtained. The effect of physical parameters like Micro-polarity and Couple stress parameter on the Drag due to oscillations is shown through graphs.

Chapter III :Rotary oscillations of a circular cylinder in a Micro-polar fluid.

The flow generated due to rotary oscillations of a circular cylinder about its axis of symmetry in a Micro-polar fluid is considered. By taking Stokesian assumptions, nonlinear convective terms of motion are dropped and hence equations are made linear. The flow field for velocity and micro-rotation components are investigated. The Skin friction acting on the cylinder is evaluated and the effects of Micro-polarity and Couple stress parameter on the Skin friction are presented in form of graphs. It is observed that for a Micro-polar fluid when the material constants satisfies the resonance condition, the Skin friction reduces to a minimum.

Chapter IV :Longitudinal oscillations of a circular cylinder in a Micro-polar fluid.

The problem of the longitudinal oscillations of a circular cylinder along its axis of symmetry in an incompressible Micro-polar fluid and the flow generated due to these oscillations in the fluid is considered. The Stokes flow is considered by neglecting nonlinear convective terms in the equations of motion on the assumption that the flow is so slow that oscillations Reynolds number is less than unity. The solution of this case cannot be obtained as limiting case of non-resonance problem. The velocity and micro-rotation components of the flow for the case of *resonance* and *non-resonance* are obtained. The Skin friction acting on the cylinder is evaluated and the effect of physical parameters like Micro-polarity and Couple stress parameter on the Skin friction due to oscillations is shown through graphs.

Chapter V :Rectilinear oscillations of a sphere in a Micro-polar fluid.

The flow of an incompressible Micro-polar fluid generated due to rectilinear oscillations of a sphere about a diameter of the sphere is considered. The flow is so slow that Oscillations Reynolds number is less than unity and hence nonlinear convective terms in the equations of motion are neglected. The stream function and Drag acting on sphere are obtained for the case of *resonance* and *non-resonance*. The effect of physical parameters like Micro-polarity and Couple stress parameter on the Drag due to oscillations is shown through graphs.

Chapter VI : Rotary oscillations of a sphere in a Micro-polar fluid.

The flow of an incompressible Micro-polar fluid generated due to rotary oscillations of a sphere about the axis of symmetry of the sphere is considered. The flow is so slow that nonlinear convective terms in the equations of motion are neglected. The toroidal velocity and Couple acting on sphere are obtained for the case of *resonance* and *non-resonance*. The effect of physical parameters like Micro-polarity and Couple stress parameter on the Couple due to oscillations is shown through graphs.

Part III deals with Couple-stress fluid flows. It consists of five Chapters 7 to 11. The problems studied in this Part III are analogous to those studied in Part II with Couple-stress fluid replacing the Micro-polar fluid.

Chapter VII : Rectilinear oscillations of a circular cylinder in a Couple-stress fluid.

The flow due to a circular cylinder oscillating rectilinearly, about its axis of symmetry in a Couple-stress fluid is considered. In this case, the flow is analyzed under Stokesian approximation. The velocities in terms of stream function of the flow are obtained. The effect of physical parameters like Reynolds number and Couple stress parameter on the Drag is analyzed through graphs.

Chapter VIII :Rotary oscillations of a circular cylinder in a Couple-stress fluid.

The flow generated due to rotary oscillations of a circular cylinder about its axis of symmetry in an incompressible Couple-stress fluid is considered. The Oscillations Reynolds number is less than unity due to flow is very slow and hence nonlinear convective terms in the equations of motion are neglected. The velocity component for the flow derived. The Skin friction acting on the cylinder is evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Skin friction due to oscillations is shown through graphs.

Chapter IX : Longitudinal oscillations of a circular cylinder in a Couple-stress fluid. The flow generated due to circular cylinder performing longitudinal oscillations along its axis of symmetry in a Couple-stress fluid is considered. Nonlinear convective terms in the equations of motion are neglected since the Oscillations Reynolds number is less than unity due to very slow flow. The velocity components in terms of stream function for the flow are derived. The Skin friction and Drag acting on the cylinder are evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Skin friction and Drag is shown through graphs.

Chapter X : Rectilinear oscillations of Sphere in a Couple-stress fluid.

The present problem, the flow arising due to rectilinear oscillations of a sphere about its axis of symmetry in a Couple-stress fluid is considered. The flow is analyzed under Stokesian approximation by ignoring nonlinear convective terms on the assumption that the Oscillations Reynolds number is less than one. The velocity components in terms of stream function of the flow are derived. The Drag acting on the sphere evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Drag are shown through graphs.

Chapter XI : Rotary oscillations of a sphere in a Couple-stress fluid.

Incompressible Couple-stress fluid flow generated due to rotary oscillations of a sphere about the axis of symmetry of the sphere is considered. The flow is so slow that nonlinear convective terms in the equations of motion are neglected. The Couple on the sphere is evaluated. The effect of Couple stress parameter and geometric parameter on the Couple are presented through graphs.

In all these chapters, the expressions for the velocity, micro-rotation for micropolar fluids and velocity field for Couple-stress fluids are obtained. The Drag/Couple/Skin friction is derived analytically and the effect of physical parameters like Reynolds number and Couple stress parameter on the Drag/Couple /Skin friction is studied graphically. It is observed that the Drag or Couple on the body will be a minimum in the case of resonance.

Chapter XII : Conclusions.

Finally, chapter twelve concentrates on the overall conclusions drawn with references to the problems discussed in the thesis. We also indicate the direction for possible future work.

Part – II

Micro-polar Fluid Flows

Chapter 2

Rectilinear oscillations of a Circular Cylinder in a Micro-polar fluid

The flow of an incompressible Micro-polar fluid generated due to rectilinear oscillations of a circular cylinder about a diameter of the cylinder is considered. The flow is so slow that Reynolds number of the flow is less than unity and hence nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered. The stream function and Drag acting on cylinder are obtained. The effect of physical parameters like Micro-polarity and Couple stress parameter on the Drag due to oscillations is shown through graphs.

2.1 Introduction

Several Stokes flow problems concerning Micro-polar fluids have been studied by researchers over the past a half a century ever since Eringen (1966) introduced the Micro-polar fluid theory. Eringen (1964) analysed simple Micro-polar fluids. Ariman (1970) studied fluids with micro-structures. Liu (1971) Initiated instability in Micro-polar Fluids. Stokes (1984) studied theories of fluids with micro-structure Later on Lukaszewicz (1999) emphasized theory and applications of Micro-polar Fluids. Lakshmana Rao et al. (1972, 1981, 1987) examined the oscillatory flows generated due to circular cylinder, spheroid and elliptic cylinder in Micro-polar fluids to determine the Drag or Couple on the oscillating body. The main thrust of the investigation being the determination of the Drag or Couple as the case may be on the oscillating body.

Ravindran (1972) examined simple oscillatory flow in polar fluids. Oscillatory flows of circular cylinder in various fluids like Micro-polar fluids, Couple-stress fluids, viscous fluids were investigated by many authors Kanwal (1955), Ariman et al.

(1967), Frater (1968), Ramkissoon et al. (1990), Rao et al. (1992), Calmelet-Eluhu et al. (1998), Anwar et al. (2004), Fetecau et al. (2006), Mehrdad Massoudi et al. (2008), Ramana Murthy et al. (2010), Nagaraju et al. (2014) by computationally or analytically. Iyengar (2001) examined incompressible viscous fluid flow of approximate sphere is performing rectilinear oscillations. Lai (1978) investigated Drag on a sphere when the sphere rectilinearly oscillates in elastico-viscous fluid.

However, in all these problems, a special case, which is branded as oscillatory flows of "Resonance" type that arise when the material parameters of the fluids are constrained in a particular form(to be stated later) have not been investigated. The rare but distinct possibility of resonance flows has been noticed by Lakshmana Rao et al. (1983, 1987). And the investigation, in this case, is mathematically more complicated than in the usual non-resonance type flows.

In this chapter, we propose to investigate this case of resonance type flow, in Micro-polar fluids, due to rectilinear oscillations of a circular cylinder about its axis of symmetry. Later on we discussed the similar problem of the Resonance type flow due to a circular cylinder in Couple-stress fluid.

2.2 Basic Equations

The field equations for velocity and micro-rotation of an incompressible Micro-polar fluid as derived by Eringen (1966) are given by:

$$\frac{\partial \rho}{\partial \tau} + div(\rho \boldsymbol{Q}) = 0 \tag{2.1}$$

$$\rho\left(\frac{\partial \bar{Q}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{Q}\right) = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{2.2}$$

$$\rho \mathcal{J}(\frac{\partial \bar{l}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{l}) = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1.\bar{l}) \tag{2.3}$$

where τ is time, ρ is density of the fluid, μ is coefficient of viscosity, k is coefficient microviscosity, J is micro-gyration coefficient and α,β, γ are coefficients of Couple stress viscosities. **Q**, ***l*** are vectors for velocity and micro-rotation vectors The constitutive equations for the stress components T_{ij} and Couple stress components M_{ij} for Micro-polar fluids are given by

$$T_{ij} = -P\delta_{ij} + \frac{1}{2}(2\mu + k)\big(u_{i,j} + u_{j,i}\big) + ke_{ijr}(w_r - l_r) \tag{2.4}$$

$$M_{ij} = \alpha l_{i,i}\delta_{i,j} + \beta l_{i,j} + \gamma l_{j,i} \tag{2.5}$$

where the permutation tensor $e_{ijk} = \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ 1 & \text{if } i, j, k \text{ are cyclic} \\ -1 & \text{if } i, j, k \text{ are anti} - \text{cyclic} \end{cases}$

and w_r = r th component of ½(curl **Q**).

neglecting the nonlinear convective terms in (2.2) and (2.3), the linearised version of the equations are given by,

$$div\bar{Q} = 0 \tag{2.6}$$

$$\rho\frac{\partial\bar{Q}}{\partial\tau} = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{2.7}$$

$$\rho J\frac{\partial\bar{l}}{\partial\tau} = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1 . \bar{l}) \tag{2.8}$$

2.3 Statement and Formulation of the Problem

A circular cylinder of radius a and of infinite length is performing rectilinear oscillations with velocity $U_0\boldsymbol{i}e^{i\sigma\tau}$ about its diameter in an infinite vat containing incompressible Micro-polar fluid. A cylindrical coordinate system (R, θ, Z) with base vectors $(\boldsymbol{e_R}, \boldsymbol{e_\theta}, \boldsymbol{e_Z})$ with origin on the axis of the cylinder is considered. Hence the fluid velocity will be in cross sectional planeof the cylinder containing the base vectors $(\boldsymbol{e_R}, \boldsymbol{e_\theta})$. The velocity and micro-rotation are assumed in the form:

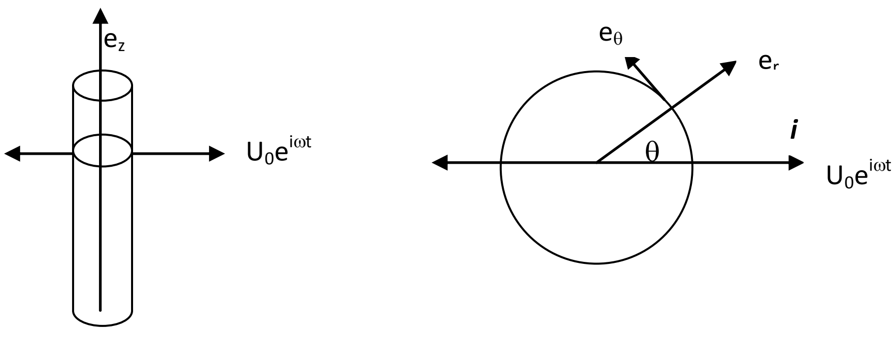


Fig 2.1 Geometry of the oscillating cylinder

$$\boldsymbol{Q} = e^{i\sigma\tau}\,(U(R,\theta)\boldsymbol{e_R} + V(R,\theta)\boldsymbol{e_\theta}) \text{and } \boldsymbol{l} = e^{i\sigma\tau}\,\mathcal{C}(R,\theta)\boldsymbol{e_z} \tag{2.9}$$

The following non-dimensional scheme is introduced.

$$\left.\begin{array}{l} \mathrm{R} = ar,\ \mathrm{U} = \mathrm{U}_0 u,\ \mathrm{V} = \mathrm{U}_0 v,\ \mathbf{Q} = \mathbf{q}\mathrm{U}_0,\ \mathcal{C} = \frac{\mathrm{C}U_0}{a}, \\ \quad \boldsymbol{l} = \boldsymbol{\upsilon}\frac{U_0}{a},\ \Psi = \psi \mathrm{U}_0 a,\ \mathrm{P} = p\rho \mathrm{U}_0^2,\ \tau = \frac{\mathrm{at}}{\mathrm{U}_0} \end{array}\right\} \tag{2.10}$$

The following are non-dimensional parameters viz, j is gyration parameter, ω is frequency parameter, s is Couple stress parameter, c is cross viscosity or Micro-polarity parameter and Re is oscillations Reynolds number for Micro-polar fluids.

$$J = \frac{\mathrm{j}\,\rho\sigma a^2}{\gamma}, \varpi = \frac{a\sigma}{\mathrm{U}_0}, s = \frac{\mathrm{k}a^2}{\gamma}, c = \frac{\mathrm{k}}{\mu + \mathrm{k}}, \epsilon = \frac{\alpha+\beta+\gamma}{\gamma}, \mathrm{Re} = \frac{\rho \mathrm{U}_0 a}{\mu} \text{and} R_0 = \frac{\rho\,\mathrm{U}_0 a}{\mu + k} \tag{2.11}$$

We can write $R_0 = Re(1 - c)$

Substituting (2.9) in (2.1) we notice that stream function ψ can be introduced as

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \ \ and \ \ v = -\frac{\partial\psi}{\partial r} \quad \text{i.e} \qquad \boldsymbol{q} = \nabla \times (\psi \boldsymbol{e}_z) \tag{2.12}$$

Using (2.9), (2.10), (2.11) in (2.7) and (2.8) we get

$$R_0\left(\frac{\partial \boldsymbol{q}}{\partial t} + \bar{q}.\nabla\bar{\mathrm{q}}\right) = -R_0.\nabla p + c\,\nabla \times \boldsymbol{\upsilon} - \nabla \times \nabla \times \boldsymbol{q} \tag{2.12a}$$

Because flow is very slow i.e. $|\bar{q}| < 1, |\nabla\bar{q}| < 1$ which implies $R_0|\bar{q}.\nabla\bar{q}| \ll 1$. Hence nonlinear convective terms can be neglected. This assumption is called Stokesian approximation. Hence we get

$$R_0\frac{\partial \boldsymbol{q}}{\partial t} = -R_0.\nabla p + c\,\nabla \times \boldsymbol{\upsilon} - \nabla \times \nabla \times \boldsymbol{q} \tag{2.13}$$

$$\frac{J}{\boldsymbol{\varpi}}\frac{\partial \boldsymbol{\upsilon}}{\partial t} = -2s\boldsymbol{\upsilon} + s\nabla \times \boldsymbol{q} - \nabla \times \nabla \times \boldsymbol{\upsilon} + \epsilon\nabla(\nabla \cdot \boldsymbol{\upsilon}) \tag{2.14}$$

We can write (2.13) and (2.14) as

$$i\varpi R_0\boldsymbol{q} = -R_0.\nabla p + c\,\nabla \times \boldsymbol{\upsilon} - \nabla \times \nabla \times \boldsymbol{q} \tag{2.15}$$

$$iJ\boldsymbol{\upsilon} = -2s\boldsymbol{\upsilon} + s\nabla \times \boldsymbol{q} - \nabla \times \nabla \times \boldsymbol{\upsilon} + \epsilon\nabla(\nabla \cdot \boldsymbol{\upsilon}) \tag{2.16}$$

Eliminating p from equation (2.15) and using (2.9) and (2.12) we get,

$$i\varpi R_0.\nabla^2\psi = c\nabla^2 C + \nabla^4\psi \tag{2.17}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ (2.18)

Substituting (2.9) and (2.12) in (2.16) we get

$$(2s + iJ)C = \nabla^2 C - s\nabla^2 \psi \quad (2.19)$$

Eliminating C from (2.18) and (2.19), we obtained an equation for stream function as

$$\nabla^2(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\psi = 0 \quad (2.20)$$

Where $\lambda_1^2 + \lambda_2^2 = (2 - c)s + i(J + \varpi R_0)$ and $\lambda_1^2\lambda_2^2 = i\varpi R_0(2s + iJ)$ (2.21)

Using (2.18) and (2.19)

$$c(2s + iJ)C = -\nabla^4\psi + (i\varpi . R_0 - sc)\nabla^2\psi \quad (2.22)$$

Using (2.21) in (2.22), the equation for C can be re-written as

$$cC = -\frac{i\varpi R_0}{\lambda_1^2\lambda_2^2}\nabla^2(\nabla^2 - \lambda_1^2 - \lambda_2^2)\psi - \nabla^2\psi \quad (2.23)$$

The solution for ψ if $\lambda_1 \neq \lambda_2$ in (2.20) is given in Lakshmana Rao et al. (1971, 1972). The solution for ψ for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \rightarrow \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation:

$$\frac{\gamma}{J} = \frac{(2\mu + k)(\mu + k)}{2\mu + 3k} \text{ and } \rho\sigma = \frac{(2\mu + k)k + \gamma\rho\sigma}{J(\mu + k)} \quad (2.24)$$

In non-dimensional form, these are given by

$$(2 - c)s = J - \varpi R_0 \text{ and } (2 - c)J = \varpi R_0(2 + c) \quad (2.25)$$

In this chapter, we are interested in the solution for ψ for the case of resonance.

We have the equation for stream function ψ as

In the case of resonance:

$$\nabla^2(\nabla^2 - \lambda^2)^2\psi = 0 \quad (2.26a)$$

In the case on non-resonance:

$$\nabla^2(\nabla^2 - \lambda_1^2)\,(\nabla^2 - \lambda_2^2)\,\psi = 0 \qquad (2.26b)$$

Th equation for the micro-rotation component C is as

In the case of resonance:

$$cC = -\frac{i\varpi R_0}{\lambda^4}\nabla^2(\nabla^2 - 2\lambda^2)\psi - \nabla^2\psi \qquad (2.27a)$$

And in the case of non-resonance:

$$cC = -\frac{i\varpi R_0}{\lambda_1^2\lambda_2^2}\nabla^2(\nabla^2 - \lambda_1^2 - \lambda_2^2)\psi - \nabla^2\psi \qquad (2.27b)$$

2.3.1 Boundary Conditions

The cylinder is oscillating in the direction of X-axis. Hence the non-dimensional velocity of the cylinder Γ after removing $e^{i\varpi t}$ is given by

$\boldsymbol{q}_\Gamma = \boldsymbol{i} = cos\theta\boldsymbol{e}_r - sin\theta\boldsymbol{e}_\theta$ which implies by no-slip condition

$$u = cos\theta \, and \, v = -sin\theta \text{ on } r=1 \qquad (2.28)$$

By hyper-stick condition $\boldsymbol{\upsilon}_\Gamma = \frac{1}{2}(\text{curl } \boldsymbol{q})_\Gamma$

$$\text{which gives } C = 0 \; on \; r = 1 \qquad (2.29)$$

2.4 Solution of the Problem

To match with the boundary conditions in (2.28) and (2.29), Stream function ψ, micro-rotation component C are assumed in the form

$$\psi = f(r)\, sin\theta \text{ and } C = g(r)\, sin\theta \qquad (2.30)$$

Substituting (2.30) in (2.26a) and (2.26b) and cancelling $sin\theta$ we get

In the case of resonance:

$$D_c^2\left(D_c^2 - \lambda^2\right)^2 f = 0 \qquad (2.31a)$$

In the case on non-resonance:

$$D_c^2(D_c^2 - \lambda_1^2)\,(D_c^2 - \lambda_2^2)\,f = 0 \tag{2.31b}$$

Substituting (2.30) in (2.27a) and (2.27b) we get

In the case of resonance:

$$cg = -\frac{i\varpi R_0}{\lambda^4} D_c^2(D_c^2 - 2\lambda^2)f - D_c^2 f \tag{2.32a}$$

In the case on non-resonance:

$$cg = -\frac{i\varpi R_0}{\lambda_1^2\lambda_2^2} D_c^2(D_c^2 - \lambda_1^2 - \lambda_2^2)f - D_c^2 f \tag{2.32b}$$

Where $D_c^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ (2.33)

Substituting (2.30) in (2.28) and (2.29), the conditions on *f* and *g* are obtained as:

$$f(1) = f'(1) = 1 \text{ and } g(1) = 0 \tag{2.34}$$

Since the equation for *f* is linear, the general solution for *f* is linear combination of individual solutions of factors in the differential operator. Hence f is taken as

$$f = a_0 f_0 + a_1 f_1 + a_2 f_2 \tag{2.34a}$$

In the case of resonance:

$$D_c^2 f_0 = 0, \quad \left(D_c^2 - \lambda^2\right)f_1 = 0 \text{ and } \left(D_c^2 - \lambda^2\right)^2 f_2 = 0 \tag{2.35a}$$

In the case of non-resonance:

$$D_c^2 f_0 = 0, \quad \left(D_c^2 - \lambda_1^2\right)f_1 = 0 \text{ and } \left(D_c^2 - \lambda_2^2\right)f_2 = 0 \tag{2.35b}$$

On solving (2.35a) and (2.35b) the solution for *f* is obtained as

In the case of resonance:

$$f(\mathrm{r}) = \frac{\mathrm{a_0}}{\mathrm{r}} + \mathrm{a_1 K_1(\lambda r)} + \mathrm{a_2 r K_1'(\lambda r)} \tag{2.36a}$$

In the case of non-resonance:

$$f(\mathrm{r}) = \frac{\mathrm{a_0}}{\mathrm{r}} + \mathrm{a_1 K_1(\lambda_1 r)} + \mathrm{a_2 K_1(\lambda_2 r)} \tag{2.36b}$$

The following results are useful to note.

In the case of resonance:

$$D_c^2 f_1 \;=\; \lambda^2 f_1 \text{ and } D_c^2 f_2 \;=(2\lambda f_1+\;\lambda^2 f_2) \tag{2.37a}$$

In the case of non-resonance:

$$\mathrm{D}_c^2 f_1 \;=\; \lambda_1^2 f_1 \quad \text{and} \mathrm{D}_c^2 f_2 \;=\; \lambda_2^2 f_2 \tag{2.37b}$$

Substituting f along with (2.37a) in (2.32a), for the case of resonance we get

$$cg \;=\; \mathrm{a}_1\left(\mathrm{i}\varpi \mathrm{R}_0 - \lambda^2\right) f_1 + \mathrm{a}_2\left(\mathrm{i}\varpi \mathrm{R}_0 - \lambda^2\right) f_2 - 2\mathrm{a}_2 \lambda f_1 \tag{2.38a}$$

Substituting f along with (2.37b) in (2.32b), for the case of non-resonance we get

$$cg \;=\; \mathrm{a}_1\left(\mathrm{i}\varpi \mathrm{R}_0 - \lambda_1^2\right) f_1 + \mathrm{a}_2\left(\mathrm{i}\varpi \mathrm{R}_0 - \lambda_2^2\right) f_2 \tag{2.38b}$$

The constants a_0, a_1, a_2 are obtained from the boundary conditions (2.34) as follows:

In the case of resonance:

$$\begin{bmatrix} 1 & K_1(\lambda) & K'_1(\lambda) \\ -1 & \lambda K'_1(\lambda) & \frac{\lambda^2+1}{\lambda} K_1(\lambda) \\ 0 & (i\varpi R_0 - \lambda^2) K_1(\lambda) & (i\varpi R_0 - \lambda^2) K'_1(\lambda) - 2\lambda K_1(\lambda) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2,39a}$$

In the case of non-resonance:

$$\begin{bmatrix} 1 & K_1(\lambda_1) & K_1(\lambda_2) \\ -1 & \lambda_1 K'_1(\lambda_1) & \lambda_2 K'_1(\lambda_2) \\ 0 & (i\varpi R_0 - \lambda_1^2) K_1(\lambda_1) & (i\varpi R_0 - \lambda_2^2) K_1(\lambda_2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2,39b}$$

Hence from (2.36), (2.38) and (2.39) f and g are completely known and hence ψ and C are known.

2.4.1 Pressure

From equation (2.15) pressure is obtained as follows.

$$dp = \nabla p.\boldsymbol{dr} = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta \tag{2.40}$$

By comparing components in equation (2.15), we obtained

$$\frac{\partial p}{\partial r} = -\frac{i\varpi}{r}\frac{\partial \psi}{\partial \theta} + \frac{c}{R_0.r}\frac{\partial C}{\partial \theta} + \frac{1}{R_0.r}\frac{\partial}{\partial \theta}(\nabla^2 \psi) \tag{2.41}$$

$$\frac{\partial p}{\partial \theta} = r\left(i\varpi\frac{\partial \psi}{\partial r} - \frac{c}{R_0}\frac{\partial C}{\partial r} - \frac{1}{R_0}\frac{\partial}{\partial r}(\nabla^2\psi)\right) \tag{2.42}$$

Substituting (2.41) and (2.42) in (2.40), we get

$$dp = \left(-\frac{i\varpi}{r}\frac{\partial \psi}{\partial \theta} + \frac{c}{R_0.r}\frac{\partial C}{\partial \theta} + \frac{1}{R_0.r}\frac{\partial}{\partial \theta}(\nabla^2\psi)\right)dr + r\left(i\varpi\frac{\partial \psi}{\partial r} - \frac{c}{R_0}\frac{\partial C}{\partial r} - \frac{1}{R_0}\frac{\partial}{\partial r}(\nabla^2\psi)\right)d\theta$$

$$dp = \frac{1}{R_0}\left[(-i\varpi f + D_c^2 f + cg)cos\theta\frac{dr}{r} + r\frac{d}{dr}(i\varpi f - D_c^2 f - cg)sin\theta\, d\theta\right] \tag{2.43}$$

Integrating on both sides of (2.43), we obtained pressure in non-dimensional form as

$$p \ = \frac{i\varpi A_0}{r}cos\theta \tag{2.44}$$

2.4.2 Drag acting on the Cylinder per length L

Drag= $D^* = aL\int_0^{2\pi}(T^*{}_{rr}\,cos\theta - T^*{}_{r\theta}\,sin\theta)\,|_{R=a}\,d\theta$ (2.45)

Required tress components are obtained as follows:

$$\text{Strain rate tensor} = E = \left[e_{ij}\right] = \frac{1}{2}\left[\nabla\bar{Q} + \nabla\bar{Q}^T\right]$$

We get strain rate tensor for this problem as

$$E = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2}\left[\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial \theta} - \frac{V}{R}\right] & 0 \\ \frac{1}{2}\left[\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial \theta} - \frac{V}{R}\right] & \frac{1}{R}\left[\frac{\partial V}{\partial \theta} + U\right] & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.46}$$

Substituting required terms in (2.4), we get the stress components as

$$T_{RR} \ = -P + (2\mu + k)\frac{\partial U}{\partial R} \tag{2.47}$$

$$T_{R\theta} \ = (\mu + k)\frac{\partial V}{\partial R} + \frac{\mu}{R}\frac{\partial U}{\partial \theta} - \mu\frac{V}{R} - kC \tag{2.48}$$

Stress components in non-dimensional form as

$$T_{rr} \ = \frac{(\mu+k)U_0}{a}\left[-pR_0 + (2-c)\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta}\right)\right] \tag{2.49}$$

$$T_{r\theta} \ = \frac{(\mu+k)U_0}{a}\left[(c-2)\frac{\partial^2\psi}{\partial r^2} + (1-c)\nabla^2\psi - cC\right] \tag{2.50}$$

Substituting (2.49) and (2.50) in (2.45) we get the Drag D^*acting on the cylinder (without the factor $e^{i\varpi t}$) is given as

$$D^* = L(\mu + k)U_0 i\varpi . R_0\pi(1 - 2a_0) \tag{2.51}$$

Dividing D^* by $L(\mu+k)U_0$,hence the non-dimensional Drag D is given by

$$D = Real\{i\varpi . R_0 . \pi\left(1 - 2a_0\right)e^{i\varpi t}\} \tag{2.52}$$

2.5 Results and Discussions

The values of λ are obtained from (2.21) by solving the following equation for x

$$x^2 - [(2 - c)s + i(J + \varpi R_0)]x + i\varpi R_0(2s + iJ) = 0 \tag{2.53}$$

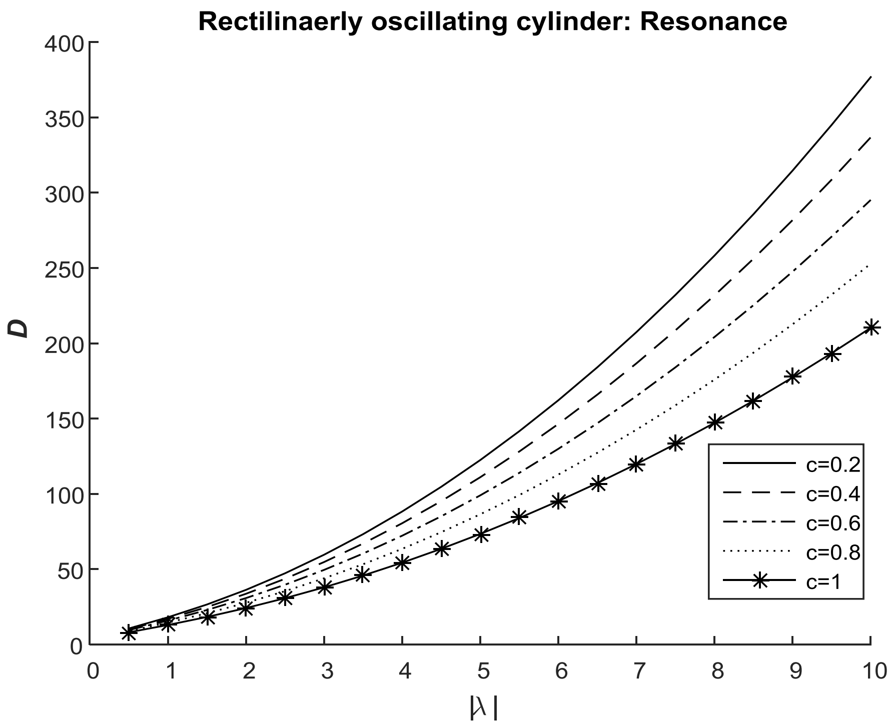
Then for resonance case

$$\lambda = \sqrt{x} = \frac{1}{\sqrt{2}}\sqrt{\left(2 - c\right)s + i(J + \varpi R_0)} \tag{2.54}$$

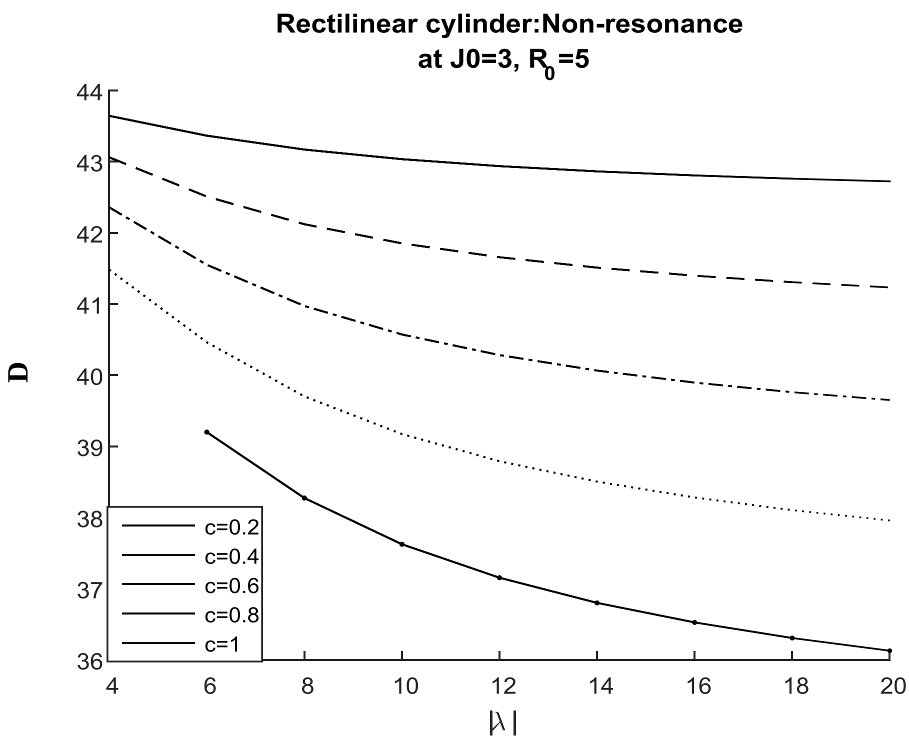
This involves 5 parameters which are related by two equations in (2.25). Hence we choose three parameters as independent. Here ϖ, R_0 and c are selected independently, with $0 \leq c \leq 1$, Re≪1 and ω≫1 such that $\varpi . R_0$ is not negligibly small (say>1). After selecting c, R_0 and ϖ, the values of s and J are obtained from (2.25) and then λ is obtained from (2.54). The values of λ are complex. These values for λ are substituted in (2.39a), (2.39b) and the constants a_0, a_1 and a_2 are obtained. Then the stream function ψ, micro-rotation component C and Drag D are obtained from (2.36), (2.38) and (2.52) respectively for both resonance and non-resonance cases. Thus obtained ψ will have complex values. To get the physical picture, these values are multiplied by $e^{i\varpi t}$ and its real part is taken.

2.5.1 Drag

From Fig 2.2, it is observed that as c increases Drag decreases for resonance and for non-resonance. But as |λ| increases, Drag increases in vide range between 30 to 400 for resonance and decreases in small interval between 36 to 44 for non-resonance.

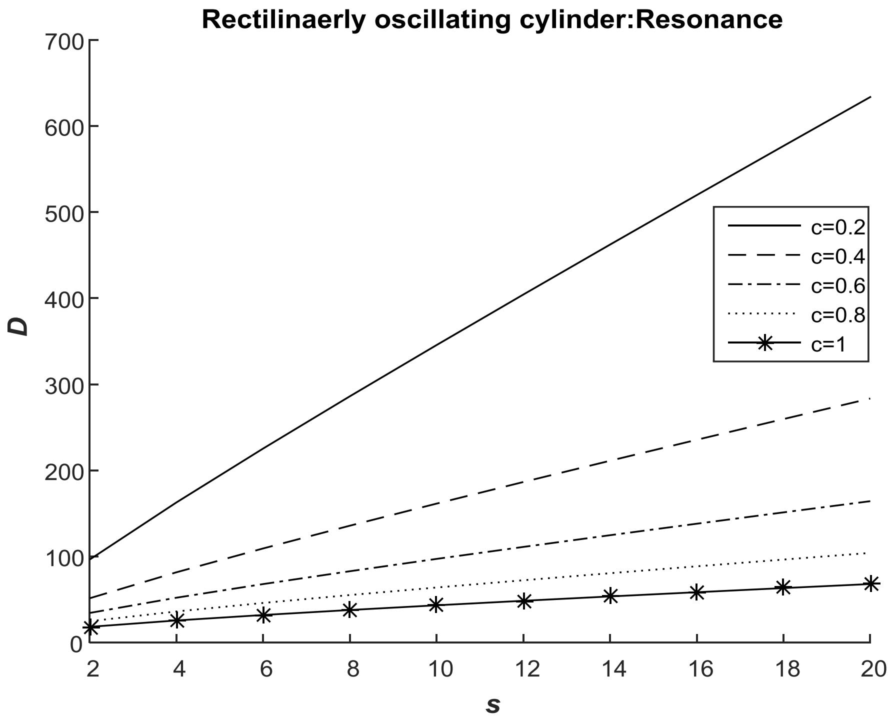


(a)

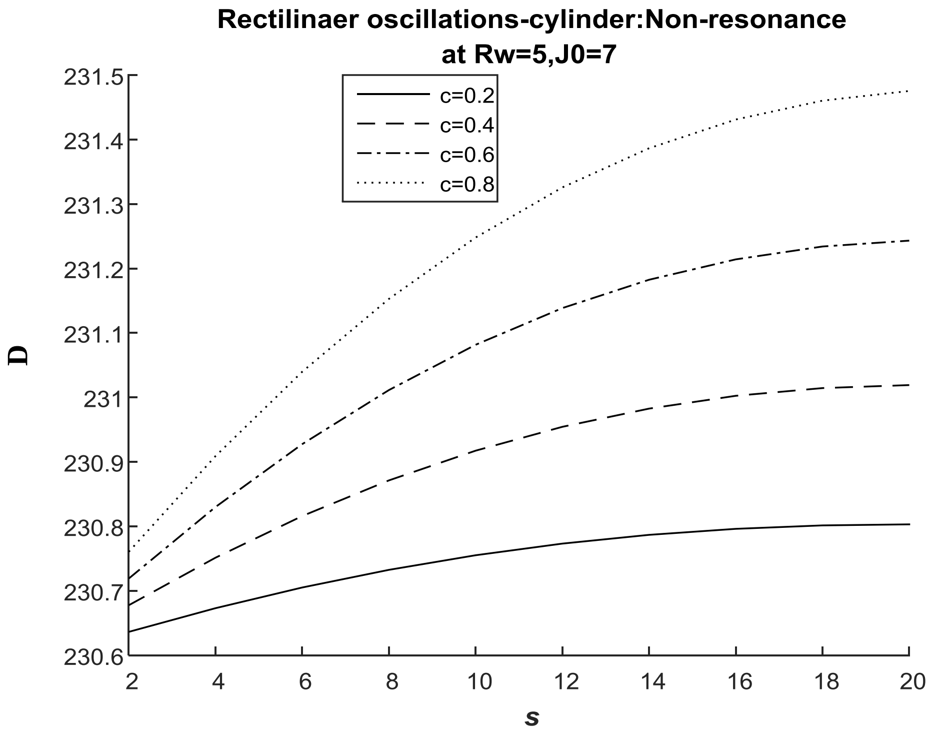


(b)

Fig 2.2 Drag Vs |λ| at different values of c for the case of a) resonance and b) non-resonance



(a)



(b)

Fig 2.3 Drag Vs Couple stress parameter s for the case of a) resonance and b) non-resonance

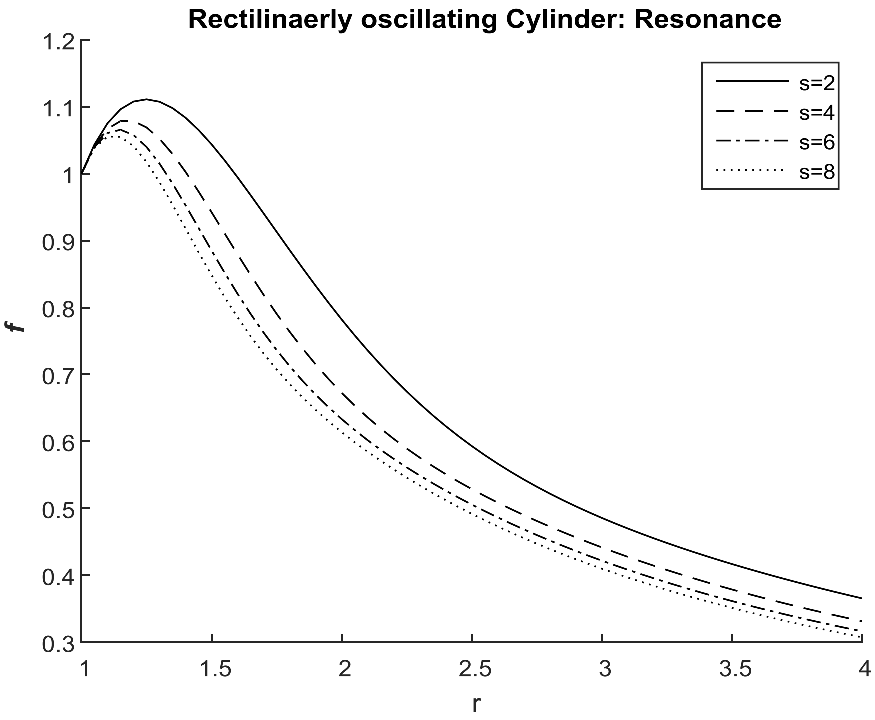
From Fig 2.3, we observe that as s increases Drag increases for resonance and non-resonance. But for resonance this variation of Drag is from 30 to 700 and non-resonance it is in a small interval between 230 to 235.

2.5.2 Stream Function

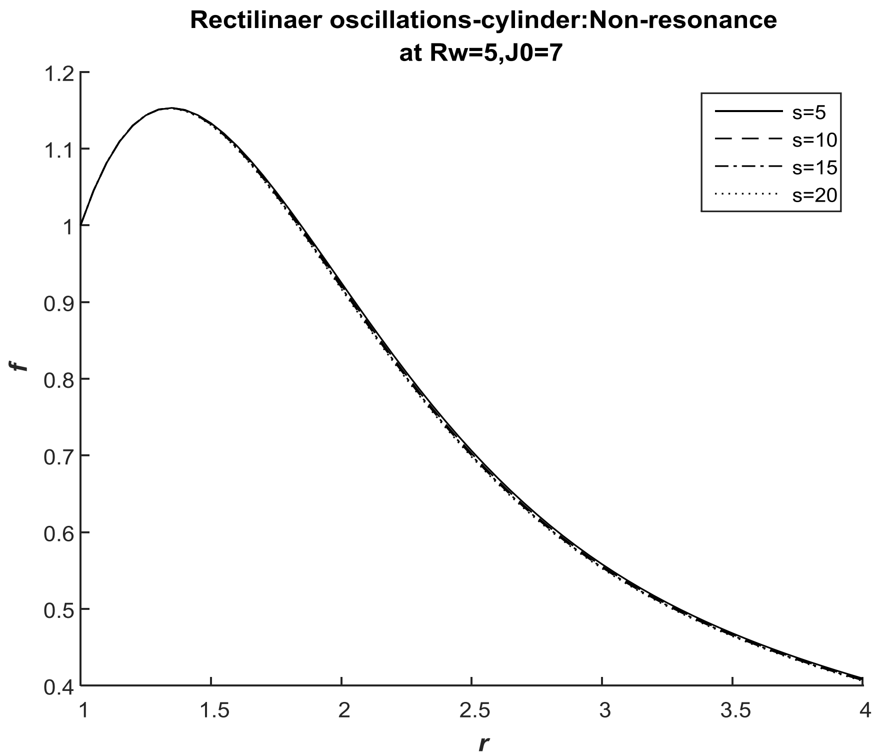
By looking the stream function f, in Fig 2.4, it is observed that the function is not effected by variation in s for non-resonance. But as s increases, stream function values decreases. From Fig 2.4 it is observed that for resonance, the peak is little lower to the peak corresponding to non-resonance and effect of s is clearly apparent. But for non-resonance, effect of s is not noticeable on stream function. As s increases, stream function values are decreasing.

From Fig 2.5, we note that the variation c is not noticeable for resonance. But for non-resonance, as c increases stream function values decreases along distance from 1.5 to 3.5 and stream function takes larger values for non-resonance at the corresponding distances.

In Fig 2.6, the contours for stream function are shown. As c increases, the values of stream function are increasing and the region of circulations becomes larger. But for non-resonance this region of circulations is still wider.

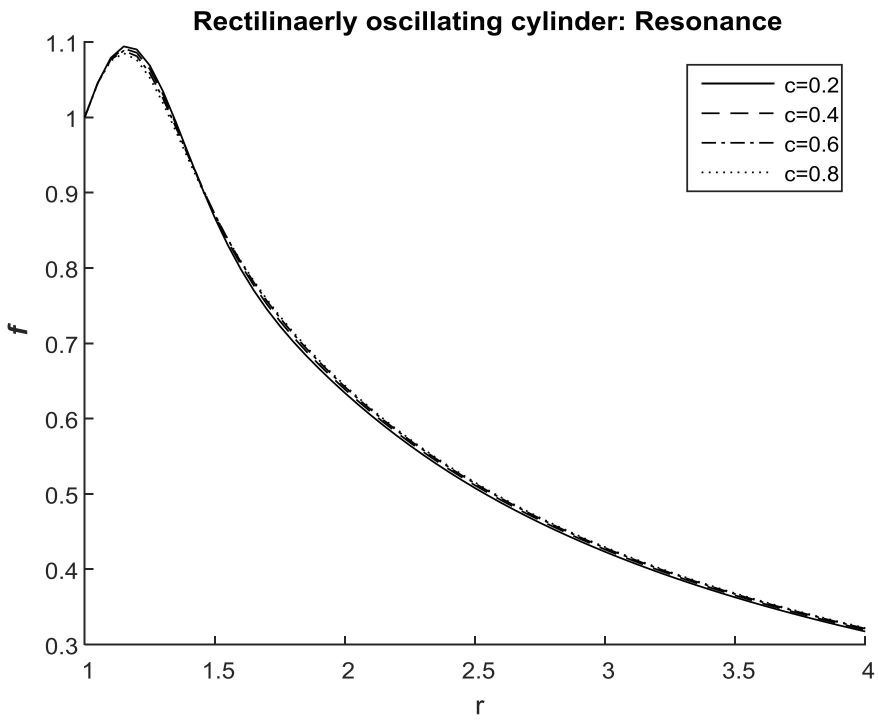


(a)

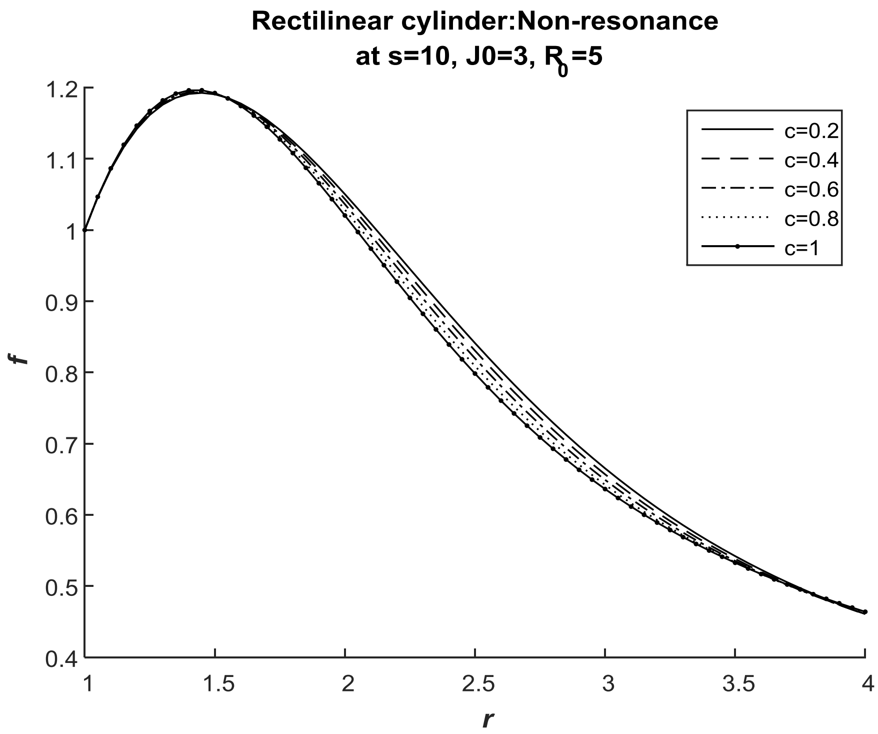


(b)

Fig 2.4 Stream function f at different values of s for the case of a) resonance and b) non-resonance

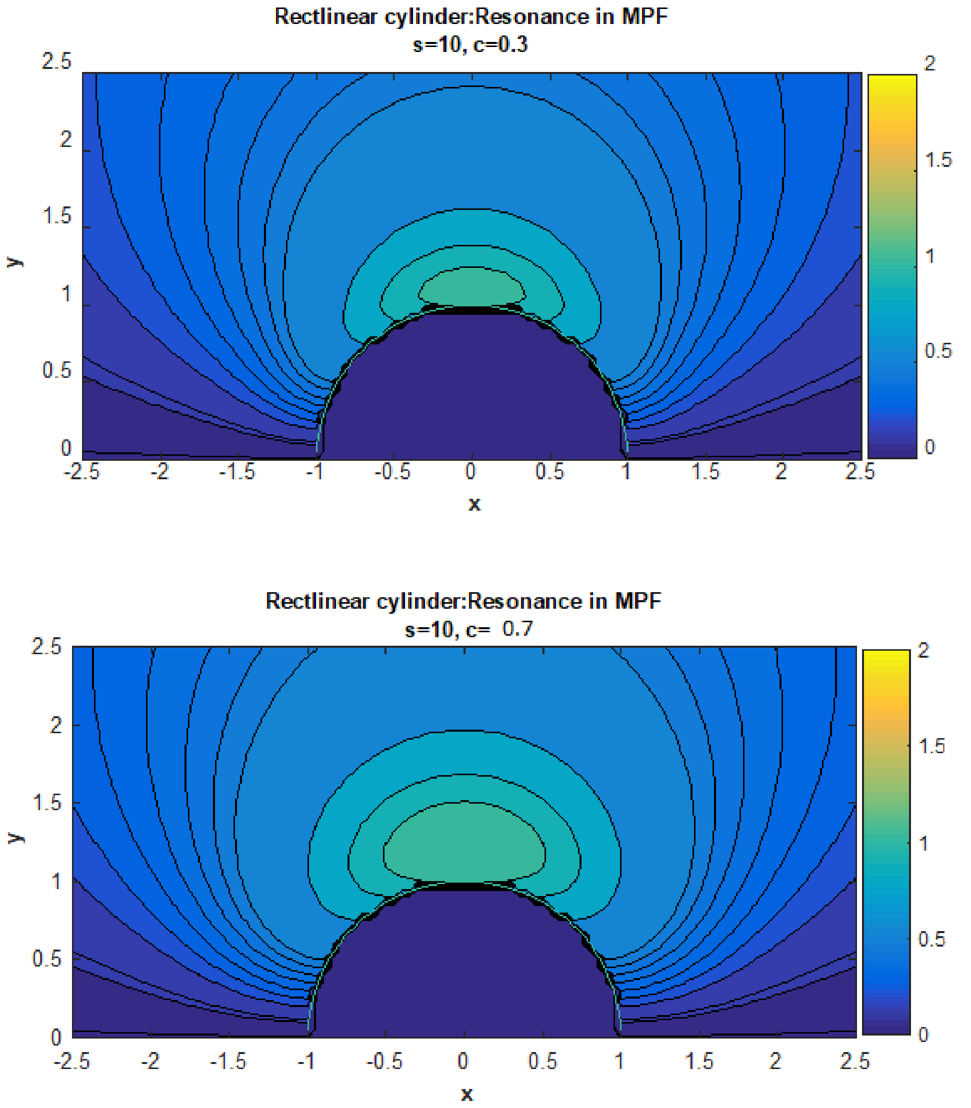


(a)

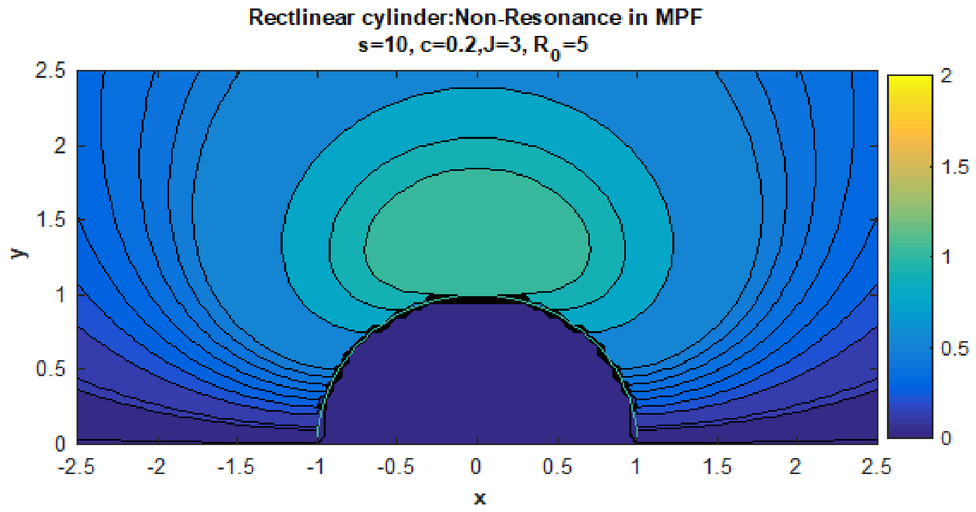


(b)

Fig 2.5 Stream function at different values of c for the case of a) resonance and b) non-resonance



(a)



(b)

Fig 2.6 Contours for stream function for the case of a) resonance and b) non-resonance

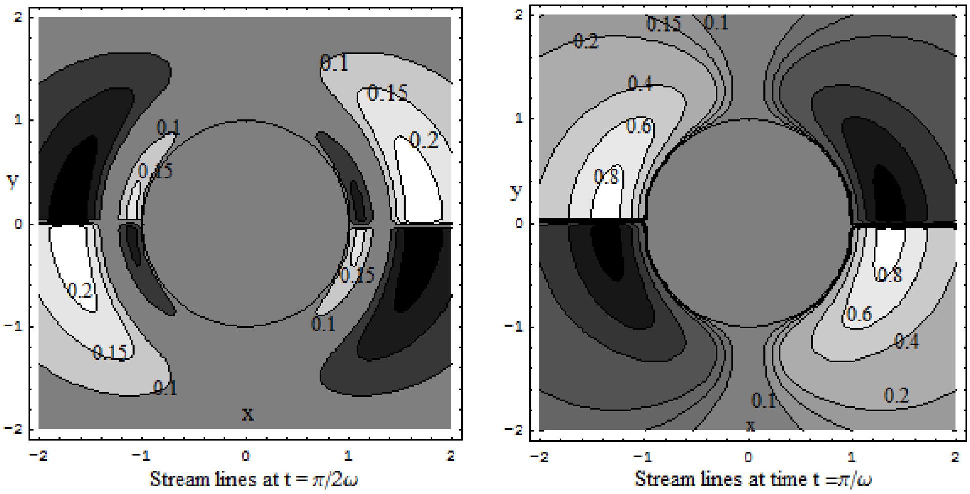


Fig 2.7 Flow pattern at different times over a half time period for non-resonance

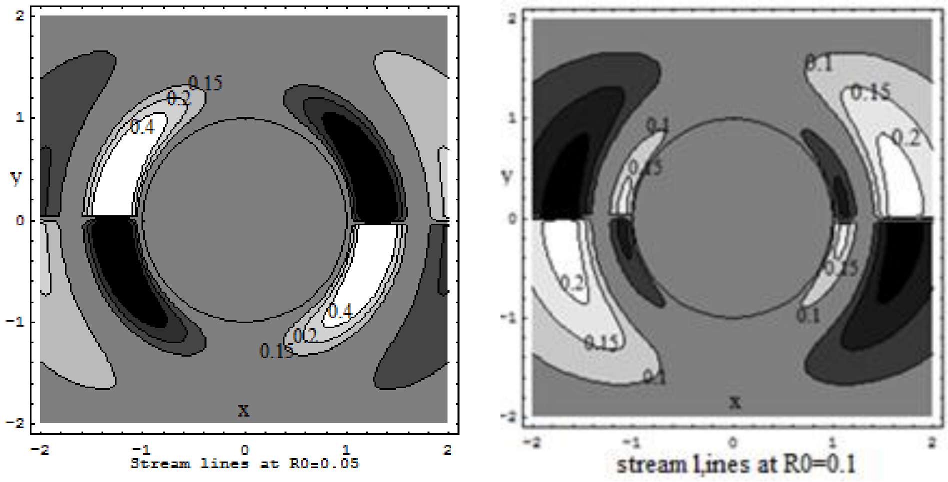


Fig 2.8 Stream lines at different values of Reynolds numbers Re for non-resonance

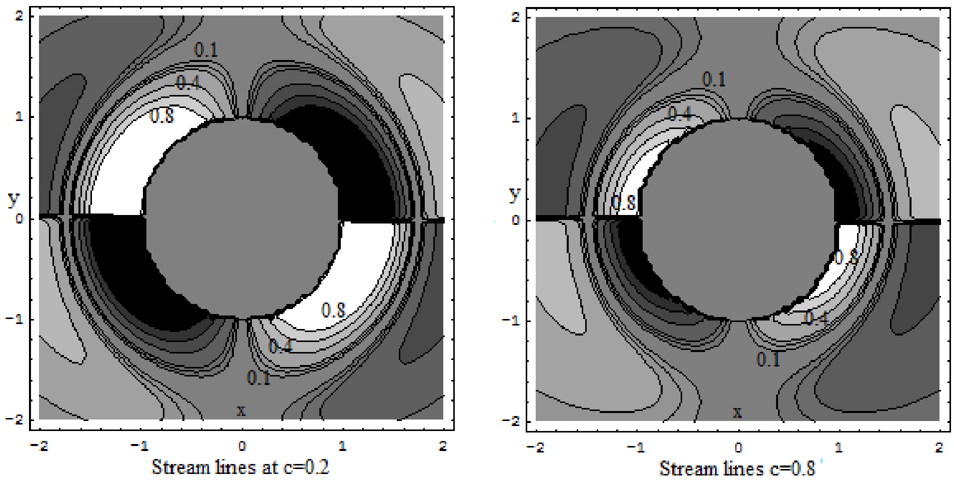


Fig 2.9 The flow pattern for different values of cross viscosity parameter *c*

From Fig 2.7, we note that as at lower times a small circulation symmetric to the line joining the poles is found. But as time increases, this region of circulations near the cylinder disappears.

From Fig 2.8, it is observed that as Reynolds number increases, flow circulation near the cylinder decreases and disappears and merges into a second circulation zone.

From Fig 2.9, it is obsereved that the variation in cross viscosity parameter *c* for small values <0.6 does not effect much the flow paterren. But as c $\rightarrow$1, the flow is sensitive to the values of c. In this case as *c* is increasing, the first zone of circulation decreases and second zone of circulations comes near to the cylinder.

Physical Reasoning:

As c$\rightarrow$0 and s$\rightarrow\infty$, we get Newtonian case. In the case of resonance c and s are connected by relation such that stream function gets lower values and hence it offers less force and hence by Drag is less. In the case of non-resonance c and s are independent and Stream function gets larger values and Drag gets high values

2.6 Conclusions

From the above observations, we conclude that

i) Drag in the case of resonance, is more than the case of non-resonance.

ii) Stream function in non-resonance has more circulations on the pole with wider region of circulations.

iii) In the non-resonance, as Reynolds number increases, the circulation regions near to the cylinder decrease and become thinner.

Chapter 3

Rotary oscillations of a Circular Cylinder in a Micro-polar fluid

The flow generated due to rotary oscillations of a circular cylinder about its axis of symmetry in a Micro-polar fluid is considered. By taking Stokesian assumptions, nonlinear convective terms of motion are dropped. In this situation, there arises a rare particular special case when material constants satisfy a condition referred to as resonance. The flow field for velocity and micro-rotation components is investigated. The Skin friction acting on the cylinder is evaluated and the effects of Micro-polarity and Couple stress parameter on the Skin friction are presented in the form of graphs. It is observed that for a Micro-polarfluid when the material constantssatisfy the resonance condition, the Skin friction reduces to a minimum.

3.1 Introduction

Many authors analysed oscillatory flows of different symmetric bodies like circular cylinder, sphere, spheroid, approximate sphere performing rotary oscillations in various non-Newtonian fluids. For example, Tekasakul et al. (1998, 2003), Iyengar et al. (2001, 2004), Anwar et al. (2004), Ashmawy (2015). In the pioneering works of Lakshmana Rao et al.(1972, 1983, 1987), flows generated due to oscillations of circular cylinder, spheroid and elliptic cylinder in Micro-polar fluids were analyzed. The aim of their analysis was to determine the Drag or Couple, as the case may be, acting on the oscillating body. Nevertheless in all these situations, as far as authors know, a special case, referred to as “Resonance type flow”(which will be defined in section3) has not been investigated till now. This type of flows can arise on every occasion when the flow is generated by oscillations in a non-Newtonian fluid. For example, this case of resonance can be observed in the papers Lakshmana Rao et al. (1971, 1972, 1983, 1987), Ramana Murthy et al. (2011), Aparna et al. (2012), Nagaraju et al. (2014). But in these papers, the case of resonance was not studied by the authors. Oscillatory flows of circular cylinder in various fluids were studied by many authors like Frater (1968), Lakshmana rao (1972). Ramkissoon et al. (1990), Rao et al. (1992), Calmelet-Eluhu et al. (1998), Anwar (2004), Fetecau et al. (2006), Mehrdad Massoudi et al. (2008), Ramana Murthy et al. (2010).

The aim of the present chapter is to study the flow due to circular cylinder performing rotary oscillations in a Micro-polar fluid when the material resonance occurs. These results may be useful in conducting experiments to examine rheological properties of Micro-polar fluids.

3.2 Basic Equations

The field equations for velocity and micro-rotation of an incompressible Micro-polar fluid as derived by Eringen (1966) are given by:

$$\frac{\partial\rho}{\partial\tau}+ div(\rho\boldsymbol{Q})=0 \tag{3.1}$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau}+\bar{Q}.\nabla_1\bar{Q}\right)=-\nabla_1 P+k\nabla_1\times\bar{l}-(\mu+k)\nabla_1\times\nabla_1\times\bar{Q} \tag{3.2}$$

$$\rho J(\frac{\partial\bar{l}}{\partial\tau}+\bar{Q}.\nabla_1\bar{l})=-2k\bar{l}+k\nabla_1\times\bar{Q}-\gamma\nabla_1\times\nabla_1\times\bar{l}+(\alpha+\beta+\gamma)\nabla_1(\nabla_1.\bar{l}) \tag{3.3}$$

whereτ is time, ρ is density of the fluid, μ is coefficient of viscosity, k is coefficient microviscosity, J is micro-gyration coefficient and α,β, γ are coefficients of Couple stress viscosities. **Q**, ***l*** are vectors for velocity and micro-rotation vectors The constitutive equations for the stress components T_{ij} and Couple stress components M_{ij} for Micro-polar fluids are given by

$$T_{ij}=-P\delta_{ij}+\frac{1}{2}(2\mu+k)\left(u_{i,j}+u_{j,i}\right)+ke_{ijr}(w_r-l_r) \tag{3.4}$$

$$M_{ij}=\alpha l_{i,i}\delta_{i,j}+\beta l_{i,j}+\gamma l_{j,i} \tag{3.5}$$

where the permutation tensor$e_{ijk}=\begin{cases}0 & \text{if } i=j \text{ or } j=k \text{ or } k=i\\ 1 & \text{if } i,j,k \text{ are cyclic}\\ -1 & \text{if } i,j,k \text{ are anti}-\text{cyclic}\end{cases}$

andw_r = r th component of ½(curl **Q**).

Neglecting the nonlinear convective terms in (3.2) and (3.3), the linearised version of the equations are given by,

$$div\bar{Q} = 0 \tag{3.6}$$

$$\rho \frac{\partial \bar{Q}}{\partial \tau} = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{3.7}$$

$$\rho \mathcal{J} \frac{\partial \bar{l}}{\partial \tau} = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1 . \bar{l}) \tag{3.8}$$

3.3 Statement and Formulation of the Problem

An infinite circular cylinder of radius a is under torsional (rotary) oscillations with velocity $V_0 \boldsymbol{e}_{\theta} e^{i\sigma\tau}$ about its axis of symmetry in an incompressible Micro-polar fluid. A polar coordinate frame (R, θ, Z) with origin on the axis of the cylinder and with base vectors $(\boldsymbol{e}_{\boldsymbol{R}}, \boldsymbol{e}_{\theta}, \boldsymbol{e}_{z})$ is taken. The flow is two dimensional and is independent of Z coordinate. Hence the velocity field is in the plane of the base vectors $(\boldsymbol{e}_{\boldsymbol{R}}, \boldsymbol{e}_{\theta})$. Micro-rotation will be parallel to $\boldsymbol{e}_{\boldsymbol{z}}$ (in general will be parallel to curl q). Hence the vectors for velocity and micro-rotation are assumed in the form:

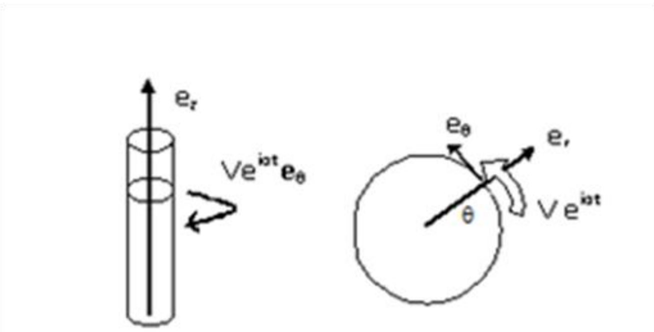


Fig 3.1 Geometry of the oscillating cylinder

$$\boldsymbol{Q} = V(R)\boldsymbol{e}_{\boldsymbol{\theta}}\boldsymbol{e}^{i\sigma\tau} \text{ and } \boldsymbol{l} = \mathcal{C}(R)\boldsymbol{e}_{z}\boldsymbol{e}^{i\sigma\tau} \tag{3.6}$$

The following non-dimensional scheme is taken. Physical quantities are on left handside with capitals. The non-dimensional variables are in small letters on RHS.

$$\left.\begin{array}{l} \mathrm{R} = ar,\ \mathrm{V} = vV_0,\ \mathbf{Q} = \mathbf{q}\mathrm{V}_0,\ \mathcal{C} = \mathrm{C}\sigma,\ \boldsymbol{l} = \upsilon\sigma \\ \quad \mathrm{P} = p\rho\mathrm{V}_0^2,\ \mathrm{P}_0 = p_0\rho\mathrm{V}_0^2 \ \text{ and } \ \tau = \frac{at}{\mathrm{V}_0} \end{array}\right\} \tag{3.7}$$

The non-dimensional parameters viz, j is gyration parameter, ϖ is frequency parameter, s is Couple stress parameter, c is cross viscosity or Micro-polarity parameter, R_0 is oscillations Reynolds number for Micro-polar fluids and Re is the usual Reynolds number are defined below.

$$J = \frac{\mathrm{j}\,\rho\sigma a^2}{\gamma},\ \varpi = \frac{a\sigma}{\mathrm{v}_0},\ s = \frac{\mathrm{k}a^2}{\gamma},\ c = \frac{\mathrm{k}}{\mu+\mathrm{k}},\ \mathrm{Re} = \frac{\rho \mathrm{v}_0 a}{\mu} \text{and} R_0 = \frac{\rho \mathrm{v}_0 a}{\mu+k} \tag{3.8}$$

By substituting velocity and micro-rotation vectors of (3.6) in the equation (3.7), we get

$$i\sigma\rho V = -\frac{P_0}{R} - k\frac{\partial \mathcal{C}}{\partial R} + (\mu + k)D^{*2}_{c}V \tag{3.9}$$

where P_0 constant pressure gradient along θ direction.

$$\text{And } D^{*2}_{c} = \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{1}{R^2} \tag{3.10}$$

Similarly equation (3) simplifies (3.8)

$$(i\sigma\rho J + 2k)\mathcal{C} = \frac{k}{R}\frac{\partial}{\partial R}(RV) + \gamma\left(D^{*2}_{c}\mathcal{C} + \frac{\mathcal{C}}{R^2}\right) \tag{3.11}$$

Using non dimensional schemes (3.7) and (3.8) in (3.9) and (3.11) we get

$$i\varpi R_0 v = -\frac{R_0 p_0}{r} - c\frac{\partial C}{\partial r} + D_c^2 v \tag{3.12}$$

$$(iJ + 2s)C = \frac{s}{r}\frac{\partial}{\partial r}(rv) + D_c^2 C + \frac{1}{r^2}C \tag{3.13}$$

$$\text{Where } D_c^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \tag{3.14}$$

$$\text{From (3.12) } cC' = [D_s^2 - i\varpi R_0]v - \frac{R_0 p_0}{r} \tag{3.15}$$

Eliminating C from (3.12) and (3.13) we get

$$(D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)v = -(iJ + 2s)R_0\frac{p_0}{r} \tag{3.16}$$

$$\text{Where } \lambda_1^2 + \lambda_2^2 = (2 - c)s + i(J + \varpi R_0) \text{ and } \lambda_1^2\lambda_2^2 = i\varpi R_0(2s + iJ) \tag{3.17}$$

The solution for v if $\lambda_1 \neq \lambda_2$ in (3.16) is given in Lakshmana Rao (1971). The solution for v for the case, $\lambda_1 = \lambda_2$cannot be obtained as a limiting case of $\lambda_1 \rightarrow \lambda_2$. This case $\lambda_1 = \lambda_2$ is called as "*Material Resonance*" or simply as "Resonance". This situation occurs if the material constants (coefficients) satisfy the relation given by

$$\frac{\gamma}{j} = \frac{(2\mu+k)(\mu+k)}{2\mu+3k} \text{ and } \quad \rho\sigma = \frac{(2\mu+k)k+\gamma\rho\sigma}{j(\mu+k)} \tag{3.18}$$

The same equations in non-dimensional form are given by

$$(2-c)s = J - R_0\varpi \ \ and \ \ (2-c)J = (2+c)\varpi R_0 \tag{3.19}$$

Our interest is to obtainv when resonance occurs. In this situation, the velocity v is given by

$$\left(D_c^2 - \lambda^2\right)^2 v = -(2s+iJ)R_0\frac{p_0}{r} \tag{3.20a}$$

For the case of non-resonance

$$(D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)v = -(iJ+2s)R_0\frac{p_0}{r} \tag{3.20b}$$

Micro-rotation C is obtained interms of v from (3.12) and (13) as below.

$$c(iJ+2s)C = \frac{1}{r}\frac{d}{dr}\ [rD_c^2 v + (cs - i\varpi R_0)rv] \tag{3.21}$$

From (3.17), we note that for non-resonance

$$2s + iJ = \frac{\lambda_1^2\lambda_2^2}{i\varpi R_0} \quad and \ iR_0\varpi - cs = \lambda_1^2 + \lambda_2^2 - \frac{\lambda_1^2\lambda_2^2}{iR_0\varpi} \tag{3.21a}$$

and for resonance,

$$2s + iJ = \frac{\lambda^4}{i\varpi R_0} \quad and \ iR_0\varpi - cs = 2\lambda^2 - \frac{\lambda^4}{iR_0\varpi} \tag{3.21b}$$

Hence $\qquad (D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)v = -\frac{\lambda^4}{i\varpi}\frac{p_0}{r}$

$$cC = -\frac{\lambda^4}{i\varpi R_0}\left(\frac{d}{dr} + \frac{1}{\mathrm{r}}\right)[D_c^2 v - 2\lambda^2 v] - \left(\frac{d}{dr} + \frac{1}{\mathrm{r}}\right)v \tag{3.21c}$$

3.3.1 Boundary Conditions

The usual no-slip condition for velocity is taken on the surface of the circular cylinder.

$$on \ \ \Gamma(\,i.e\ r = 1),\ v\ = 1 \tag{3.22}$$

By hyper-stick condition for micro-rotation component C on Γ is given by,

$C_{\Gamma} = \frac{1}{2} Curl\, \boldsymbol{Q}_{\Gamma}$ along z direction, which yields that

$$on\ \Gamma\ (i.e\ \ r = 1),\quad C = 1 \tag{3.23}$$

3.4 Solution of the Problem

Velocity v is obtained in the form

$$v = a_1 v_1 + a_2 v_2 - \frac{p_0}{ri\varpi} \tag{3.24}$$

where, for resonance, $(D_c^2 - \lambda^2)v_1 = 0$ and $(D_c^2 - \lambda^2)^2 v_2 = 0$ (3.25a)

and for non-resonance, $(D_c^2 - \lambda_1^2)v_1 = 0$ and $(D_c^2 - \lambda_2^2)v = 0$ (3.25b)

(3.25a) will yields the solutions as for the case of resonance,

$v_1 = K_1(\lambda r)$ and $v_2 = rK_1^{'}(\lambda r)$ (3.26a)

(3.25b) will yields the solutions as for the case of non-resonance,

$v_1 = K_1(\lambda_1 r)$ and $v_2 = K_1(\lambda_2 r)$ (3.26b)

The results given below are important to observe.

In the case of resonance:

$\mathrm{D}_c^2 v_1 = \lambda^2 v_1$ and $\mathrm{D}_c^2 v_2 = 2\lambda v_1 + \lambda^2 v_2$ (3.27a)

In the case of non-resonance:

$\mathrm{D}_c^2 v_1 = \lambda_1^2 v_1$ and $\mathrm{D}_c^2 v_2 = \lambda_2^2 v_2$ (3.27b)

From (3.21), micro-rotation is given by

In the case of resonance:

$$\frac{\lambda^4}{i\varpi R_0} cC = \left(\frac{d}{dr} + \frac{1}{\mathrm{r}}\right)\left(\left(\frac{\lambda^4}{i\varpi R_0} - \lambda^2\right) v + a_2 2\lambda v_1\right) \tag{3.28a}$$

For the case of resonance, this reduces to:

$$cC = \left(\frac{iR_0\varpi}{\lambda^2} - 1\right)\left\{a_1\lambda K_0(\lambda r) - a_2\frac{1+\lambda^2 r^2}{\lambda r}K_1(\lambda r)\right\} - a_2\frac{2iR_0\varpi}{\lambda^2}K_0(\lambda r)$$

In the case of non-resonance:

$$cC = -\{a_1\left(\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi} - \lambda_2^2\right)\lambda_1 K_0(\lambda_1 r) + a_2\left(\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi} - \lambda_1^2\right)\lambda_2 K_0(\lambda_2 r) \qquad (3.28b)$$

The coefficients a_1 , a_2 are obtained with the help of boundary conditions (3.22) and (3.23) as below

In the case of resonance:

$$\begin{bmatrix} K_1(\lambda) & K_1'(\lambda) \\ c_0\lambda K_0(\lambda) & -\frac{(1+\lambda^2)}{\lambda}K_1(\lambda) - 2\lambda^2 K_0(\lambda) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1+\frac{p_0}{i\varpi} \\ \frac{c\lambda^2}{i\varpi R_0 - \lambda^2} \end{bmatrix} \qquad (3.29a)$$

In the case of non-resonance:

$$\begin{bmatrix} K_1(\lambda_1) & K_1(\lambda_2) \\ \left(1-\frac{\lambda_1^2}{i\varpi R_0}\right)\lambda_2 K_0(\lambda_1) & \left(1-\frac{\lambda_2^2}{i\varpi R_0}\right)\lambda_1 K_0(\lambda_2) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1+\frac{p_0}{i\varpi} \\ \frac{c\lambda_1^2\lambda_2^2}{i\varpi R_0} \end{bmatrix} \qquad (3.29b)$$

Hence from (3.29) we can calculate a_1 and a_2 and hence velocity v and micro-rotation C are known.

3.4.1 Skin friction acting on the cylinder per length L

Skin friction acting on the circular cylinder is

$$c_f = \frac{2T_{r\theta}}{\rho V_0^2} \qquad (3.30)$$

$T_{r\theta}$ is obtained as follows

From (3.4), for Micro-polar fluids stress component is

$$T_{ij} = -P\delta_{ij} + \frac{1}{2}(2\mu + k)\left(q_{i,j} + q_{j,i}\right) + ke_{ijr}(w_r - l_r)$$

In cylindrical co-ordinate system,

$$u_{i,j} + u_{j,i} = \begin{bmatrix} 2\frac{\partial U}{\partial R} & \frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R} & \frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z} \\ \frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R} & \frac{2}{R}\left(U + \frac{\partial V}{\partial \theta}\right) & \frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z} \\ \frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z} & \frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z} & 2\frac{\partial W}{\partial Z} \end{bmatrix} \tag{3.31}$$

For this present problem, $$u_{i,j} + u_{j,i} = \begin{bmatrix} 0 & \frac{\partial V}{\partial R} - \frac{V}{R} & 0 \\ \frac{\partial V}{\partial R} - \frac{V}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.32}$$

Substituting (3.32) in (3.4) we get

$$T_{R\theta} = (\mu + k)\frac{\partial V}{\partial R} - \mu\frac{V}{R} - kC \tag{3.33}$$

By using non-dimensional scheme (3.7) and (3.8), we get

$$T_{rz} = \frac{(\mu + k)V_0}{a}\left\{\frac{dv}{dr} - (1 - c)\frac{v}{r} - cC\right\}e^{i\varpi t} \tag{3.34}$$

Substituting (3.34) in (3.30), the Skin friction acting on the circular cylinder (after removing the factor $e^{i\varpi t}$) is obtained as

$$c_f = \frac{2}{R_0}\left(\frac{dv}{dr} - 1\right) \tag{3.35}$$

which for the case of resonance gives

$$c_f = \frac{2}{R_0}\left[a_1\lambda K_1'(\lambda) + a_2\frac{(1+\lambda^2)}{\lambda}K_1(\lambda) + \frac{p_0}{i\varpi} - 1\right] \tag{3.36a}$$

and the case of non-resonance case,

$$c_f = \frac{2}{R_0}\left[a_1\lambda_1 K_1'(\lambda_1) + a_2\lambda_2 K_1'(\lambda_2) + \frac{p_0}{i\varpi} - 1\right] \tag{3.36b}$$

3.5 Results and Discussions

In the case of material resonance, the value of λcannot be assumed randomly. In this case, the values of λ are obtained from (3.17) by solving the following equation for x.

$$x^2 - [(2-c)s + i(j + \varpi R_0)]x + i\varpi R_0(ij + 2s) = 0 \tag{3.37}$$

Hence the values of λ for resonance are given by

$$\lambda = \sqrt{x} = \sqrt{\frac{(2-c)s + i(j + \varpi.R_0)}{2}} \tag{3.38}$$

The above equation (3.37) consists of 5 parameters related by two equations as in (3.19). Hence three parameters can be chosen arbitrary, i.e. independent. Here ϖ, R_0 and c are taken arbitrarily, with $0 \leq c \leq 1$, $R_0 \ll 1$ and $\varpi \gg 1$ so that $\varpi.R_0$ is not negligibly small (say>1). With this choice of values of R_0, the convective terms can be neglected by keeping local time derivative as it is. By assuming the values of c, R_0 and ϖ, the values of s and J are obtained from (3.19) and then the value of λ is obtained from (3.38). In the case of non-resonance, all 5 parameters are independent. The values of the constants a_1 and a_2 are obtained by substituting the values of λ(complex in general) in (3.38). The solutions obtained here are in agreement with the results of [10] as a special case when the longitudinal oscillations are not present. *It can be noted that the case of resonance will not occur for viscous fluids*.

By keeping |λ| fixed, the restriction on the parameters c, s increases. We can observe from the Fig 3.2 that the values of s for the case of resonance are much smaller than the case of non-resonance for a particular |λ| value.

3.5.1 Skin friction

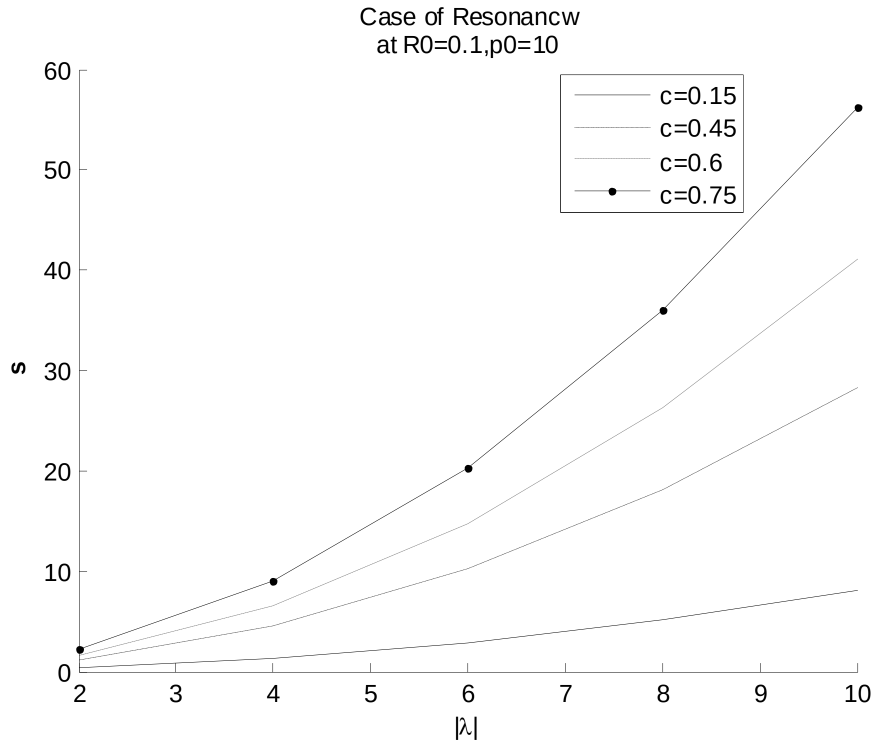
Skin friction (after removing the oscillation factor $e^{i\sigma t}$) acting on the surface of the cylinder is shown in Fig 3.3 and 3.4. From Fig 3.3, we note that when the Reynolds number R_0 is small, Skin friction is high and as R_0 increases Skin friction drastically decreases. Again the Skin friction for non-resonance case is much higher the Skin friction in the case of resonance. From Fig 3.4, when |λ| is fixed, in the case of non-resonance, the Skin friction is very high and is almost a constant for a given cross viscosity parameter c. But in the case of resonance, the skin fiction increases as |λ| increases. Whether |λ| is fixed or not, in any case, the Skin friction for the case of resonance is much less than the case of non-resonance. This is one important observation which may be useful for industrial applications. By varying the concentration of additives, the material parameters can be adjusted in such way that resonance case can be created and Skin friction on the surfaces can be reduced drastically.

3.5.2 Velocity

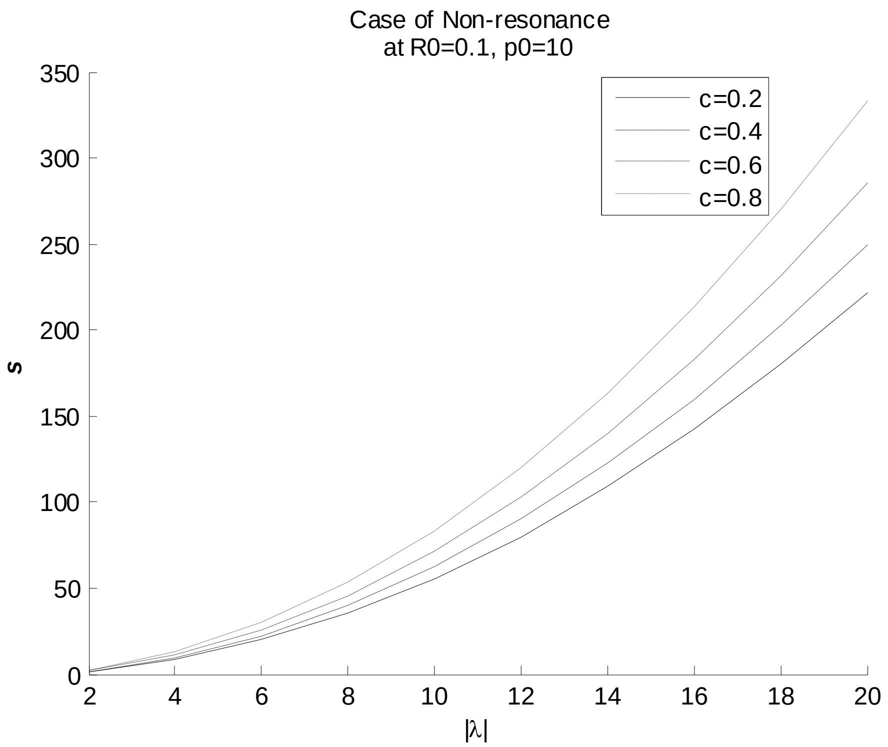
Velocity profiles (after removing the oscillation factor $e^{i\sigma t}$) are shown in the Fig 3.5 and Fig 3.6. In Fig 3.5, it is observed that the variation in velocity is very negligible with respect to the case of non-resonace, i.e the velocity field is almost same for the case of resonance and non-resonance. But we can observe that in both cases as Reynolds number increases, velocity increases. But from Fig 3.6, it can be observed that the velocity is numerically much higher in case of non-resonance than in the case of resonance. This may be reason that Skin friction is high in the case of non-resonance.

3.5.3 Micro-rotation

From Fig 3.7, it can be observed that in the case of non-resonance, micro-rotation near to the cylinder is more than one and decreases to a minimum and then goes to zero as distance r increases. But in the case of resonance, micro-rotation never exceeds one and decreases as Reynolds number increases and goes to zero very fast. In Fig 3.8, it can be observed that, in the case of non-resonance, micro-rotation shoots up near to the surface of the cylinder when $|\lambda|$ is fixed. But in the case of resonance, micro-rotation drastically decreases as $|\lambda|$ increases and vanishes very near to the surface. This may be reason that Skin friction is much smaller in the case of resonance.

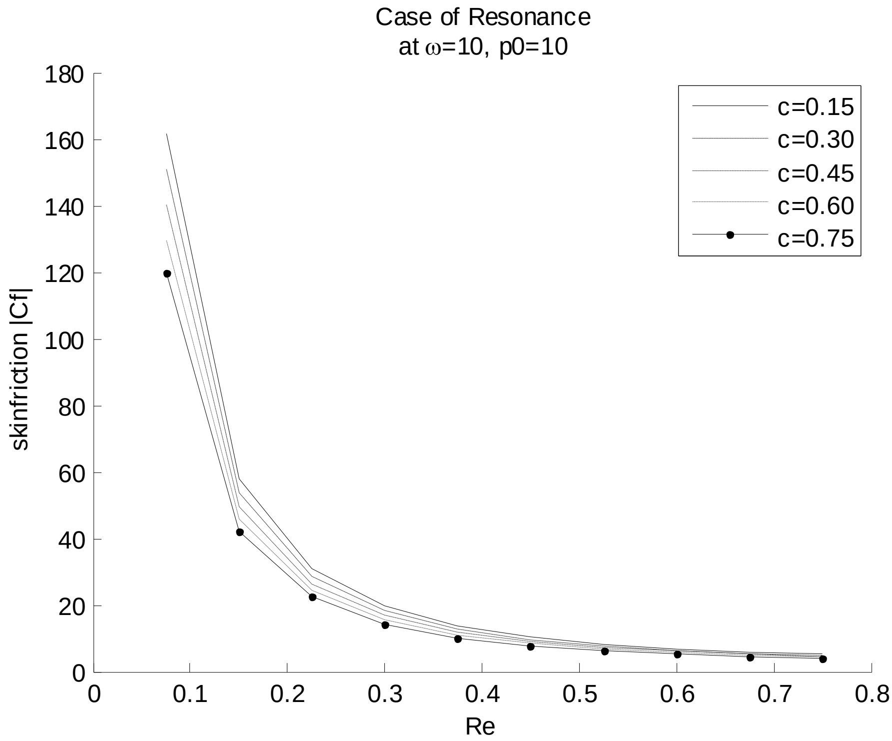


(a)

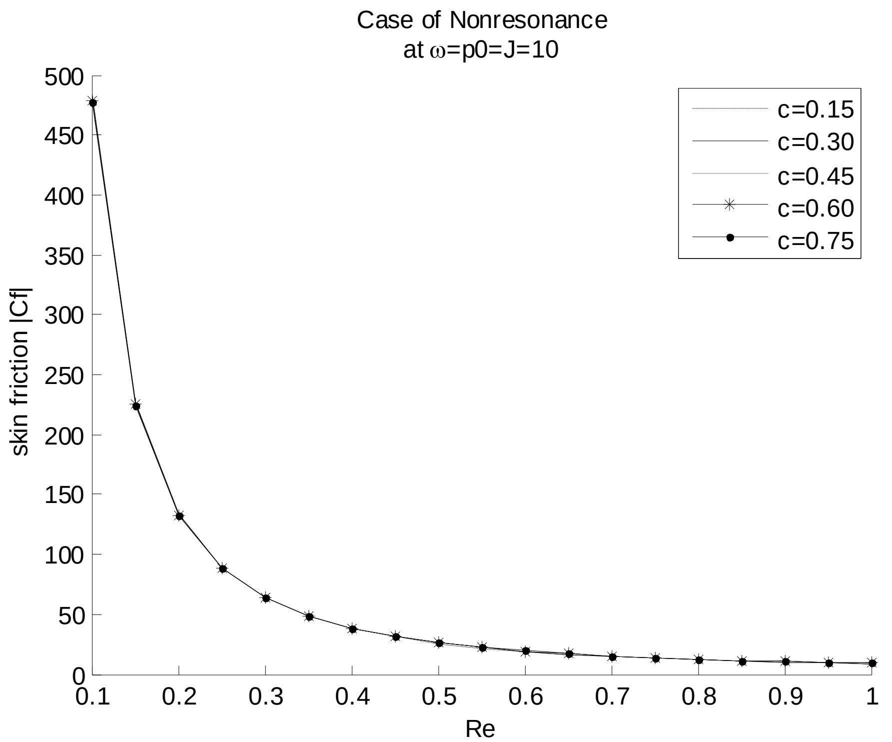


(b)

Fig 3.2 Couple-stress parameter s Vs length (or geometric) parameter $|\lambda|$ in the the case of (a) resonance and (b) non- resonance.

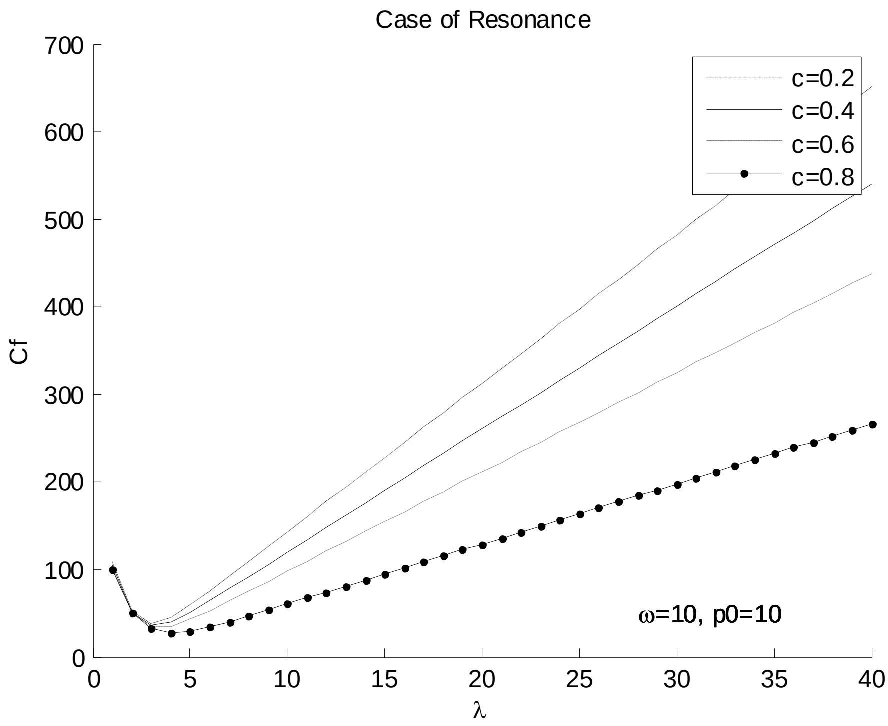


(a)

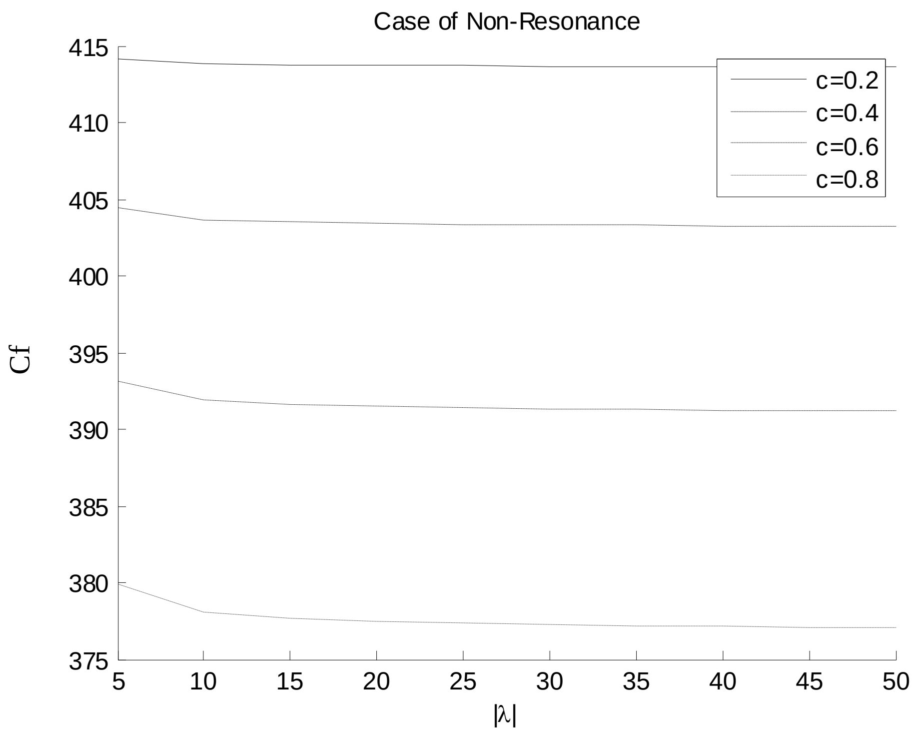


(b)

Fig 3.3 Skin friction Vs Reynolds number in the the case of (a) resonance and (b) non- resonance.

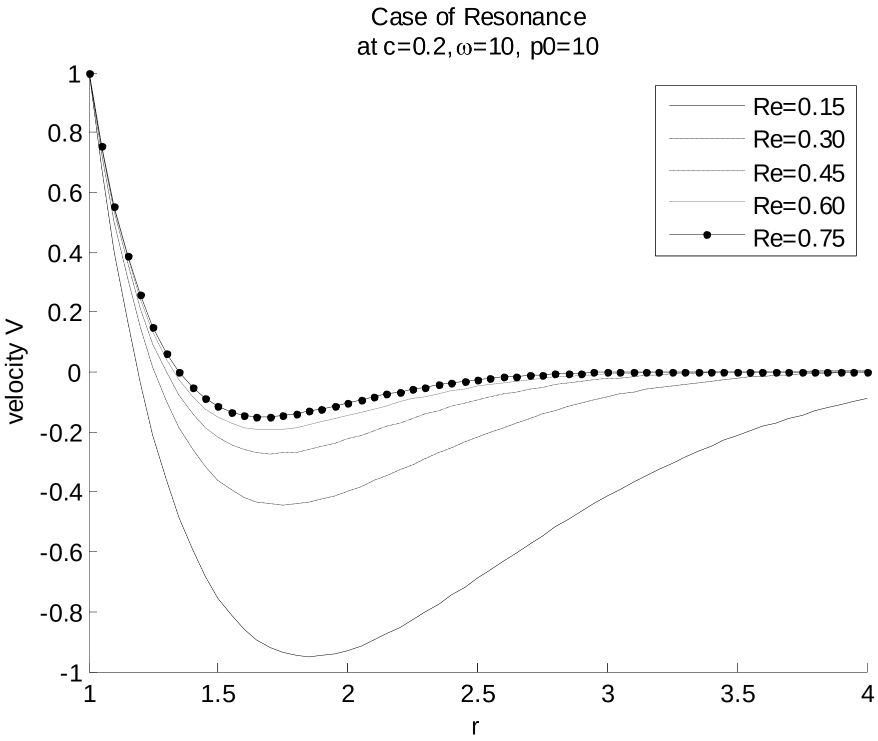


(a)

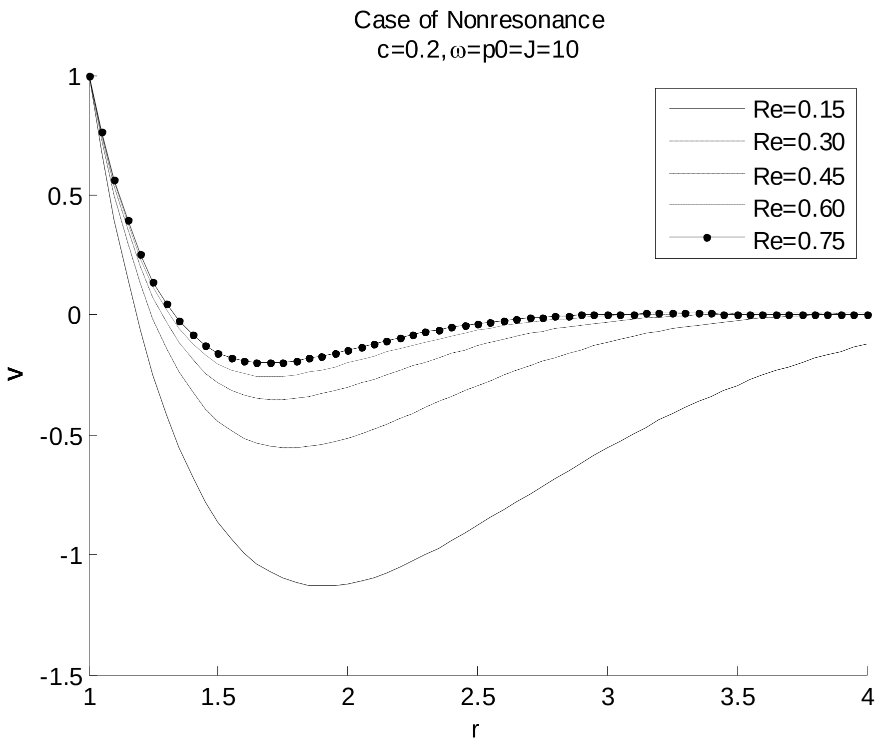


(b)

Fig 3.4 Skin friction Vs length (or geometric) parameter $|\lambda|$ in the case of (a) resonance and (b) non- resonance.

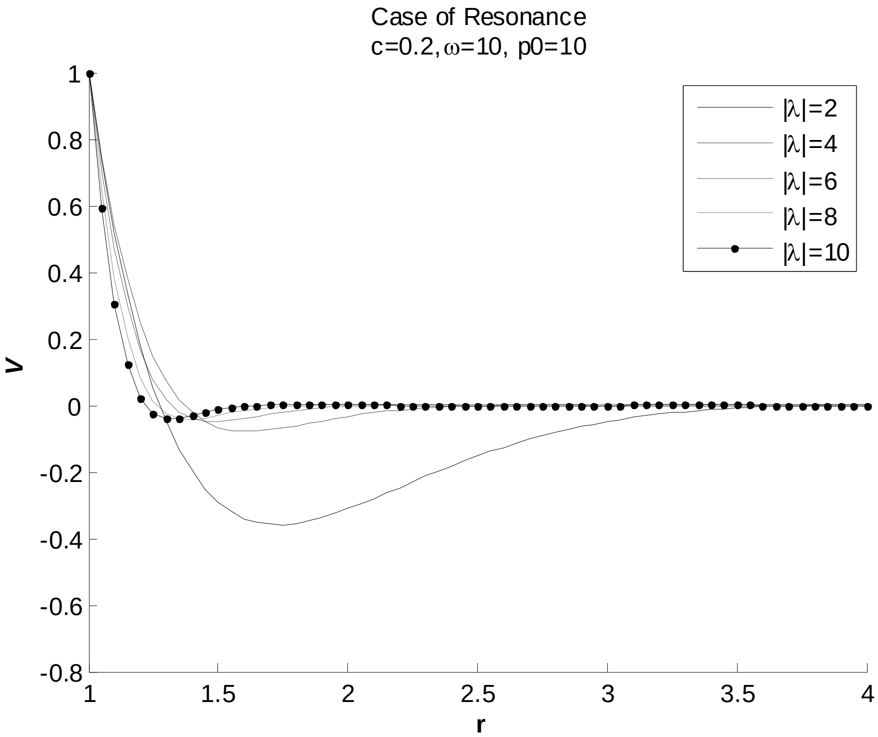


(a)

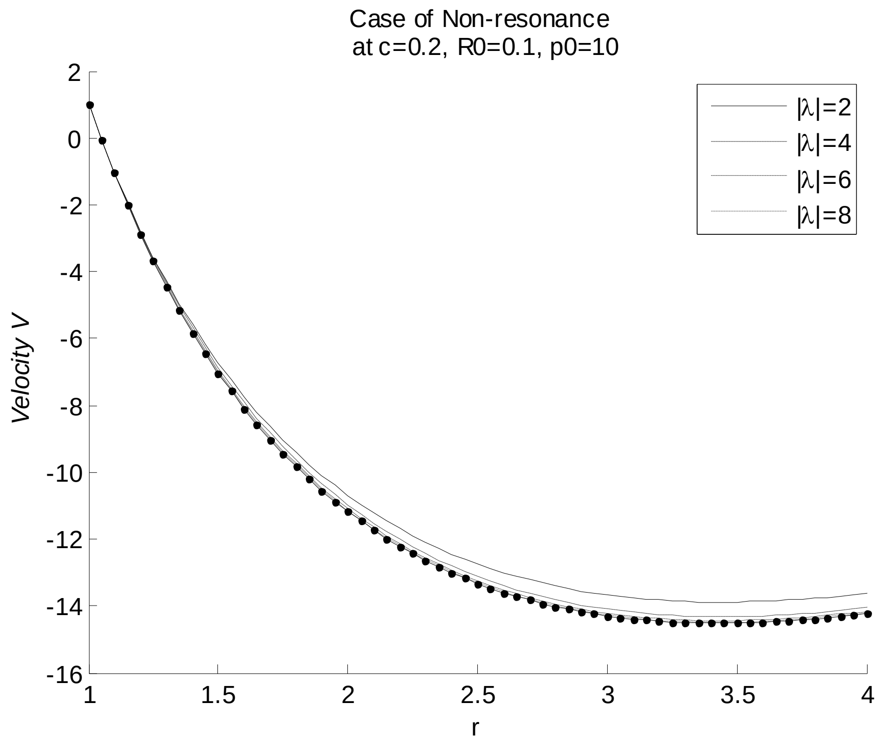


(b)

Fig 3.5 Real value of velocity Vs distance r at different Reynolds numbers for the case of (a) resonance and (b) non- resonance.

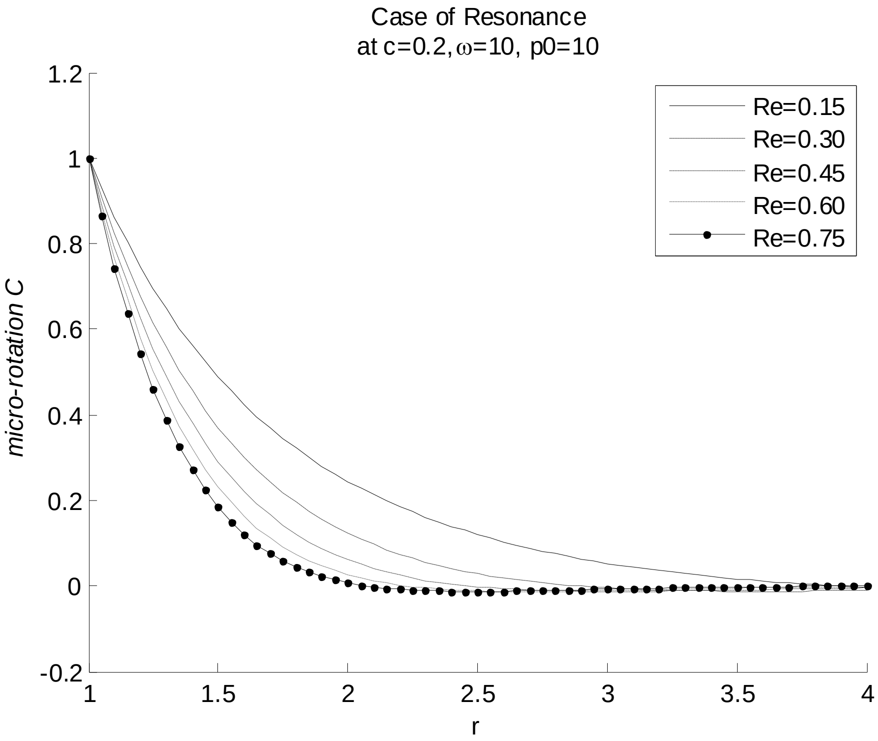


(a)

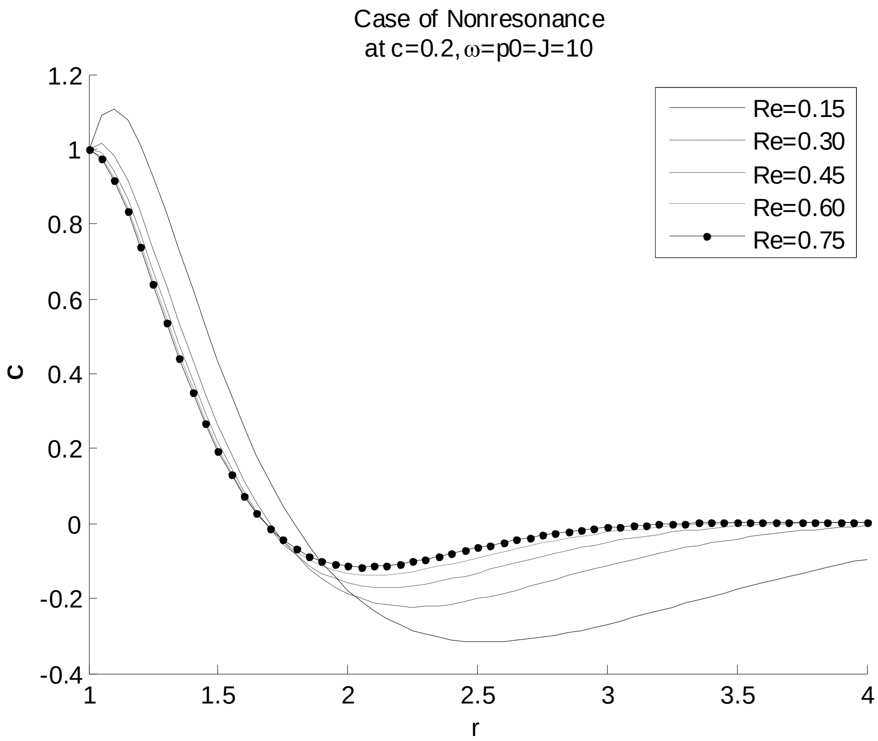


(b)

Fig 3.6 Real values of velocity at different values of length parameter $|\lambda|$ for the case of (a) resonance and (b) non- resonance.

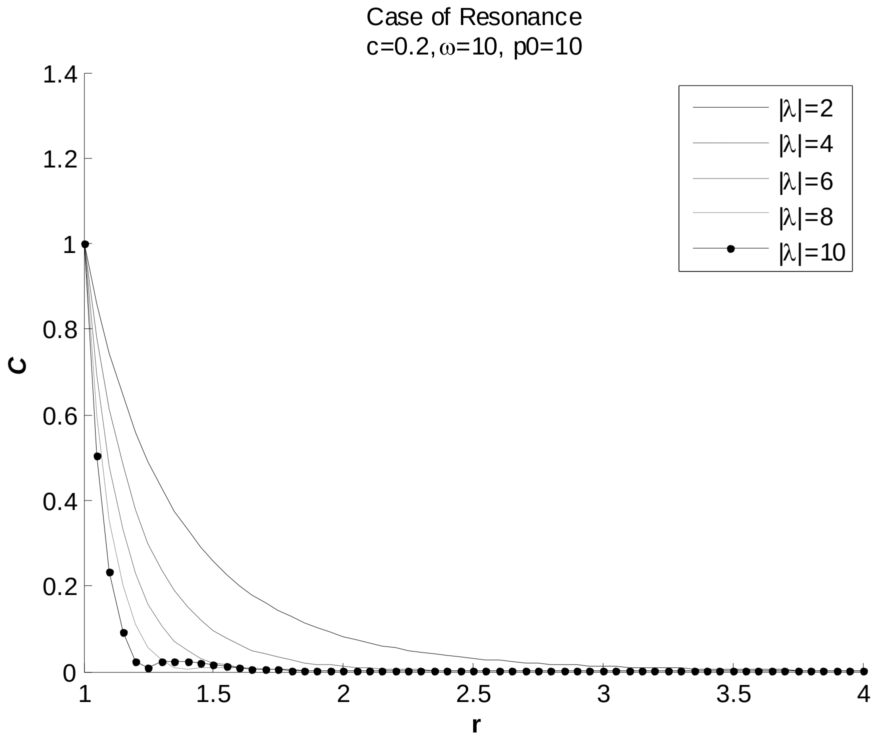


(a)

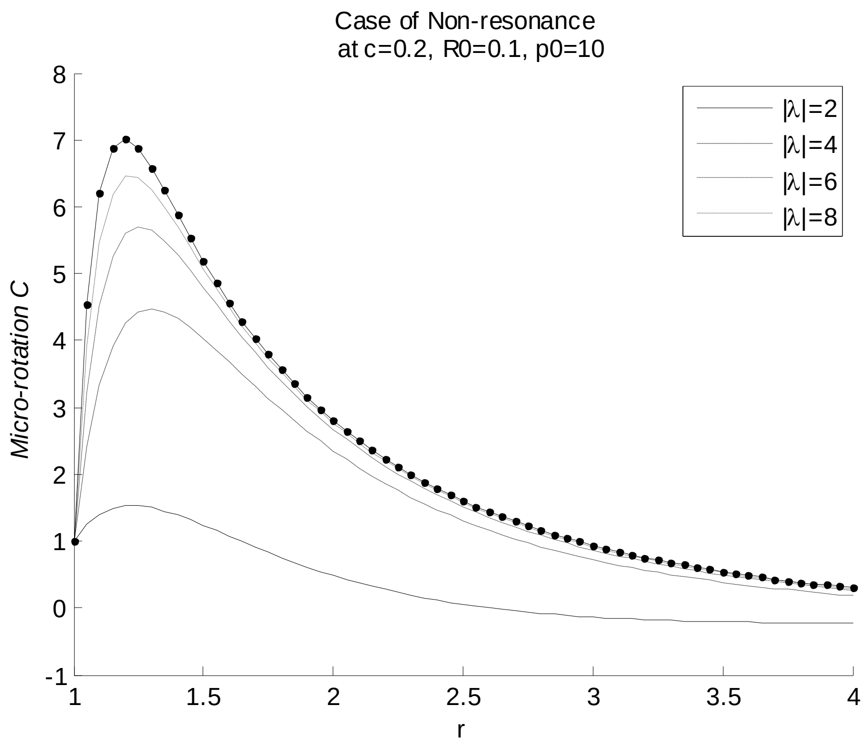


(b)

Fig 3.7 Real part of Micro-rotation at different values of Reynolds numbers for the case of (a) resonance and (b) non- resonance.



(a)



(b)

Fig 3.8 Real part of Micro-rotation at different values of length parameter $|\lambda|$ for the case of (a) resonance and (b) non- resonance.

3.6 Conclusions

In this chapter, rotary oscillations of a circular cylinder about its axis in an incompressible micro-polar fluid are considered, when the physical parameters satisfy a “resonance” condition. Clearly there is much difference between the case of non-resonance and resonance. It is observed that (after removing the oscillation factor $e^{i\sigma t}$).

1. Skin friction is much smaller in the case of resonance than in the case of non-resonance.
2. The velocity field is similar in the case of non-resonance and resonance.
3. The micro-rotation field vanishes quickly near to the surface in the case of resonance. But in the case of non-resonance, micro-rotation shoots up near to the cylinder.

Chapter 4

Longitudinal oscillations of a Circular Cylinder in Micro-polar fluid

The problem of the longitudinal oscillations of a circular cylinder along its axis of symmetry in an incompressible Micro-polar fluid and the flow generated due to these oscillations in the fluid is considered. The Stokes flow is considered by neglecting nonlinear convective terms in the equations of motion on the assumption that the flow is so slow that Reynolds number is less than unity. Here we get a rare, but distinct special case referred to as *resonance* in which material constants are inter related in a particular way. The velocity and micro-rotation components of the flow for the case of *resonance* and *non-resonance* are obtained. The Skin friction acting on the cylinder is evaluated and the effect of physical parameters like Micro-polarity and Couple stress parameter on the Skin friction due to oscillations is shown through graphs.

4.1 Introduction

There is a vast literature available on Stokesian flows generated in Micro-polar fluids over the past half a century ever since the classical theory of Micro-polar fluids was formulated by Eringen (1966). Ariman (1967, 1970) studied Micro-polar fluid flows between two concentric cylinders and fluids with micro-structures. Eringen (1964, 1990) studied theory of simple Micro-polar fluids, theory of thermo-microstretch fluids and bubbly liquids. Liu (1971) initiated instability in Micro-polar fluids. Ramkissoon (1976, 1977) examined Micro-polar fluid flow of axially symmetric body. Ravidran (1972) studied simple oscillatory flows in polar fluids. Later Vijay Kumar Stokes (1984) explained the theories of fluids with micro-structures in this book. Oscillatory flows of circular cylinder in various fluids were studied by many authors like Frater (1968), Lakshmana rao (1972). Ramkissoon et al.

(1990), Rao et al. (1992), Calmelet-Eluhu et al. (1998), Anwar (2004), Fetecau et al. (2006), Mehrdad Massoudi et al. (2008), Ramana Murthy et al. (2010).

With the aim of obtaining Drag or Couple, Lakshmana Rao et al. in (1971, 1972, 1983, 1987) studied the oscillatory flows in the case of a circular cylinder, sphere, spheroid and elliptic cylinder in incompressible Micro-polar fluids. However, in all these above problems, a special case, named as "Resonance" type flow that arises when the material parameters of the fluids are related in a special form (will be defined later) have not been investigated until recently. The rare but distinct possibility of this type of resonance flows has been noticed in Lakshmana Rao et al. in (1983, 1987). This case arises in the papers of Ramana Murthy (2011), Aparna (2012), Nagaraju (2014), but the case of Resonance was not attempted by the authors. In the present chapter, we propose to investigate this case of resonance type flow, in Micro-polar fluids, due to longitudinal oscillations of a circular cylinder along its axis of symmetry.

4.2 Basic Equations

The basic equations of motion for incompressible Micro-polar fluids as introduced by Eringen (1966), are given by

$$\frac{\partial \rho}{\partial \tau} + div(\rho \bar{Q}) = 0 \tag{4.1}$$

$$\rho\left(\frac{\partial \bar{Q}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{Q}\right) = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{4.2}$$

$$\rho j\left(\frac{\partial \bar{l}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{l}\right) = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1 . \bar{l}) \tag{4.3}$$

The constitutive equations for the stress components T_{ij} and Couple stress components M_{ij} for an incompressible Micro-polar fluid are given by

$$T_{ij} = -P\delta_{ij} + \frac{1}{2}(2\mu + k)\left(u_{i,j} + u_{j,i}\right) + ke_{ijr}(w_r - l_r) \tag{4.4}$$

$$M_{ij} = \alpha l_{i,i}\delta_{i,j} + \beta l_{i,j} + \gamma l_{j,i} \tag{4.5}$$

where w_r = r th component of ½(curl $\bar{Q}$) and e_{ijr} is permutation tensor = 0 if any two indices are equal and = 1 if i,j, r are cyclic and = –1 if i,j,r are acyclic. Neglecting the nonlinear convective terms in (6.2) and (6.3), the linearised version of the equations are given by,

$$div\bar{Q} = 0 \tag{4.6}$$

$$\rho \frac{\partial \bar{Q}}{\partial \tau} = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{4.7}$$

$$\rho \mathrm{j} \frac{\partial \bar{l}}{\partial \tau} = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1 . \bar{l}) \tag{4.8}$$

4.3 Statement and Formulation of the Problem

A circular cylinder of radius a and, of infinite length is performing longitudinal oscillations with velocity $W_0 \boldsymbol{e_z} e^{i\sigma\tau}$ along its axis of symmetry in an infinite vat containing incompressible Micro-polar fluid. A cylindrical coordinate system (R, θ, Z) with base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ with origin on the axis of the cylinder is considered. Since the flow is generated by these oscillations, the fluid velocity will be in cross sectional plane of the cylinder containing the base vectors $(\boldsymbol{e_R}, \boldsymbol{e}_z)$. We assume the flow is axially symmetric and hence the velocity and micro-rotation are assumed in the following form:

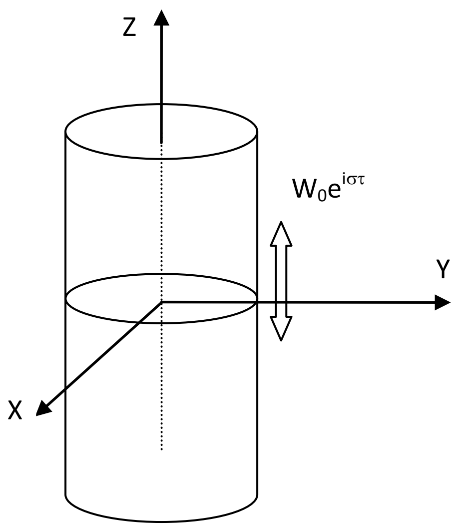


Fig 4.1 Geometry of the oscillating cylinder

$$\boldsymbol{Q} = \boldsymbol{q}W_0 = e^{i\sigma\tau} W(R)\boldsymbol{e_z} \text{ and } \boldsymbol{l} = \boldsymbol{v}\frac{W_0}{a} = e^{i\sigma\tau}\mathcal{B}(R)\boldsymbol{e}_\theta \quad (4.9)$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate the corresponding non-dimensional quantities.

$$\mathrm{R} = ar, \mathrm{W} = w\mathrm{w}_0, \mathcal{B} = \mathrm{B}\frac{W_0}{a}, \mathrm{P} = p\rho\mathrm{w}_0^2, \mathrm{Z} = a\mathrm{z}, \ \tau = \frac{a\mathrm{t}}{\mathrm{w}_0} \quad (4.10)$$

The following non-dimensional parameters are introduced viz, J is gyration parameter, ϖ is frequency parameter, s is Couple stress parameter, c is cross viscosity or Micro-polarity parameter and R_0 is oscillations Reynolds number for Micro-polar fluids.

$$J = \frac{\mathrm{j}\rho\mathrm{w}_0 a}{\gamma}, \ \varpi = \frac{a\sigma}{\mathrm{w}_0}, \ s = \frac{\mathrm{k}a^2}{\gamma}, \ c = \frac{\mathrm{k}}{\mu+\mathrm{k}} \text{ and } R_0 = \frac{\rho w_0 a}{\mu+k} \quad (4.11)$$

Substituting this non-dimensional scheme (4.10) and non-dimensional parameters (4.11), the equations of motion (4.7) and (4.8) are reduced to

$$R_0\frac{\partial\bar{q}}{\partial t} = -R_0\nabla p + c\nabla\times\bar{v} - \nabla\times\nabla\times\bar{q} \quad (4.12)$$

$$\frac{j}{\varpi}\frac{\partial\bar{v}}{\partial t} = -2s\bar{v} + s\nabla\times\bar{q} - \nabla\times\nabla\times\bar{v} \quad (4.13)$$

Further, by the choice of the velocity field in (4.9), the equations of motion are reduced to

$$iR_0\varpi w = -R_0p_0 + \frac{c}{r}\frac{d}{dr}(rB) + \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) \quad (4.14)$$

$$iJB = -2sB - s\frac{dw}{dr} + \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rB)\right) \quad (4.15)$$

where $p_0 = \frac{dp}{dz}$ is constant pressure gradient along z direction.

From (4.14)

$$-\frac{c}{r}\frac{d}{dr}(rB) = -R_0p_0 + w'' + \frac{1}{r}w' - i\varpi R_0 w \quad (4.16)$$

From (4.15)

$$(2s + ij)B = -sw' + D_c^2 B \quad (4.17)$$

where $D_c^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ (4.18)

Differentiating on (4.16) to eliminate p_0, we get

$$-cD_c^2 B = \frac{d}{dr}\left(w'' + \frac{1}{r}w'\right) - i\varpi R_0 w' \quad (4.19)$$

Substituting (4.19) in (4.17), we get

$$c(2s + iJ)B = -\frac{d}{dr}\left(w'' + \frac{1}{r}w'\right) + (i\varpi R_0 - cs)w' \quad (4.20)$$

We can write (4.19) and (4.20) as

$$-cD_c^2 B = D_c^2 w' - i\varpi R_0 w' \quad (4.21)$$

$$c(2s + iJ)B = -D_c^2 w' + (i\varpi R_0 - cs)w' \quad (4.22)$$

Eliminating B from (4.21) and (4.22) we get

$$(D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)w' = 0 \quad (4.23)$$

Where $\lambda_1^2 + \lambda_2^2 = (2 - c)s + i(J + \varpi R_0)$ and $\lambda_1^2\lambda_2^2 = i\varpi R_0(2s + iJ)$ (4.24)

The solution for w' if $\lambda_1 \neq \lambda_2$ in (4.23) is given in Nagaraju (2014) (which can be obtained as a special case). The solution for w′ for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \to \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation:

$$\frac{\gamma}{j} = \frac{(2\mu + k)(\mu + k)}{2\mu + 3k} \text{ and } \rho\sigma = \frac{(2\mu + k)k + \gamma\rho\sigma}{j(\mu + k)} \quad (4.25)$$

In non-dimensional form, these conditions are given by

$$(2 - c)s = J - R_0\varpi \text{ and } (2 - c)J = (2 + c)\varpi R_0 \quad (4.26)$$

In this chapter, we are interested in the solution for *w* and B for the case of *resonance*. In this case, the equations for *w* and B are given by

$$(D_c^2 - \lambda^2)^2 w' \text{ and } cB = -\frac{i\varpi R_0}{\lambda^4}(D_c^2 - 2\lambda^2)w' - w' \quad (4.27a)$$

and for the case of non-resonance

$$\left(D_c^2 - {\lambda_1}^2\right)\left(D_c^2 - {\lambda_2}^2\right)w' = 0 \text{ and } cB = -\frac{i\varpi R_0}{\lambda_1^2\lambda_1^2}(D_c^2 - \lambda_1^2 - \lambda_2^2)w' - w' \qquad (4.27b)$$

Substituting these equations in (4.14), we get

In the case of resonance:

$$iR_0\varpi w = -R_0 p_0 - \frac{i\varpi R_0}{\lambda^4 r}\frac{d}{dr}\{r(D_c^2 - 2\lambda^2)w'\} \qquad (4.28a)$$

In the case of non-resonance:

$$iR_0\varpi w = -R_0 p_0 - \frac{i\varpi R_0}{\lambda_1^2\lambda_2^2 r}\frac{d}{dr}\{r(D_c^2 - \lambda_1^2 - \lambda_2^2)w'\} \qquad (4.28b)$$

4.3.1 Boundary conditions

By no-slip condition, (the non-dimensional velocity of a fluid particle on the circular cylinder Γ is same as the velocity of cylinder i.e w=1) and by hyper-stick condition, (the micro-rotation vector of a particle on the cylinder is ½ of angular velocity of the particle on the cylinder i.e $B = \frac{1}{2}(Curl\, Q_\Gamma)_\theta$ (where the suffix represents the component along that direction θ) i.e B=0 and hence we have;

$$On r = 1; \quad w = 1 \text{ and} B = 0 \qquad (4.29)$$

4.4 Solution of the Problem

Since the equation for w' is linear, the general solution for w' is linear combination of individual solutions of factors in the differential operator. Solution for w' is assumed in the form

$$w' = a_1 w'_1 + a_2 w'_2 \qquad (4.30)$$

Where w'_1 *and* $w'_2$are satisfies the following equations

In the case of resonance:

$$(D_c^2 - \lambda^2)w'_1 = 0 \text{ and } (D_c^2 - \lambda^2)^2 w'_2 = 0 \qquad (4.31a)$$

In the case of non-resonance:

$$(D_c^2 - \lambda_1^2)w'_1 = 0 \text{ and } (D_c^2 - \lambda_2^2)w'_2 = 0 \tag{4.31b}$$

These will yields the solutions as below

In the case of resonance:

$$w'_1 = K_1(\lambda r) \text{ and } w'_2 = rK'_1(\lambda r) \tag{4.32a}$$

In the case of non-resonance:

$$w'_1 = K_1(\lambda_1 r) \text{ and } w'_2 = K_1(\lambda_2 r) \tag{4.32b}$$

The following results are useful to note.

$$D_c^2 w'_1 = \lambda^2 w'_1 \text{ and } D_c^2 w'_2 = (2\lambda w'_1 + \lambda^2 w'_2) \tag{4.33a}$$

The following results are useful to note in case of non resonance.

$$D_c^2 w'_1 = \lambda_1^2 w'_1 \text{ and } D_c^2 w'_2 = \lambda_2^2 w'_2 \tag{4.33b}$$

Substituting (4.32a) in (4.28a) we get w for the case of resonance as

$$w = -\frac{p_0}{i\varpi} - \frac{a_1}{\lambda} K_0(\lambda r) + \frac{a_2}{\lambda^2}\left(\lambda r K_1(\lambda r) + K_0(\lambda r)\right) \tag{4.34a}$$

Substituting (4.32b) in (4.28b) we get w for the case of non-resonance as

$$w = -\frac{p_0}{i\varpi} - a_1 \frac{K_0(\lambda_1 r)}{\lambda_1} - a_2 \frac{K_0(\lambda_2 r)}{\lambda_2} \tag{4.34b}$$

Substituting (4.32a) in (4.27a) we get B for the case of resonance as

$$cB = \left(\frac{i\varpi R_0}{\lambda^2} - 1\right)(a_1 K_1(\lambda r) + a_2 r K'_1(\lambda r)) - \frac{2a_2 i\varpi R_0}{\lambda^3} K_1(\lambda r) \tag{4.35a}$$

Substituting (4.32b) in (4.27b) we get B for the case of non-resonance as

$$cB = \left(\frac{i\varpi R_0}{\lambda_1^2} - 1\right) a_1 K_1(\lambda_1 r) + \left(\frac{i\varpi R_0}{\lambda_2^2} - 1\right) a_2 K_1(\lambda_2 r) \tag{4.35b}$$

The constants a_1, a_2 are obtained from the boundary conditions (4.29) as follows:

In the case of resonance:

$$\begin{bmatrix} -\lambda K_0(\lambda) & \lambda K_1(\lambda) + K_0(\lambda) \\ K_1(\lambda) & K'_1(\lambda) + \frac{2i\varpi R_0}{\lambda(\lambda^2 - i\varpi R_0)} K_1(\lambda) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \lambda^2 \left(1 - \frac{ip_0}{\varpi}\right) \\ 0 \end{bmatrix} \tag{4.36a}$$

In the case of non-resonance:

$$\begin{bmatrix} -\frac{K_0(\lambda_1)}{\lambda_1} & -\frac{K_0(\lambda_2)}{\lambda_2} \\ c_1 K_1(\lambda_1) & c_2 K_1(\lambda_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{ip_0}{\varpi} \\ 0 \end{bmatrix} \tag{4.36b}$$

Where $c_1 = \frac{i\varpi R_0}{\lambda_1^2} - 1$ and $c_2 = \frac{i\varpi R_0}{\lambda_2^2} - 1$ (4.37)

From (4.36a) and (4.36b), we can calculate $a_1$and a_2 for both the cases.

Hence velocity component w and micro-rotation component B are known.

4.4.1 Skin friction acting on the cylinder per length L

Skin friction acting on the circular cylinder is

$$c_f = \frac{2T_{rz}}{\rho w_0^2} \tag{4.38}$$

T_{rz} obtained as follows:

In cylindrical co-ordinate system,

$$\mathrm{E} = \nabla \mathrm{Q} + (\nabla \mathrm{Q})^{\mathrm{T}} = \begin{bmatrix} 2\frac{\partial U}{\partial R} & \frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R} & \frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z} \\ \frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R} & \frac{2}{R}\left(U + \frac{\partial V}{\partial \theta}\right) & \frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z} \\ \frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z} & \frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z} & 2\frac{\partial W}{\partial Z} \end{bmatrix} \tag{4.39}$$

For this present problem, $E = \begin{bmatrix} 0 & 0 & \frac{\partial W}{\partial R} \\ 0 & 0 & 0 \\ \frac{\partial W}{\partial R} & 0 & 0 \end{bmatrix}$ (4.40)

Substituting (4.40) in (4.4) we get

$$T_{RZ} = (\mu + k)\frac{\partial W}{\partial R} + kB \tag{4.41}$$

By using non-dimensional scheme (4.10), we get

$$T_{rz} = \frac{(\mu + k)W_0}{a}\left\{\frac{dw}{dr} + cB\right\}e^{i\varpi t} \tag{4.42}$$

Substituting (4.42) in (4.38), we get the Skin friction acting on the circular cylinder (after deleting the factor $e^{i\varpi t}$) is obtained as:

$$c_f = \frac{2}{R_o}\left(\frac{dw}{dr}\right)_{r=1} \quad (4.43)$$

In the case of resonance:

$$c_f = \frac{2}{R_0}[a_1 K_1(\lambda) + a_2 K'_1(\lambda)] \quad (4.44a)$$

In the non-resonance case:

$$c_f = \frac{2}{R_0}[a_1 \mathrm{K}_1(\lambda_1) + \mathrm{a}_2 K_1(\lambda_2)] \quad (4.44b)$$

4.5 Results and Discussions

For resonance case, the value of λ cannot be taken randomly. In the case of resonance the values of λ are obtained from (4.26) by solving the following equation for x.

$$x^2 - [(2-c)s + i(J + \varpi R_0)]x + i\varpi R_0(iJ + 2s) = 0 \quad (4.45)$$

Then in resonance case, the values of λ are given by

$$\lambda = \sqrt{x} = \sqrt{\frac{(2-c)s + i(J + \varpi.R_0)}{2}} \quad (4.46)$$

This equation involves 5 parameters which are related by two equations in (4.26). Hence we choose three parameters as an independent. Here ϖ, R_0 and c are selected independently, with $0 \leq c \leq 1$, $R_0 \ll 1$ and $\varpi \gg 1$ such that $\varpi.R_0$ is not negligibly small (say>1). For this range of values of R_0, the nonlinear convective terms can be neglected but local derivative is retained. After selecting c, R_0 and ϖ, the values of s and J are obtained from (4.26) and then λ is obtained from (4.46). In the case of non-resonance, all 5 parameters are independent. The values of λ are complex. These values for λ are substituted in (4.36a) and (4.36b) and then constants $a_1 and a_2$ are obtained.

4.5.1 Velocity

Velocity w in the direction Z axis is computed by using Eq. (30.1).

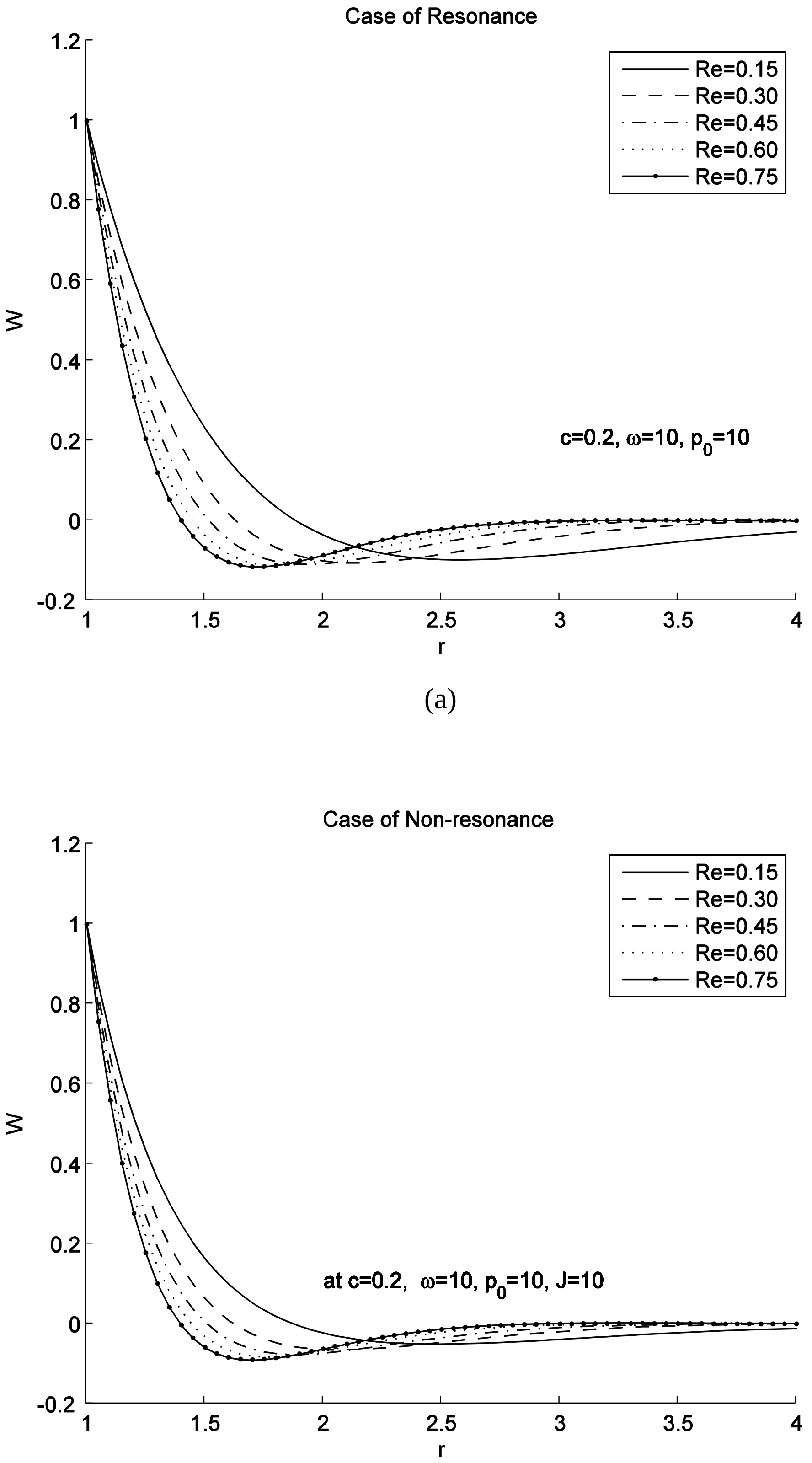
From Fig 4.2, it can be observed that as Reynolds number increases, the velocity w decreases near to the cylinder and then increases slightly and tends to zero at a distance four times the radius of the cylinder. In the case of resonance, the velocity w becomes zero at a longer distance than in the case of non-resonance. *Hence observe that an increase in Reynolds number, in the case of resonance, decreases the velocity w near to the cynilder and velocity vanishes at a longer distance than in the case of non-resonance.*

Similarly, from Fig 4.3, as Micro-polarity parameter *c* increases, in the case of resonance, velocity w increases in the range of distance 0.5 to 2.5. Whereas the effect of Micro-polarity *c* is negligible in the case of non-resonance. *Hence, the conclusion is that micropolarity parameter increases the velocity in the case of resonance and has no effect on velocity in non-resonance case.*

4.5.2 Micro-rotation

It is observed that Micro-rotation component B is positive, in the case of resonance, and becomes zero at a long distance from the cylinder. From Fig 4.4, we notice that as, Reynolds number increases, in the case of resonance, micro-rotation increases near to the cylinder. But in the case of non-resonance, as Reynolds number increases, micro-rotation increases from negative values to positive values and then soon becomes zero. *It can be concluded that in the case of resonance, micro-rotation vanishes at a long distance from the cylinder and in the case of non-resonace, it vanishes relatively near to the cylinder.*

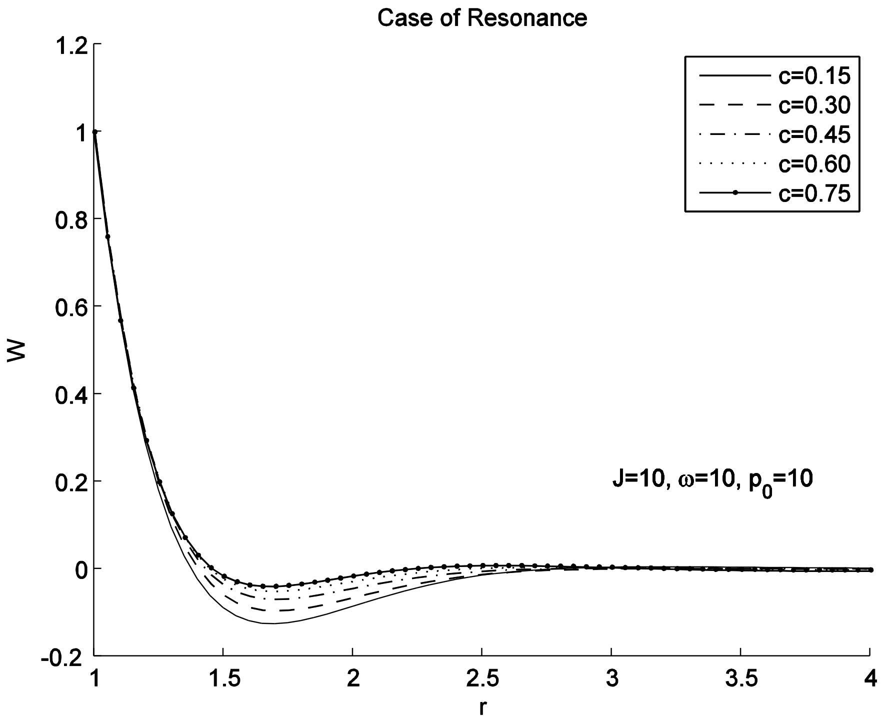
From Fig 4.5, it is observed that, in the case of resonance, as Micro-polarity increases, micro-rotation increases and is always positive. But in the case of non-resonance, micro-rotation decreases and increases from negative values to positive values and the effect of Micro-polarity on micro-rotation is almost negligible.



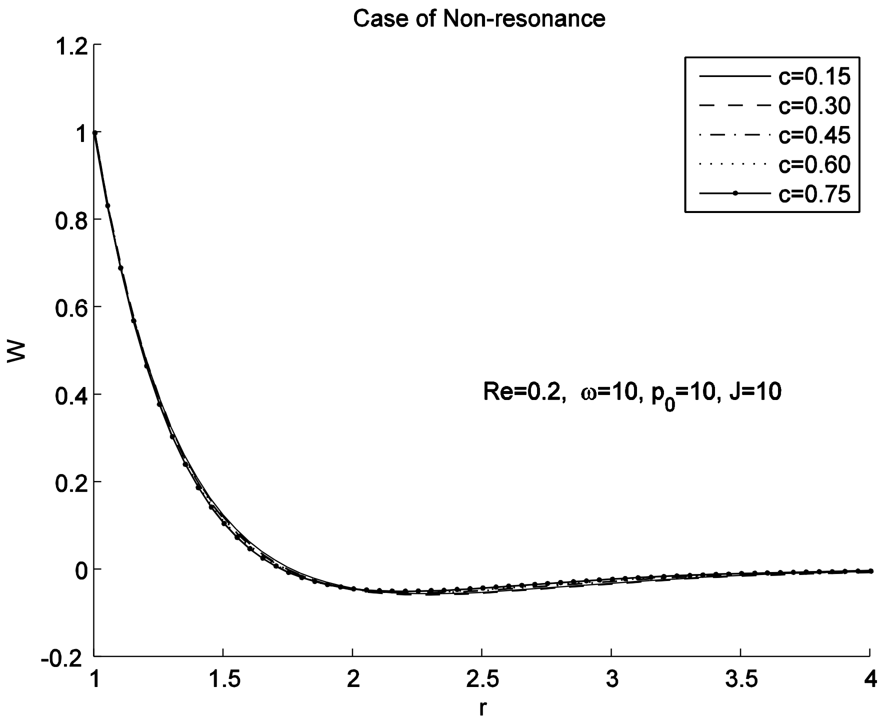
(a)

(b)

Fig 4.2 Velocity at different values of Reynolds numbers for the case of a) resonance and b) non-resonance

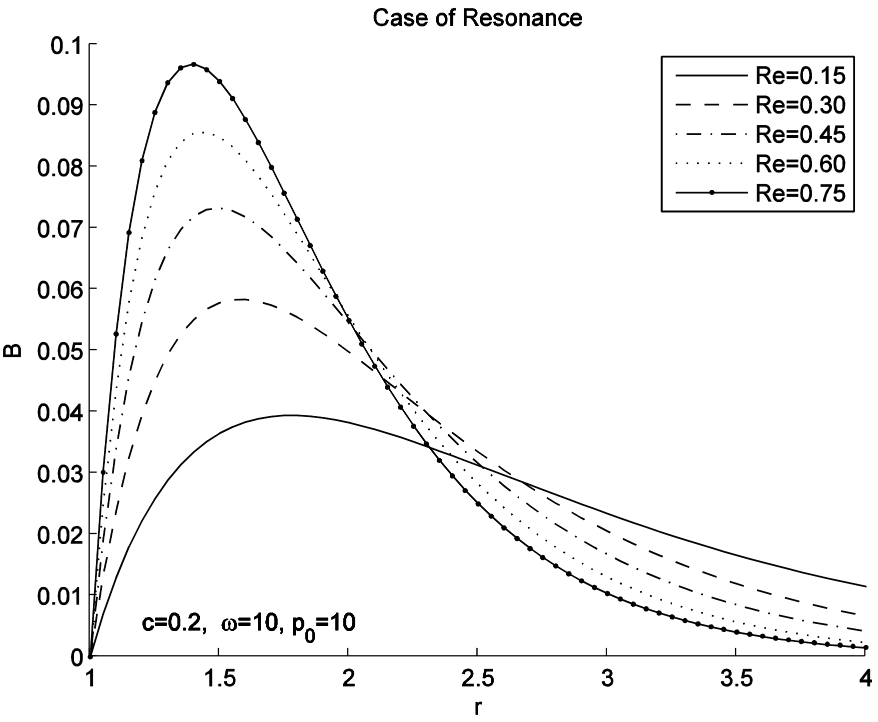


(a)

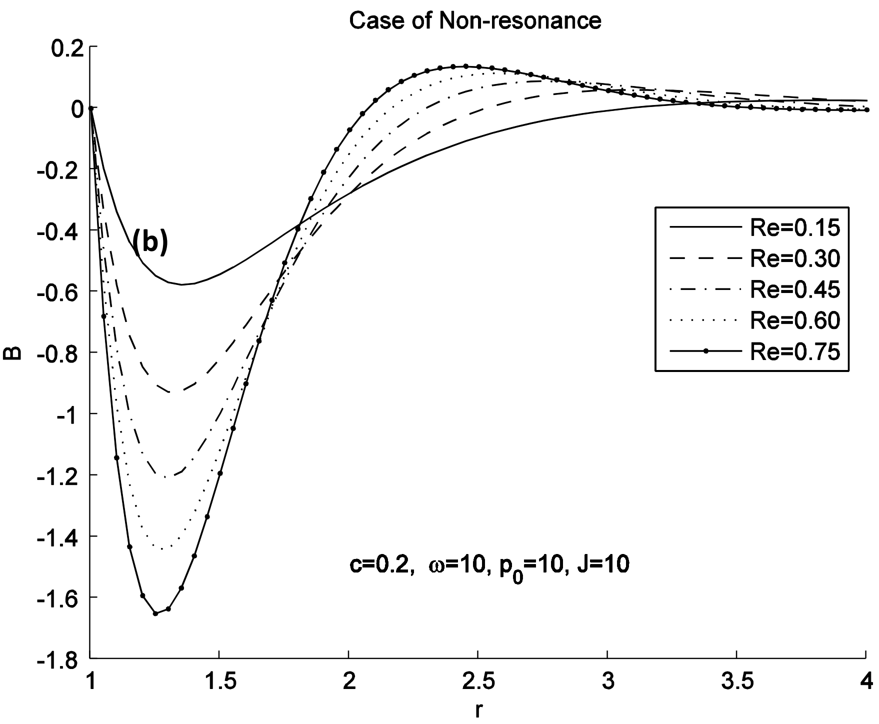


(b)

Fig 4.3 Velocity at different values of Micro-polarity parameter *c* for the case of a) resonance and b) non-resonance

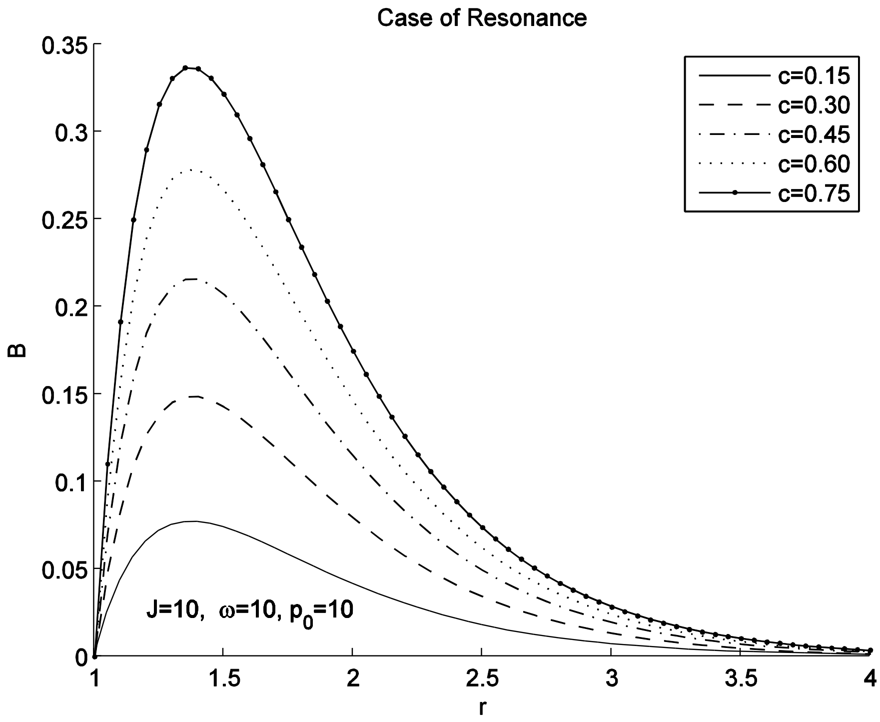


(a)

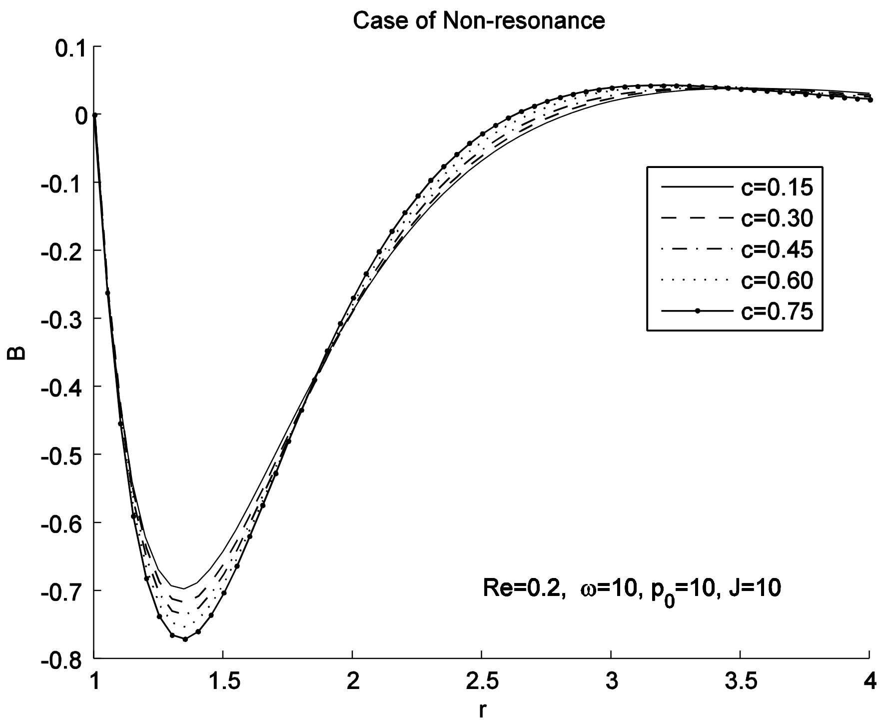


(b)

Fig 4.4 Micro-rotation at different values of Reynolds number for the case of a) resonance and b) non-resonance



(a)



(b)

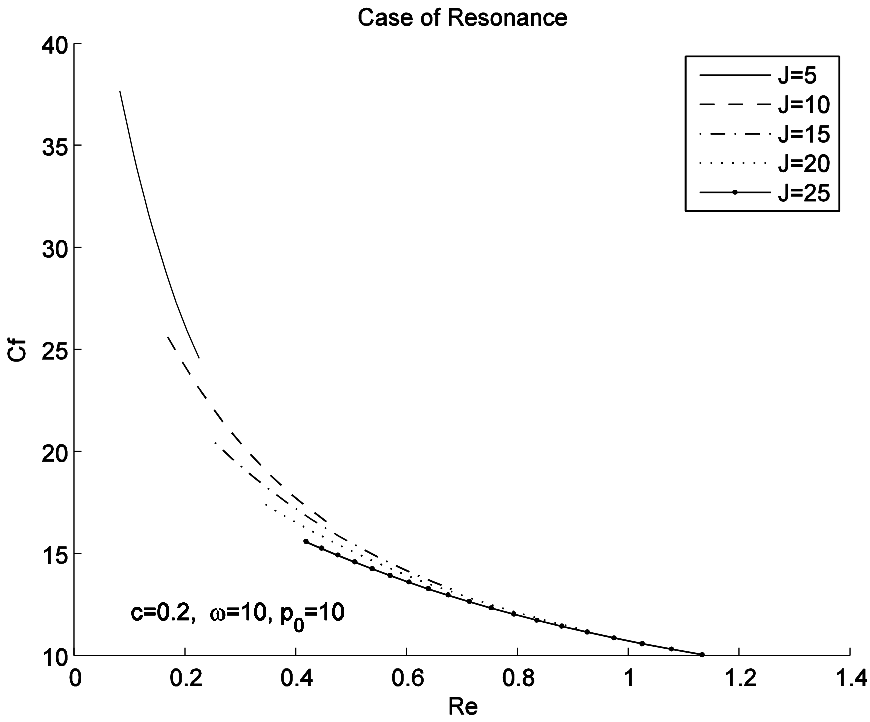
Fig 4.5 Micro-rotation at different values of Micro-polarity parameter c for the case of a) resonance and b) non-resonance

4.5.3 Skin friction

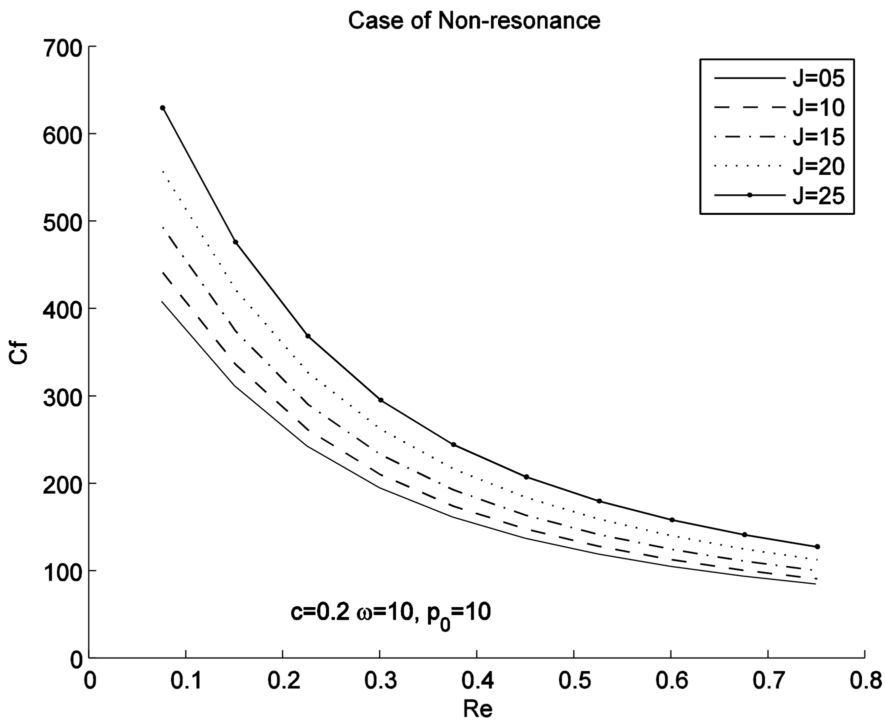
From Fig 4.6, we can observe that resonance flow occurs only a particular range of gyration parameter J and Re. Skin friction is much smaller in the case of resonance and as Gyration parameter increases, Skin friction decreases. But in the case of non-resonance, as gyration parameter increases, Skin friction also increases. Resonance decreases the Skin friction drastically to a low value (from 500 in non-resonance case to 30 in resonance) *It is noticed that as gyration parameter j increases, in the case of resonance, Skin friction decreases and in the case resonance, Skin friction increases.*

From the Fig 4.7, it is observed that Skin friction is not affected much by variation in Micro-polarity in the case of resonance. But opposite to this in the case of non-resonance, as Micro-polarity increases, Skin friction decreases drastically. From Fig 4.8, as Reynolds number increases, Skin friction decreases. This is expected, since in the formula Eq. (4.34), Reynolds number is in the denominator. It is very interesting to note that the Skin friction in the case of resonance is much smaller than in the case of non-resonance.

Hence the conclusion is that as Reynolds number or Micro-polarity increases, Skin friction decreases but the case of resonance offers less resistance for the flow and hence Skin friction is very much lesser than the case of non-resonance.

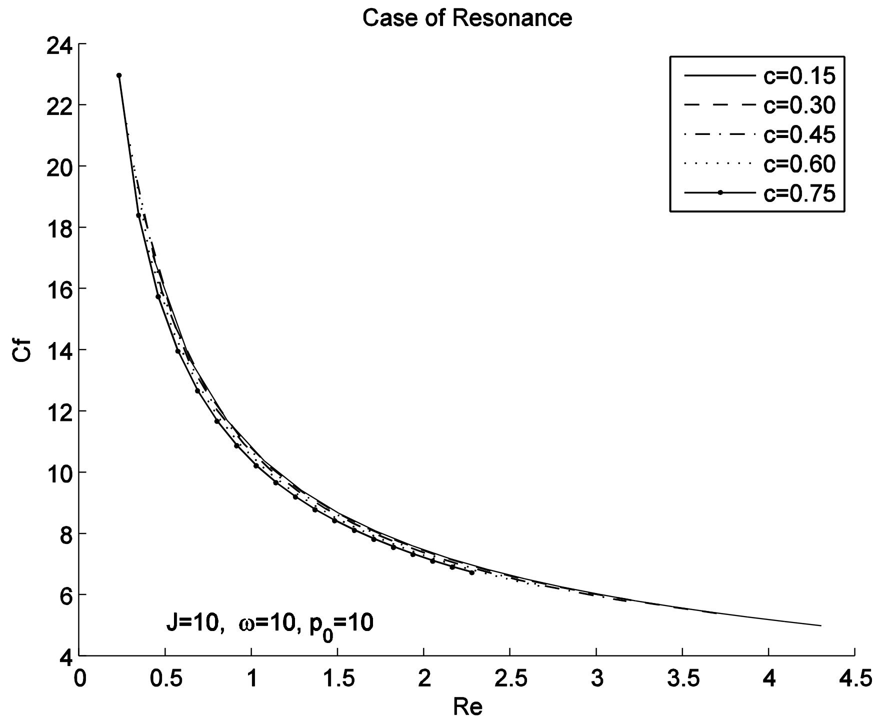


(a)



(b)

Fig 4.6 Skin friction at different values of gyration parameter J for the case of a) resonance and b) non-resonance



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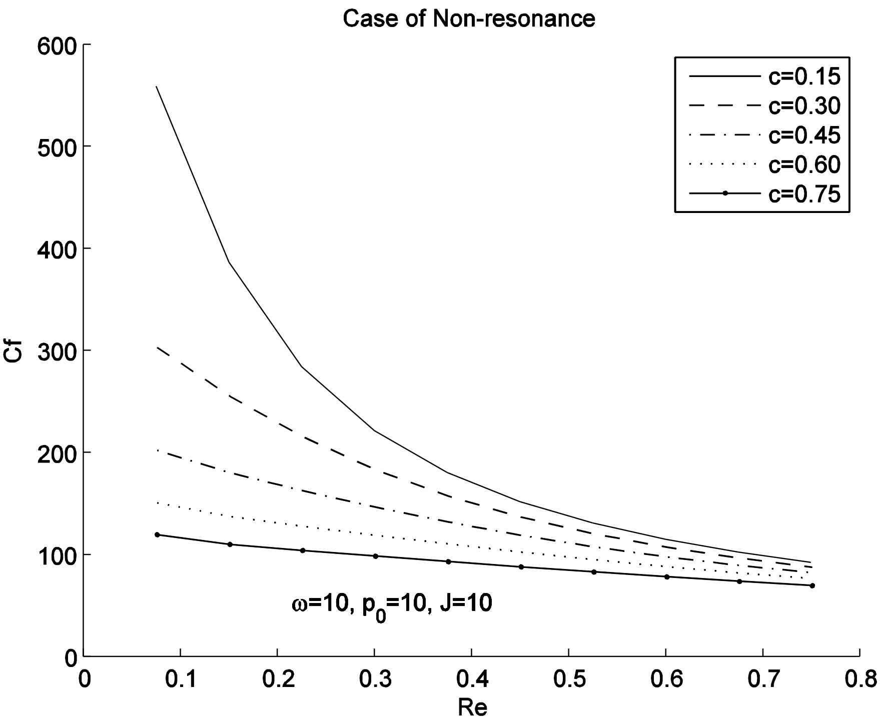
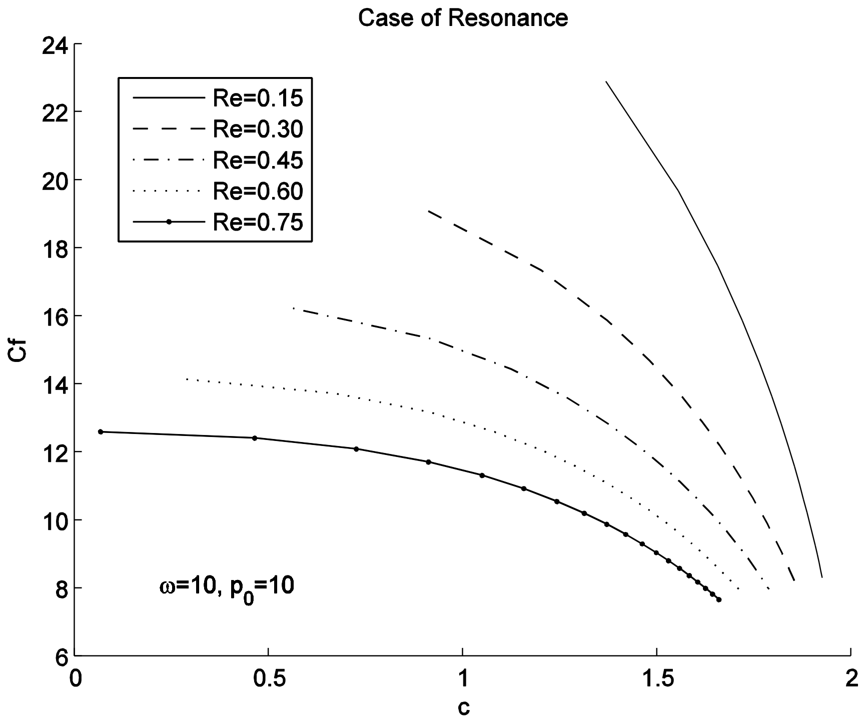
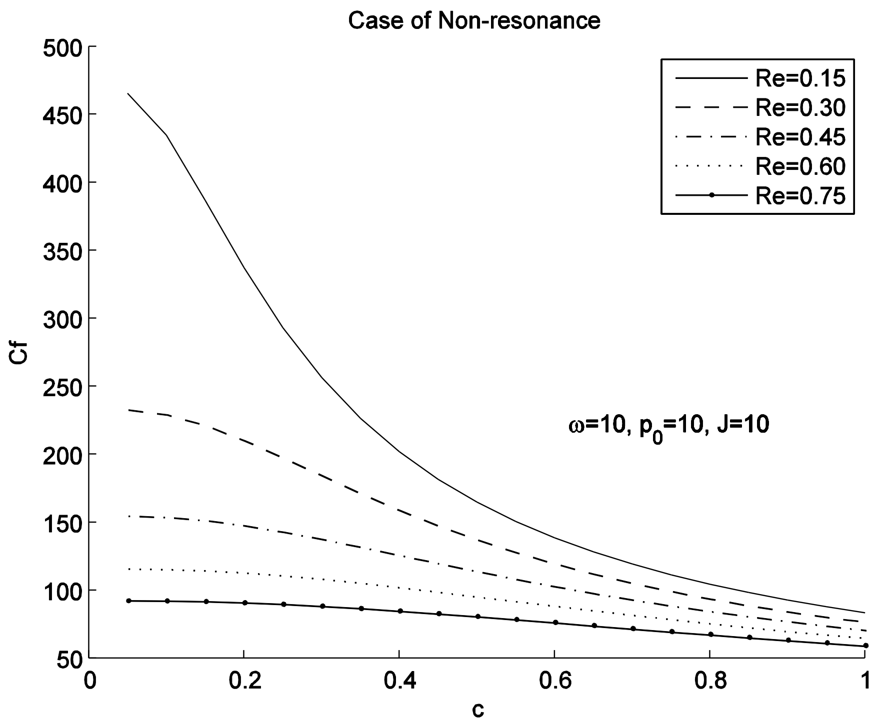


(b)

Fig 4.7 Skin friction at different values of Micro-polarity *c* for the case of a) resonance and b) non-resonance



(a)



(b)

Fig 4.8 Skin friction at different values of Reynolds number Re for the case of a) resonance and b) non-resonance

4.6 Conlusions

1. Case of resonance makes the micro-rotation as unidirectional (i.e positive only). In non-resonance micro-rotation raises from negative values to positive values and then vanishes.
2. The case of resonance offers less resistance to the flow and hence decreases Skin friction.

These two observations are very important to focus our attention on the case of "resonance". They may have Industrial application, for producing a suitable Micro-polar fluid to get minimum Skin friction.

Chapter 5

Rectilinear oscillations of Sphere in a Micro-polar fluid

The flow of an incompressible Micro-polar fluid generated due to rectilinear oscillations of a sphere about a diameter of the sphere is considered. The flow is so slow that the Reynolds number is less than unity and hence nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered. The stream function and Drag acting on sphere are obtained. The effect of physical parameters like Micro-polarity and Couple stress parameter on the Drag due to oscillations is shown through graphs.

5.1 Introduction

Lakshmana Rao et al. (1970) studied slow stationary flow of a Micro-polar fluid past a sphere. Lakshmana Rao et al. (1971, 1981, 1987) studied the oscillatory flows generated due to oscillations of sphere, spheroid and elliptic cylinder in Micro-polar fluids, with the aim of determining of the Drag or Couple on the oscillating body.

Ravindran (1972) studied simple oscillatory flow in polar fluids. Frater (1967 and 1968) studied oscillatory flows in elastico-viscous fluid, and evaluated Drag on sphere, damping force on a body. Analytical and Computational studies in Couple stress fluid flows examined by Lakshmana Rao et al. (1980). Iyengar et al. (1993, 2001 and 2004) examined oscillatory flows due to oscillating of approximate sphere, two concentric spheres in Micro-polar fluid and approximate sphere in viscous fluid. Lai et al. (1978) examined an elastico viscous fluid flow of sphere performing rectilinear oscillations and evaluated Drag on a sphere. The problems regarding oscillatory flows in various fluids generated due to rectilinear oscillations of different symmetric bodies like circular cylinder, elliptic cylinder, sphere, approximate sphere, sphereoid have been studied by many researchers. Some are quoted here as example, Liu (1978), Lakshmana Rao (1981, 1987), Iyengar (2001). Stimson et al. (1926) examined the viscous fluid motion of two spheres. Verma et al. (1971) studied slow oscillatory flow past a fixed porous sphere. Aparna et al. (2012) examined the flow of micropolar fluid due to rotary oscillations of a permeable sphere. Ashmawy (2015 and 2016) examined oscillatory flows of composite sphere in a concentric spherical cavity and spherical particle moving in a Couple-stress fluid.

In this chapter we intend to investigate this case of resonance type flow due to rectilinear oscillations of a sphere about its axis of symmetry in Micro-polar fluids. Later on we discussed similar problem in Couple-stress fluids.

5.2 Basic Equations

The basic equations of motion for incompressible Micro-polar fluids as introduced by Eringen (1966), are given by

$$\frac{\partial \rho}{\partial \tau} + div(\rho \boldsymbol{Q}) = 0 \tag{5.1}$$

$$\rho\left(\frac{\partial \bar{Q}}{\partial \tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{5.2}$$

$$\rho J\left(\frac{\partial \bar{l}}{\partial \tau} + \bar{Q}.\nabla_1\bar{l}\right) = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1.\bar{l}) \tag{5.3}$$

The constitutive equations for the stress components T_{ij} and Couple stress components M_{ij} for an incompressible Micro-polar fluid are given by

$$T_{ij} = -P\delta_{ij} + \frac{1}{2}(2\mu + k)\left(u_{i,j} + u_{j,i}\right) + ke_{ijr}(w_r - l_r) \tag{5.4}$$

$$M_{ij} = \alpha l_{i,i}\delta_{i,j} + \beta l_{i,j} + \gamma l_{j,i} \tag{5.5}$$

where the permutation tensor $e_{ijk} = \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ 1 & \text{if } i, j, k \text{ are cyclic} \\ -1 & \text{if } i, j, k \text{ are anti} - \text{cyclic} \end{cases}$

and w_r = r th component of ½(curl **Q**).

Neglecting the nonlinear convective terms in (6.2) and (6.3), the linearised version of the equations are given by,

$$div\bar{Q} = 0 \tag{5.6}$$

$$\rho \frac{\partial \bar{Q}}{\partial \tau} = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{5.7}$$

$$\rho \mathcal{J} \frac{\partial \bar{l}}{\partial \tau} = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1 . \bar{l}) \tag{5.8}$$

5.3 Statement and Formulation of the Problem

A sphere of radius a is performing rectilinear oscillations with velocity $U_0 \boldsymbol{k} e^{i\sigma\tau}$ about its diameter in an infinite vat containing incompressible Micro-polar fluid. Spherical coordinate system (R, θ, ϕ) with base vectors $(\boldsymbol{e_R}, \boldsymbol{e}_\theta, \boldsymbol{e_\phi})$ with origin at the centre of the sphere and Z axis along direction of oscillations of the sphere is considered. The flow is axially symmetric, hence the velocity field is independent of toroidal coordinate ϕ and the flow will be in cross sectional plane of the sphere containing the base vectors $(\boldsymbol{e_R}, \boldsymbol{e}_\theta)$. The velocity and micro-rotation are assumed in the form:

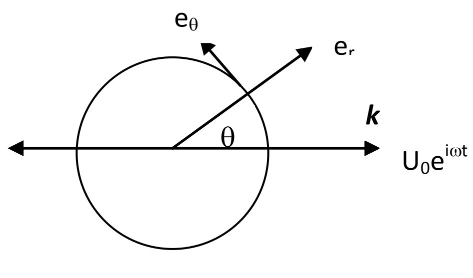


Fig 5.1 Geometry of the oscillating Sphere

$$\boldsymbol{Q} = e^{i\sigma\tau}\{U(R,\theta)\boldsymbol{e_r} + V(R,\theta)\boldsymbol{e_\theta}\} \text{ and } \boldsymbol{l} = e^{i\sigma\tau}\left(\frac{C(R,\theta)}{\boldsymbol{h_3}}\right)\boldsymbol{e}_\phi \tag{5.9}$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$\left.\begin{aligned} R = ar,\ \boldsymbol{Q} = U_0\boldsymbol{q},\ U = U_0 u,\ \mathrm{V} = \mathrm{U}_0 v, \boldsymbol{l} = \frac{U_0}{a}\boldsymbol{\upsilon} \\ \mathcal{C} = \frac{W_0}{a}C, \Psi = \psi U_0 a\ \ P = p\rho U_0^2,\ \tau = \frac{at}{U_0} \end{aligned}\right\} \tag{5.10}$$

The following are non-dimensional parameters viz, j is gyration parameter, ω is frequency parameter, s is Couple stress parameter, c is cross viscosity or Micro-polarity parameter and Re is oscillations Reynolds number for Micro-polar fluids.

$$J = \frac{\mathrm{j}\rho\sigma a^2}{\gamma},\ \varpi = \frac{\mathrm{a}\sigma}{\mathrm{U}_0},\ s = \frac{\mathrm{k}a^2}{\gamma},\ c = \frac{\mathrm{k}}{\mu+\mathrm{k}},\ \ \mathrm{Re} = \frac{\rho \mathrm{U}_0 a}{\mu}, \mathrm{R}_0 = \frac{\rho \mathrm{U}_0 a}{\mu+k} = Re(1-c) \tag{5.11}$$

By the choice of velocity field in (5.9) and incompressibility condition in (5.1), we notice that stream function ψ can be introduced as

$$u = \frac{1}{r^2 Sin\theta}\frac{\partial\psi}{\partial\theta} and v = -\frac{1}{rSin\theta}\frac{\partial\psi}{\partial r}\quad \text{i.e}\qquad \boldsymbol{q} = \nabla\times\left(\frac{\psi}{h_3}\boldsymbol{e}_\emptyset\right) \tag{5.12}$$

Using non-dimensional scheme (5.10) and (5.11) in (5.7) and (5.8) we get

$$R_0\frac{\partial\boldsymbol{q}}{\partial t} = -R_0.\nabla p + c\,\nabla\times\boldsymbol{\upsilon} - \nabla\times\nabla\times\boldsymbol{q} \tag{5.13}$$

$$\frac{J}{\boldsymbol{\varpi}}\frac{\partial\boldsymbol{\upsilon}}{\partial t} = -2s\boldsymbol{\upsilon} + s\nabla\times\boldsymbol{q} - \nabla\times\nabla\times\boldsymbol{\upsilon} + \epsilon\nabla(\nabla\cdot\boldsymbol{\upsilon}) \tag{5.14}$$

We can write (5.13) and (5.14) as

$$i\varpi R_0\boldsymbol{q} = -R_0.\nabla p + c\,\nabla\times\boldsymbol{\upsilon} - \nabla\times\nabla\times\boldsymbol{q} \tag{5.15}$$

$$iJ\boldsymbol{\upsilon} = -2s\boldsymbol{\upsilon} + s\nabla\times\boldsymbol{q} - \nabla\times\nabla\times\boldsymbol{\upsilon} + \epsilon\nabla(\nabla\cdot\boldsymbol{\upsilon}) \tag{5.16}$$

By taking curl to (5.15) pressure p can be eliminated and then using (5.9) and (5.12) we get,

$$i\varpi R_0.E^2\psi = cE^2C + E^4\psi \tag{5.17}$$

$$\text{where } \mathrm{E}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{cot\theta}{r^2}\frac{\partial}{\partial\theta} \tag{5.18}$$

similarly by using (5.9) and (5.12) in (5.16), we get

$$(2s + iJ)C = E^2C - sE^2\psi \tag{5.19}$$

Substitute (5.17) in (5.19), we get

$$c(2s + iJ)C = -E^4\psi + (i\varpi R_0 - cs)E^2\psi \tag{5.20}$$

Taking E^2 operation to (5.20) and then eliminating E^2C using (5.17) we get,

$$E^2(E^2 - \lambda_1^2)\,(E^2 - \lambda_2^2)\,\psi = 0 \tag{5.21}$$

Where $\lambda_1^2 + \lambda_2^2 = (2 - c)s + i(J + \varpi R_0)$ and $\lambda_1^2\lambda_2^2 = i\varpi R_0(2s + iJ)$ (5.22)

Using (5.22) in (5.20), the equation for C can be re-written as

$$cC = -\frac{i\varpi R_0}{\lambda_1^2\lambda_2^2}\mathrm{E}^2(\mathrm{E}^2 - \lambda_1^2 - \lambda_2^2)\psi - \mathrm{E}^2\psi \tag{5.23}$$

The solution for ψ if $\lambda_1 \neq \lambda_2$ in (5.21) is given in Lakshmana Rao et al. (1971, 1972). The solution for ψ for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \to \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation:

$$\frac{\gamma}{J} = \frac{(2\mu + k)(\mu + k)}{2\mu + 3k} \text{ and } \quad \rho\sigma = \frac{(2\mu + k)k + \gamma\rho\sigma}{J(\mu + k)} \tag{5.24}$$

In non-dimensional form, these conditions are given by

$$(2 - c)s = J - \varpi.R_0 \ and \ (2 - c)J = \varpi.R_0(2 + c) \tag{5.25}$$

In this chapter, we are interested in the solution for ψ for the case of resonance.

We have the equation for stream function ψ as

In the case of resonance:

$$\mathrm{E}^2(\mathrm{E}^2 - \lambda^2)^2\,\psi = 0 \tag{5.26a}$$

In the case on non-resonance:

$$\mathrm{E}^2(\mathrm{E}^2 - \lambda_1^2)\,(\mathrm{E}^2 - \lambda_2^2)\,\psi = 0 \tag{5.26b}$$

And we have equation for the micro-roation component C as

In the case of resonance:

$$cC = -\frac{i\varpi R_0}{\lambda^4} \mathrm{E}^2(\mathrm{E}^2 - 2\lambda^2)\psi - E^2\psi \tag{5.27a}$$

In the case on non-resonance:

$$cC = -\frac{i\varpi R_0}{\lambda_1^2\lambda_2^2} \mathrm{E}^2(\mathrm{E}^2 - \lambda_1^2 - \lambda_2^2)\psi - \mathrm{E}^2\psi \tag{5.27b}$$

5.3.1 Boundary Conditions

The sphere is oscillating in the direction of Z-axis. Hence the non-dimensional velocity of the sphereΓ after removing $e^{i\varpi t}$ is given by

$\boldsymbol{q}_{\Gamma} = \boldsymbol{i} = cos\theta\boldsymbol{e}_r - sin\theta\boldsymbol{e}_\theta$ which implies by no-slip condition

$$u = cos\theta \, and \, v = -sin\theta \text{ on } r=1 \tag{5.28}$$

By hyper-stick condition $\boldsymbol{\upsilon}_{\Gamma} = \frac{1}{2}(\mathrm{curl}\, \boldsymbol{q})_{\Gamma}$

which reduces to $C = 0 \; on \; r = 1$ (5.29)

5.4 Solution of the Problem

To match with the boundary conditions in (5.28) and (5.29), Stream function ψ, micro-rotation component C are assumed in the form

$$\psi = f(r)\sin^2\theta \; and \; C = g(r)\sin^2\theta \tag{5.30}$$

Substituting (5.30) in (5.26a) and (5.26b) and cancelling $\sin^2\theta$ we get

In the case of resonance:

$$D_s^2\left(D_s^2 - \lambda^2\right)^2 f = 0 \tag{5.31a}$$

In the case on non-resonance:

$$D_s^2(D_s^2 - \lambda_1^2)(D_s^2 - \lambda_2^2) f = 0 \tag{5.31b}$$

Substituting (5.30) in (5.27a) and (5.27b) we get

In the case of resonance:

$$cg = -\frac{i\varpi R_0}{\lambda^4} D_s^2(D_s^2 - 2\lambda^2)f - D_s^2 f \tag{5.32a}$$

In the case on non-resonance:

$$cg = -\frac{i\varpi R_0}{\lambda_1^2\lambda_2^2} D_s^2(D_s^2 - \lambda_1^2 - \lambda_2^2)f - D_s^2 f \tag{5.32b}$$

Where $D_s^2 = \frac{d^2}{dr^2} - \frac{2}{r^2}$ (5.33)

Substituting (2.30) in (2.28) and (2.29), the conditions on f and g are obtained as:

$$f(1) = \frac{1}{2},\ f'(1) = 1 \text{and } g(1) = 0 \tag{5.34}$$

Since the equation for f is linear, the general solution for f is linear combination of individual solutions of factors in the differential operator. Hence f is taken as

$$f = a_0 f_0 + a_1 f_1 + a_2 f_2 \tag{5.34a}$$

In the case of resonance:

$$D_s^2 f_0 = 0,\ (D_s^2 - \lambda^2)f_1 = 0 \text{ and } (D_s^2 - \lambda^2)^2 f_2 = 0 \tag{5.35a}$$

In the case of non-resonance:

$$D_s^2 f_0 = 0,\quad (D_s^2 - \lambda_1^2)f_1 = 0 \text{ and } \quad (D_s^2 - \lambda_2^2)f_2 = 0 \tag{5.35b}$$

On solving (5.35a) and (5.35b) the solution for f is obtained as

In the case of resonance:

$$f(\mathrm{r}) = \frac{\mathrm{a}_0}{\mathrm{r}} + \mathrm{a}_1\sqrt{r}K_{\frac{3}{2}}(\lambda r) + \mathrm{a}_2 r^{\frac{3}{2}}K'_{\frac{3}{2}}(\lambda r) \tag{5.36a}$$

In the case of non-resonance:

$$f(\mathrm{r}) = \frac{\mathrm{a}_0}{\mathrm{r}} + \mathrm{a}_1\sqrt{r}K_{\frac{3}{2}}(\lambda_1 r) + \mathrm{a}_2\sqrt{r}K_{\frac{3}{2}}(\lambda_2 r) \tag{5.36b}$$

The following results are useful to note.

In the case of resonance:

$$D_s^2 f_1 \; = \; \lambda^2 f_1 \text{ and } D_s^2 f_2 \; = \; 2\lambda f_1 + \lambda^2 f_2 \tag{5.37a}$$

In the case of non-resonance:

$$\mathrm{D}_s^2 f_1 \; = \; \lambda_1^2 f_1 \quad \text{and} \mathrm{D}_s^2 f_2 \; = \; \lambda_2^2 f_2 \tag{5.37b}$$

Substituting *f* along with (5.37a) in (5.32a), we get

$$cg \; = \; \mathrm{a}_1\left(\mathrm{i}\varpi\mathrm{R}_0 - \lambda^2\right)f_1 + \mathrm{a}_2\left(\mathrm{i}\varpi\mathrm{R}_0 - \lambda^2\right)f_2 - 2\mathrm{a}_2\lambda f_1 \tag{5.38a}$$

Substituting *f* along with (5.37b) in (5.32b), we get

$$cg \; = \; \mathrm{a}_1\left(\mathrm{i}\varpi\mathrm{R}_0 - \lambda_1^2\right)f_1 + \mathrm{a}_2\left(\mathrm{i}\varpi\mathrm{R}_0 - \lambda_2^2\right)f_2 \tag{5.38b}$$

The constants a_0, a_1, a_2 are obtained from the boundary conditions (5.34) as follows:

In the case of resonance:

$$\begin{bmatrix} 1 & K_{\frac{3}{2}}(\lambda) & K'_{\frac{3}{2}}(\lambda) \\ -1 & \frac{1}{2}K_{\frac{3}{2}}(\lambda) + \lambda K'_{\frac{3}{2}}(\lambda) & \frac{1}{2}K'_{\frac{3}{2}}(\lambda) + \left(\frac{9}{4\lambda} + \lambda\right)K_{\frac{3}{2}}(\lambda) \\ 0 & (i\varpi R_0 - \lambda^2)K_{\frac{3}{2}}(\lambda) & (i\varpi R_0 - \lambda^2)K'_{\frac{3}{2}}(\lambda) - 2\lambda K_{\frac{3}{2}}(\lambda) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \tag{5.39a}$$

In the case of non-resonance:

$$\begin{bmatrix} 1 & K_{\frac{3}{2}}(\lambda_1) & K_{\frac{3}{2}}(\lambda_2)) \\ -1 & \frac{1}{2}K_{\frac{3}{2}}(\lambda_1) + \lambda_1 K'_{\frac{3}{2}}(\lambda_1) & \frac{1}{2}K_{\frac{3}{2}}(\lambda_2) + \lambda_2 K'_{\frac{3}{2}}(\lambda_2) \\ 0 & (i\varpi R_0 - \lambda_1^2)K_{\frac{3}{2}}(\lambda_1) & (i\varpi R_0 - \lambda_2^2)K_{\frac{3}{2}}(\lambda_2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \tag{5.39b}$$

Hence from (5.36), (5.38) and (5.39) *f* and g are completely known and hence stream function ψ and micro-rotation component C are known.

5.4.1 Pressure

From equation (5.15) pressure is obtained as follows.

$$dp = \nabla p.\boldsymbol{dr} = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial \theta}d\theta \tag{5.40}$$

By comparing components in equation (5.15), we obtained

$$\frac{\partial p}{\partial r} = \frac{1}{R_0}\left\{\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(-i\varpi R_0\psi + cC + E^2\psi)\right\} \tag{5.41}$$

$$\frac{\partial p}{\partial\theta} = \frac{-1}{R_0}\left\{\frac{1}{sin\theta}\frac{\partial}{\partial r}(-i\varpi R_0\psi + cC + E^2\psi)\right\} \tag{5.42}$$

Substituting (5.41) and (5.42) in (5.40), we get

$$dp = \frac{1}{R_0}\left\{\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(-i\varpi R_0\psi + cC + E^2\psi)dr - \frac{1}{sin\theta}\frac{\partial}{\partial r}(-i\varpi R_0\psi + cC + E^2\psi)d\theta\right\}$$

$$dp = \frac{1}{R_0}\left[(-i\varpi R_0 f + D_s^2 f + cg)2cos\theta\frac{dr}{r^2} + \frac{d}{dr}(i\varpi R_0 f - D_s^2 f - cg)sin\theta\, d\theta\right] \tag{5.43}$$

Integrating on both sides of (5.43), we obtained pressure in non-dimensional form as

$$p = \frac{i\varpi A_0}{r^2}cos\theta \tag{5.44}$$

5.4.2 Drag acting on the Sphere of radius a

$$\text{Drag}= D^* = 2\pi a^2\int_0^\pi (T^*_{rr}\,cos\theta - T^*_{r\theta}\,sin\theta)sin\theta\,|_{R=a}d\theta \tag{5.45}$$

Required tress components are obtained as follows:

$$\text{Strain rate tensor} = E = [e_{ij}] = \frac{1}{2}\,[\nabla\bar{Q} + \nabla\bar{Q}^T]$$

$$E=\begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2R}\left[\frac{\partial U}{\partial\theta} + R\frac{\partial V}{\partial R} - V\right] & \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial U}{\partial\phi} + R\frac{\partial W}{\partial R} - W\right] \\ \frac{1}{2R}\left[\frac{\partial U}{\partial\theta} + R\frac{\partial V}{\partial R} - V\right] & \frac{1}{R}\left[\frac{\partial V}{\partial\theta} + U\right] & \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial V}{\partial\phi} + \frac{\partial W}{\partial\theta} - WCot\theta\right] \\ \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial U}{\partial\phi} + R\frac{\partial W}{\partial R} - W\right] & \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial V}{\partial\phi} + \frac{\partial W}{\partial\theta} - WCot\theta\right] & \frac{1}{R}\left[\frac{1}{Sin\theta}\frac{\partial W}{\partial\phi} + VCot\theta + U\right] \end{bmatrix}$$

We get strain rate tensor for this problem as

$$E = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2R}\left[\frac{\partial U}{\partial\theta} + R\frac{\partial V}{\partial R} - V\right] & 0 \\ \frac{1}{2R}\left[\frac{\partial U}{\partial\theta} + R\frac{\partial V}{\partial R} - V\right] & \frac{1}{R}\left[\frac{\partial V}{\partial\theta} + U\right] & 0 \\ 0 & 0 & \frac{1}{R}[VCot\theta + U] \end{bmatrix} \tag{5.46}$$

Substituting required terms in (5.4), we get the stress components as

$$T_{RR} = -P + (2\mu + k)\frac{\partial U}{\partial R} \tag{5.47}$$

$$T_{R\theta} = (\mu + k)\frac{\partial V}{\partial R} + \frac{\mu}{R}\frac{\partial U}{\partial\theta} - \mu\frac{V}{R} - \frac{kC}{Rsin\theta} \tag{5.48}$$

Stress components in non-dimensional form as

$$T_{rr} = \frac{(\mu+k)U_0}{a}\left[-pR_0 + (2-c)\left(\frac{1}{r^2 sin\theta}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{2}{r^3 sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \tag{5.49}$$

$$T_{r\theta} = \frac{(\mu+k)U_0}{a}.\frac{1}{rsin\theta}\left[(c-2)\frac{\partial^2\psi}{\partial r^2} - (c-2).\frac{1}{r}\frac{\partial\psi}{\partial r} + (1-c)\mathrm{E}^2\psi - cC\right] \tag{5.50}$$

Substituting (5.30) in (5.49) and (5.50) we get

$$T_{rr} = \frac{(\mu+k)U_0}{a}\left[-pR_0 + (2-c)\left(\frac{2f'}{r^2} - \frac{4f}{r^3}\right)cos\theta\right] \tag{5.51}$$

$$T_{r\theta} = \frac{(\mu+k)U_0}{a}.\frac{1}{r}\left[(c-2)\left(\frac{2f}{r^2} - \frac{f'}{r}\right) - D_s^2 f - cg\right]sin\theta \tag{5.52}$$

On boundary *r=1,* stress components are

$$T_{rr} = -\frac{(\mu+k)U_0}{a}pR_0 \tag{5.53}$$

$$T_{r\theta} = -\frac{(\mu+k)U_0}{a}D_s^2 fsin\theta \tag{5.54}$$

Substituting (5.53) and (5.54) in (5.45), we get the Drag on the sphere (for resonance and non-resonance cases without the factor $e^{i\varpi t}$) as

$$Drag = D^* = \frac{4\pi(\mu+k)U_0 a}{3}R_0 i\varpi(1-3A_0) \tag{5.55}$$

Dividing D^* by $6\pi(\mu+k)U_0 a$, hence the non-dimensional Drag D is given by

$$\mathrm{D} = \mathrm{Real}\left\{\frac{2}{9}\mathrm{R}_0 \mathrm{i}\varpi(1-3\mathrm{A}_0)\right\} \tag{5.56}$$

5.5 Results and Discussions

The values of λ are obtained from (5.22) by solving the following equation for x

$$x^2 - [(2-c)s + i(J+\varpi R_0)]x + i\varpi R_0(2s+iJ) = 0 \tag{5.57}$$

Then for resonance case

$$\lambda = \sqrt{\mathrm{x}} = \sqrt{\frac{[(2-c)s + \mathrm{i}(J+\varpi \mathrm{R}_0)]}{2}} \tag{5.58}$$

This involves 5 parameters which are related by two equations in (5.25). Hence we choose three parameters as independent. Here ϖ, R_0 and c are selected independently, with $0 \le c \le 1$, Re≪1 and ω≫1 such that $\varpi . R_0$ is not negligibly small (say>1). After selecting c, R_0 and ϖ, the values of s and J are obtained from (5.25) and then λ is obtained from (5.58). The values of λ are complex. These values for λ are substituted in (5.39a), (5.39b) and the constants a_0, a_1 and a_2 are obtained. Then the stream function ψ, micro-rotation component C and Drag D are obtained from (5.36), (5.38) and (5.56) respectively for both resonance and non-resonance cases. Thus obtained ψ will have complex values. To get the physical picture, these values are multiplied by $e^{i\omega t}$ and its real part is taken.

5.5.1 Drag

From Fig 5.2, it is observed that as |λ| is increasing, the Drag D is almost constant. But for resonance, the values of Drag are small in comparison with non-resonance. The variation of Drag with s is same with |λ| but the corresponding s values are very large in the range of 100s.

From Fig 5.3 Drag variation with J, the gyration parameter is shown. We note that for resonance Drag is smaller than the case of nonresonance. In nonresonance, behaviour of Drag is not constant and range of Drage in limited to 85 to 95 only.

5.5.2 Stream Function

From Fig 5.4, we note that as c increases the values of stream function are also increasing for resonance. But for non-resonance, as c increases, values of stream function are decreasing. Effect of c is not very much effective for non-resonance. From Fig 5.5, we notice that the effect of ***s*** on stream function is not much for non-resonance. Again the stream function takes larger values than the case of resonance.

From Fig 5.6, we note that for resonance, formation of circulations is not observed. Since for resonance no circulations are seen we take stream lines for the case of non-resonance for the variation of Micro-polarity parameter. Since for resonance no circulations are seen we take stream lines for the case of non-resonance for the variation of Micro-polarity parameter *c*. this is shown in Fig 5.7. As *c* increases, the circulations started near the sides of sphere spread in region and by value. At large values of c near to 0.8 or more, only near the sphere we see black region i.e only near to the sphere values are less than 0.5.

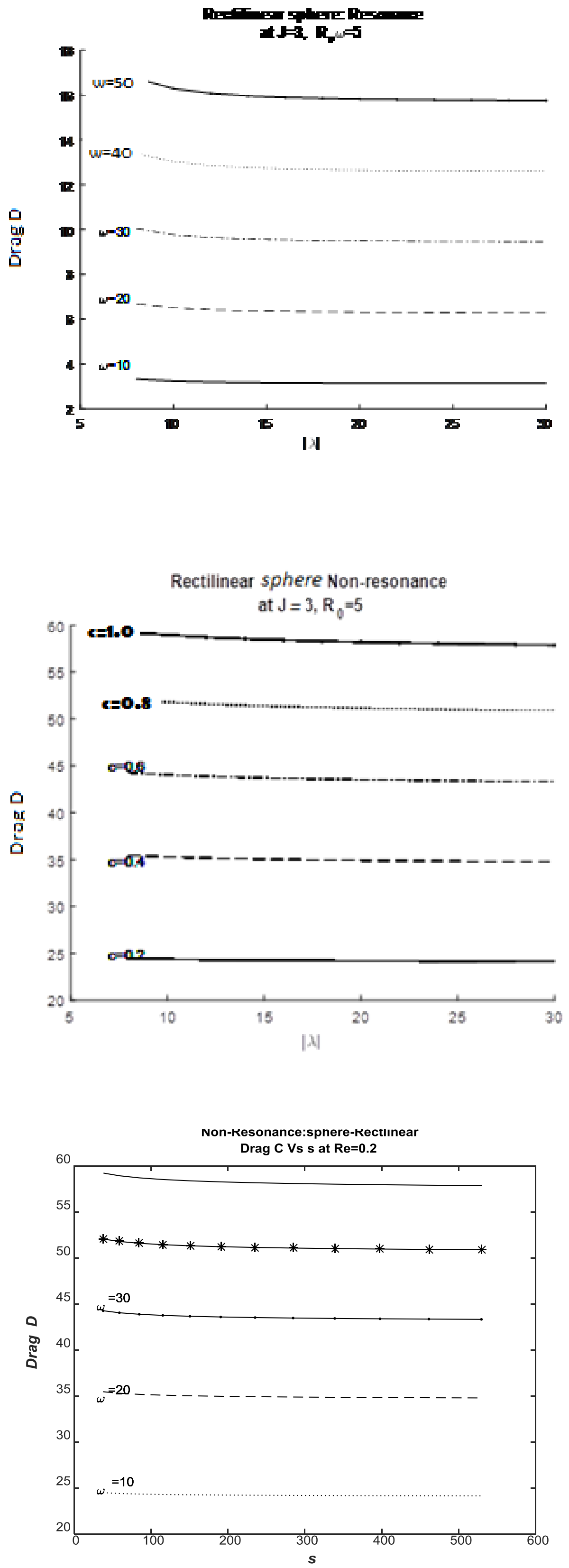
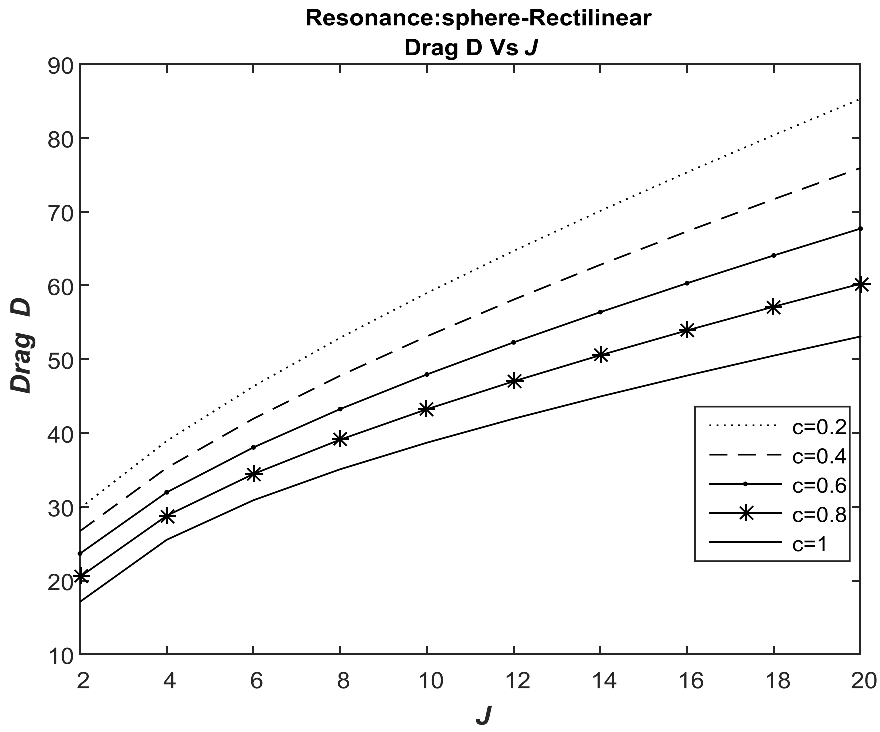
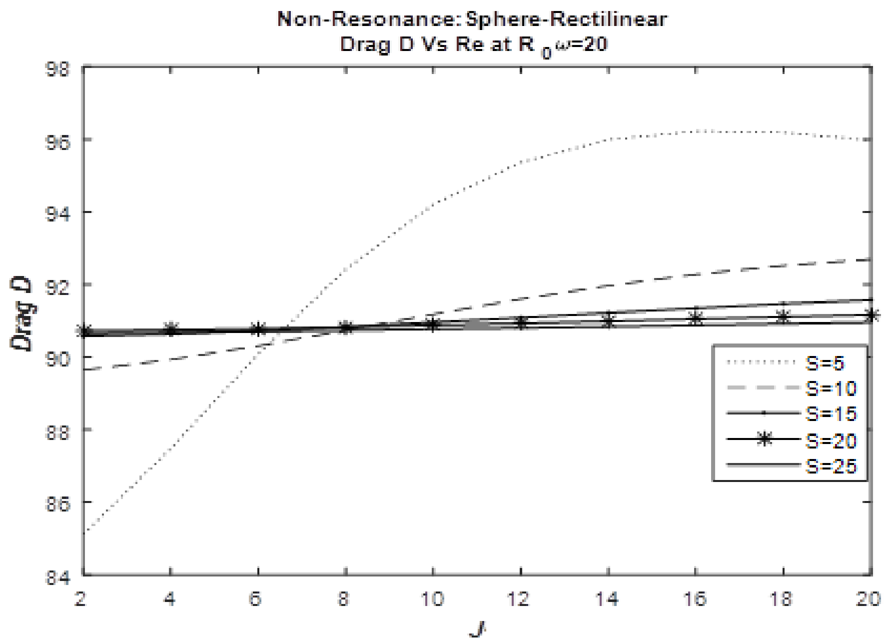


Fig 5.2 Drag Vs |λ| or ***s*** for resonance and non-resonance

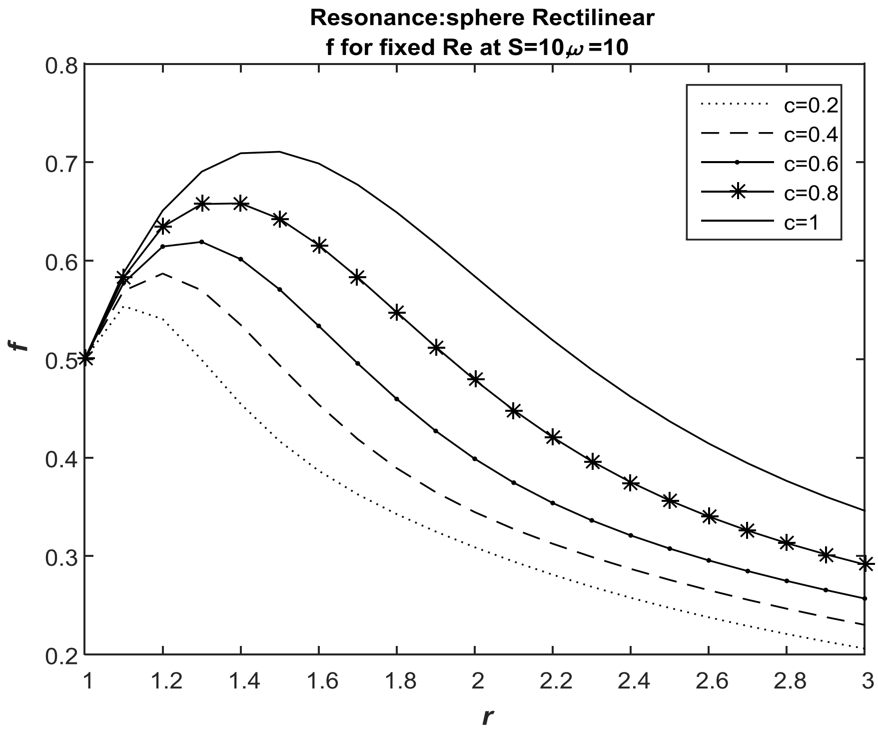


(a)

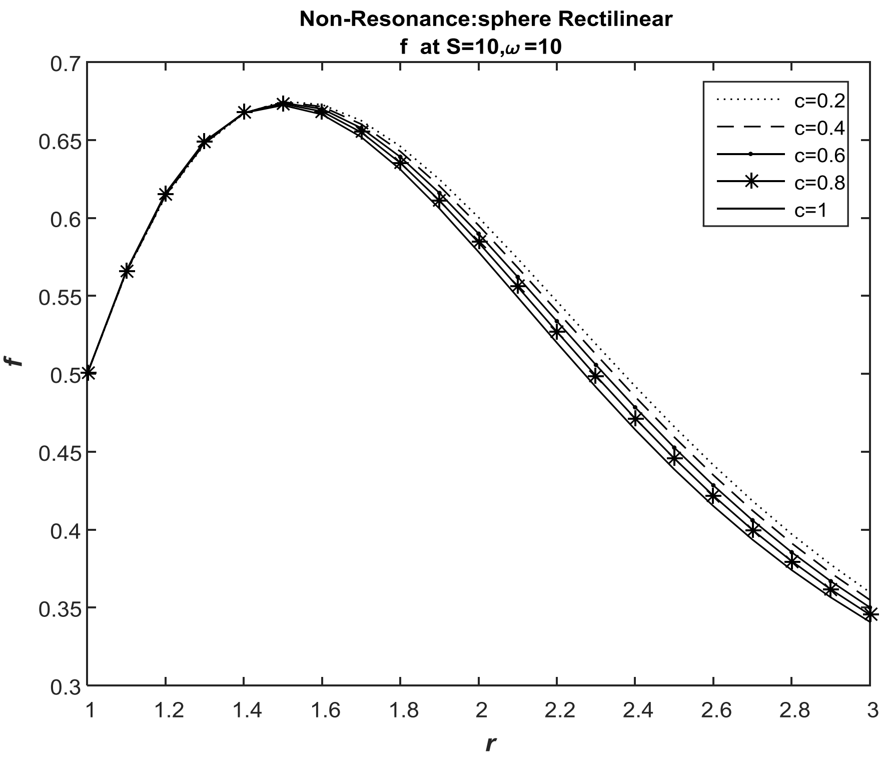


(b)

Fig 5.3 Drag Vs J for case of the case of a) resonance and b) non-resonance



(a)



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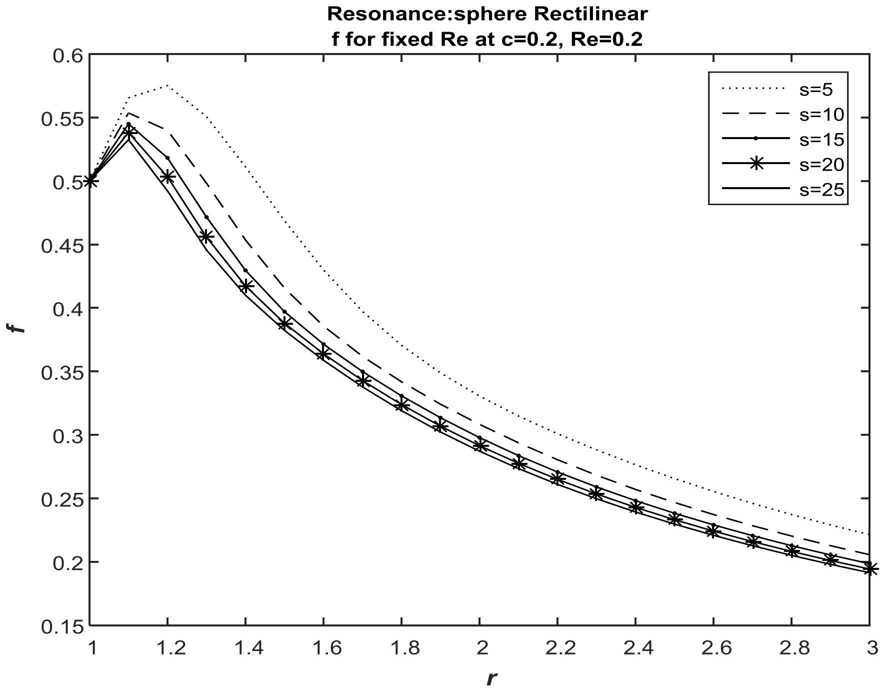
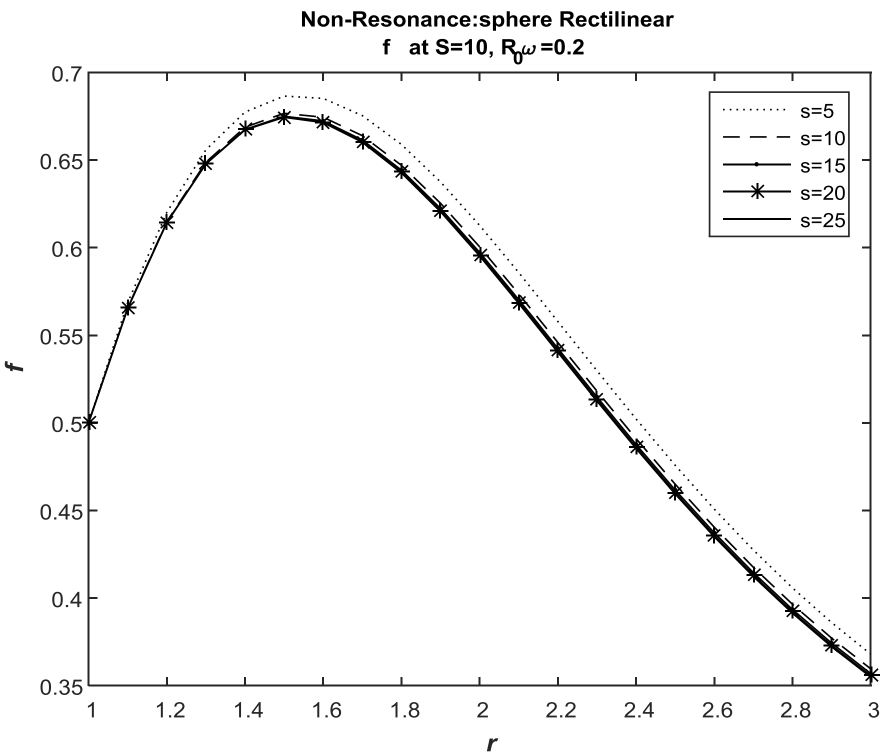
(b)

Fig 5.4 Stream function at different values of c for the case of

a) resonance and b) non-resonance



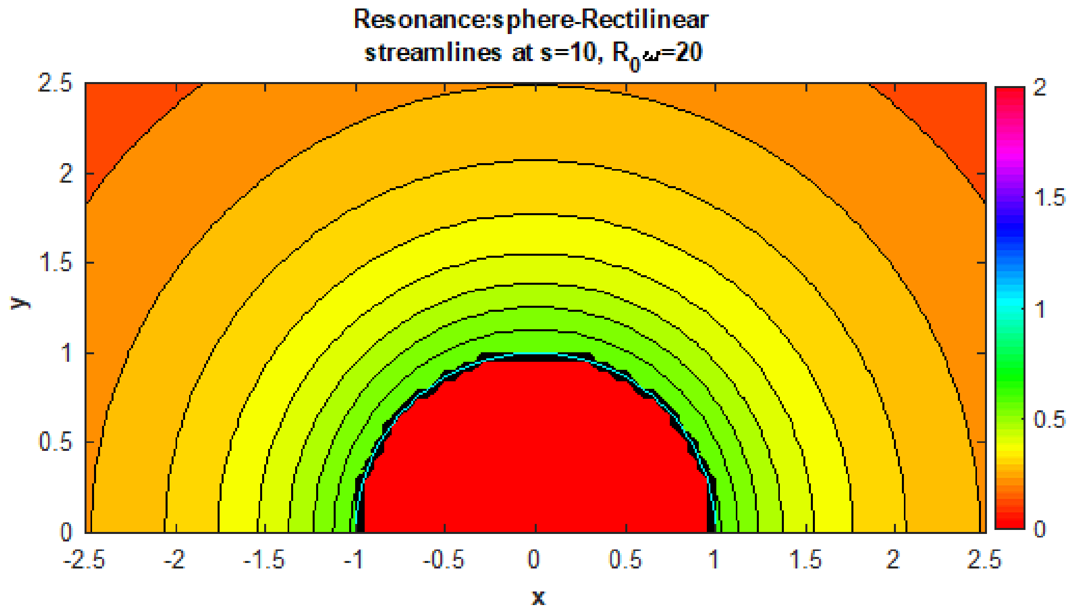
(a)



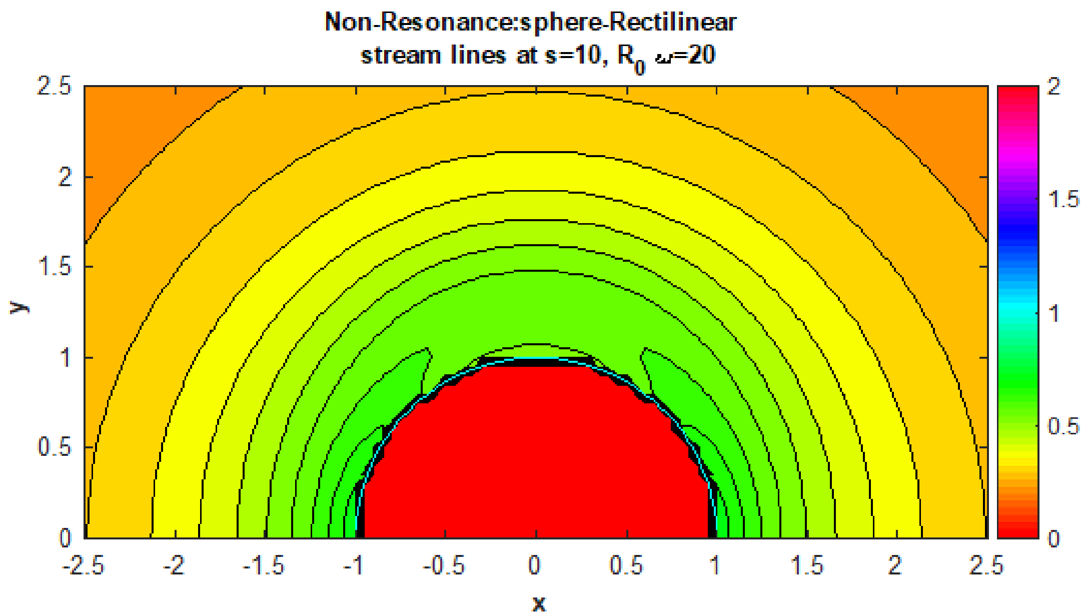
(b)

Fig 5.5 Stream function at different values of ***s*** for the case of

a) resonance and b) non-resonance



(a)



(b)

Fig 5.6 Stream lines for the case of a) resonance and b) non-resonance

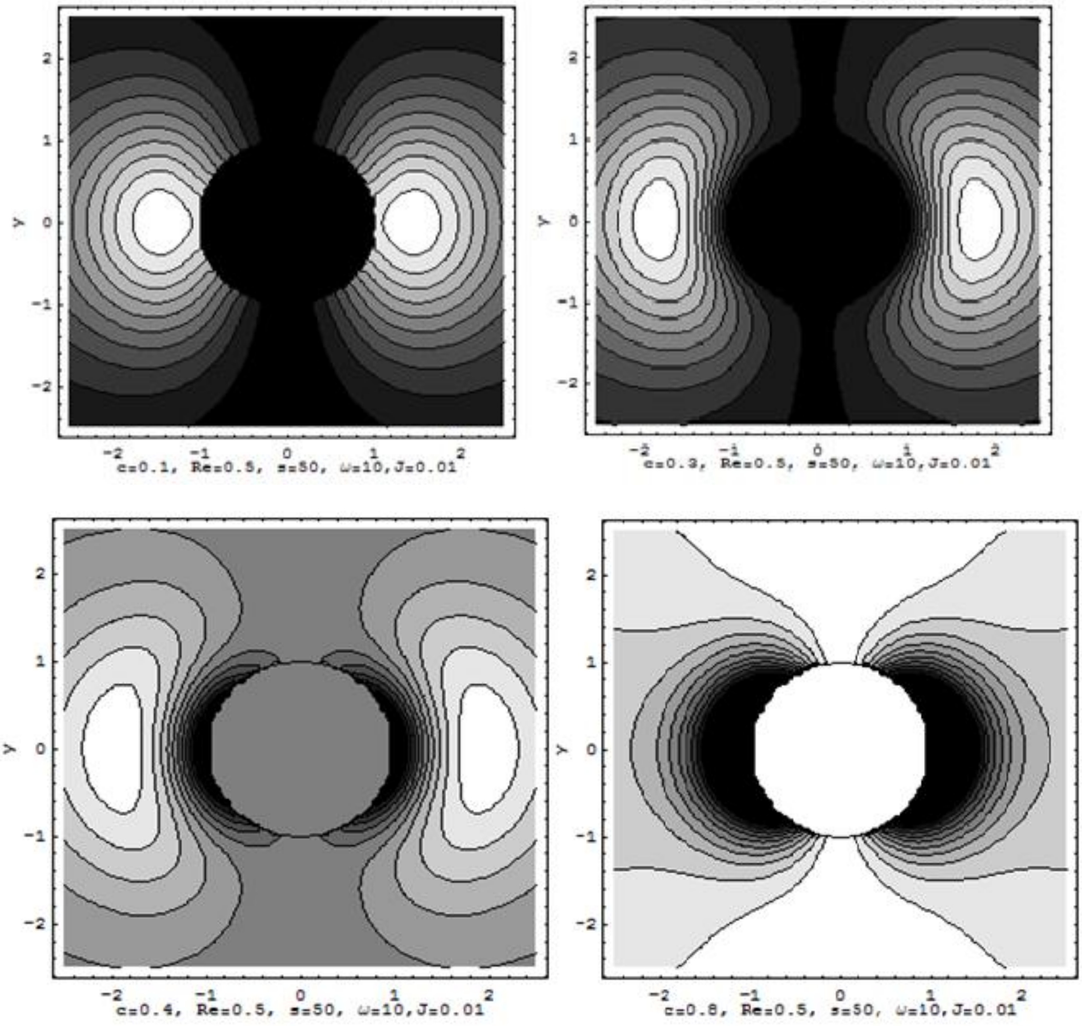


Fig 5.7. Stream lines at different values of Micro-polarity parameter *c* for *non-resonance*.

5.6 Conclusions

From the above graphs and observations we can conclude that

i) Drag in the case of resonance is much smaller than the case of non-resonance. i.e resonance offers less friction or helps to reduce the friction.

ii) For resonance, circulations near sphere disappear. For non-resonance, the effect of Micro-polarity parameter changes the pattern of stream function according as c is small or big.

Chapter – 6

Rotary oscillations of a sphere in a Micro-polar fluid

The flow of an incompressible micro-polar fluid generated due to rotary oscillations of a sphere about the axis of symmetry of the sphere is considered. The flow is so slow that nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered.

6.1 Introduction

Several such flow problems concerning with micro-polar fluids have been studied by many authors over the past five decades, ever since Eringen (1966) introduced the micro-polar fluid theory. And we have vast literature regarding micro-polar fluid theory as I introduced in previous chapters. Stokes (1968, 1971) studied effects of couple stresses in fluids on hydro magnetic channel flows and creeping flow past a sphere. Frater (1967, 1968) studied the elastico-viscous fluid flows generated due to oscillations of sphere and evaluated drag and damping force on the body. Ariman et al. (1967) examined micro-polar fluid flow between two concentric cylinders. Iyengar (1993, 2001) investigated flow of approximate sphere in incompressible micro-polar fluid and in incompressible viscous fluid. Lakshman Rao et al. (1971, 1981, 1983) have studied the micro-polar fluid flows generated due to oscillations of different symmetric bodies like sphere and spheroid. These problems were attempted to obtained drag or couple on the symmetric body.

The problems related to oscillatory Stokes flow are very common in non-Newtonian fluid flow and are of much interest to the investigators. Ravindran (1972) **s**tudied simple oscillatory flow in polar fluids. Verma (1971) studied oscillatory fluid flow past a fixed porous sphere. There is a vast literature regarding problems of oscillatory flows of sphere in different fluids. For example, Stimson (1926), Frater (1967, 1968), Lakshman Rao (1970, 1971), Lai(1978). Many researchers examined oscillatory flows of different objects generated due to rotary oscillations. Lakshmana Rao et al. (1983), Tekasakul et al. (1998, 2003), Iyengar et al. (2001, 2004), Aparna et al. (2012), Ashmawy (2015) are some of problems related to rotary oscillatory flows. Anwar (2004) studied micro-polar fluid flow of circular cylinder rotating and oscillating.

In all these problems, some authors found that a distinct flow exists which is technically termed as resonance flow and there lies a relation between material constants (to be stated later). Till now this has not been investigated by many researchers. This case arises in Lakshmana Rao (1971, 1981, 1983), but resonance case was not attempted by the authors. Aparna (2012) examined oscillatory fluid flow of permeable sphere oscillating rotary oscillations in an incompressible micro-polar fluid. In all above problems, the case of resonance if exists was not studied. In this chapter we propose to investigate this case of resonance type flow, in micro-polar fluids, due to rotary oscillations of a sphere about its axis of symmetry. Later the similar case investigated in couple-stress fluid.

6.2 Basic Equations

The basic equations of motion for incompressible micro-polar fluids as introduced by Eringen (1966), are given by

$$\frac{\partial \rho}{\partial \tau} + div(\rho \boldsymbol{Q}) = 0 \tag{6.1}$$

$$\rho\left(\frac{\partial \bar{Q}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{Q}\right) = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{6.2}$$

$$\rho J(\frac{\partial \bar{l}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{l}) = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1.\bar{l}) \tag{6.3}$$

The constitutive equations for the stress components T_{ij} and couple stress components M_{ij} for an incompressible micro-polar fluid are given by

$$T_{ij} = -P\delta_{ij} + \frac{1}{2}(2\mu + k)\left(u_{i,j} + u_{j,i}\right) + ke_{ijr}(w_r - l_r) \tag{6.4}$$

$$M_{ij} = \alpha l_{i,i}\delta_{i,j} + \beta l_{i,j} + \gamma l_{j,i} \tag{6.5}$$

where the permutation tensor $e_{ijk} = \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ 1 & \text{if } i, j, k \text{ are cyclic} \\ -1 & \text{if } i, j, k \text{ are anti}-\text{cyclic} \end{cases}$

and w_r = r th component of ½(curl **Q**).

Neglecting the nonlinear convective terms in (6.2) and (6.3), the linearised version of the equations are given by,

$$div\bar{Q} = 0 \tag{6.6}$$

$$\rho \frac{\partial \bar{Q}}{\partial \tau} = -\nabla_1 P + k\nabla_1 \times \bar{l} - (\mu + k)\nabla_1 \times \nabla_1 \times \bar{Q} \tag{6.7}$$

$$\rho \mathcal{J} \frac{\partial \bar{l}}{\partial \tau} = -2k\bar{l} + k\nabla_1 \times \bar{Q} - \gamma\nabla_1 \times \nabla_1 \times \bar{l} + (\alpha + \beta + \gamma)\nabla_1(\nabla_1 . \bar{l}) \tag{6.8}$$

6.3 Statement and Formulation of the Problem

A sphere of radius *a* is performing rotary oscillations with velocity $W_0 e^{i\sigma\tau}$ about its axis of symmetry in an infinite vat containing incompressible micro-polar fluid. Spherical coordinate system (R, θ, ϕ) with base vectors ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$) with origin at the centre of the sphere is considered. The flow is axially symmetric, hence it is independent of toroidal coordinate ϕ. The velocity and micro-rotation are assumed in the form:

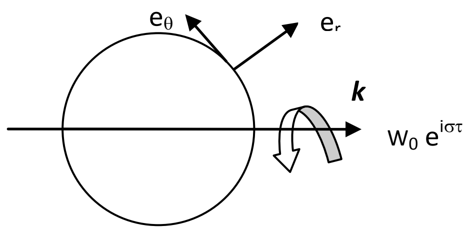


Fig 6.1 Geometry of the oscillating Sphere

$$\boldsymbol{Q} = e^{i\sigma\tau} W(R,\theta)\boldsymbol{e}_\phi \text{ and } \boldsymbol{l} = e^{i\sigma\tau} \{\mathcal{A}(R,\theta)\boldsymbol{e}_r + \mathcal{B}(R,\theta)\boldsymbol{e}_\theta\} \tag{6.9}$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$\left.\begin{array}{l} R = ar,\ \boldsymbol{Q} = W_0\boldsymbol{q},\ W = W_0 w,\ \boldsymbol{l} = \frac{W_0}{a}\boldsymbol{\upsilon} \\ \mathcal{A} = \frac{W_0}{a}A,\ \mathcal{B} = \frac{W_0}{a}B,\ P = p\rho W_0^2,\ \tau = \frac{at}{W_0} \end{array}\right\} \tag{6.10}$$

The following are non-dimensional parameters viz, J is gyration parameter, ϖ is frequency parameter, *s* is couple stress parameter, *c* is cross viscosity or micro-polarity parameter and Re is oscillations Reynolds number for micro-polar fluids.

$$\left.\begin{array}{l} \mathrm{J} = \frac{\mathrm{j}\rho\sigma a^2}{\gamma},\ \varpi = \frac{a\sigma}{W_0},\ s = \frac{\mathrm{k}a^2}{\gamma},\ c = \frac{\mathrm{k}}{\mu+\mathrm{k}} \\ \mathrm{Re} = \frac{\rho \mathrm{W}_0 a}{\mu},\ \mathrm{R}_0 = \frac{\rho \mathrm{W}_0 a}{\mu+k} = Re(1-c),\ \varepsilon = \frac{\alpha+\beta+\gamma}{\gamma} \end{array}\right\} \tag{6.11}$$

By the choice of the velocity field in (6.9) and non-dimensional scheme (6.10) and (6.11), the equations of motion (6.2) and (6.3) are reduced to

$$i\varpi R_0\boldsymbol{q} + R_0\boldsymbol{q}.\nabla\boldsymbol{q} = -R_0.\nabla p + c\,\nabla\times\boldsymbol{v} - \nabla\times\nabla\times\boldsymbol{q} \tag{6.12}$$

$$ij\boldsymbol{v} + \frac{J}{\varpi}\boldsymbol{q}.\nabla\boldsymbol{v} = -2s\boldsymbol{v} + s\nabla\times\boldsymbol{q} - \nabla\times\nabla\times\boldsymbol{v} + \epsilon\nabla(\nabla\cdot\boldsymbol{v}) \tag{6.13}$$

Neglecting the nonlinear convective terms in (6.12) and (6.13), the linearised version of the equations are given by,

$$i\varpi R_0\boldsymbol{q} = -R_0.\nabla p + c\,\nabla\times\boldsymbol{v} - \nabla\times\nabla\times\boldsymbol{q} \tag{6.14}$$

$$(iJ + 2s)\boldsymbol{v} = s\nabla\times\boldsymbol{q} - \nabla\times\nabla\times\boldsymbol{v} + \epsilon\nabla(\nabla\cdot\boldsymbol{v}) \tag{6.15}$$

Let us consider $\nabla\times\boldsymbol{v} = \frac{G}{h_3}\boldsymbol{e}_\phi$ and $\nabla\cdot\boldsymbol{v} = F(r,\theta)$ (6.16)

Now assuming (is known as swirl),

$$\boldsymbol{q} = we_\phi = \frac{\zeta}{h_3}e_\phi \tag{6.17}$$

Using (6.16) and (6.17) in (6.14) we get

$$(E^2 - iR_0\varpi)\zeta = -cG \tag{6.18}$$

where $\mathrm{E}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{cot\theta}{r^2}\frac{\partial}{\partial\theta}$ (6.19)

Using (6.16) and (6.17) in (6.15) we get

$$(iJ + 2s)\boldsymbol{v} = s\nabla\times\left(\frac{\zeta}{h_3}e_\phi\right) - \nabla\times\left(\frac{G}{h_3}\boldsymbol{e}_\phi\right) + \varepsilon\nabla F \tag{6.20}$$

By comparing the components in (6.20) we get

$$(2s + iJ)A = \frac{s}{r^2 sin\theta}.\frac{\partial\zeta}{\partial\theta} - \frac{1}{r^2 sin\theta}.\frac{\partial G}{\partial\theta} + \varepsilon\frac{\partial F}{\partial r} \quad (6.21)$$

$$(2s + iJ)B = -\frac{s}{r sin\theta}.\frac{\partial\zeta}{\partial r} + \frac{1}{r sin\theta}.\frac{\partial G}{\partial r} + \frac{\varepsilon}{r}\frac{\partial F}{\partial\theta} \quad (6.22)$$

To eliminate G taking divergence to (6.20) we get

$$(\nabla^2 - p_1^2)F = 0 \quad (6.23)$$

with $p_1{}^2 = \frac{2s + iJ}{\varepsilon}$ (6.24)

with $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}.\frac{\partial}{\partial r} + \frac{1}{r^2}(\frac{\partial^2}{\partial\theta^2} + \cot\theta.\frac{\partial}{\partial\theta})$ (6.25)

Taking Curl to (6.20) we get

$$(E^2 - (2s + iJ))G = sE^2\zeta \quad (6.26)$$

Taking $(E^2 - (2s + iJ))$ on both sides of (6.18) then substituting (6.26)

$$(E^2 - \lambda_1^2)\,(E^2 - \lambda_2^2)\,\zeta = 0 \quad (6.27)$$

Where $\lambda_1^2 + \lambda_2^2 = (2 - c)s + i(J + \varpi R_0)$ and $\lambda_1^2\lambda_2^2 = i\varpi R_0(2s + iJ)$ (6.28)

The solution for ζ if $\lambda_1 \neq \lambda_2$ in (6.27) is given in Lakshmana Rao et al. (1971). The solution for ζ for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \to \lambda_2$. This case is referred to as"*Resonance*". This resonance occurs if the material coefficients follow the following relation:

$\frac{\gamma}{j} = \frac{(2\mu + k)(\mu + k)}{2\mu + 3k}$ and $\rho\sigma = \frac{(2\mu + k)k + \gamma\rho\sigma}{J(\mu + k)}$ (6.29)

in non-dimensional form

$(2 - c)s = J - R_0\varpi$ and $(2 - c)J = \varpi R_0(2 + c)$ (6.30)

In this chapter we are interested in the solution for ζ for the case of resonance. In this case the equations for ζ is given by

$$(E^2 - \lambda^2)^2\,\zeta = 0 \quad (6.31a)$$

In the case of non-resonance the equations for ζ is given by

$$(E^2 - \lambda_1^2)\,(E^2 - \lambda_2^2)\,\zeta = 0 \quad (6.31b)$$

6.3.1 Boundary Conditions

By no-slip condition, the non-dimensional swirl ζ and by hyper-stick condition, the micro-rotation components A and B are given by

$$\zeta = \sin^2\theta \text{ and } A = cos\theta \, , B = - \, sin\theta \text{ on r=1} \quad (6.32)$$

6.4 Solution of the Problem

To match with the boundary conditions in (6.32), The swirl function ζ and the divergence function F are assumed in the form

$$\zeta = f(r)\sin^2\theta \text{ and } F = g(r)cos\theta \quad (6.33)$$

Substituting (6.33) in (6.27) we get

$$(D_s^2 - \lambda_1^2)\,(D_s^2 - \lambda_2^2)f = 0 \quad (6.34)$$

Where $D_s^2 = \frac{d^2}{dr^2} - \frac{2}{r^2}$ (6.35)

Substituting (6.33) in (6.23) we get

$$\frac{d^2g}{dr^2} + \frac{2}{r}\frac{dg}{r} - \frac{2}{r^2}g = 0 \quad (6.36)$$

Equation for *f* for the case of resonance is given by

$$(D_s^2 - \lambda_1^2)\,(D_s^2 - \lambda_2^2)f = 0 \quad (6.37a)$$

In the case of non-resonance

$$(D_s^2 - \lambda^2)^2 f = 0 \quad (6.37b)$$

The solutions for *f(r)* is given by

In the case of resonance:

$$f(\mathrm{r}) = \mathrm{a}_1\sqrt{r}K_{\frac{3}{2}}(\lambda r) + \mathrm{a}_2 r^{\frac{3}{2}}K'_{\frac{3}{2}}(\lambda r) = a_1 f_1 + a_2 f_2 \quad (6.38a)$$

In the case of non-resonance:

$$f(\mathrm{r}) = \mathrm{a}_1\sqrt{r}K_{\frac{3}{2}}(\lambda_1 r) + \mathrm{a}_2\sqrt{r}K_{\frac{3}{2}}(\lambda_2 r) = a_1 f_1 + a_2 f_2 \quad (6.38b)$$

The solutions for *g(r)* by (6.36) is given

$$g(r) = \frac{a_3}{\sqrt{r}}K_{\frac{3}{2}}(p_1 r) = a_3 g_3 \quad (6.39)$$

We can write (6.20) as

$$(iJ + 2s)\boldsymbol{v} = \nabla \times \left(\frac{s\zeta - G}{h_3}e_\phi\right) + \varepsilon\nabla F \quad (6.40)$$

By substituting (6.18) in (6.40) we get

$$(iJ + 2s)\boldsymbol{v} = \frac{1}{c}\nabla \times \left((E^2 - (iR_0\varpi - cs))\frac{\zeta}{h_3}e_\phi\right) + \varepsilon\nabla F \tag{6.41}$$

By comparing the components in (6.41)

$$c(2s + iJ)A = \frac{1}{r^2 sin\theta}.\frac{\partial}{\partial\theta}(E^2 - (iR_0\varpi - cs))\zeta + c\varepsilon\frac{\partial F}{\partial r} \tag{6.42}$$

$$c(2s + iJ)B = -\frac{1}{rsin\theta}.\frac{\partial}{\partial r}(E^2 - (iR_0\varpi - cs))\zeta + \frac{c\varepsilon}{r}\frac{\partial F}{\partial\theta} \tag{6.43}$$

Substituting (6.33) in (6.42) and (6.43) we get

$$c(2s + iJ)A = \frac{2cos\theta}{r^2}(D_s^2 - (iR_0\varpi - cs))f + c\varepsilon g' cos\theta \tag{6.44}$$

$$c(2s + iJ)B = -\frac{sin\theta}{r}.\frac{d}{dr}(D_s^2 - (iR_0\varpi - cs))f - \frac{c\varepsilon}{r}g sin\theta \tag{6.45}$$

Now assuming $A = \bar{A}(r)cos\theta \; and \; B = \bar{B}(r)sin\theta$ (6.46)

Now (6.44) and (6.45) becomes

In the case of resonance:

$$\left.\begin{aligned} c\frac{\lambda^4}{iR_0\varpi}\bar{A} &= \frac{2}{r^2}\left(D_s^2 - 2\lambda^2 + \frac{\lambda^4}{iR_0\varpi}\right)f + c\varepsilon g' \\ c\frac{\lambda^4}{iR_0\varpi}\bar{B} &= -\frac{1}{r}.\frac{d}{dr}\left(D_s^2 - 2\lambda^2 + \frac{\lambda^4}{iR_0\varpi}\right)f - \frac{c\varepsilon}{r}g \end{aligned}\right\} \tag{6.47a}$$

In the case of non-resonance:

$$\left.\begin{aligned} c\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}\bar{A} &= \frac{2}{r^2}\left(D_s^2 - \lambda_1^2 - \lambda_2^2 + \frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}\right)f + c\varepsilon g' \\ c\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}\bar{B} &= -\frac{1}{r}.\frac{d}{dr}\left(D_s^2 - \lambda_1^2 - \lambda_2^2 + \frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}\right)f - \frac{c\varepsilon}{r}g \end{aligned}\right\} \tag{6.47b}$$

We denote $\Delta_1(x) = 1 + \frac{xK_{\frac{1}{2}}(x)}{K_{\frac{3}{2}}(x)}$ (6.48)

We notice that

$$\left.\begin{aligned} \frac{d}{dr}\left[\sqrt{r}K_{\frac{3}{2}}(\lambda r)\right] &= -\frac{K_{\frac{3}{2}}(\lambda r)}{\sqrt{r}}\Delta_1(\lambda r) \\ \frac{d}{dr}\left[\frac{K_{\frac{3}{2}}(\lambda r)}{\sqrt{r}}\right] &= -\frac{K_{\frac{3}{2}}(\lambda r)}{r^{\frac{3}{2}}}(1 + \Delta_1(\lambda r)) \\ \frac{d}{dr}\left[r^{\frac{3}{2}}K'_{3/2}(\lambda r)\right] &= \frac{1}{\lambda\sqrt{r}}\left[\left(\lambda^2 r^2 + \frac{3}{2}\right)K_{\frac{3}{2}}(\lambda r) - \frac{\lambda r}{2}K_{\frac{1}{2}}(\lambda r)\right] \end{aligned}\right\} \tag{6.49}$$

Substituting (6.38a), (6.38b), (6.39) and (6.49) in (6.47a) and (6.47b), we get,

In the case of resonance:

$$\frac{c\lambda^4}{iR_0\varpi}\bar{A} = \frac{2}{r^2}\frac{\lambda^2(\lambda^2 - iR_0\varpi)}{iR_0\varpi}f + \frac{4\lambda a_2}{r^2}f_1 - \frac{\varepsilon c a_3}{r^{\frac{3}{2}}}K_{\frac{3}{2}}(p_1 r)[1 + \Delta_1(p_1 r)] \tag{6.50a}$$

$$\frac{c\lambda^4}{iR_0\varpi}\bar{B} = \frac{\lambda^2(\lambda^2 - iR_0\varpi)}{iR_0\varpi r^{\frac{3}{2}}}\left[a_1 K_{\frac{3}{2}}(\lambda r)\Delta_1(\lambda r) + \frac{a_2}{\lambda}\left(\left(\lambda^2 r^2 + \frac{3}{2}\right)K_{\frac{3}{2}}(\lambda r) - \frac{\lambda r}{2}K_{\frac{1}{2}}(\lambda r)\right)\right]$$

$$+2\lambda a_2\frac{K_{\frac{3}{2}}(\lambda r)}{r\sqrt{r}}\Delta_1(\lambda r) + \frac{c\varepsilon a_3}{r^{\frac{3}{2}}}K_{\frac{3}{2}}(p_1 r) \tag{6.51a}$$

In the case of non-resonance:

$$c\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}\bar{A} = \frac{2}{r}\left[\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}(a_1 f_1 + a_2 f_2) - (a_1\lambda_2^2 f_1 + a_2\lambda_1^2 f_2)\right] + c\varepsilon a_3 g'_3 \tag{6.50b}$$

$$c\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}\bar{B} = -\frac{1}{r}\frac{d}{dr}\left[\frac{\lambda_1^2\lambda_2^2}{iR_0\varpi}(a_1 f_1 + a_2 f_2) - (a_1\lambda_2^2 f_1 + a_2\lambda_1^2 f_2)\right] - \frac{\boldsymbol{c\varepsilon}}{\boldsymbol{r}} a_3 g_3 \tag{6.51b}$$

Now the condition at r=1 are given by

$$f(1) = 1,\ \bar{A} = 1, \bar{B} = -1 \tag{6.52}$$

Substituting the above formulae (6.49), we get on r=1;

$$a_1 K_{\frac{3}{2}}(\lambda_1) + a_2 K_{\frac{3}{2}}(\lambda_2) = 1 \tag{6.53}$$

$$\frac{2a_1}{\lambda_1^2}K_{\frac{3}{2}}(\lambda_1) + \frac{2a_2}{\lambda_2^2}K_{\frac{3}{2}}(\lambda_2) + \frac{a_3 c\varepsilon}{\lambda_1^2\lambda_2^2}.K_{\frac{3}{2}}(p_1)[1 + \Delta_1(p_1)] = \frac{2-c}{iR_0\varpi} \tag{6.54}$$

$$a_1.\frac{\lambda_1^2 - iR_0\varpi}{c\lambda_1^2}.K_{\frac{3}{2}}(\lambda_1)\Delta(\lambda_1) + a_2.\frac{\lambda_2^2 - iR_0\varpi}{c\lambda_2^2}.K_{\frac{3}{2}}(\lambda_2)\Delta(\lambda_2) - a_3\varepsilon\frac{iR_0\varpi}{\lambda_1^2\lambda_2^2}.K_{\frac{3}{2}}(p_1) = -1 \tag{6.55}$$

The constants a_1, a_2 and a_3 are obtained from the boundary conditions (6.32) or (6.52) as follows:

In the case of resonance:

$$\begin{bmatrix} K_{\frac{3}{2}}(\lambda) & K'_{\frac{3}{2}}(\lambda) & 0 \\ 0 & 4\lambda K_{\frac{3}{2}}(\lambda) & c\varepsilon K_{\frac{3}{2}}(p_1)(1+\Delta_1(p_1) \\ K_{\frac{3}{2}}(\lambda)\Delta_1(\lambda) & \frac{2iR_0\varpi}{\lambda(\lambda^2 - iR_0\varpi)}K_{\frac{3}{2}}(\lambda)\Delta_1(\lambda) - \frac{1}{\lambda}\left[\left(\lambda^2 + \frac{3}{2}\right)K_{\frac{3}{2}}(\lambda) - \frac{\lambda}{2}K_{\frac{1}{2}}(\lambda)\right] & c\varepsilon K_{\frac{3}{2}}(p_1).\frac{iR_0\varpi}{\lambda^2(\lambda^2 - iR_0\varpi)} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \frac{\lambda^2}{iR_{0\varpi}}((c-2)\lambda^2 + 2iR_0\varpi) \\ -1 \end{bmatrix} \tag{6.56a}$$

In the case of non-resonance:

$$\begin{bmatrix} K_{\frac{3}{2}}(\lambda_1) & K_{\frac{3}{2}}(\lambda_2) & 0 \\ \frac{2}{\lambda_1^2}K_{\frac{3}{2}}(\lambda_1) & \frac{2}{\lambda_2^2}K_{\frac{3}{2}}(\lambda_2) & -\varepsilon\frac{iR_0\varpi}{\lambda_1^2\lambda_2^2}K_{\frac{3}{2}}(p_1)(1+\Delta_1(p_1)) \\ K_{\frac{3}{2}}(\lambda_1)\Delta_1(\lambda_1)c_1 & K_{\frac{3}{2}}(\lambda_2)\Delta_1(\lambda_2)c_2 & \varepsilon\frac{iR_0\varpi}{\lambda_1^2\lambda_2^2}K_{\frac{3}{2}}(p_1) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2-c}{iR_{0\varpi}} \\ -1 \end{bmatrix} \tag{6.56b}$$

Hence from (6..56a) and (6.56b), we can calculate a_1, a_2 and a_3.And hence ζ and F are known.

6.4.1 Couple Acting on the Sphere of radius a

Couple acting on the sphere due to Cauchy's stresses is

$$C_1^* = 2\pi a^3 \int_0^{\pi} (T^*{}_{r\emptyset} \sin^2\theta)\,|_{R=a}\, d\theta \tag{6.57}$$

$$T_{r\emptyset} = (2\mu + k)E_{r\phi} + k\epsilon_{132}(\omega_2 - v_2) \tag{6.58}$$

$$\omega_2 = -\frac{1}{2sin\theta}\frac{\partial\zeta}{\partial r} = -\frac{1}{2}f'(r)sin\theta \tag{6.59}$$

$$E_{r\phi} = \frac{1}{2r}\left(f'(r) - \frac{2f}{r}\right)sin\theta \tag{6.60}$$

After substituting (6.59) and (6.60) in (6.58) we get

On r=1, $T_{r\phi} = [(\mu + k)f'(1) - (2\mu + k)f(1) + kB(1)]sin\theta$ (6.61)

Substituting (6.61) in (6.57), we get

$$C^* = \frac{8}{3}\pi a^2 W_0(\mu + k)(f'(1) - 2) \tag{6.62}$$

For resonance, non-dimensional Couple is

$$C^* = \frac{8}{3}\pi a^2 W_0(\mu + k)\left\{a_1 K_{\frac{3}{2}}(\lambda)\Delta_1(\lambda) + \frac{a_2}{\lambda}\left[\left(\lambda^2 + \frac{3}{2}\right)K_{\frac{3}{2}}(\lambda) - \frac{\lambda}{2}K_{\frac{1}{2}}(\lambda)\right] - 2\right\} \tag{6.63a}$$

For non-resonance, non-dimensional Couple is

$$C^* = \frac{8}{3}\pi a^2 W_0(\mu + k)\left\{a_1 K_{\frac{3}{2}}(\lambda_1)\Delta_1(\lambda_1) + a_2 K_{\frac{3}{2}}(\lambda_2)\Delta_1(\lambda_2) - 2\right\} \tag{6.63b}$$

Dividing by $4\pi\mu a^2 W_0$ we dimensional Couple as

For resonance case,

$$C^* = \frac{2}{3}.\frac{1}{1-c}\left\{a_1 K_{\frac{3}{2}}(\lambda)\Delta_1(\lambda) + \frac{a_2}{\lambda}\left[\left(\lambda^2 + \frac{3}{2}\right)K_{\frac{3}{2}}(\lambda) - \frac{\lambda}{2}K_{\frac{1}{2}}(\lambda)\right] - 2\right\} \tag{6.64a}$$

For non-resonance case,

$$C^* = \frac{2}{3}.\frac{1}{1-c}\left\{a_1 K_{\frac{3}{2}}(\lambda_1)\Delta_1(\lambda_1) + a_2 K_{\frac{3}{2}}(\lambda_2)\Delta_1(\lambda_2) - 2\right\} \tag{6.64b}$$

6.5 Results and Discussions

For resonance case, the value of λ cannot be taken randomly. In the case of resonance the values of λ are obtained from (6.28) by solving the following equation for x.

$$x^2 - [(2-c)s + i(J + \varpi R_0)]x + i\varpi R_0(iJ + 2s) = 0 \tag{6.65}$$

Then in resonance case, the values of λ are given by

$$\lambda = \sqrt{x} = \sqrt{\frac{(2-c)s + i(J + \varpi.R_0)}{2}} \tag{6.66}$$

This equation involves 5 parameters which are related by two equations in (6.28). Hence we choose three parameters as an independent. Here ϖ, R_0 and c are selected independently, with $0 \leq c \leq 1$, $R_0 \ll 1$ and $\varpi \gg 1$ such that $\varpi.R_0$ is not negligibly small (say>1). For this range of values of R_0, the nonlinear convective terms can be neglected but local derivative is retained. After selecting c, $R_0$and ϖ, the values of s and Jare obtained from (6.28) and then λ is obtained from (6.66). In the case of non-resonance, all 5 parameters are independent. The values of λ are complex. These values for λ are substituted in (4.38a) and (4.38b) and then constants $a_1 and a_2$are obtained.

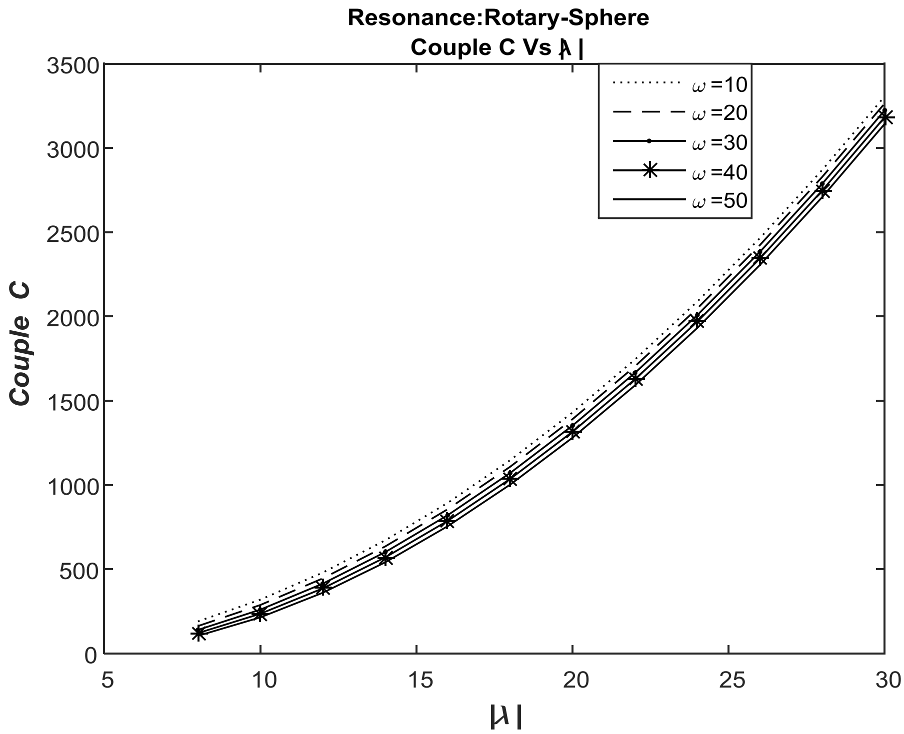
6.5.1 Couple

Couple is effected by all the five parameters. The effects of Reynolds number Re, couple stress parameter s and Gyro-viscosity parameter J on the couple are shown in the figures.

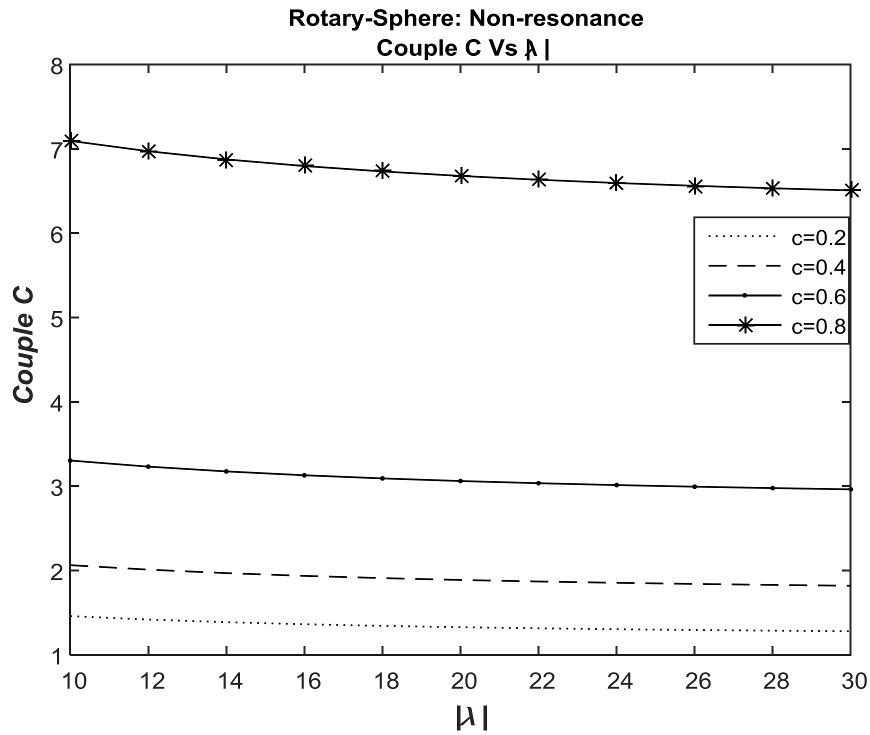
From Fig 6.2, it is observed that as |λ| increases, for resonance, couple increases drastically and takes very large values. Bur for non-resonance couple decreases within a small interval and is almost constant.

From Fig 6.3, we note that in the case of non-resonance, variation of couple stress parameter s will not effect couple. But for the case of resonance, as s increases couple also increases. In both cases of resonance and non-resonance, as c increases, couple also increases.

From Fig 6.4, we note that frequency parameter effects the couple very much. As ϖ increases, couple decreases.

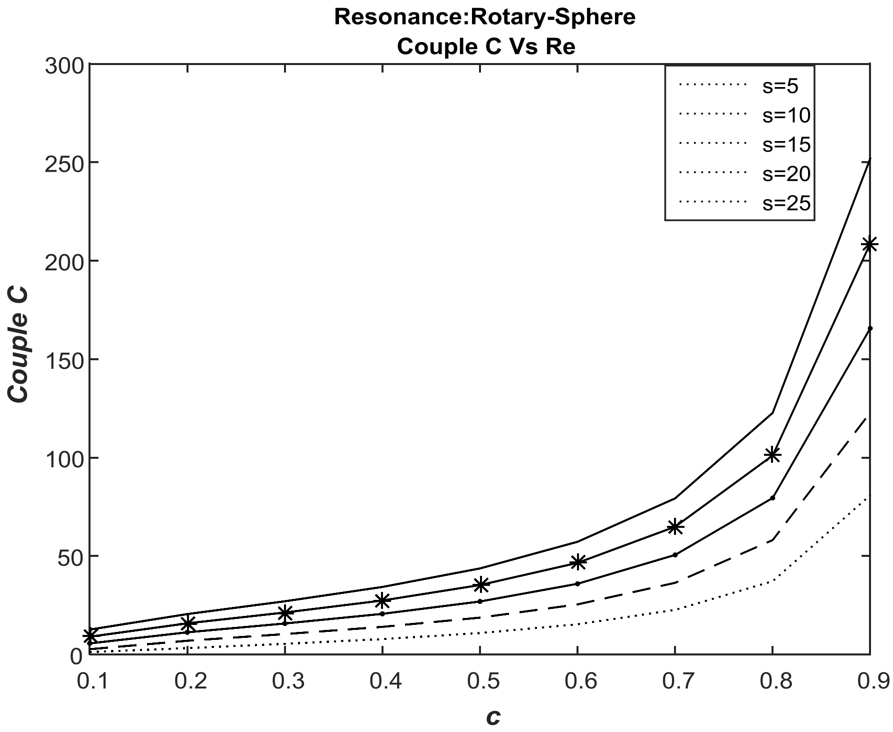


(a)

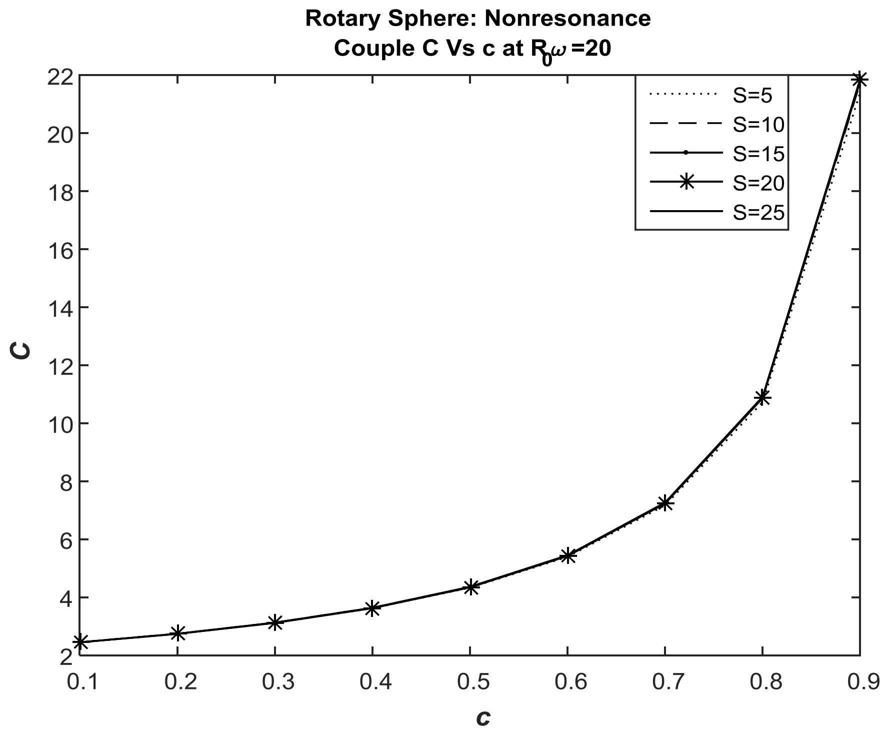


(b)

Fig 6.2 Couple Vs $|\lambda|$ at different values of c for the case of a) resonance and b) non-resonance



(a)



(b)

Fig 6.3 Couple Vs cross-viscosity parameter for the case of a) resonance and b) non-resonance

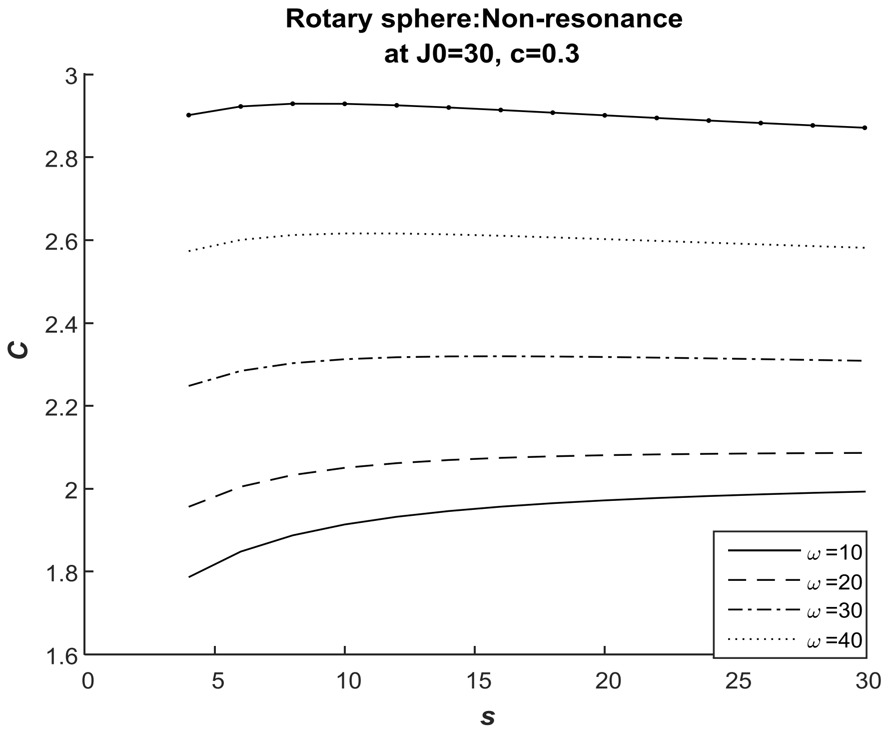


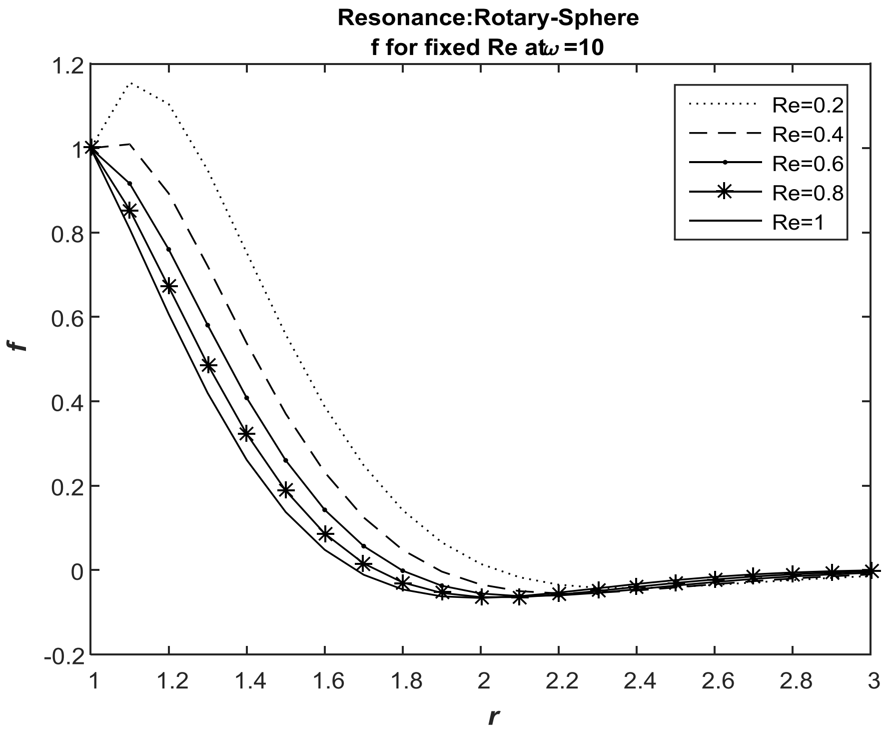
Fig 6.4 Couple vs couple stress parameter s for non-resonance

6.5.2 Velocity

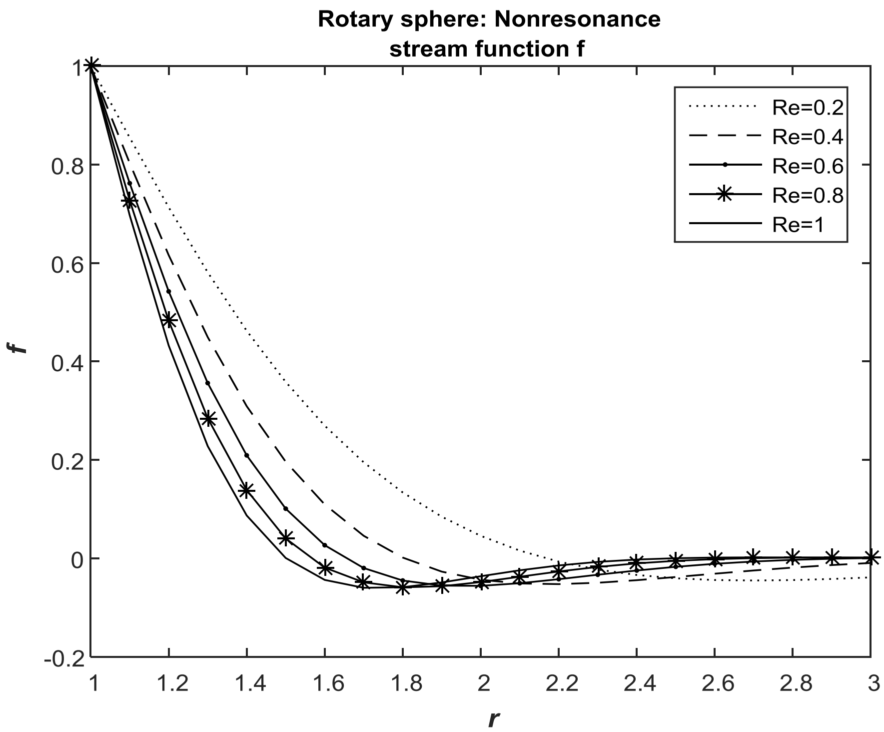
The toroidal velocity *W* is found in terms of swirl ς which is defined in terms of function f.

Form Fig 6.5, we observe that for resonance velocity raises more than 1 and negative in a small range of r for small values of Reynolds number Re. But for non-resonance, velocity is always less than 1 and negative for larger range of r.

From Fig 6.6 and Fig 6.7, it is observed that as couple stress parameter *s* and micro-polarity parameter cdo not show effect on velocity in the case of non-resonance. But in the case of resonance as s increases, velocity decreases and as c increases, velocity also increases.

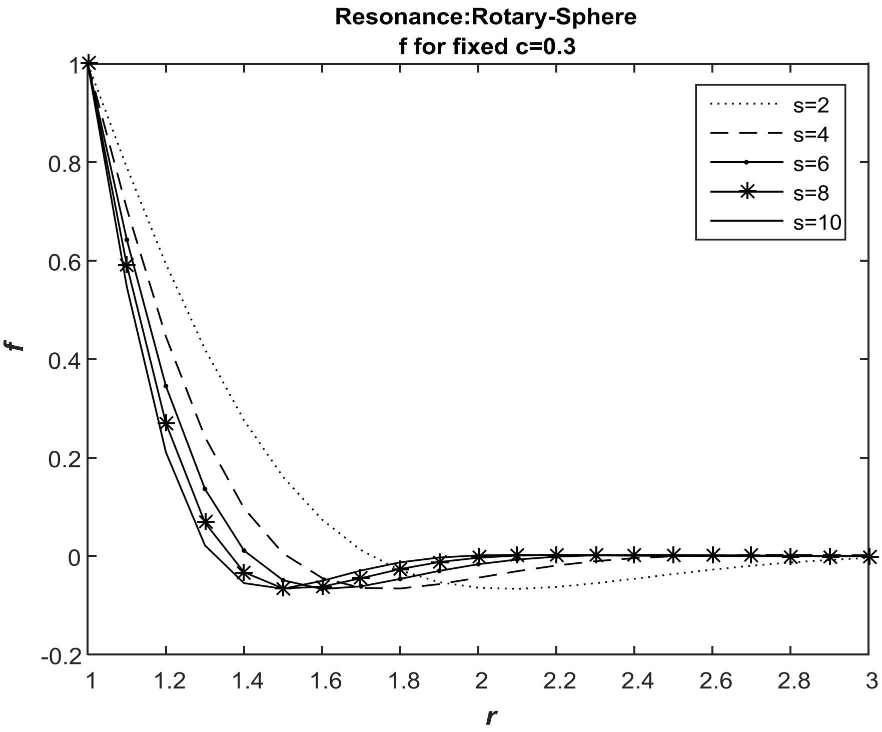


(a)

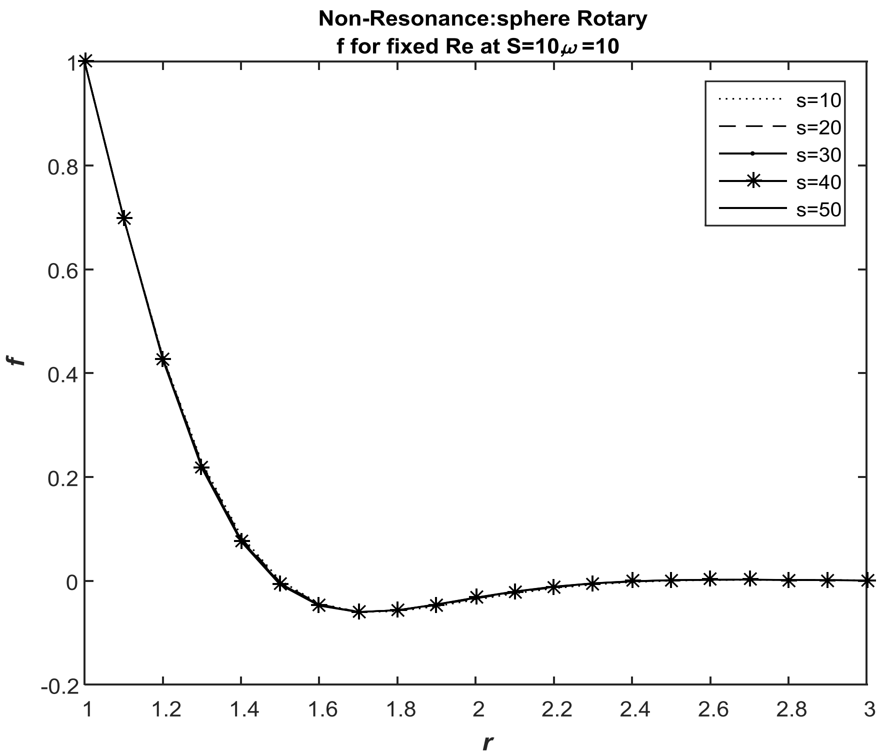


(b)

Fig 6.5 Velocity f at different values of Re for the case of a) resonance and b) non-resonance

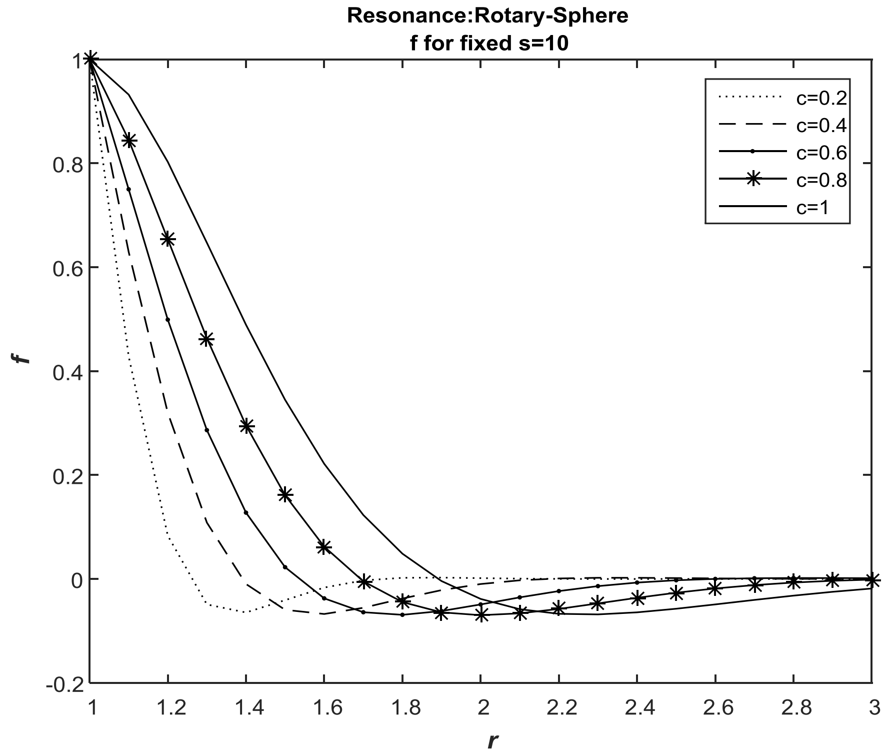


(a)

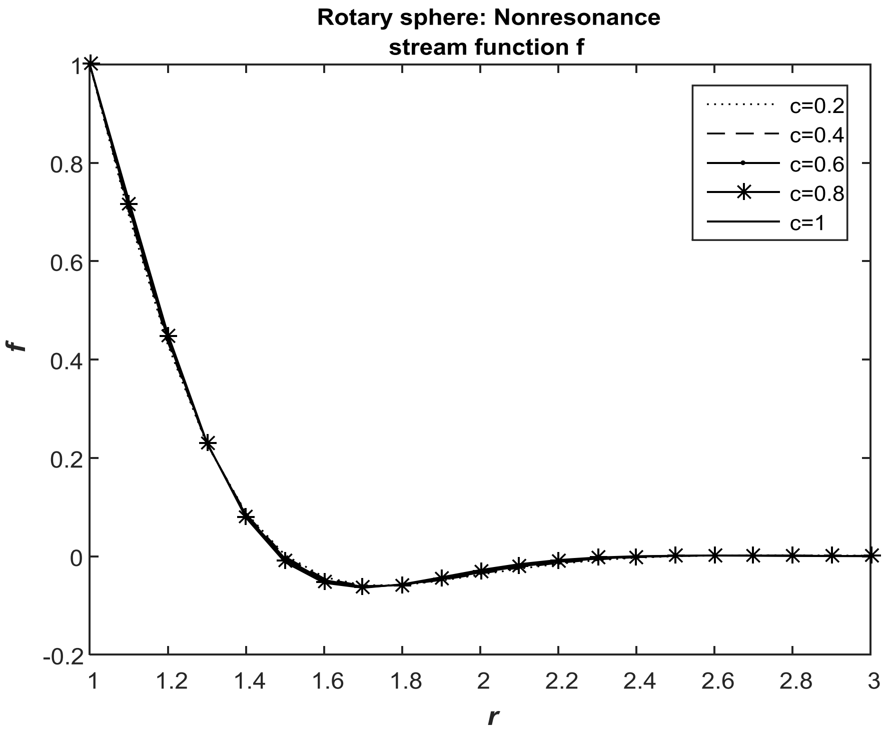


(b)

Fig 6.6 Velocity f for variations in s for the case of a) resonance and b) non-resonance

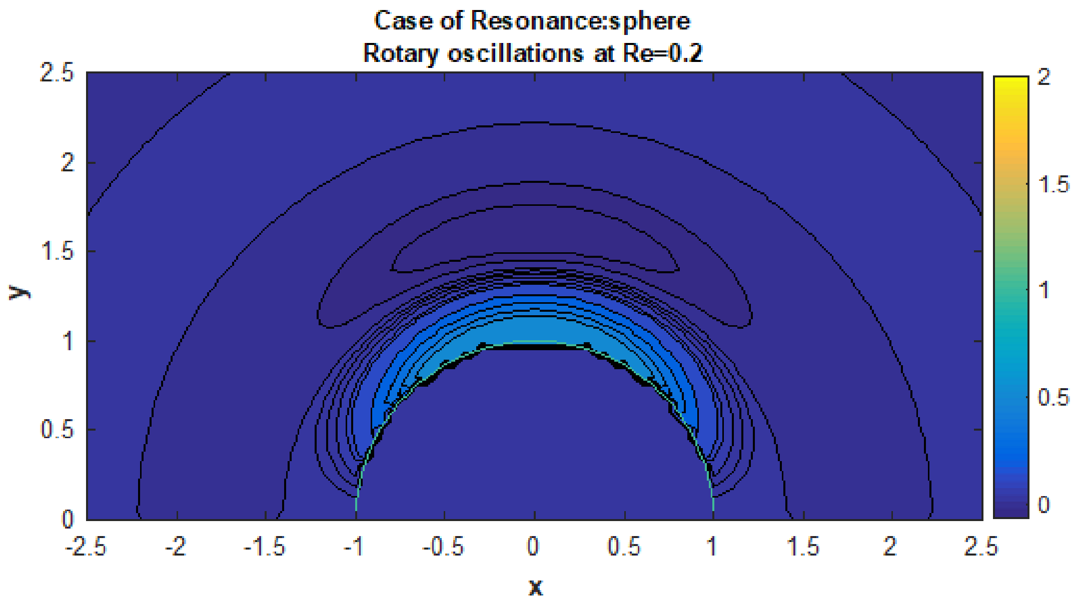


(a)

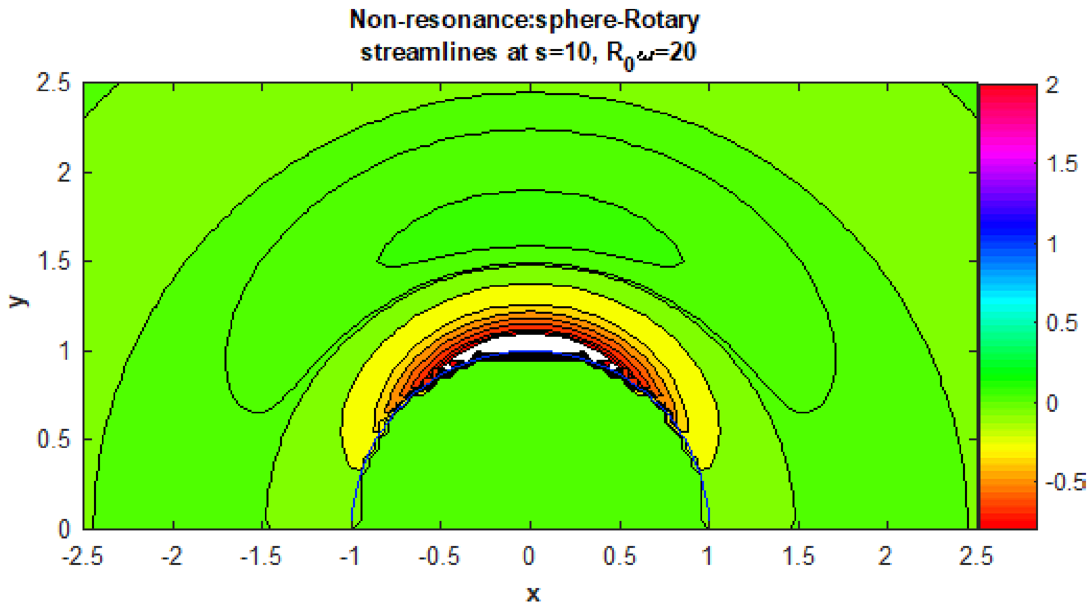


(b)

Fig 6.7 Velocity f for variations in c for the case of a) resonance and b) non-resonance



(a)



(b)

Fig 6.8 Velocity contours for the case of a) resonance and b) non-resonance

From Fig 6.8 of the contour lines of velocity, we confirm the observations made above in fig6.5 i.e for non-resonance near to the cap or pole of the sphere toroidal velocity forms circulations with negative values (red and yellow colour). For resonance, all circulations are positive.(blue colour).

6.6 Conclusions

From the above observations we conclude that in the case of rotary oscillations

i) For resonance couple is very high and for non-resonance it is low.

ii) For resonance, velocity is positive in the entire range. For non-resonance velocity at the pole is negative.

Part – III

Couple-stress Fluid Flows

Chapter 7

Rectilinear oscillations of a Circular Cylinder in a Couple-stress fluid

The flow due to a circular cylinder oscillating rectilinearly about its axis of symmetry in a Couple-stress fluid is considered. There occurs a rare but an important special case referred to as Resonance flow. The material constants satisfy a specific relation called resonance condition. In this case, the flow is analyzed under Stokesian approximation. The velocity component of the flow is derived. The effect of physical parameters like Reynolds number and Couple stress parameter on the Drag is analyzed through graphs.

7.1 Introduction

The flow problems in Couple stress fluids have been attracting many researchers due their Mathematical simplicity and beauty and importance in many applications. Oscillatory flows of circular cylinder in various fluids like Micro-polar fluids, Couple-stress fluids, viscous fluids were investigated by many authors Kanwal (1955), Ariman et al. (1967), Ramkissoon et al. (1990), Rao et al. (1992), Calmelet-Eluhu et al. (1998), Anwar et al. (2004), Fetecau et al. (2006), Mehrdad Massoudi et al. (2008), Ramana Murthy et al. (2010), Nagaraju et al. (2014) by computationally or analytically. An incompressible viscous flow due to rectilinear oscillations of an approximate sphere was studied by Iyengar et al. (2001). Oscillatory flow of a sphere due to rectilinear oscillations in an elastic-viscous fluid was investigated by Lai et al. (1978). In these papers, the authors analyzed Drag on the object.

In this chapter, we propose to investigate this case of resonance type flow, in Couple-stress fluids, due to rectilinear oscillations of a circular cylinder about its axis of symmetry. In chapter 2 the similar problem of the Resonance type flow due to a circular cylinder in Micro-polar fluid is investigated.

7.2 Basic Equations

The basic equations of motion for an incompressible Couple stress fluid introduced by Stokes (1966) are given by:

$$div\bar{Q} = 0 \tag{7.1}$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{7.2}$$

where $\bar{Q}$ is fluid velocity vector, ρ is density, τ is time, μ is viscosity coefficient.

By neglecting nonlinear convective terms in (7.2) we get

$$\rho\frac{\partial\bar{Q}}{\partial\tau} = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{7.3}$$

The (Cauchy's) stress tensor T and Couple stress tensor M satisfy the constitutive equations as below:

$$T = -PI + \lambda(\nabla_1.Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2}I \times (\nabla_1.M) \tag{7.4}$$

$$M = mI + 2\eta\nabla_1(\nabla_1 \times Q) + 2\eta'[\nabla_1(\nabla_1 \times Q)]^T \tag{7.5}$$

7.3 Statement and Formulation of the Problem

A circular cylinder of radius a and of infinite length is performing rectilinear oscillations with velocity $U_0 e^{i\sigma\tau}$ about its diameter in an incompressible Couple-stress fluid. A cylindrical coordinate system (R, θ, Z) with origin on the axis of the cylinder is considered. The fluid flow is assumed to be in cross-sectional plane with the base vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$. The velocity and pressure are assumed as:

$$\boldsymbol{Q} = (U(R,\theta)\boldsymbol{e_r} + V(R,\theta)\boldsymbol{e_\theta}) \text{ and } P_0 = Pe^{i\sigma\tau} \tag{7.6}$$

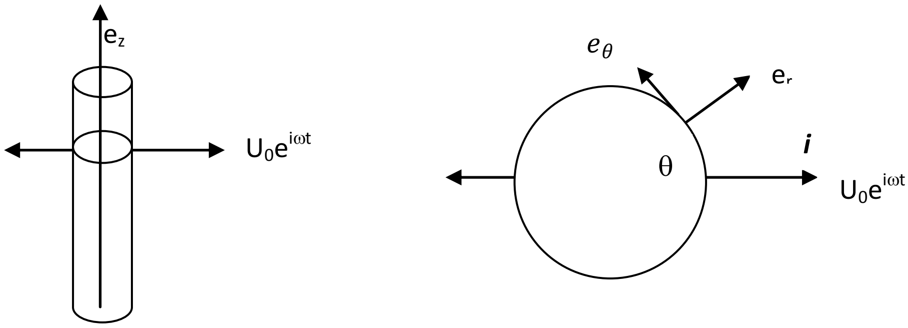


Fig 7.1 Geometry of the oscillating cylinder

The following non-dimensional scheme is introduced.

$$\mathrm{R} = ar,\ \mathrm{U} = \mathrm{U}_0 u,\ \mathrm{V} = \mathrm{U}_0 v,\ \mathbf{Q} = \mathbf{q}\mathrm{U}_0 e^{i\sigma\tau},\ \mathrm{P} = p\rho\mathrm{U}_0^2 e^{i\sigma\tau},\ \tau = \frac{\mathrm{at}}{U_0} \tag{7.7}$$

The following are non-dimensional parameters viz, ω is frequency parameter, S is Couple stress parameter, and Re is oscillations Reynolds number for Couple-stress fluids.

$$\varpi = \frac{\sigma \mathrm{a}}{\mathrm{U}_0},\ s = \frac{\mu a^2}{\eta},\ \mathrm{Re} = \frac{\rho\mathrm{U}_0 a}{\mu}\ ,\ \mathrm{Re}.\varpi = \frac{\rho\sigma a^2}{\mu} \tag{7.8}$$

Substituting (7.6) in (7.1) we notice that stream function ψ can be introduced as

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta} and v = -\frac{\partial\psi}{\partial r} \quad \text{i.e} \qquad \boldsymbol{q} = \nabla \times (\psi \boldsymbol{e}_z) \tag{7.9}$$

Using (7.6), (7.7) and (7.8) in (7.3) we get

$$Re.S\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{t}} = -Re.S.\nabla p_0 - S\nabla\times\nabla\times\boldsymbol{q} - \nabla\times\nabla\times\nabla\times\nabla\times\boldsymbol{q} \tag{7.10}$$

Using (7.6), we get

$$Re.S.i\varpi\boldsymbol{q} = -Re.S.\nabla p - S\,\nabla\times\ \nabla\times\boldsymbol{q} -\ \nabla\times\ \nabla\times\nabla\times\nabla\times\boldsymbol{q} \tag{7.11}$$

To eliminate pressure, applying curl to (7.11) and substituting (7.9) we get,

$$\nabla^2(\nabla^2 - \lambda_1^2)\ (\nabla^2 - \lambda_2^2)\ \psi = 0 \tag{7.12}$$

$$\text{Where } \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{7.13}$$

$$\lambda_1^2 + \lambda_2^2 = S \text{ and } \lambda_1^2\lambda_2^2 = \ Re.S.i\varpi \tag{7.14}$$

The solution for ψ if $\lambda_1 \neq \lambda_2$ in (7.12) is given in Lakshmana Rao et al. (1971). The solution for ψ for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \to \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation in dimensional form.

$$S = 4Re.\, i\varpi \tag{7.15}$$

In this chapter we are interested in the solution for ψ for the case of resonance $\lambda_1 = \lambda_2 = \lambda$.

In this case of resonance, the equations for ψ is given by

$$\nabla^2 \left(\nabla^2 - \lambda^2\right)^2 \psi = 0 \tag{7.16a}$$

For the case of non-resonance, the equations for ψ is given by

$$\nabla^2 (\nabla^2 - \lambda_1^2)\, (\nabla^2 - \lambda_2^2)\, \psi = 0 \tag{7.16b}$$

7.3.1 Boundary Conditions

The cylinder is oscillating in the direction of X axis. Hence the non-dimensional velocity of cylinder Γ after removing $e^{i\varpi t}$ is given by

$\boldsymbol{q}_\Gamma = \boldsymbol{i} = \cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta$ which implies by no-slip condition

$$u = \cos\theta \text{ and } v = -\sin\theta \text{ on } r = 1 \tag{7.17}$$

By hyper-stick condition $\boldsymbol{\upsilon}_\Gamma = \frac{1}{2}(\text{curl}\, \boldsymbol{q})_\Gamma = 0$ on r=1 (7.18)

7.4 Solution of the Problem

To match with the boundary conditions, stream function ψ is assumed in the form

$$\psi = f(r) \sin\theta \tag{7.19}$$

Substituting (7.19) in (7.16), we get an equation for f in the case of Resonance as

$$D_c^2 \left(D_c^2 - \lambda^2\right)^2 f = 0 \tag{7.20a}$$

In the case of non-resonance equation for f is

$$D_c^2\left(D_c^2 - \lambda_1^2\right)\left(D_c^2 - \lambda_2^2\right)f = 0 \tag{7.20b}$$

With $D_c^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ (7.21)

From the boundary conditions in (7.17) and (7.18), the conditions on f are obtained as:

B.C-1: $f(1) = 1$ (7.22)

B.C-2: $f'(1) = 1$ (7.23)

B.C-3: $D_c^2 f = 0 \; on \; r = 1$ (7.24)

Since the equation for f is linear, the general solution for f is linear combination of individual solutions of factors in the differential operator. Hence f is taken as

$$f = A_0 f_0 + A_1 f_1 + A_2 f_2 \tag{7.25}$$

In the case of resonance,

$$D_c^2 f_0 = 0, \; \left(D_c^2 - \lambda^2\right)f_1 = 0 \text{ and } \left(D_c^2 - \lambda^2\right)^2 f_2 = 0 \tag{7.26a}$$

In the case of non-resonance,

$$D_c^2 f_0 = 0, \; \left(D_c^2 - \lambda_1^2\right)f_1 = 0 \text{ and } \left(D_c^2 - \lambda_2^2\right)f_2 = 0 \tag{7.26b}$$

On solving (7.26a), the solution for f for the case of resonance is obtained as

$$f(\mathrm{r}) = \frac{\mathrm{A}_0}{\mathrm{r}} + \mathrm{A}_1\mathrm{K}_1(\lambda\mathrm{r}) + \mathrm{A}_2 rK'_1(\lambda\mathrm{r}) \tag{7.27a}$$

On solving (7.26b), the solution for f for the case of non-resonance is obtained as

$$f(\mathrm{r}) = \frac{\mathrm{A}_0}{\mathrm{r}} + \mathrm{A}_1\mathrm{K}_1(\lambda_1\mathrm{r}) + \mathrm{A}_2\mathrm{K}_1(\lambda_2\mathrm{r}) \tag{7.27b}$$

We notice that, for the case of resonance

$$D_c^2 f_0 = 0, \; D_c^2 f_1 = \lambda^2 f_1 \text{ and } D_c^2 f_2 = \lambda^2(2f_1 + f_2) \tag{7.28a}$$

This implies that at r=1, $D_c^2 f = \lambda^2\left(A_1 f_1 + A_2(2f_1 + f_2)\right) = 0$

this reduces to $1 - A_0 + 2A_2 f_1 = 0 \; or \; A_0 - 2A_2 K_1(\lambda) = 1$

In the case of non-resonance

$$D_c^2 f_0 \;=\; 0,\;\; D_c^2 f_1 \;=\; \lambda_1^2 f_1 \quad \text{and} \quad D_c^2 f_2 \;=\; \lambda_2^2 f_2 \qquad (7.28b)$$

The constants A_0 ,A_1 , A_2 are obtained from the boundary conditions (7.23), (7.24) and (7.25) in matrix form for the case of resonance as:

$$\begin{bmatrix} 1 & K_1(\lambda) & \lambda K'_1(\lambda) \\ -1 & \lambda K'_1(\lambda) & \frac{\lambda^2+1}{\lambda} K_1(\lambda) \\ 1 & 0 & -2K_1(\lambda) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (7.29a)$$

For the case of non-resonance,

$$\begin{bmatrix} 1 & K_1(\lambda_1) & K_1(\lambda_2) \\ -1 & \lambda_1 K'_1(\lambda_1) & \lambda_2 K'_1(\lambda_2) \\ 0 & \lambda_1^2 K_1(\lambda_1) & \lambda_2^2 K_1(\lambda_2) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad (7.29b)$$

On solving the equation (7.29a) and (7.29b) for A_0, A_1, A_2 we get completely f and hence ψ for both the cases.

7.4.1 Pressure

$$dp = \nabla p.\boldsymbol{dr} = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta \qquad (7.30)$$

By comparing components in equation (7.11), pressure is obtained as follows.

$$Re.S \frac{\partial p}{\partial r} = -Re.Si\varpi \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \frac{S}{r}\frac{\partial}{\partial \theta}(\nabla^2 \psi) - \frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^4 \psi) \qquad (7.31)$$

$$Re.S \frac{\partial p}{\partial \theta} = Re.Si\varpi r \frac{\partial \psi}{\partial r} - Sr \frac{\partial}{\partial r}(\nabla^2 \psi) + r \frac{\partial}{\partial r}(\nabla^4 \psi) \qquad (7.32)$$

Substituting (7.31) and (7.32) in (7.30) and integrating, we get pressure in non-dimensional form

$$p \;= \frac{i\omega A_0}{r} cos\theta \qquad (7.33)$$

7.4.2 Drag acting on the Cylinder per length L

$$\text{Drag} = D^* = aL \int_0^{2\pi} (T^*_{rr} cos\theta - T^*_{r\theta} sin\theta)\,|_{R=a} d\theta \qquad (7.34)$$

Required tress components are obtained as follows:

$$\text{Strain rate tensor} = E = \left[e_{ij}\right] = \frac{1}{2}\ [\nabla\bar{Q} + \nabla\bar{Q}^T]$$

We get strain rate tensor for this problem as

$$E = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2}\left[\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial \theta} - \frac{V}{R}\right] & 0 \\ \frac{1}{2}\left[\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial \theta} - \frac{V}{R}\right] & \frac{1}{R}\left[\frac{\partial V}{\partial \theta} + U\right] & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.35}$$

Form (7.5) Couple stress tensor M obtained as

$$M = \begin{bmatrix} m & 0 & 2\eta\frac{\partial C}{\partial R} \\ 0 & m & 2\eta\frac{1}{R}\frac{\partial C}{\partial \theta} \\ 2\eta'\frac{\partial C}{\partial R} & 2\eta'\frac{1}{R}\frac{\partial C}{\partial \theta} & m \end{bmatrix} \tag{7.36}$$

Where $\nabla_1 \times Q = C\bar{e}_z$ (7.37)

And $\nabla_1 . M = m_3\bar{e}_Z$ (7.38)

Where $m_3 = 2\eta\nabla^2 C$ (7.39)

$$\mathrm{I} \times (\nabla_1 . M) = \begin{bmatrix} 0 & -2\eta\nabla^2 C & 0 \\ 2\eta\nabla^2 C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.40}$$

By substituting (7.35) and (7.40) in (7.4) and simplifying we get

$$T_{RR} = -P + 2\mu\frac{\partial U}{\partial R} \tag{7.41}$$

$$T_{R\theta} = \mu\ \left[\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial \theta} - \frac{V}{R}\right] - \eta\nabla^2 C \tag{7.42}$$

In non-dimensional form are

$$T_{rr} = \frac{\mu U_0}{a}\left[-Re.p + 2\left(\frac{f'}{r} - \frac{f}{r^2}\right)cos\theta\right] \tag{7.41a}$$

At r=1, $T_{rr} = -\frac{\mu U_0}{a}(Re.i\varpi A_0)cos\theta$ (7.43)

$$T_{r\theta} = \frac{\mu U_0}{a}\left[-D_c^2 f + \frac{1}{S}D_c^4 f + 2\left(\frac{f'}{r} - \frac{f}{r^2}\right)\right]sin\theta \tag{7.42a}$$

At r=1, $T_{r\theta} = \frac{\mu U_0}{a}\frac{1}{S}D_c^4 f sin\theta = \frac{\mu U_0}{a}\frac{\lambda^4}{S}\left(1 - A_0 + 4A_2K_1(\lambda)\right)sin\theta$ (7.44)

We note that at r=1; $D_c^4 f = (D_c^2 - \lambda^2)^2 f - \lambda^4 f = \lambda^4 (A_0 - 1)$

Substituting (7.43) and (7.44) in (7.34) we get the Drag D^* on the cylinder (without the factor $e^{i\omega t}$) is given as for resonance

$$D^* = L\mu U_0 i\pi\varpi Re(1 - 2A_0) = -L\mu U_0 i\pi\varpi Re(1 + 4A_2 K_1(\lambda)) \quad (7.45a)$$

For non-resonance

$$D^* = L\mu U_0 i\pi\varpi Re(1 - 2A_0) = -L\mu U_0 i\pi\varpi Re\left(A_0 + \frac{\lambda_1^2}{\lambda_2^2} A_1 K_1(\lambda_1) + \frac{\lambda_2^2}{\lambda_1^2} A_2 K_1(\lambda_2)\right)$$

(7.45b)

By dividing $L\mu U_0$, we get the entire Drag in non-dimensional form for both cases as

$$D = i\pi\varpi Re(1 - 2A_0) \quad (7.46)$$

7.5 Results and Discussions

The roots of $x^2 - Sx + i\varpi ReS = 0$ are taken as the values of λ^2.

$$\text{Hence}\lambda = \sqrt{x} = \begin{cases} \sqrt{\dfrac{S \pm \sqrt{S^2 - 4S.Re.i\omega}}{2}} & \text{for non resonance} \\ \sqrt{\dfrac{S}{2}} & \text{for resonance} \end{cases} \quad (7.47)$$

Here ϖ, S and Re are choosen independently, with Re≪1 and ω≫1 such that ω.Re is not negligibly small (say>1) then λ is obtained from (7.47). Then A_0, A_1 and A_2 and hence ψ and Drag are obtained. To get physical quantities, the corresponding real part of the quantities are taken.

7.5.1 Stream function

In Fig 7.2 stream function f at different values of Reynolds number Re is shown. For resonance we notice that stream function takes smaller values than the case of non-resonance and vanishes at relatively nearer to the cylinder than in the case of non-resonance. (i.e stream function vanishes at larger distances from the cylinder.)

In Fig 7.3 stream function f at different values of frequency parameter ϖ is shown. For resonance we notice that stream function takes smaller values than the case of non-resonance and vanishes at relatively nearer to the cylinder than in the case of non-resonance. (i.e stream function vanishes at larger distances from the cylinder as in the case of Fig 7.2 for Reynolds number Re.)

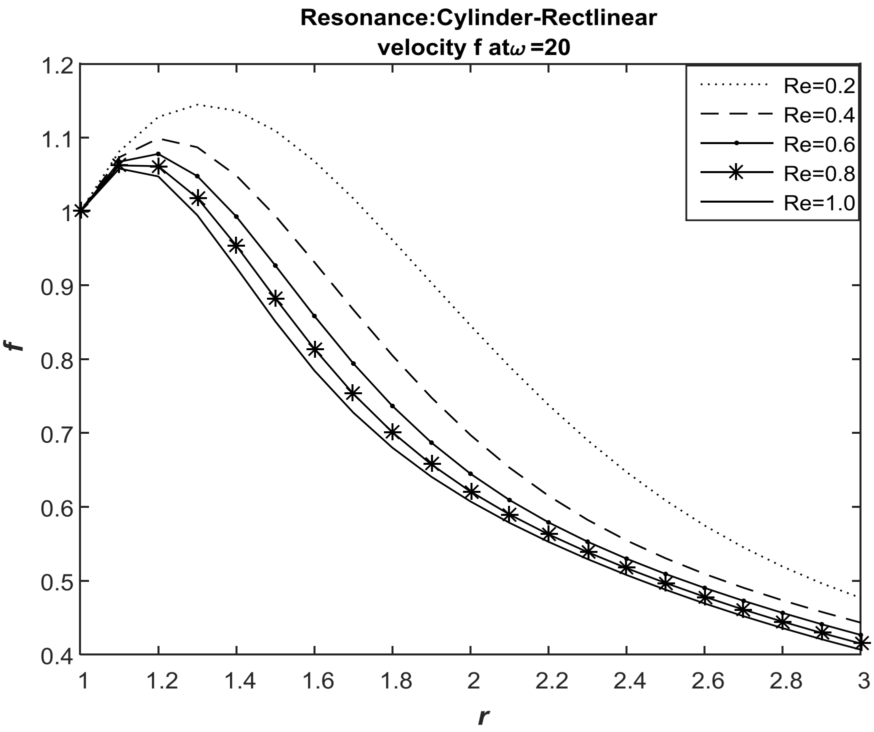
In Fig 7.4 we notice that for non-resonance internal circulations near to the pole of cross sectional circle are found. As we move into the center of this internal circulations, the value of stream function increases (goes from red to blue). For resonance this internal circulations are not found. As we move from pole of the cross sectional circle, the values of the stream function are decreasing (goes from green to yellow.)

7.5.2 Drag

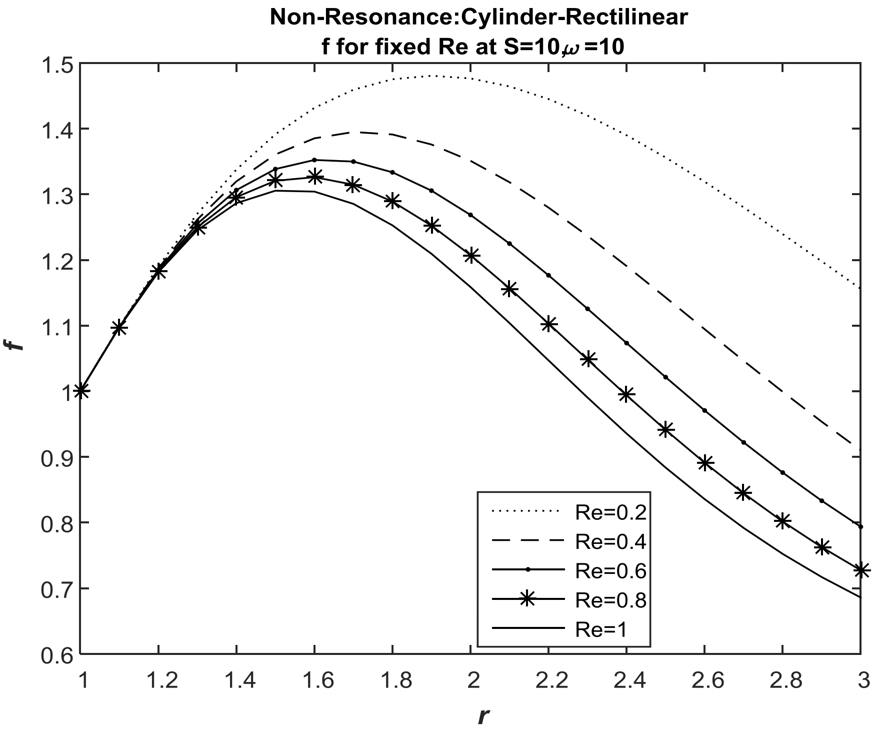
Drag has two different formulas in (7.45). But by using boundary conditions, it will reduce to one simple formula for resonance and non-resonance. From fig 7.5, we observe that when $|\lambda|$ is fixed, we get only one curve for resonance, since Re.ϖ is fixed. The distinct feature in this is that as $|\lambda|$ increases, Drag also increases for resonance but for non-resonance opposite behavior is observed. i.e as $|\lambda|$ increases, for non-resonance, Drag decreases and becomes constant for large values of $|\lambda|$ at a particular ϖ. Again, when $|\lambda|$ is less than 4, abrupt behavior is observed. This tendency increases as ϖ increases. Mathematically this happens because, at this value of ϖ, the value of S goes to negative values.

From Fig 7.6, we observe that as Re increases, Drag also increases, but for resonance Drag is lesser than in the case of non-resonance.

From Fig 7.7, we notice that as S increases, Drag increases for resonance. But for non-resonance as S increases, Drag decreases. But the values of Drag are smaller in comparison with the case of non-resonance. At particular value of S, for resonance, Drag is almost constant.

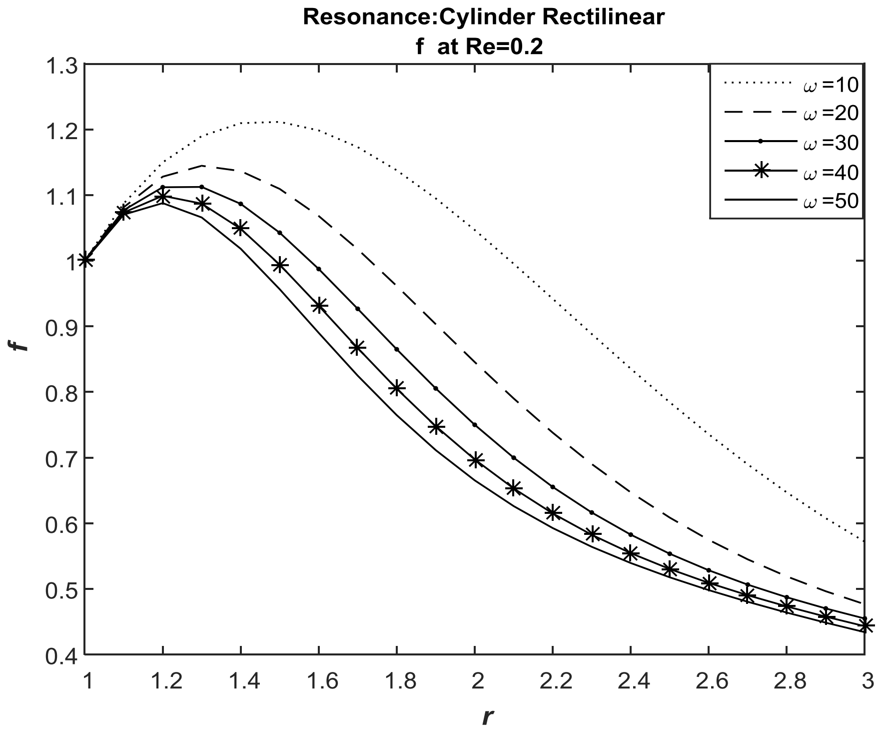


(a)



(b)

Fig.7.2 Stream function f at different values of Reynolds number for the case of a) resonance and b) non-resonance



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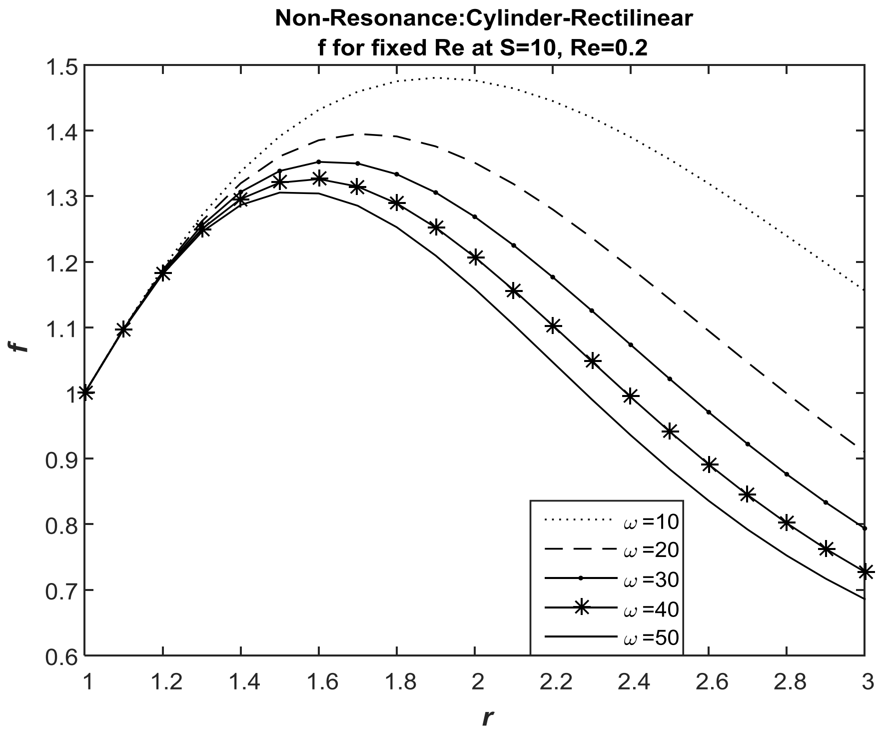

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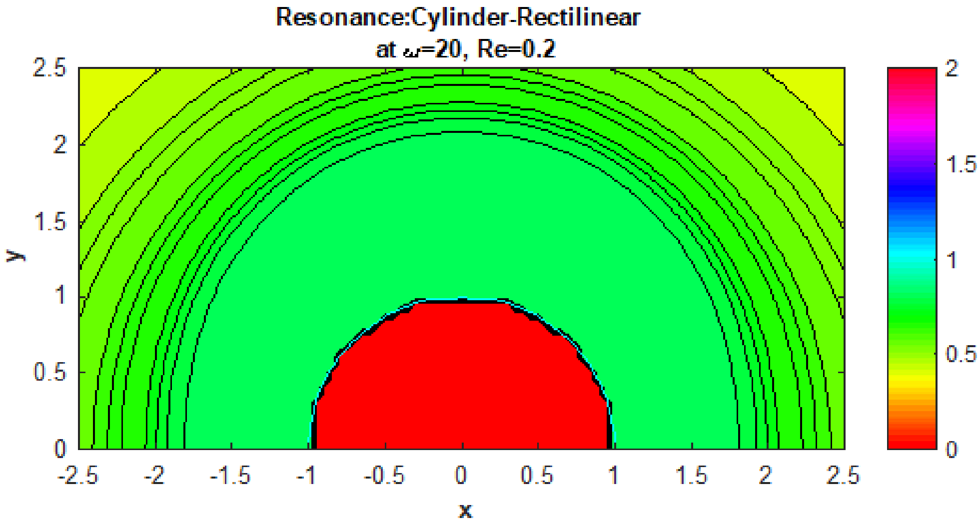
Fig 7.3 Stream function f at fixed Reynolds number for the case of a) resonance and b) non-resonance



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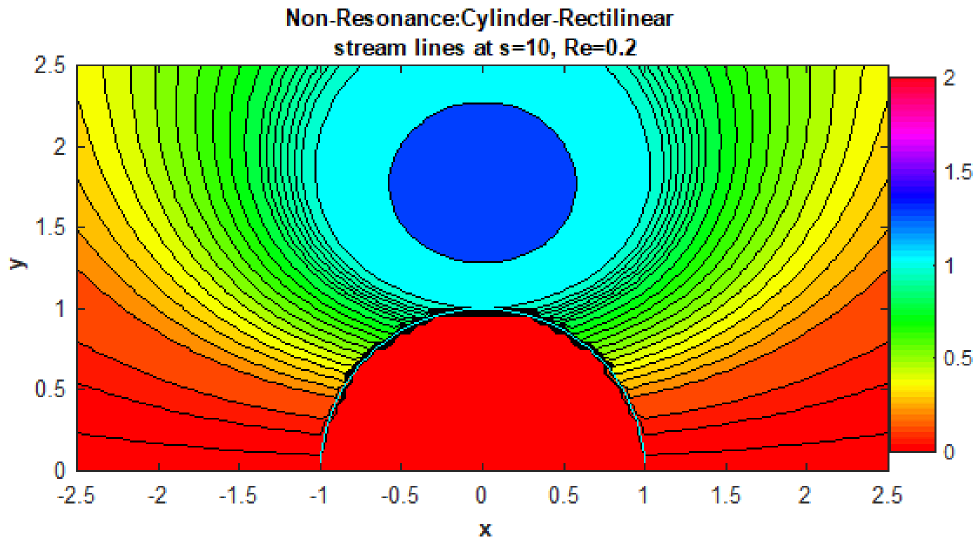
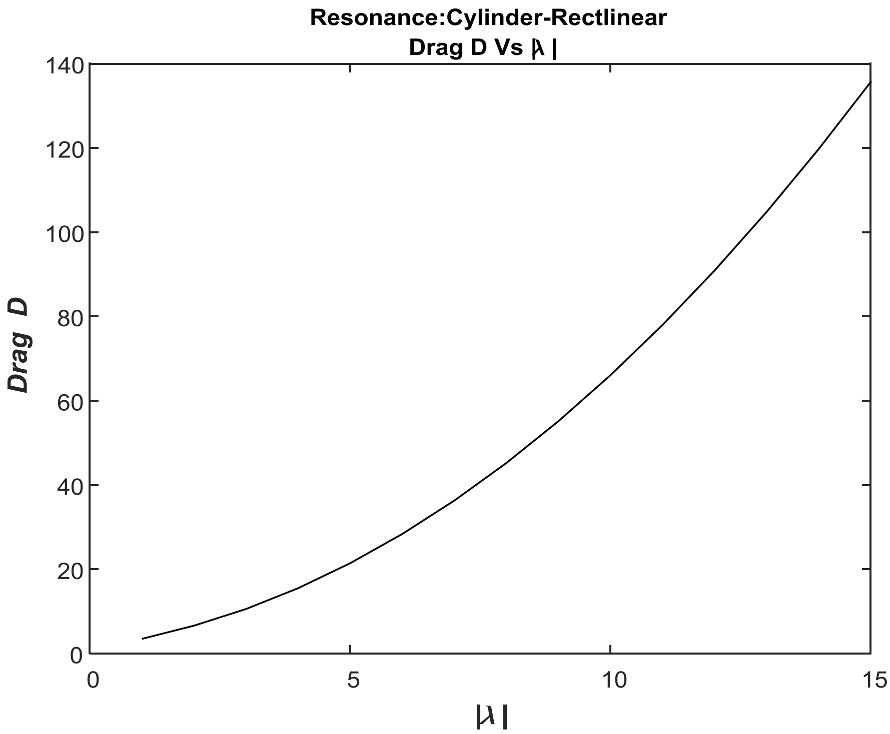
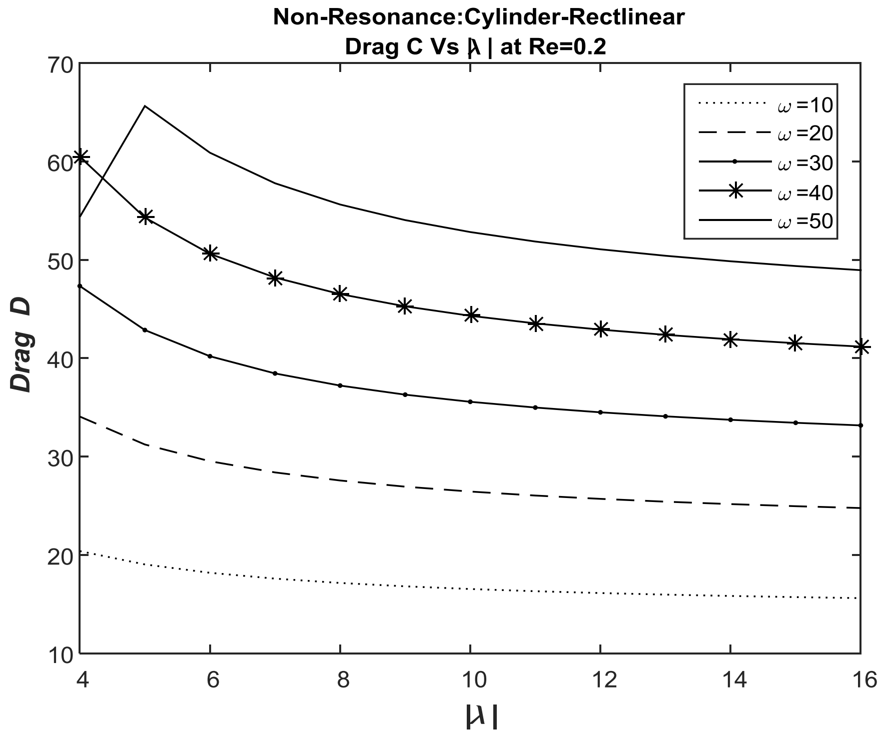


(b)

Fig 7.4 Stream lines at Re=0.2 for the case of a) resonance and b) non-resonance

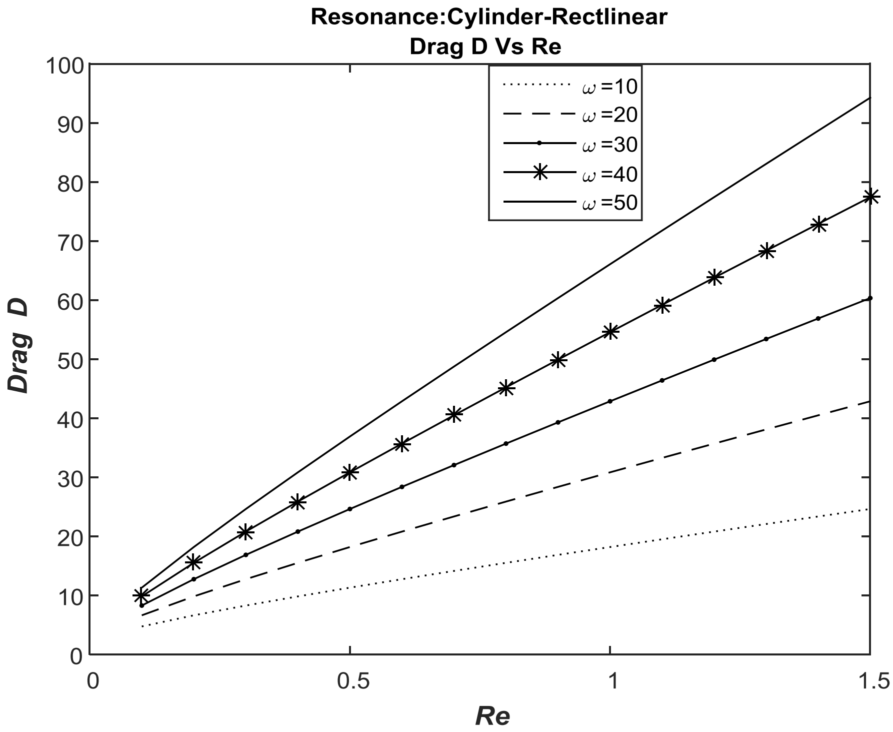


(a)

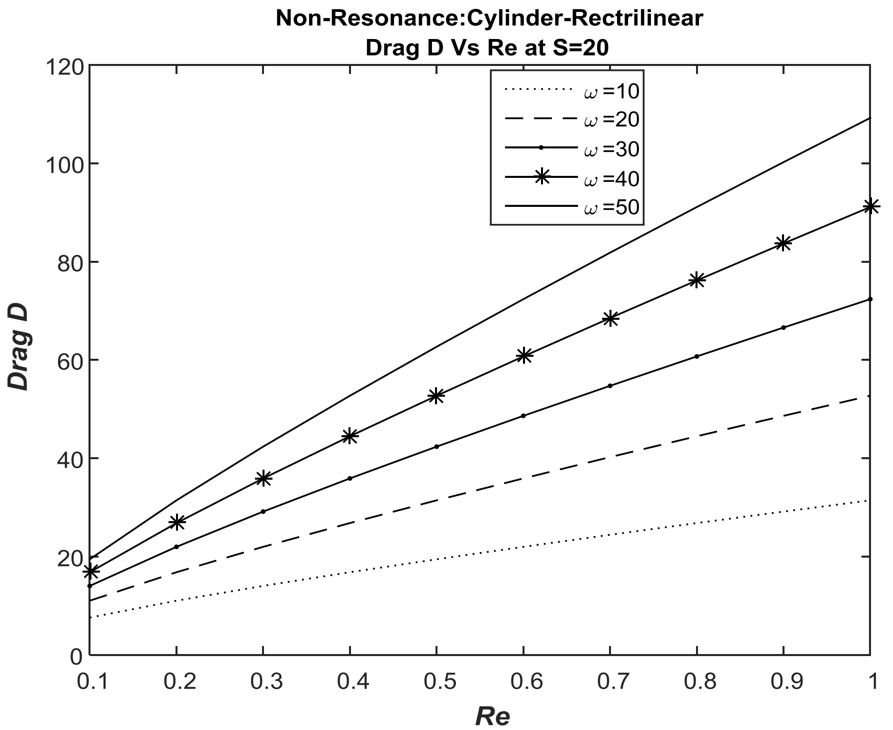


(b)

Fig 7.5 Drag Vs | λ| at different values of frequency parameter for the case of a) resonance and b) non-resonance

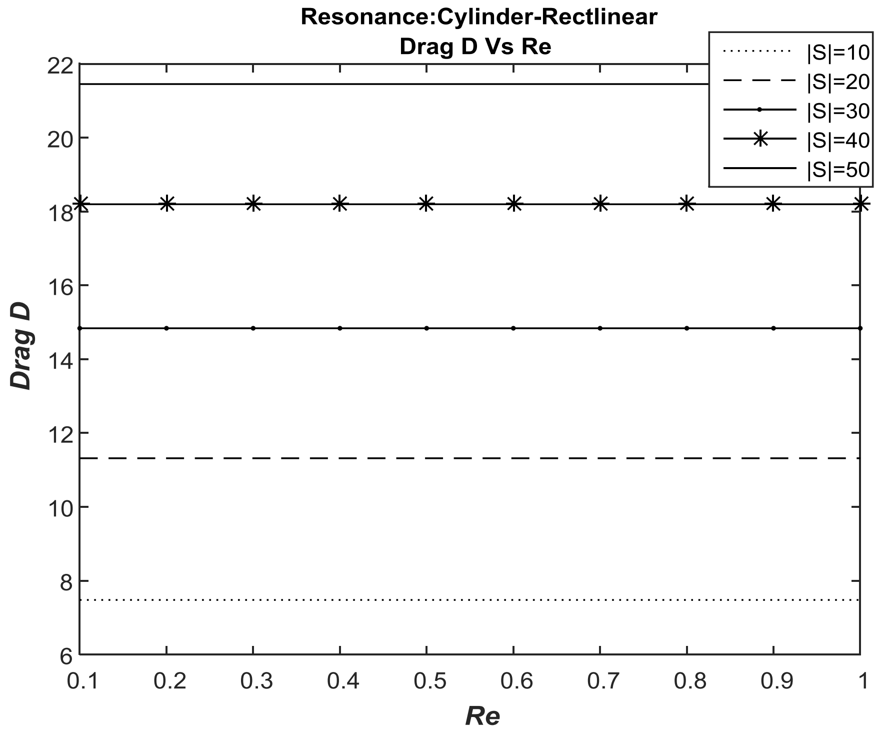


(a)

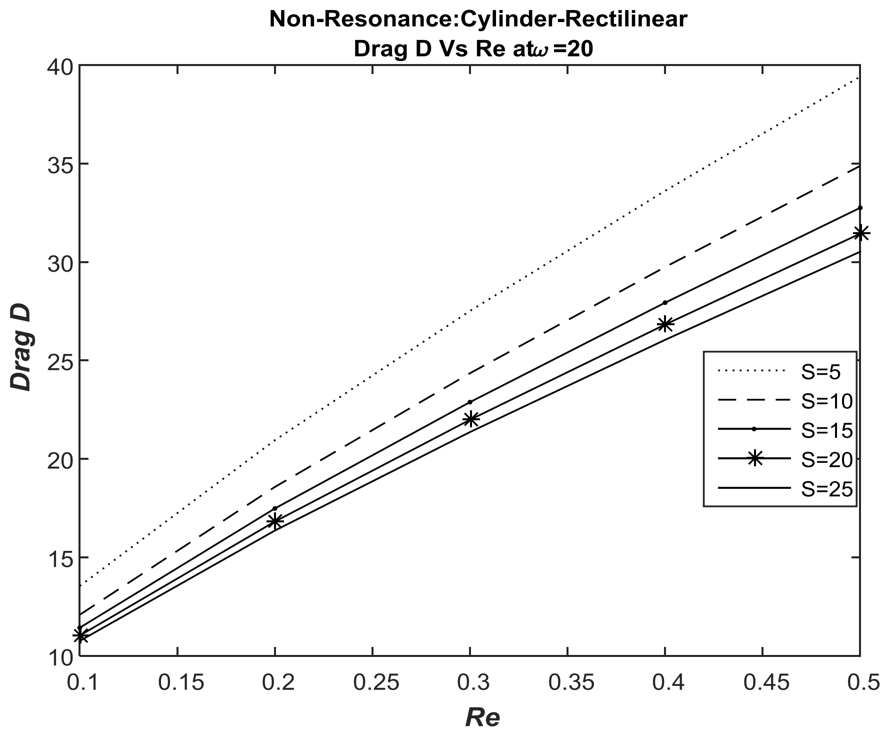


(b)

Fig 7.6 Drag Vs Reynolds number Re for different values of ω for the case of a) resonance and b) non-resonance



(a)



(b)

Fig 7.7 Drag Vs Re at different values of S for the case of a) resonance and b) non-resonance

7.6 Conclusions

We observe that when resonance occurs:

i) Stream function values decrease from a high value to a low value from the cylinder and vanishes near to the cylinder.

ii) Drag takes less values in comparison with non-resonance for variation in ϖ and S.

when there is non resonance:

i) Stream function forms a circulation near to the surface of the cylinder. as move go near to the center of the circulation, the value of stream function increases

ii) Drag is very high in comparison to resonance for the variation of ϖ and S.

Chapter 8

Rotary oscillations of a Circular Cylinder in a Couple-stress fluid

The flow generated due to rotary oscillations of a circular cylinder about its axis of symmetry in an incompressible Couple-stress fluid is considered. The Reynolds number for the flow is less than unity due to very slow flow and hence nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered. The velocity component for the flow derived. The Skin friction acting on the cylinder is evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Skin friction due to oscillations is shown through graphs.

8.1 Introduction

Many authors investigated the flow of Couple-stress fluids in cylindrical geometry. Ariman et al. (1967) studied Couple-stress fluids and flow of Micro-polar fluids between two concentric cylinders. Kanwal (1955) studied viscous fluid flow of axisymmetric bodies generated due to rotary and longitudinal oscillations. Frater (1968) evaluated Drag on a circular cylinder oscillating in an elastico-viscous fluid. Ravindran (1972) Studied simple oscillatory flow in polar fluids. Soundalgekar et al. (1974) analysed effects of Couple stresses on the oscillatory flow past an infinite plate with constant suction. Lakshmana Rao et al. (1972, 1983, 1987) studied the oscillatory flows of circular cylinder, spheroid and elliptic cylinder in incompressible Micro-polar fluids, the main thrust of the investigation being the determination of the Drag or Couple as the case may be on the oscillating body. Lakshmana Rao et al. (1980) examined Couple-stress fluid flows by analytically and computationally. Iyengar et al. (2001, 2004) studied oscillatory flow of Micro-polar fluid generated by the rotary oscillations of approximate sphere and two concentric spheres. Calmelet-Eluhu et al. (1998) studied Micro-polar fluid flow of circular cylinder generated due to longitudinal and torsional oscillations. Fetecau et al. (2006) found solutions for the motion of second grade fluid due to longitudinal and torsional oscillations of circular cylinder. Anwar et al. (2004), Aparna et al. (2012) examined rotary oscillations of circular cylinder, permeable sphere in an incompressible Micro-polar fluid. Mehrdad Massoudi et al. (2008) numerically studied the motion of second grade fluid due to longitudinal and torsional oscillations of a cylinder. Ramkissoon et al. (1990), Rao et al. (1992), Ramana Murthy et al. (2010), Nagaraju et al. (2014) studied oscillatory flows of circular cylinder due to performing longitudinal and torsional oscillations in viscous fluid, Couple-stress fluid, Micro-polar fluid.

In this chapter we propose to investigate incompressible Couple-stress fluid flow due to Circular Cylinder performing rotary oscillations.

8.2 Basic Equations

The basic equations of an incompressible Couple stress fluid introduced by Stokes (1966) are given by:

$$div\bar{Q} = 0 \tag{8.1}$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{8.2}$$

where $\bar{Q}$ is fluid velocity vector, ρ is density, τ is time, μ is viscosity coefficient.

By neglecting non linear convective terms in (8.2) we get

$$\rho\frac{\partial\bar{Q}}{\partial\tau} = -\nabla_1 P - \mu\nabla_1 \times \nabla_1 \times \bar{Q} - \eta\nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{8.3}$$

For Couple stress fluids, the stress components T_{ij} and Couple stress tensor M satisfy the following constitutive equations.

$$T = -PI + \lambda(\nabla_1.Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2}I \times (\nabla_1.M) \tag{8.4}$$

$$M = mI + 2\eta\nabla_1(\nabla_1 \times Q) + 2\eta'[\nabla_1(\nabla_1 \times Q)]^T \tag{8.5}$$

8.3 Statement and Formulation of the Problem

A circular cylinder of radius a and of infinite length is performing rotary oscillations with velocity $V_0 e^{i\sigma\tau}\boldsymbol{e_\theta}$ about its axis of symmetry in an infinite vat containing incompressible Couple-stress fluid. A cylindrical coordinate system (R, θ, Z) with base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_\theta, \boldsymbol{e}_Z)$ with origin on the axis of the cylinder is considered. Hence the fluid velocity will be in cross sectional plane of the cylinder containing the base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_\theta)$. The velocity is assumed in the form:

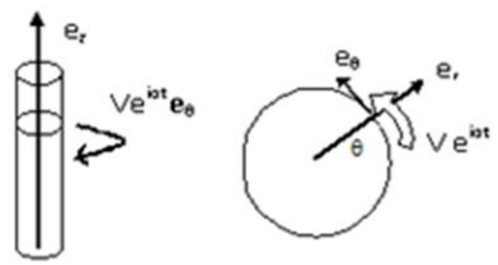


Fig 8.1 Geometry of the oscillating cylinder

$$\boldsymbol{Q} = V(R)\boldsymbol{e_\theta}\boldsymbol{e}^{i\sigma\tau} \tag{8.6}$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$R = ar,\ \ V = v.a\sigma,\ \ \boldsymbol{Q} = \boldsymbol{q}\mathrm{v}_0,\ \ P = p\rho \mathrm{v}_0^2,\ \ \tau = \frac{at}{\mathrm{v}_0} \tag{8.7}$$

The following are non-dimensional parameters viz, ϖ is frequency parameter, S is Couple stress parameter and Re is Reynolds number for Couple-stress fluids.

$$\varpi = \frac{\sigma a}{\mathrm{V}_0},\ S = \frac{\mu a^2}{\eta},\ \mathrm{Re} = \frac{\rho \mathrm{V}_0 a}{\mu} \text{which gives } \mathrm{Re}.\varpi = \frac{\rho\sigma a^2}{\mu} \tag{8.8}$$

By the choice of velocity field in (8.6) the equations of motion (8.3) is reduced to

$$i\sigma\rho V = -\frac{P_0}{R} + \mu D_c^2 V - \eta D_c^4 V \tag{8.9}$$

where $\frac{\partial P}{\partial \theta} = P_0$

Using non dimensional scheme (8.7) and (8.8) in (8.9) we get

$$D_c^4 v - S D_c^2 v + i\varpi Re.sv = -\frac{p_0}{r} Re.S \tag{8.10}$$

This equation (8.10) can be written as

$$(D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)v = -\frac{p_0}{r} Re.S \tag{8.11}$$

where $D_c^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ (8.12)

$\lambda_1^2 + \lambda_2^2 = S$ and $\lambda_1^2 \lambda_2^2 = i\varpi Re.S$ (8.13)

The solution for v if $\lambda_1 \neq \lambda_2$ in (8.11) is given in Lakshmana Rao et al. (1971). The solution for v for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \rightarrow \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation in dimensional form.

$$2\lambda^2 = S = 4i\varpi Re \tag{8.14}$$

Now the equations for v for the case of resonance is given by

$$(D_c^2 - \lambda^2)^2 v = -\frac{p_0}{r} Re.S \tag{8.15a}$$

For the case of non-resonance

$$(D_c^2 - \lambda_1^2)\ (D_c^2 - \lambda_2^2)\ v = -\frac{p_0}{r} Re.S \tag{8.15b}$$

8.3.1 Boundary Conditions

By no-slip condition, the non-dimensional velocity of the circular cylinder Γ is given by $v = 1$ (8.16)

By hyper-stick condition,

Curl $Q_\Gamma = 2\bar{e}_z$ which yields on r=1, $\frac{\partial v}{\partial r} = 1$ (8.17)

8.4 Solution of the Problem

The solution of (8.15a), velocity function v is assumed in the form

$$v = A_1 v_1 + A_2 v_2 - \frac{1}{\lambda^4 r} p_0 Re.s \quad (8.18a)$$

With $(D_c^2 - \lambda^2)v_1 = 0$ and$(D_c^2 - \lambda^2)^2 v_2 = 0$ (8.19a)

these will yield the solutions as

$$v_1 = K_1(\lambda r) \text{ and } v_2 = rK_1^{'}(\lambda r) \quad (8.20a)$$

For the case of non-resonance, the corresponding solution will be

$$v = A_1 v_1 + A_2 v_2 - \frac{1}{\lambda_1^2 \lambda_2^2 r} p_0 \, Re.s \quad (8.18b)$$

With $\left(D_c^2 - \lambda_1^2\right)v_1 = 0$ and $\left(D_c^2 - \lambda_2^2\right)v_2 = 0$ (8.19b)

these will yield the solutions as for non-resonance case as

$$v_1 = K_1(\lambda_1 r) \text{ and } v_2 = K_1(\lambda_2 r) \quad (8.20b)$$

The following results are useful to note.

$$D_c^2 v_1 = \lambda^2 v_1 and D_c^2 v_2 = 2\lambda v_1 + \lambda^2 v_2 \quad (8.21a)$$

In case of non-resonance,

$$D_c^2 v_1 = \lambda_1^2 v_1 and D_c^2 v_2 = \lambda_2^2 v_2 \quad (8.21b)$$

The constants$A_1, A_2$are obtained by applying the boundary conditions (8.16) and (8.17) to (8.18a) as follows:

$$\begin{bmatrix} K_1(\lambda) & K'_1(\lambda) \\ \lambda K'_1(\lambda) & \frac{(1+\lambda^2)}{\lambda} K_1(\lambda) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{ip_0}{\varpi} \\ 1 + \frac{ip_0}{\varpi} \end{bmatrix} \quad (8.22a)$$

In the case of non-resonance, the conditions for A_1, A_2 are given by

$$\begin{bmatrix} K_1(\lambda_1) & K_1(\lambda_2) \\ \lambda_1 K'_1(\lambda_1) & \lambda_2 K'_1(\lambda_2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{ip_0}{\varpi} \\ 1 + \frac{ip_0}{\varpi} \end{bmatrix} \quad (8.22b)$$

Hence we can calculate $A_1 and A_2$ from (8.22a) and (8.22b) for both the cases.

Hence velocity v is known.

8.4.1 Skin friction acting on the Cylinder per length L

Skin friction acting on the circular cylinder is given by

$$c_f = \frac{2T_{r\theta}}{\rho v_0^2}. \tag{8.23}$$

For Couple stress fluids, the constitutive equations for stress and Couple stresses are given by (8.4) and (8.5) as

$$T = -PI + \lambda(\nabla_1 . Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2} I \times (\nabla_1 . M)$$

And $\quad M = mI + 2\eta\nabla_1(\nabla_1 \times Q) + 2\eta'[\nabla_1(\nabla_1 \times Q)]^T$

Strain rate tensor is given by

$$E = \frac{1}{2}(\nabla_1 \bar{Q} + \nabla_1 \bar{Q}^T) = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2}(\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R}) & \frac{1}{2}(\frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z}) \\ \frac{1}{2}(\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R}) & \frac{1}{2}(U + \frac{\partial V}{\partial \theta}) & \frac{1}{2}(\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z}) \\ \frac{1}{2}(\frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z}) & \frac{1}{2}(\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z}) & \frac{\partial W}{\partial Z} \end{bmatrix} \tag{8.24}$$

For this present problem, we get strain rate tensor as

$$E = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial V}{\partial R} - \frac{V}{R}) & 0 \\ \frac{1}{2}(\frac{\partial V}{\partial R} - \frac{V}{R}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.25}$$

$$M = \begin{bmatrix} m & 0 & 2\eta\frac{\partial C}{\partial R} \\ 0 & m & 0 \\ 2\eta'\frac{\partial C}{\partial R} & 0 & m \end{bmatrix} where\ C = \frac{\partial V}{\partial R} + \frac{V}{R} \tag{8.26}$$

$$\nabla_1 . M = \frac{2\eta}{R}\frac{\partial}{\partial R}\left(R\frac{\partial C}{\partial R}\right)\bar{e}_Z \tag{8.27}$$

$$\text{And } I \times (\nabla_1 . M) = \begin{bmatrix} 0 & -\frac{2\eta}{R}\frac{\partial}{\partial R}\left(R\frac{\partial C}{\partial R}\right) & 0 \\ \frac{2\eta}{R}\frac{\partial}{\partial R}\left(R\frac{\partial C}{\partial R}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.28}$$

By substituting (8.26), (8.27) and (8.28) in (8.4) and simplifying we get

$$T_{R\theta} = \left\{\mu\left(\frac{\partial V}{\partial R} - \frac{V}{R}\right) - \eta\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\left(\frac{\partial V}{\partial R} + \frac{V}{R}\right)\right)\right\} e^{i\varpi t} \tag{8.29}$$

Using non dimensional scheme (8.7) and (8.8) in (8.29) we get

$$T_{r\theta} = \frac{V_0\eta}{a^3}\left[S\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\,(rD_c^2 v)\right]. \tag{8.30}$$

The Skin friction acting on the circular cylinder (after deleting the factor $e^{i\varpi t}$) is obtained as:

$$c_f = \frac{1}{Re.S}\left[S\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - \frac{1}{r}D_c^2 v - \frac{\partial}{\partial r}D_c^2 v\right]$$

On r=1, $c_f = \frac{-1}{Re.S}\left[D_c^2 v + \frac{\partial}{\partial r}D_c^2 v\right]$ (8.31)

We note that for Resonance:

$$D_c^2 v = \lambda^2\left\{A_1K_1(\lambda r) + \frac{A_2}{\lambda}\left(2K_1(\lambda r) + \lambda r K'_1(\lambda r)\right)\right\}$$

$$\frac{d}{dr}D_c^2 v = \lambda^2\left\{A_1\lambda K'_1(\lambda r) + \frac{A_2}{\lambda}\left(2\lambda K'_1(\lambda r) + r\lambda^2 K''_1(\lambda r) + \lambda K'_1(\lambda r)\right)\right\}$$

$$= \lambda^2\left\{A_1\lambda K'_1(\lambda r) + \frac{A_2}{\lambda}\left(2\lambda K'_1(\lambda r) + \frac{(1 + r^2\lambda^2)K_1(\lambda r)}{r}\right)\right\}$$

Hence on r=1;

$$D_c^2 v = \lambda^2\left\{A_1K_1(\lambda) + \frac{A_2}{\lambda}\left(2K_1(\lambda) + \lambda K'_1(\lambda)\right)\right\}$$

$$\frac{d}{dr}D_c^2 v = \lambda^2\left\{A_1\lambda K'_1(\lambda) + \frac{A_2}{\lambda}\left(2\lambda K'_1(\lambda) + (1 + \lambda^2)K_1(\lambda)\right)\right\}$$

Using the boundary conditions and $\lambda K_1'(\lambda) + K_1(\lambda) = -\lambda K_0(\lambda)$ we have ;

In the resonance case, the Skin friction is given by

$$c_f = \frac{2\lambda^2}{Re.S}\left[1 - A_2K_0(\lambda)\right] \tag{8.32a}$$

In the non-resonance case, the Skin friction is given by

$$c_f = \frac{1}{Re.S}\left[\lambda_1^3 K_0(\lambda_1)\right)A_1 + \lambda_2^3 K_0(\lambda_2))A_2] \quad (8.32b)$$

8.5 Results and Discussions

The roots of $x^2 - Sx + i\varpi ReS = 0$ are taken as the values of λ^2.

Hence for non-resonance $\lambda = \sqrt{\frac{S\pm\sqrt{S^2-4S.Re.i\omega}}{2}}$ and for resonance $\lambda = \sqrt{\frac{S}{2}}$ (8.33)

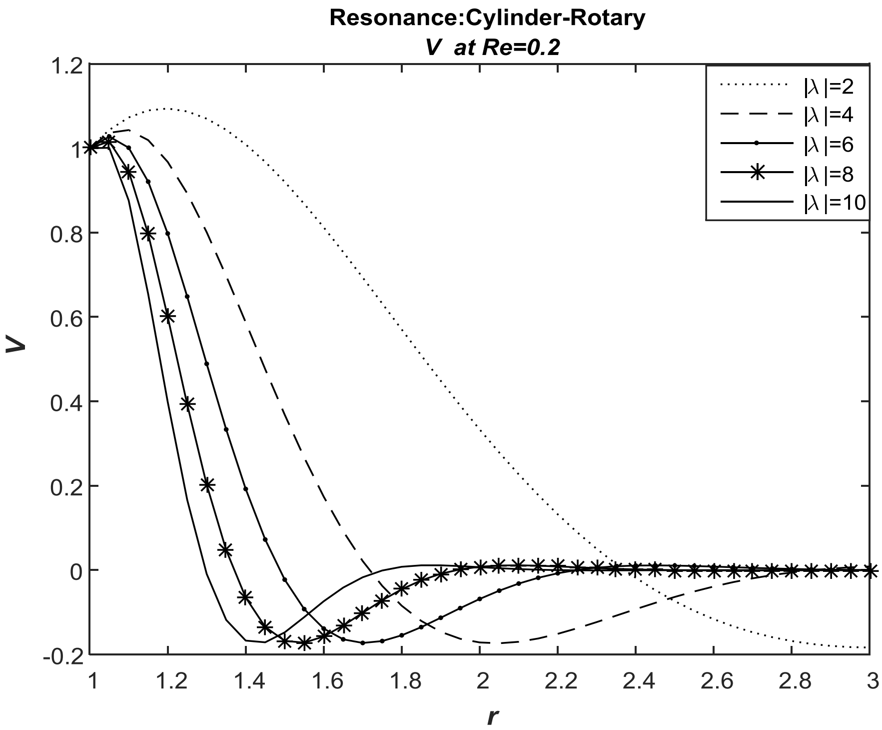
Here ϖ *and* Re are choosen independently, with Re≪1 and ω≫1 such that ω.Re is not negligibly small (say>1) then λ is obtained from (8.33). Then A_1 and A_2 and hence V and Drag are obtained. To get physical quantities, the corresponding real part of the quantities are taken.

8.5.1 Velocity

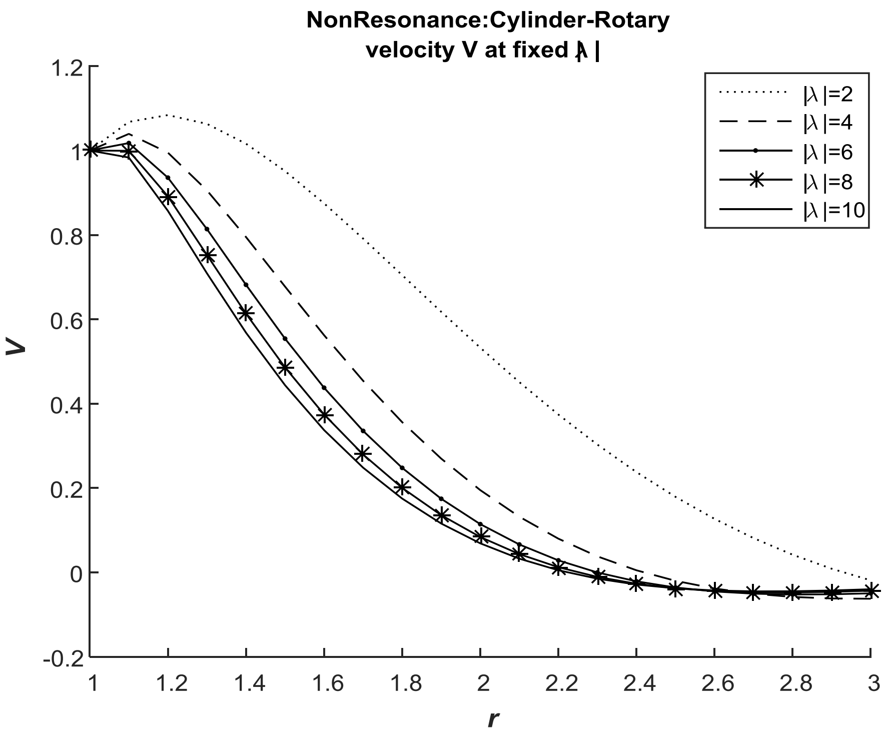
When |λ| is fixed, for resonance Re and ϖ cannot vary independently. From Fig 8.2, for resonance we observe that as |λ| increases, velocity drastically decreases near to the cylinder and takes negative values. But for non-resonance, as |λ| increases, velocity decreases slowly and takes positive values only.

From Fig 8.3, we note that as ϖ increases, velocity decreases near to the cylinder. But for resonance, this variation in velocity is drastic at r=1 (near to the cylinder) and near r=2. Velocity takes first increases and then decreases to negative vales and again increases and goes to zero.

From Fig.8.4, for resonance the behavior of velocity is same as in fig 8.3 for variation in ϖ. But for variation in Re, the change in velocity is clear even as near as 1.5 times the radius of the cylinder.

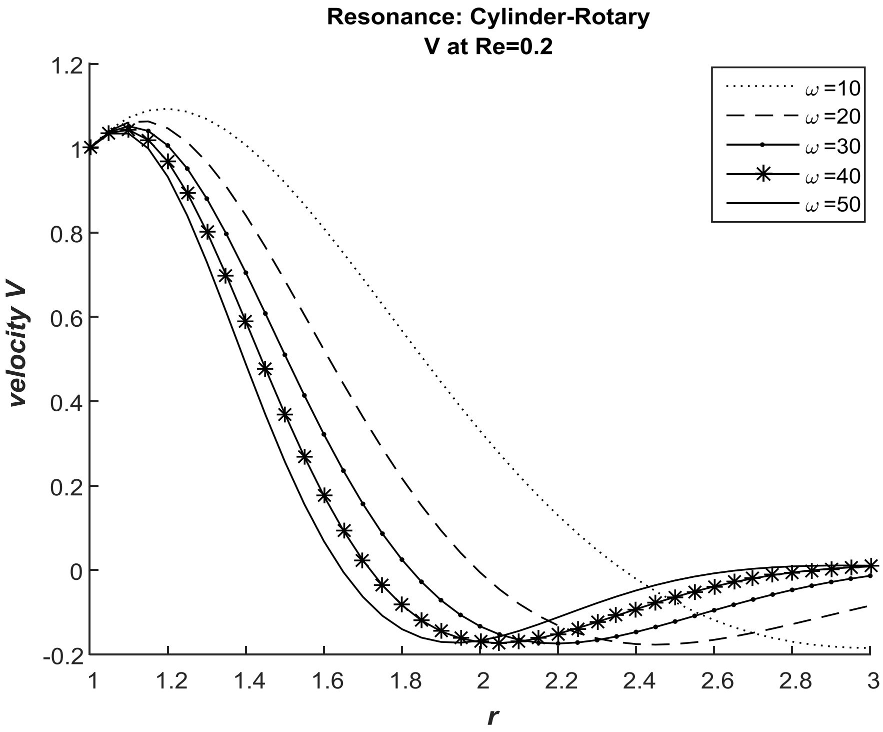


(a)

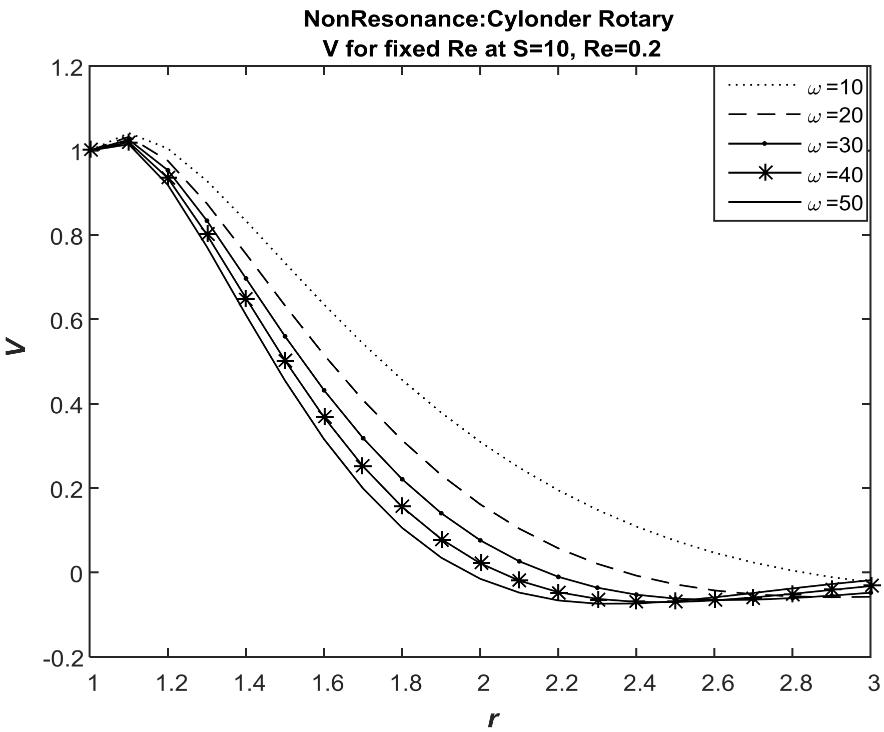


(b)

Fig 8.2 Velocity at fixed values of |λ| for the case of (a) resonance and (b) non-resonance

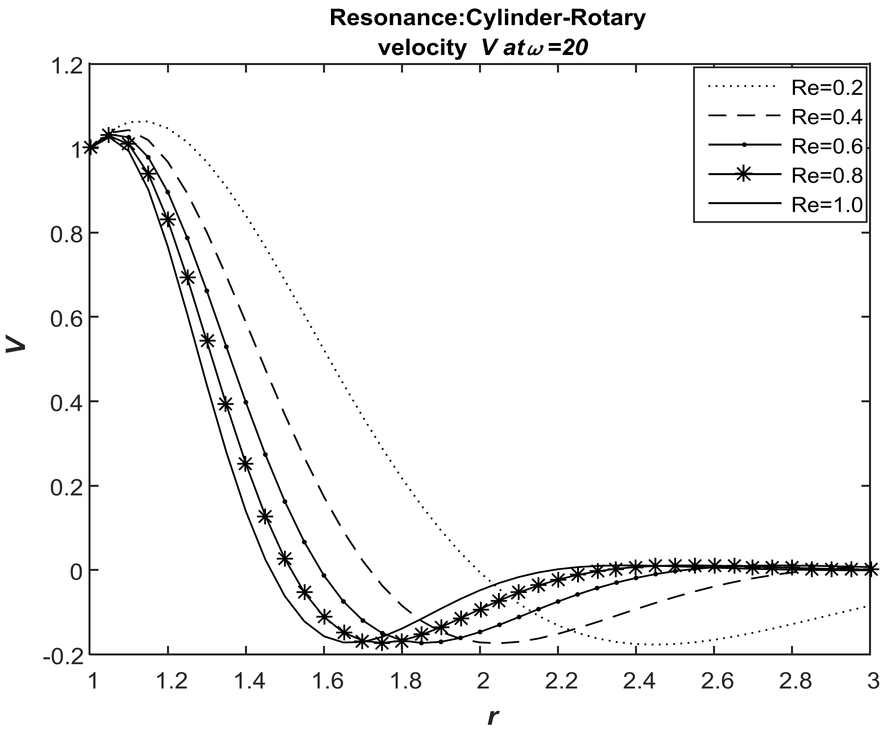


(a)

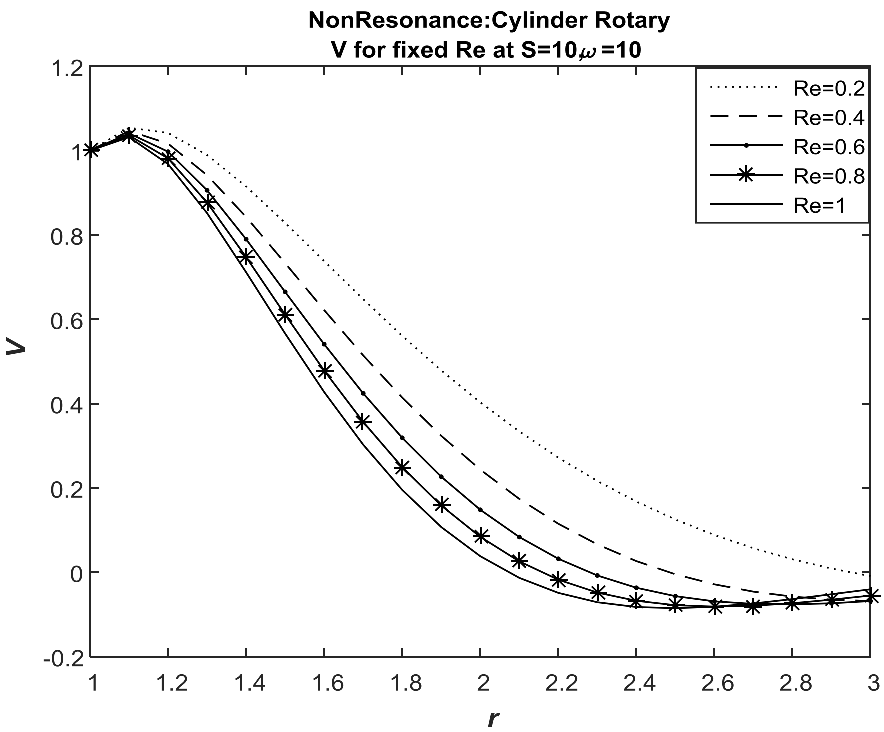


(b)

Fig 8.3 Velocity at fixed values of ϖ for the case of (a) resonance and (b) non-resonance



(a)



(b)

Fig 8.4 Velocity at different values of Re for the case of (a) resonance and (b) non-resonance

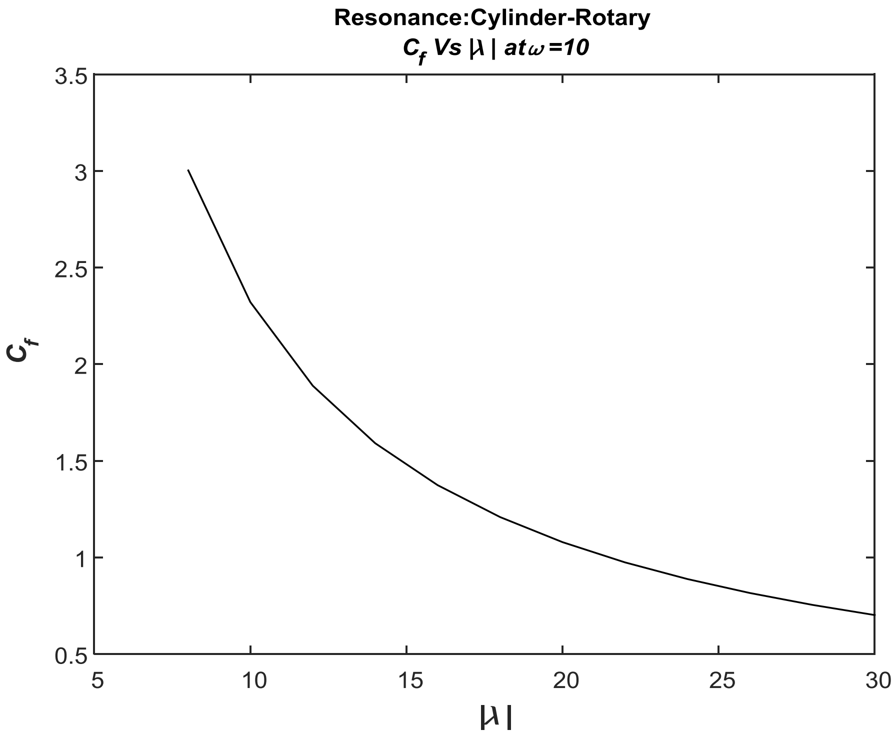
8.5.2 Skin friction

From Fig 8.5, we note that as $|\lambda|$ increases, Skin friction decreases. But for resonance, Skin friction is very small in comparison with the values of Skin friction in the case of non-resonance.

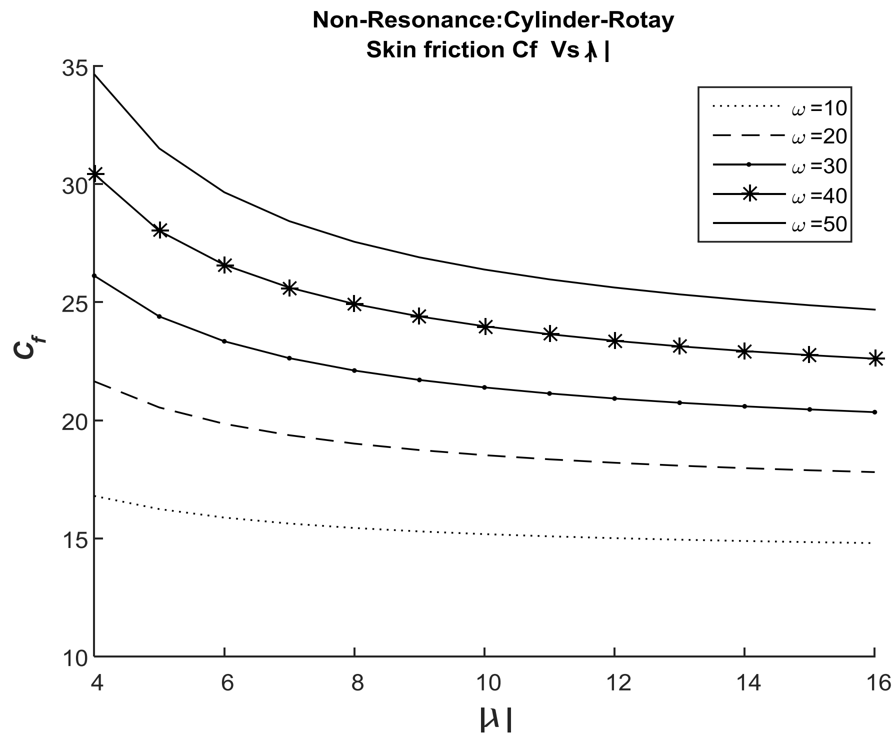
From Fig 8.6, the Skin friction is almost same for resonance and non-resonance.

Form Fig 8.7, we notice that Skin friction at lower values of Re, is more for resonance but at values near to Re=1, Skin friction for resonance is less than the corresponding values of non-resonance case.

From Fig 8.8, we notice that as ϖ increases, skin-friction also increases. But when Re increases, skin-friction decreases, since $Re\varpi$ comes as a unit in the Skin friction. In this case for resonance and non-resonance, Skin friction is almost same. This is because ϖ does not appear explicitly in the formula for skin-friction.

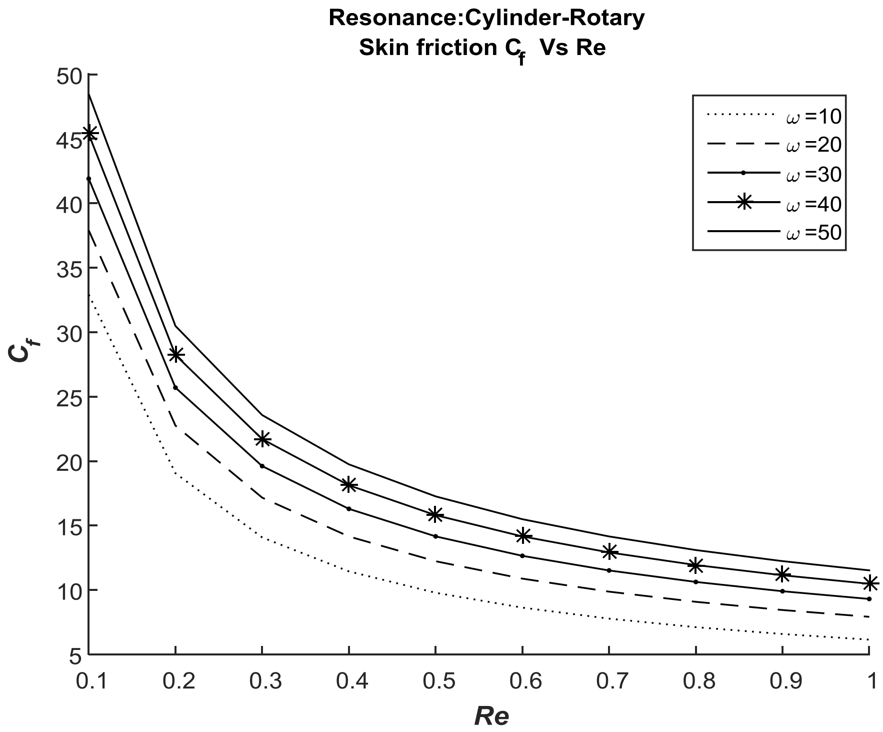


(a)

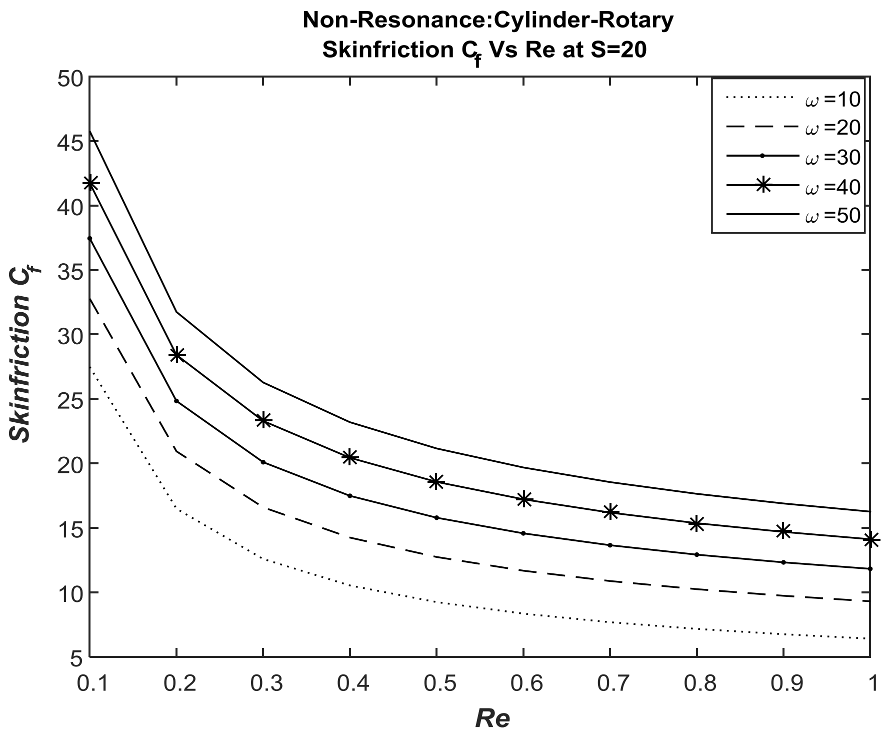


(b)

Fig 8.5 Skin friction Vs |λ| for the case of (a) resonance and (b) non-resonance

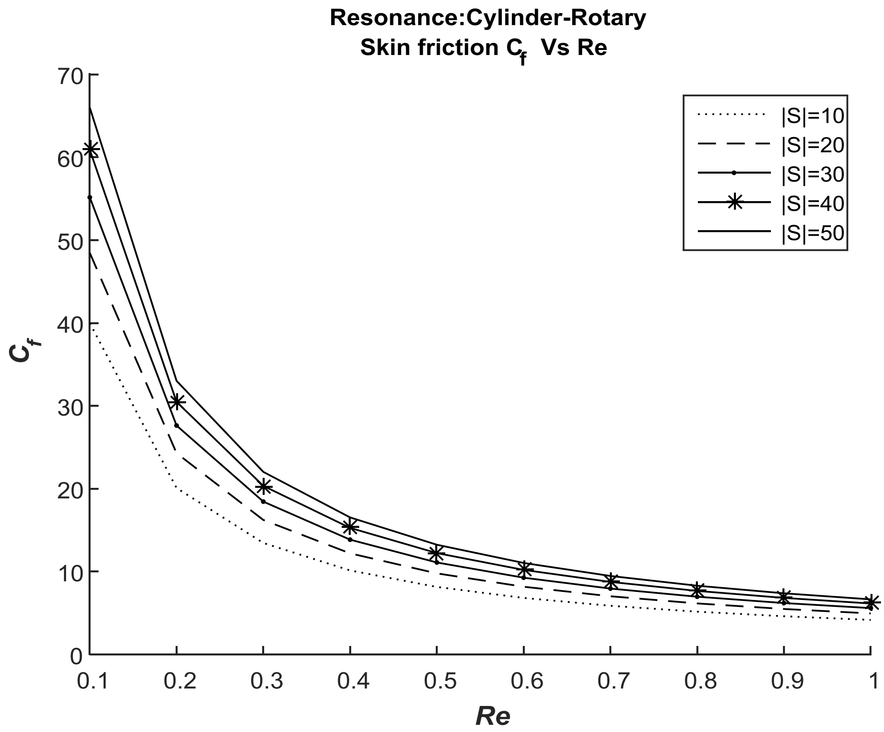


(a)

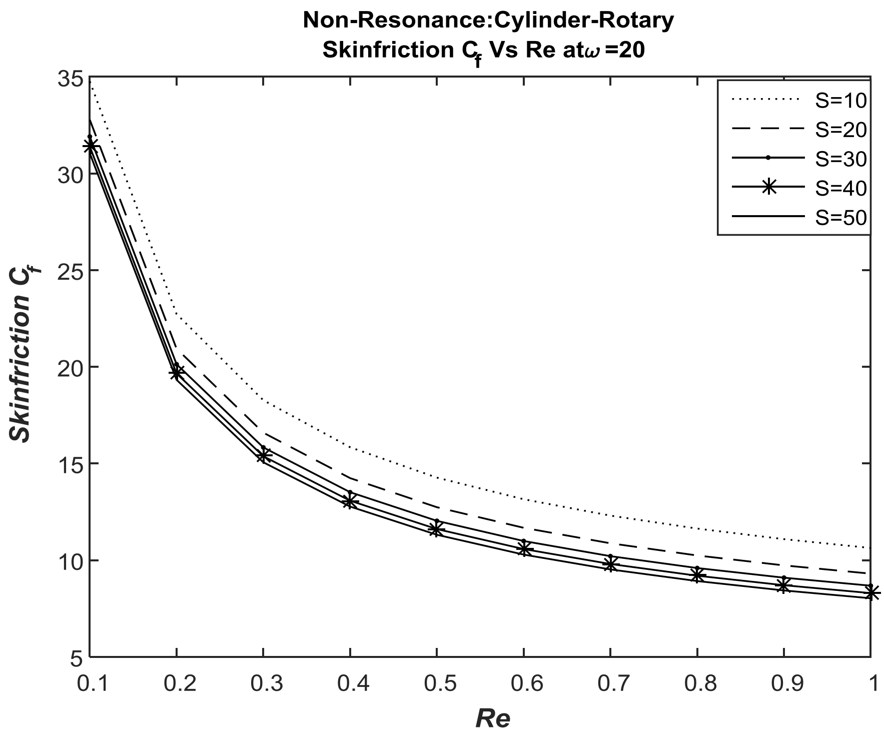


(b)

Fig 8.6 Skin friction Vs Re for the case of (a) resonance and (b) non-resonance

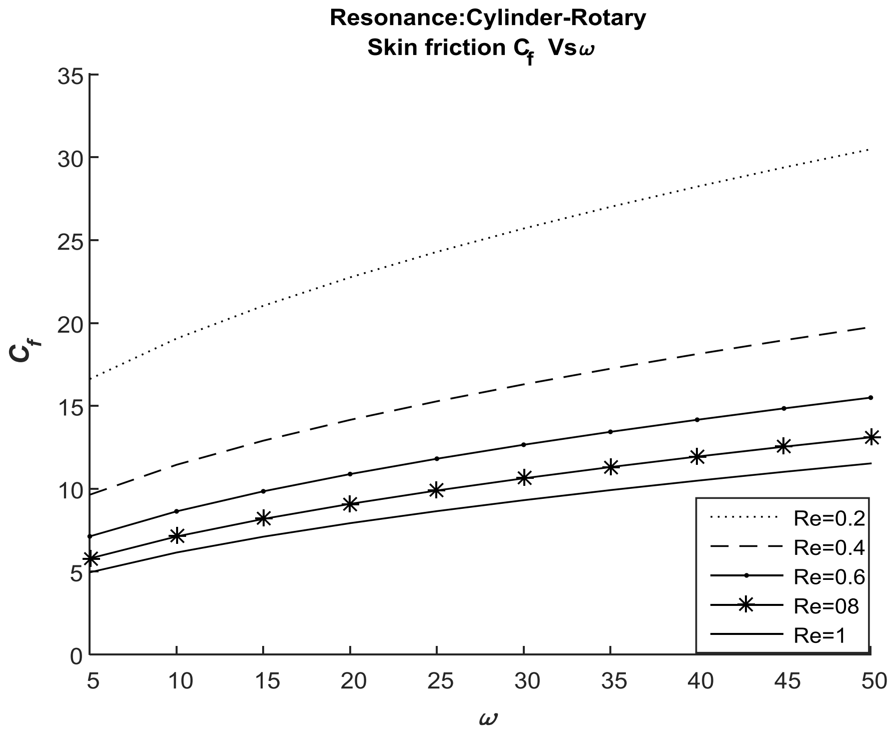


(a)

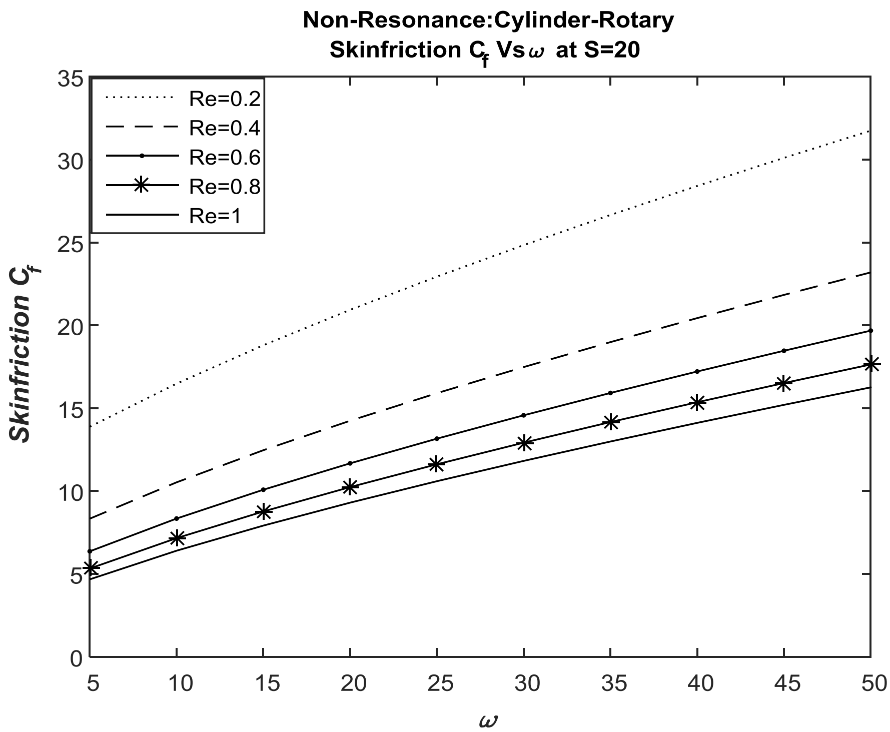


(b)

Fig 8.7 Skin friction Vs Reynolds number Re at different values of Couple stress parameter S, for the case of (a) resonance and (b) non-resonance.



(a)



(b)

Fig 8.8 Skin friction Vs frequency parameter ϖ different values of Reynolds number Re for the case of (a) resonance and (b) non-resonance.

8.6 Conclusions

We observe that for resonance,

i) When $|\lambda|$ is fixed, the skin-friction decreases drastically in comparison with non-resonance**.**

ii) Velocity changes from positive values to negative values near to the cylinder**.** for non-resonance, velocity takes positive values only.

Chapter 9

Longitudinal oscillations of a Circular Cylinder in Couple-stress fluid

This chapter aims at the flow generated due to Circular cylinder performing longitudinal oscillations along its axis of symmetry in a Couple-stress fluid. There arises a rare but distinct case which is referred to as Resonance flow. In this special case material constants satisfy a resonance condition. Nonlinear convective terms in the equations of motion are neglected since the Oscillations Reynolds number is less than unity due to very slow flow. The velocity component for the flow is derived. The Skin friction acting on the cylinder is evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Skin friction are shown through graphs.

9.1 Introduction

Several researchers examined the flow of non-Newtonian fluids in cylindrical geometry. Frater (1968) studied an elastic-viscous fluid flow of circular cylinder performing oscillations and obtained Drag on a circular cylinder. Ravindran (1972) studied simple oscillatory flow in polar fluids. Ariman (1967) analysed Couple-stress fluid flows and Micro-polar fluid flows between two concentric cylinders. Lakshmana Rao (1980) studied Couple-stress fluid flows by analytically and computationally. Lakshmana Rao et al. (1972, 1987) in studied the oscillatory flows due to circular cylinder and elliptic cylinder in an incompressible Micro-polar fluid, the main thrust of the investigation being the determination of the Drag or Couple as the case may be on the oscillating body.

The flows due to longitudinal and torsional oscillations of various objects like cylinder, rod, sphere in various fluids were investigated by different authors Kanwal et al. (1955), Casarella et al. (1969), Rajagopal (1983), Ramkissoon et al. (1990, 1991), Rao et al. (1992), Bandelli et al. (1994), Pontrelli (1997), Calmelet-Eluhu et al. (1998), Akyildiz (1998), Fetecau et al. (2006), Owen et al. (2006), Mehrdad Massoudi et al. (2008), Ramana Murthy et al. (2010), Nagaraju et al. (2014) by numerically or analytically. In all these problems authors evaluated Drag or Couple acting on the body. Ramana Murthy et al. (2009, 2010, 2011) studied a flow of Micro-polar fluid under transverse magnetic field with suction. The rare but distinct possibility of resonance flows has been noticed and the investigation.

In this chapter, we propose to investigate this case of resonance type flow, in Couple-stress fluids, due to longitudinal oscillations of a circular cylinder about its axis of symmetry. In chapter 4 similar oscillatory flow in Micro-polar fluid was discussed.

9.2 Basic Equations

The basic equations of an incompressible Couple stress fluid introduced by Vijay Kumar Stokes (1966) are given by:

$$div\bar{Q} = 0 \tag{9.1}$$

$$\rho\left(\frac{\partial\bar{Q}}{\partial\tau} + \bar{Q}.\nabla_1\bar{Q}\right) = -\nabla_1 P - \mu\nabla_1\times\nabla_1\times\bar{Q} - \eta\nabla_1\times\nabla_1\times\nabla_1\times\nabla_1\times\bar{Q} \tag{9.2}$$

where $\bar{Q}$ is fluid velocity vector, ρ is density, τ is time, μ is viscosity coefficient.

By neglecting non linear convective terms in (9.2) we get

$$\rho\frac{\partial\bar{Q}}{\partial\tau} = -\nabla_1 P - \mu\nabla_1\times\nabla_1\times\bar{Q} - \eta\nabla_1\times\nabla_1\times\nabla_1\times\nabla_1\times\bar{Q} \tag{9.3}$$

For Couple stress fluids, the stress components T_{ij} and Couple stress tensor M satisfy the following constitutive equations.

$$T = -PI + \lambda(\nabla_1.Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2}I\times(\nabla_1.M) \tag{9.4}$$

$$M = mI + 2\eta\nabla_1(\nabla_1\times Q) + 2\eta'[\nabla_1(\nabla_1\times Q)]^T \tag{9.5}$$

9.3 Statement and Formulation of the Problem

A circular cylinder of radius a and of infinite length is performing longitudinal oscillations with velocity $W_0 e^{i\sigma\tau}$ along its axis of symmetry in an infinite vat containing incompressible Couple-stress fluid. A cylindrical coordinate system (R, θ, Z) with base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_\theta, \boldsymbol{e}_{\boldsymbol{Z}})$ with origin on the axis of the cylinder is considered. Since the flow is axially symmetric, the fluid velocity will be in cross-sectional plane of the cylinder containing the base vectors $(\boldsymbol{e}_R, \boldsymbol{e}_Z)$. The velocity is assumed in the form:

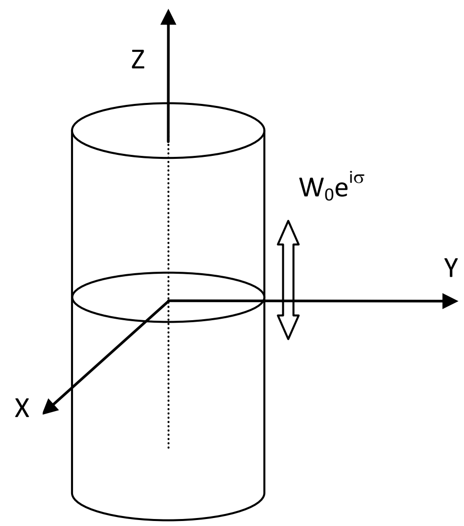


Fig 9.1 Geometry of the oscillating cylinder

$$\boldsymbol{Q} = W(R)\boldsymbol{e}_{\boldsymbol{z}} e^{i\sigma\tau} \tag{9.6}$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$R = ar, W = w\mathrm{W}_0, \boldsymbol{Q} = \boldsymbol{q}W_0 \,, P_0 = p_0\rho a\sigma^2, and \tau = \frac{at}{W_0} \tag{9.7}$$

The non-dimensional parameters ϖ frequency parameter, s Couple stress parameter and R_0 oscillations Reynolds number for Couple-stress fluids are defined as below.

$$\varpi = \frac{a\sigma}{\mathrm{w}_0},\; S = \frac{\mu a^2}{\eta}, \quad \mathrm{Re} = \frac{\rho \mathrm{w}_0 a}{\mu} \text{sothat } \varpi . Re = \frac{\rho\sigma a^2}{\mu} \tag{9.8}$$

By the choice of the velocity field in (9.6), by taking curl the equation (9.3) is reduced to

$$i\sigma\rho W' = \mu D_c^2 W' - \eta D_c^4 W' \tag{9.9}$$

Where $D_c^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$ (9.10)

Using non dimensional scheme (9.7) and (9.8) in (9.9) we get

$$D_c^4 w' - SD_c^2 w' + i\varpi Re.Sw' = 0 \tag{9.11}$$

This equation (9.11) can be written as

$$(D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)w' = 0 \tag{9.12}$$

Where $\lambda_1^2 + \lambda_2^2 = S$ and $\lambda_1^2\lambda_2^2 = i\varpi Re.S$ (9.13)

The solution for w' in (9.12) if $\lambda_1 \neq \lambda_2$ is given in Lakshmana Rao et al. (1972) for a similar case of Micro-polar fluids. The solution for w' for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \to \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation in non-dimensional form as;

$$2\lambda^2 = S = 4i\varpi Re \tag{9.14}$$

In this case of resonance, the equations for w' is given by

$$(D_c^2 - \lambda^2)^2 w' = 0 \tag{9.15a}$$

In the case of non-resonance, the equation for w' is given by

$$(D_c^2 - \lambda_1^2)(D_c^2 - \lambda_2^2)w' = 0 \tag{9.15b}$$

By the choice of the velocity field in (9.6), the equation (9.3) is reduced to

$$i\sigma\rho W = -\frac{\partial P}{\partial Z} + \eta\left\{\left(\frac{\partial}{\partial R} + \frac{1}{R}\right)\left(-D_c^2 + \frac{\mu}{\eta}\right)W'\right\} \tag{9.16}$$

Using non dimensional scheme (9.7) and (9.8), the above equation (9.16) reduces to

$$i\varpi Re.Sw = -p_0 Re.S - \left(\frac{d}{dr} + \frac{1}{r}\right)(D_c^2 - S)w' \tag{9.17}$$

This can be written for the case of resonance as

$$i\varpi Re.Sw = -p_0.Re.S - \left(\frac{d}{dr} + \frac{1}{r}\right)(D_c^2 - 2\lambda^2)w' \tag{9.18a}$$

And in the case of non-resonance as

$$i\varpi Re.Sw = -p_0.Re.S - \left(\frac{d}{dr} + \frac{1}{r}\right)(D_c^2 - \lambda_1^2 - \lambda_2^2)w' \tag{9.18b}$$

9.3.1 Boundary conditions

By no-slip condition, the non-dimensional velocity on the circular cylinder Γ is given by

i.e $w = 1\ on\ r = 1$ (9.19)

By hyper-stick condition, Curl $Q_\Gamma = 0$

which yields $\frac{dw}{dr} = 0$ on boundary $r = 1$

i.e $w' = 0\ on\ r = 1$ (9.20)

9.4 Solution of the Problem

Solution for (9.15a) or (19.15b), the general solution for w' is linear combination of individual solutions of factors in the differential operator. Hence w' is assumed in the form

$$w' = A_1 w'_1 + A_2 w'_2 \tag{9.21}$$

Where w_1' and w_2' satisfies the following equations for the case of resonance

$$(D_c^2 - \lambda^2)w'_1 = 0 \text{ and } (D_c^2 - \lambda^2)^2 w'_2 = 0 \tag{9.22a}$$

In the case of non-resonance

$$\left(D_c^2 - \lambda_1^2\right)w_1' = 0 \text{ and } \left(D_c^2 - \lambda_2^2\right)w_2' = 0 \tag{9.22b}$$

Thus the solutions for (9.22a) and (9.22b) as follows

In the case of resonance:

$w'_1 = K_1(\lambda r)$ and $w'_2 = \lambda r K_1'(\lambda r)$ (9.23a)

In the case of non-resonance:

$w'_1 = K_1(\lambda_1 r)$ and $w'_2 = K_1(\lambda_2 r)$ (9.23b)

Hence from (9.21)

In the case of resonance:

$w' = \mathrm{A}_1\mathrm{K}_1(\lambda \mathrm{r}) + \mathrm{A}_2\mathrm{r}\,\mathrm{K}_1'(\lambda\mathrm{r})$ (9.24a)

In the case of non-resonance:

$w'(\mathrm{r}) = \mathrm{A}_1\mathrm{K}_1(\lambda_1\mathrm{r}) + \mathrm{A}_2\mathrm{K}_1(\lambda_2\mathrm{r})$ (9.24b)

The following results are useful to note.

$D_c^2 w_1' = \lambda^2 w_1'$ and $D_c^2 w_2' = 2\lambda w_1' + \lambda^2 w_2'$ (9.25a)

The following results are useful to note in case of non resonance.

$D_c^2 w'_1 = \lambda_1^2 w'_1$ and $D_c^2 w'_2 = \lambda_2^2 w'_2$ (9.25b)

$xK_n'(x) + nK_n(x) = -xK_{n-1}(x)$ and $x^2K_n''(x) + xK_n'(x) = (n^2 + x^2)K_n(x)$ (9.25c)

The condition on $w(1) = 1$ can be obtained from (9.18a) and (9.18b) as

For resonance:

$$\varpi Re.S\left(i + \frac{p_0}{\varpi}\right)w(r) = -\left(\frac{d}{dr} + \frac{1}{r}\right)(D_c^2 - 2\lambda^2)w'(r)$$

$$= -\left(\frac{d}{dr} + \frac{1}{r}\right)(A_1\lambda^2 w'_1 + A_2(\lambda^2 w_2' + 2\lambda w_1') - 2\lambda^2(A_1 w'_1 + A_2 w_2'))$$

$$= \lambda^2(A_1\left(w_1'' + \frac{w_1'}{r}\right) + A_2\left(w_2'' + \frac{w_2'}{r}\right)) - 2\lambda A_2\left(w_1'' + \frac{w_1'}{r}\right)$$

We simplify 2nd term as below:

$$w_2'' + \frac{w_2'}{r} = \lambda r K_1''(\lambda r) + 2K_1'(\lambda r) = \frac{1}{x}(x^2K_1'' + 2xK_1') \text{ with } x = \lambda r$$

$$= \frac{(1+x^2)K_1(x) + xK_1^{'}}{x} = \frac{(K_1 + xK_1^{'})}{x} + xK_1 = -K_0 + xK_1$$

Now we have

$$\varpi Re.S\left(i + \frac{p_0}{\varpi}\right)w(r) = (A_1\lambda^2 - 2\lambda A_2)(-\lambda K_0(\lambda r)) + A_2\lambda^2(\lambda r K_1(\lambda r) - K_0(\lambda r))$$

Evaluating this at r=1, we get

$$\varpi Re.S\left(i + \frac{p_0}{\varpi}\right) = -A_1\lambda^3 K_0(\lambda) + A_2\lambda^2(\lambda K_1(\lambda) + K_0(\lambda))$$

Or $$\lambda^2\left(1 - \frac{ip_0}{\varpi}\right) = -A_1\lambda K_0(\lambda) + A_2(\lambda K_1(\lambda) + K_0(\lambda))$$

For non-resonance:

$$\varpi Re.S\left(i + \frac{p_0}{\varpi}\right)w(r) = -\left(\frac{d}{dr} + \frac{1}{r}\right)(D_c^2 - \lambda_1^2 - \lambda_2^2)w'(r)$$

$$= -\left(\frac{d}{dr} + \frac{1}{r}\right)(-A_1\lambda_2^2 {w'}_1 - A_2\lambda_1^2 w_2^{'})$$

$$= -A_1\lambda_2^2\lambda_1 K_0(\lambda_1 r) - A_2\lambda_1^2\lambda_2 K_0(\lambda_2 r)$$

Evaluating this at r=1, we get

$$\varpi Re.S(i + p_0) = -\lambda_1\lambda_2(A_1\lambda_2 K_0(\lambda_1) + A_2\lambda_1 K_0(\lambda_2))$$

Or $$\lambda_1\lambda_2(1 - ip_0) = -(A_1\lambda_2 K_0(\lambda_1) + A_2\lambda_1 K_0(\lambda_2))$$

Now the constants A_1, A_2 are obtained by applying the boundary conditions as follows:

$$\begin{bmatrix} -\lambda K_0(\lambda) & \lambda K_1(\lambda) + K_0(\lambda) \\ \mathrm{K}_1(\lambda) & \mathrm{K}_1{}'(\lambda) \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \lambda^2(1 - \frac{ip_0}{\varpi}) \\ 0 \end{bmatrix} \tag{9.26a}$$

In the case of non-resonance, the conditions for A_1, A_2 are given by

$$\begin{bmatrix} -\lambda_2 K_0(\lambda_1) & -\lambda_1 K_0(\lambda_2) \\ \mathrm{K}_1(\lambda_1) & \mathrm{K}_1(\lambda_2) \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\lambda_2(1 - \frac{ip_0}{\varpi}) \\ 0 \end{bmatrix} \tag{9.26b}$$

From (9.26a) and (9.26b) we can calculate A_1 and A_2 for both the cases.

By using w' from (9.18a) we get w for the case of resonance as

$$w(r) = \frac{ip_0}{\varpi} - \frac{A_1}{\lambda} K_0(\lambda r) + \frac{A_2}{\lambda^2}\left(\lambda r K_1(\lambda r) + K_0(\lambda r)\right) \quad (9.27a)$$

By using w' from (9.18b) we get w for the case of non-resonance as

$$w = \frac{ip_0}{\varpi} - \frac{K_0(\lambda_1 r)}{\lambda_1} A_1 - \frac{K_0(\lambda_2 r)}{\lambda_2} A_2 \quad (9.27b)$$

9.4.1 Skin friction acting on the cylinder per length L

Skin friction acting on the circular cylinder $c_f = \frac{2T_{rz}}{\rho W_0^2}$ (9.28)

The Drag acting on the cylinder (r=a) per unit length (z=0 to z=1) is given by

$$\mathrm{D} = \int \boldsymbol{T_n}.\boldsymbol{k}\ \mathrm{r d\theta dz} = \mathrm{T_{rz}}.2\pi a = \mathrm{c}_f\ \pi a \rho W_0^2 \quad (9.29)$$

Hence Drag is given in terms of Skin friction.

For Couple stress fluids, the constitutive equations for stress and Couple stresses are given by (9.4) and (9.5).

The strain rate tensor E is given by $E = \frac{1}{2}(\nabla_1 \bar{Q} + \nabla_1 \bar{Q}^T)$

In cylindrical co-ordinate system, E is given as below.

$$E = \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2}\left(\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R}\right) & \frac{1}{2}\left(\frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z}\right) \\ \frac{1}{2}\left(\frac{\partial V}{\partial R} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{V}{R}\right) & \frac{1}{2}\left(U + \frac{\partial V}{\partial \theta}\right) & \frac{1}{2}\left(\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z}\right) \\ \frac{1}{2}\left(\frac{\partial W}{\partial R} + \frac{\partial U}{\partial Z}\right) & \frac{1}{2}\left(\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial Z}\right) & \frac{\partial W}{\partial Z} \end{bmatrix} \quad (9.30)$$

For this present problem, $E = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial W}{\partial R} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\partial W}{\partial R} & 0 & 0 \end{bmatrix}$ (9.31)

Form (9.5) Couple stress tensor M is obtained as

$$M = \begin{bmatrix} m & 2\eta\frac{\partial B}{\partial R} - 2\eta'\frac{B}{R} & 0 \\ 2\eta'\frac{\partial \boldsymbol{B}}{\partial R} - 2\eta\frac{B}{R} & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad (9.32)$$

where $B = -\frac{\partial W}{\partial R} = -W'$ (9.33)

$$\nabla_1 . M = -2\eta D_c^2 W' \bar{e}_\theta \tag{9.34}$$

And $I \times (\nabla_1 . M) = \begin{bmatrix} 0 & 0 & -2\eta D_c^2 W' \\ 0 & 0 & 0 \\ 2\eta D_c^2 W' & 0 & 0 \end{bmatrix}$ (9.35)

By substituting (9.31) and (9.35) in (9.4) and simplifying we get

$$T_{RZ} = \left\{\mu \frac{\partial W}{\partial R} - \eta D_c^2 W' \right\} e^{i\varpi t} \tag{9.36}$$

Using non dimensional scheme (9.7) and (9.8) in (9.36) we get

$$T_{rz} = \frac{\sigma\eta}{a^2} [S w' - D_c^2 w'] e^{i\varpi t} \tag{9.37}$$

The Skin friction acting on the circular cylinder (after deleting the factor $e^{i\varpi t}$) is obtained as:

$$c_f = -\frac{\varpi}{S.Re} \left\{ S \frac{\partial w}{\partial r} - D_c^2 w' \right\}_{r=1} = \frac{\varpi}{S.Re} D_c^2 w' |_{r=1} \tag{9.38}$$

In the resonance case, the Skin friction is given by

$$c_f = \frac{\varpi}{SRe} 2\lambda A_2 K_1(\lambda) \tag{9.39a}$$

In the non-resonance case, the Skin friction is given by

$$c_f = \frac{\varpi}{S.Re} \{A_1 \lambda_1^2 \mathrm{K}_1(\lambda_1) + A_2 \lambda_2^2 \mathrm{K}_1(\lambda_2)\} \tag{9.39b}$$

9.5 Results and Discussions

The roots of $x^2 - Sx + i\varpi ReS = 0$ are taken as the values of λ^2. Hence

$\lambda = \sqrt{x} = \sqrt{\frac{S \pm \sqrt{S^2 - 4S.Re.i\omega}}{2}}$ for nonresonance, $\lambda = \sqrt{S/2}$ for resonance (9.40)

Here ϖ *and* Re are choosen independently, with $Re \ll 1$ and $\omega \gg 1$ such that $\omega.Re$ is not negligibly small (say>1) then λ is obtained from (9.40). Then A_1 and A_2 and hence w and Skin friction are obtained. To get physical quantities, the corresponding real part of the quantities are taken.

In this chapter analytical expressions for velocity component w and Skin friction c_f are obtained as (9.27a), (9.27b), (9.39a) and (9.39b). The numerical results are presented in the form of graphs for different Re, ϖ, S values from Fig 9.2 to 9.7. In every figure upper graph (a) is for the case of resonance and lower graph (b) for non-resonance case.

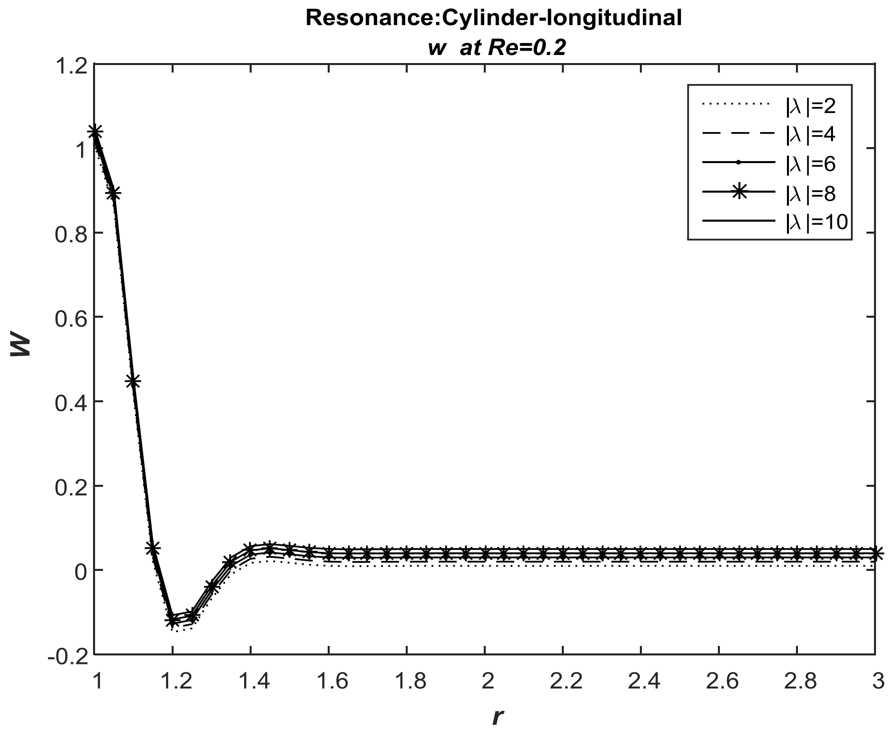
9.5.1 Velocity

When we fix $|\lambda|$ in the case of resonance, it means we fix $|S|$ also. Hence in this case, variation of Re and ω cannot be found separately. But for non-resonance, since there are two λ values namely λ_1 andλ_2, Re and ω can vary independently even if $|\lambda|$ is fixed. Because of this reason, if $|\lambda|$ is fixed, velocity and Skin friction show distinct behavior for the case of resonance.

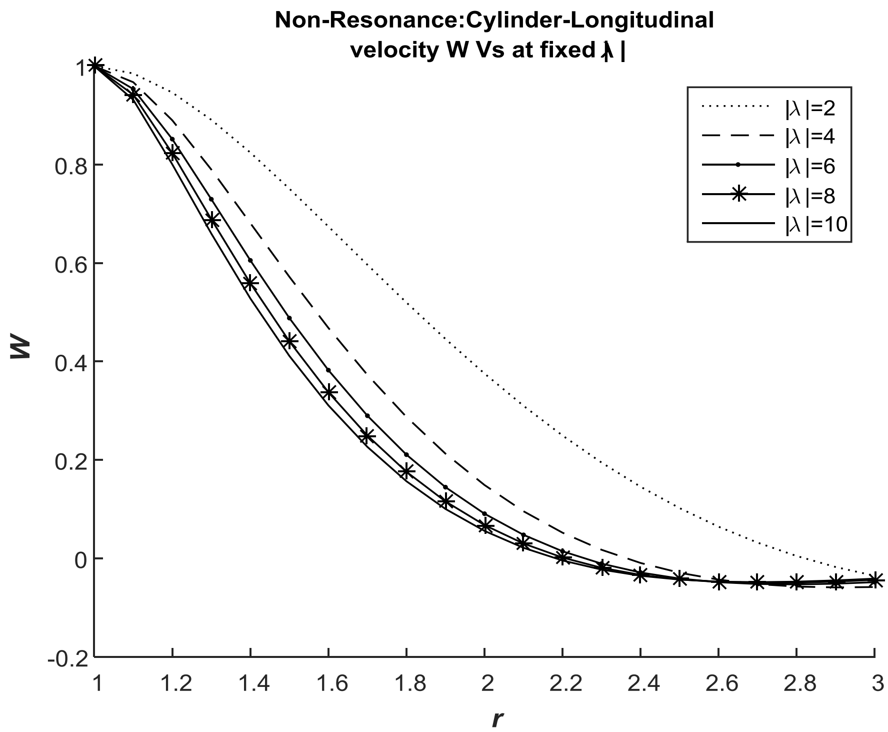
In Fig 9.2, we observe that in the case of resonance, velocity goes to negative values (flow reversal takes place) near to the cylinder and within a short range vanishes. But in the case of non-resonance, the small flow reversal takes place at a larger distance and flow then flow vanishes.

In Fig 9.3 we observe that in the case of resonance, at different values of frequency parameter ϖ, velocity takes negative values in a larger range of r and vanishes. But in the case of non-resonance, velocity becomes negative in a short range of r and vanishes. As ϖ increases, the velocity curve goes nearer to the cylinder in both the cases and this is clearly visible for the case of resonance.

In Fig 9.4 we observe that in the case of resonance, at different values of Reynolds numbers Re, velocity takes negative values in a larger range of r and vanishes. But in the case of non-resonance, velocity becomes negative in a short range of r and vanishes. As Re increases, the velocity curve is nearer to the cylinder in both the cases and in particular for resonance.

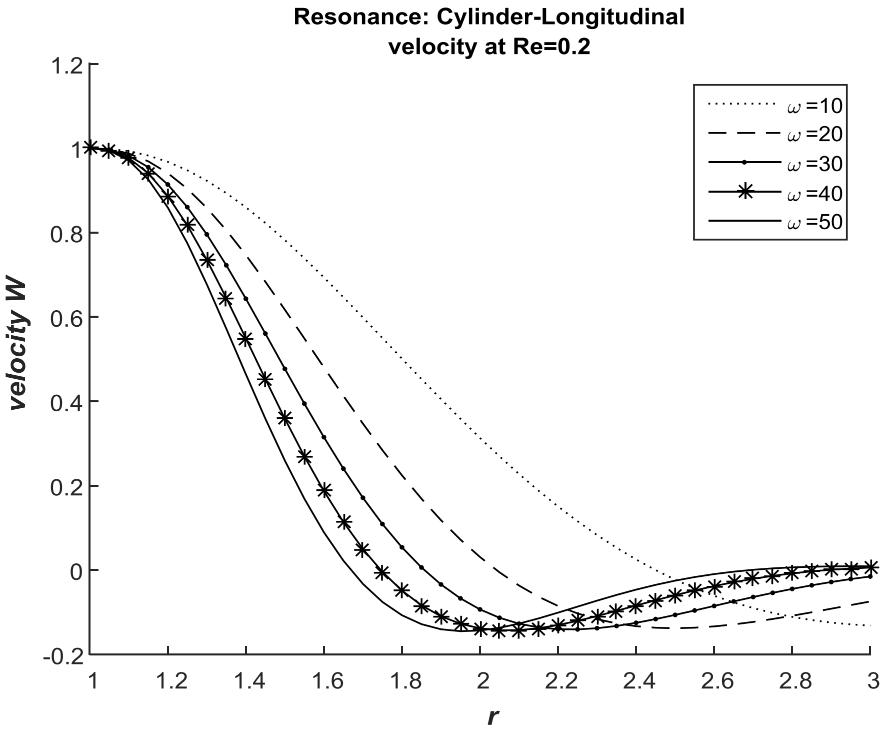


(a)

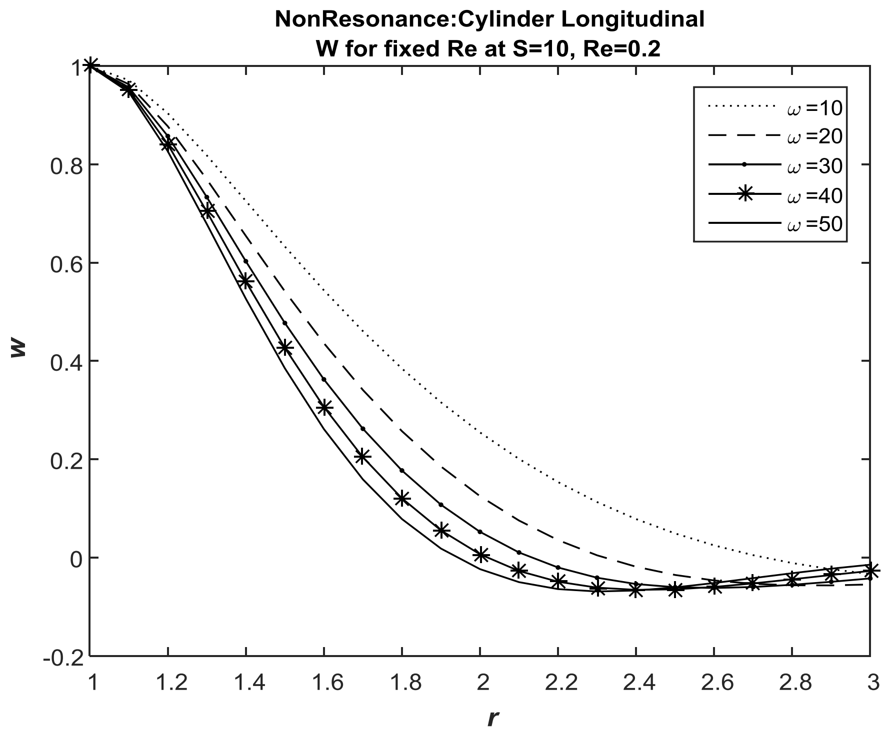


(b)

Fig 9.2 Velocity for fixed values of |λ| for the case of (a) resonance and (b) non-resonance

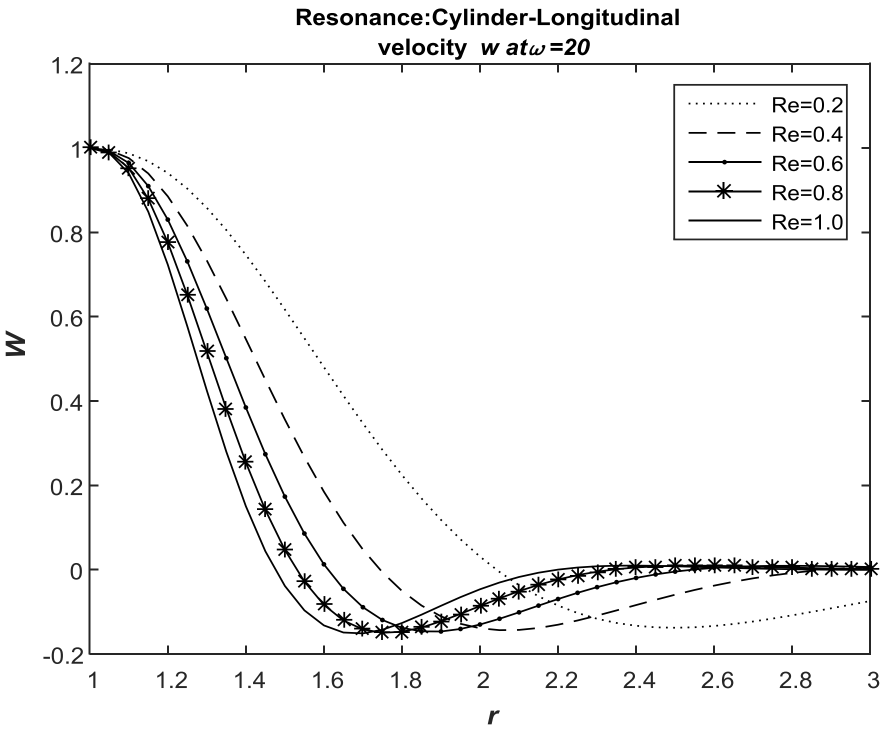


(a)

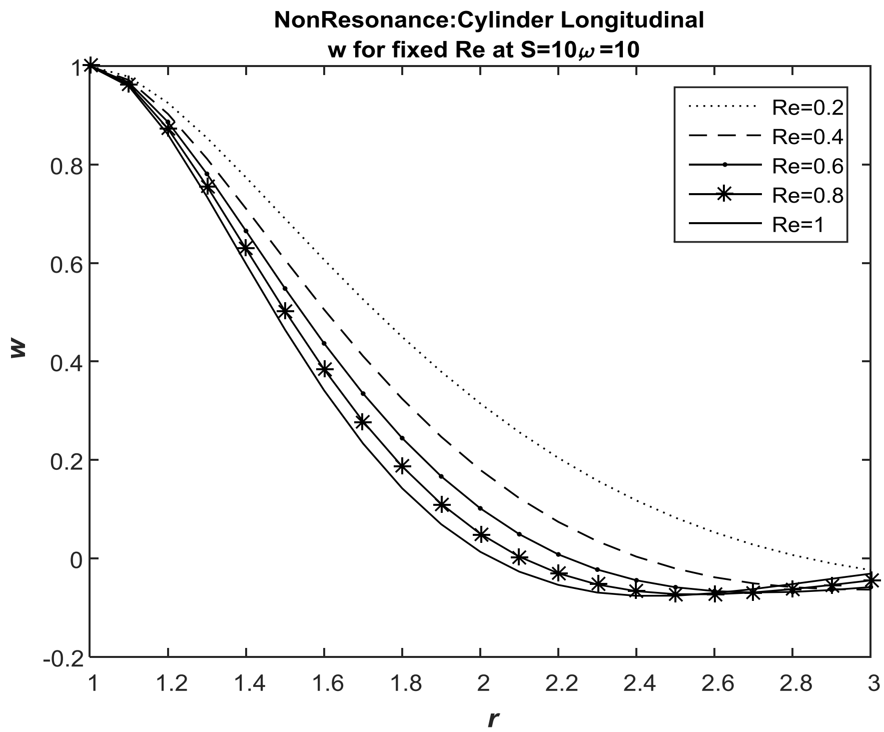


(b)

Fig 9.3 Velocity w at different values of ϖ for the case (a) resonance and (b) non-resonance



(a)



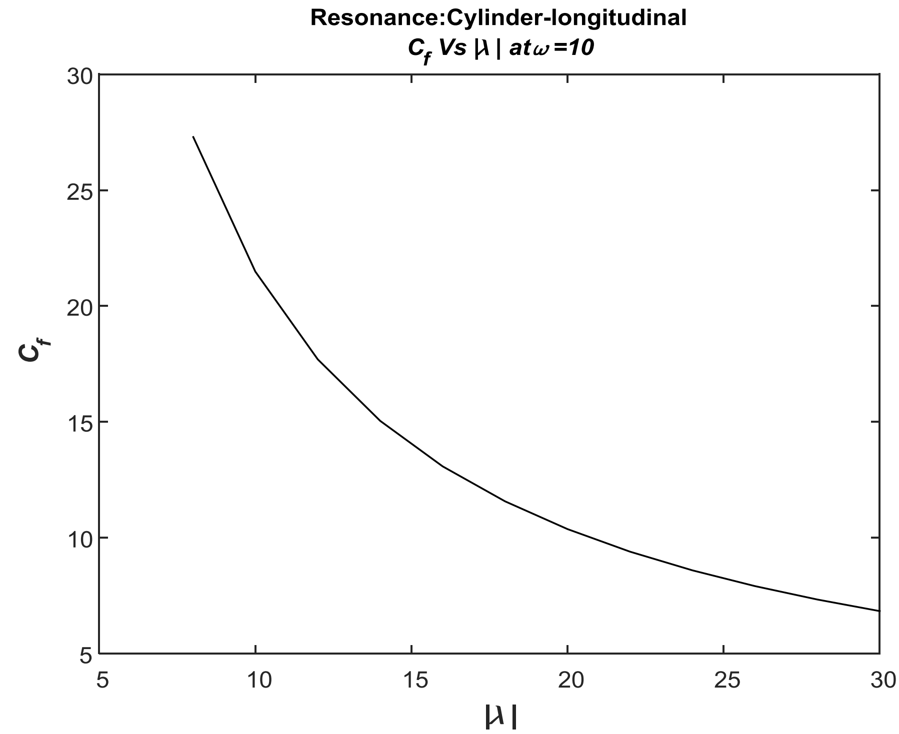
(b)

Fig 9.4 Velocity *w* at different values of Re for the case of (a) resonance and (b) non-resonance

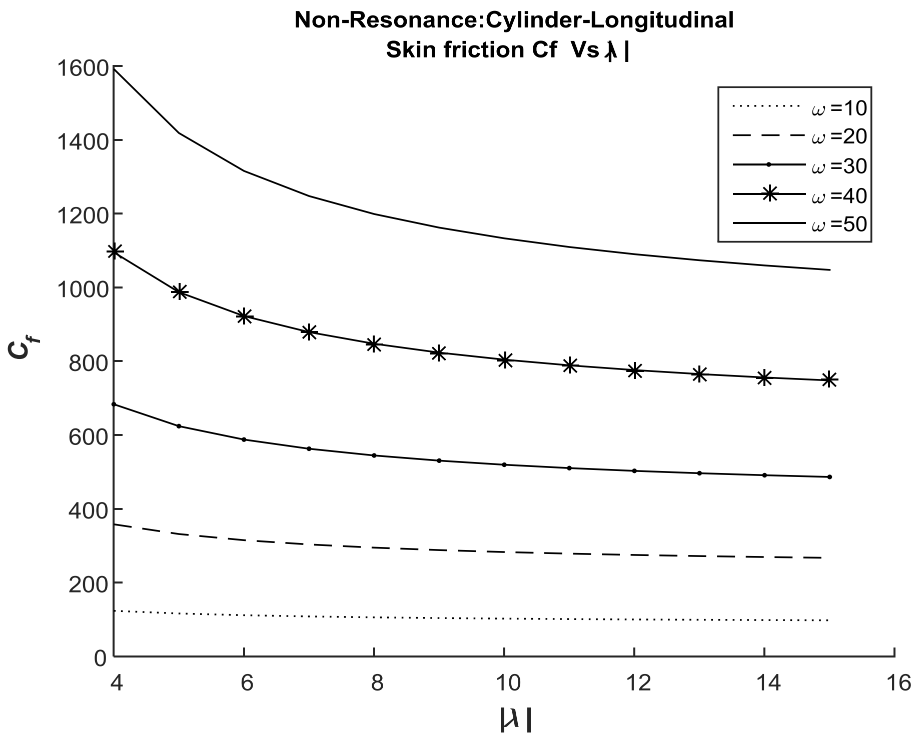
9.5.2 Skin friction

In Fig 9.5 effect of $|\lambda|$ on Skin friction is shown. Since $|\lambda|$ is fixed, there will be no variations in ϖ and Re for the case of resonance. But for non-resonance ϖ or Re can vary. We can observe that the Skin friction is very smaller for resonance than in the case of non-resonance. For small values of $|\lambda| <4$ Skin friction is very high and is not shown in figures.

In Fig 9.6, we observe that, the Skin friction for resonance and non-resonance is almost same. We note that as frequency parameter increases, Skin friction increases. But in Fig 9.7, we observe that as Re increases, Skin friction also decreases. This behavior is opposite to the effect of frequency parameter ϖ. This is because, the product of Re and ϖ is constant and this product come as an unit for calculation of λ. The Skin friction for the case of resonance is lesser than the case of non-resonance.

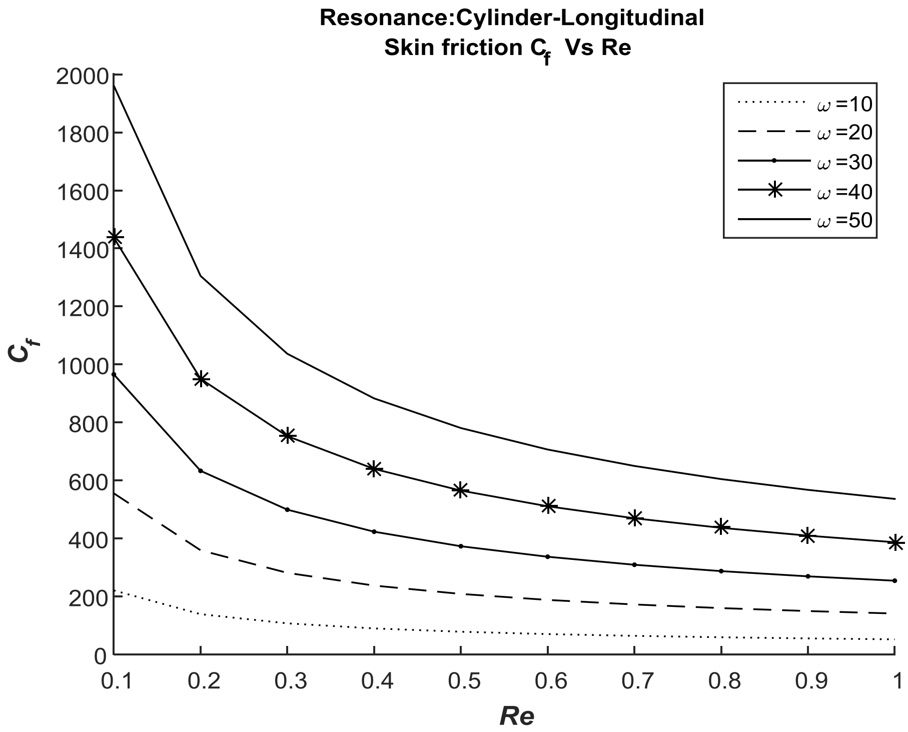


(a)

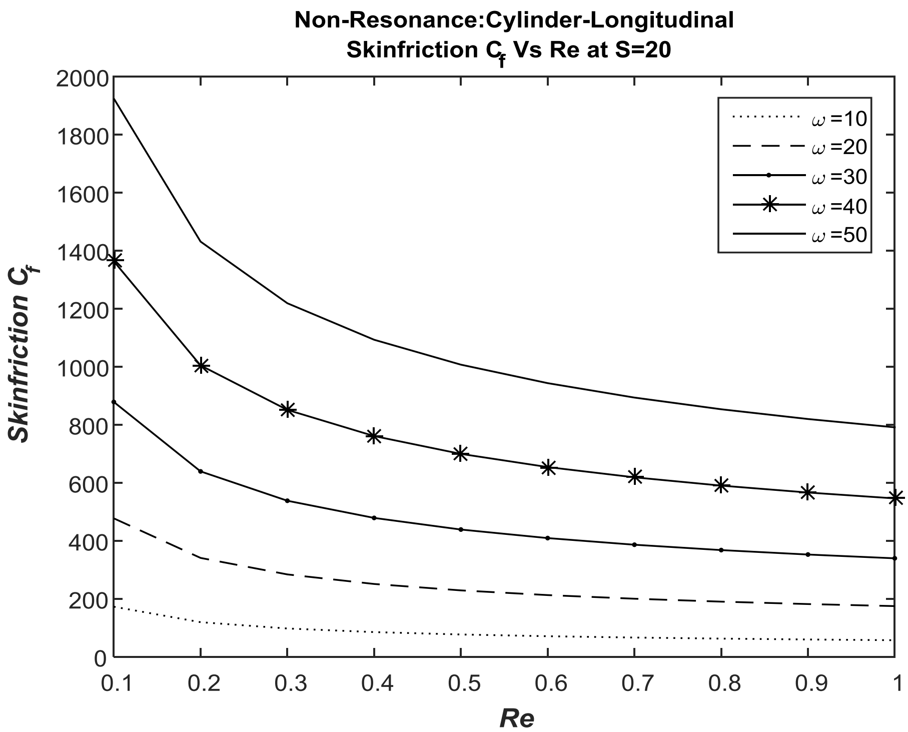


(b)

Fig 9.5 Skin friction Vs |λ| for the case of (a) resonance and (b) non-resonance

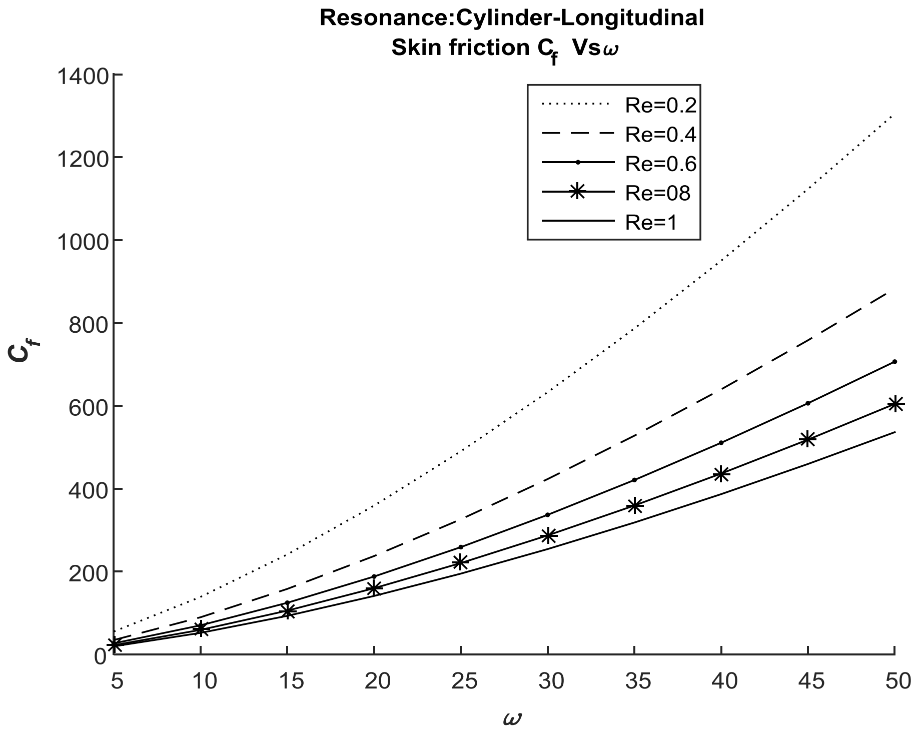


(a)

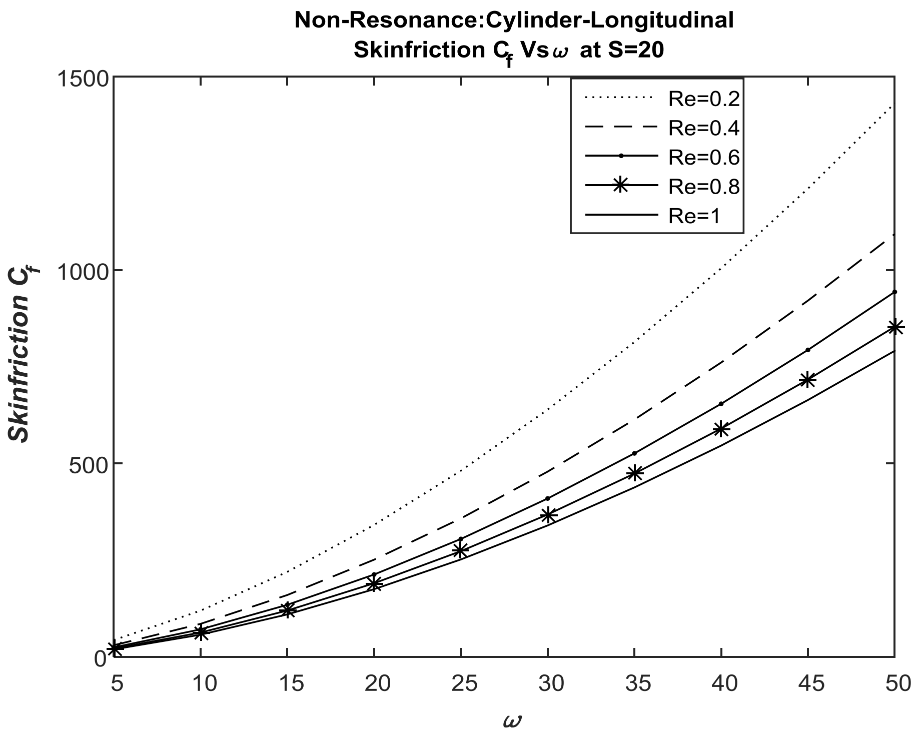


(b)

Fig 9.6 Skin friction Vs Reynolds number Re for the case of (a) resonance and (b) non-resonance



(a)



(b)

Fig 9.7 Skin friction Vs frequency ϖ for the case of (a) resonance and (b) non-resonance

9.6 Conclusions

In the case resonance we observe that

i) Velocity changes are very high near to the cylinder and vanishes far from the cylinder.

ii) Skin friction increases as frequency parameter increases and decreases as Reynolds number increases.

iii) When $|\lambda|$ is fixed Skin friction will be minimum (reduces to very low values)

In the case of non-resonance

i) Velocity changes occur far from the cylinder in comparison with resonance and vanish relatively near to the cylinder.

ii) Skin friction is of same order as in the case of resonance when frequency parameter is fixed. But takes greater values when Reynolds number is fixed.

iii) When $|\lambda|$ is fixed Skin friction will be very high.

Chapter 10

Rectilinear oscillations of Sphere in a Couple-stress fluid

The present chapter deals with the flow arising due to rectilinear oscillations of a sphere about its axis of symmetry in a Couple-stress fluid. Due these oscillations, there occurs a attenuate but an important appropriate case which is referred to as Resonance flow. In this case material constants are related by a resonance condition. The flow is analyzed under Stokesian approximation by ignoring nonlinear convective terms, under the assumption that the Reynolds number is less than one due to very slow flow. The velocity components of the flow in terms of stream function are derived. The Drag acting on the sphere evaluated and the effect of physical parameters like Reynolds number and Couple stress parameter on the Drag are shown through graphs.

10.1 Introduction

Several researchers investigated the flow of non-Newtonian fluids in Spherical geometry. Vijay Kumar Stokes (1968, 1971) analysed effects of Couple-stresses in fluids on hydromagnetic channel flows and on the creeping flow past a Sphere. Frater (1967, 1968) studied oscillatory flows in elastico-viscous fluid, and evaluated Drag on sphere, damping force on a body. Lakshmana Rao et al. (1970) studied slow stationary flow of a Micro-polar fluid past a sphere. Analytical and Computational studies in Couple stress fluid flows examined by Lakshmana Rao et al. (1980). Lakshmana Rao et al. (1971, 1981, 1987) studied the oscillatory flows generated due to oscillations of sphere, spheroid and elliptic cylinder in Micro-polar fluids, with the aim of determining of the Drag or Couple on the oscillating body. Lai et al. (1978) examined an elastic-viscous fluid flow of sphere performing rectilinear oscillations and evaluated Drag on a sphere.

Iyengar et al. (1993, 2001, 2004) examined oscillatory flows due to oscillating of approximate sphere, two concentric spheres in Micro-polar fluid and approximate sphere in viscous fluid. Stimson et al. (1926) examined the viscous fluid motion of two spheres. Verma et al. (1971) studied slow oscillatory flow past a fixed porous sphere. Aparna et al. (2012) examined the flow of micro-polar fluid due to rotary oscillations of a permeable sphere. Ashmawy (2015, 2016) examined oscillatory flows of composite sphere in a concentric spherical cavity and spherical particle moving in a Couple-stress fluid.

In this chapter we intend to investigate this case of resonance type flow due to rectilinear oscillations of a sphere about its axis of symmetry in Couple-stress fluids. In chapter 4 similar case investigated in Micro-polar fluids.

10.2 Basic Equations

The governing equations of an incompressible Couple stress fluid introduced by Stokes (1966) are given by:

$$div\, \bar{Q} = 0 \tag{10.1}$$

$$\rho\left(\frac{\partial \bar{Q}}{\partial \tau} + \bar{Q}.\nabla_1 \bar{Q}\right) = -\nabla_1 P - \mu \nabla_1 \times \nabla_1 \times \bar{Q} - \eta \nabla_1 \times \nabla_1 \times \nabla_1 \times \nabla_1 \times \bar{Q} \tag{10.2}$$

where $\bar{Q},\ \rho,\ \tau,\ \mu,\ \eta\ and\ P$ are fluid velocity, density, time, viscosity coefficient, Couple stress viscosity and pressure respectively and ∇_1 is dimensional gradient operator . For Couple stress fluids, the stress tensor T and Couple stress tensor M satisfy the following constitutive equations.

$$T = -PI + \lambda(\nabla_1 . Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2} I \times (\nabla_1 . M) \tag{10.3}$$

$$M = mI + 2\eta \nabla_1(\nabla_1 \times Q) + 2\eta' [\nabla_1(\nabla_1 \times Q)]^T \tag{10.4}$$

10.3 Statement and Formulation of the Problem

A Sphere of radius a is performing rectilinear oscillations with velocity $U_0 e^{i\sigma\tau}$ about its diameter in an infinite vat containing incompressible Couple-stress fluid. A spherical coordinate system $(R, \theta, \emptyset)$ with base vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\emptyset)$ with origin at the centre of the sphere is considered. The flow is axially symmetric and hence the fluid velocity will be independent of $\emptyset$ and will be in plane containing the base vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$. The velocity is assumed in the form:

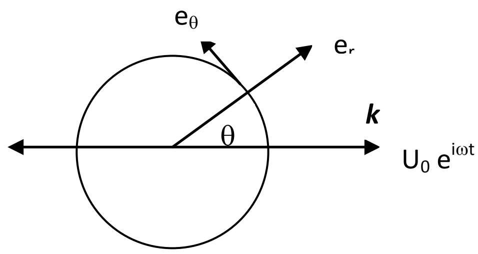


Fig 10.1 Geometry of the oscillating Sphere

$$\boldsymbol{Q} = e^{i\sigma\tau}\left(U(R,\theta)\boldsymbol{e_r} + V(R,\theta)\boldsymbol{e_\theta}\right) \tag{10.5}$$

The following non-dimensional scheme is introduced.

$$\mathrm{R} = ar;\ \mathrm{U} = \mathrm{U}_0 u;\ \ \mathrm{V} = \mathrm{U}_0 v;\ \ \mathbf{Q} = \mathbf{q}\mathrm{U}_0;\ \ \mathrm{P} = p\rho \mathrm{U}_0^2,\ \ \tau = \frac{\mathrm{at}}{U_0} \tag{10.6}$$

The following are non-dimensional parameters ϖ is frequency parameter, S is Couple stress parameter and Re is Reynolds number for Couple-stress fluids.

$$\varpi = \frac{\sigma \mathrm{a}}{\mathrm{U}_0},\ s = \frac{\mu a^2}{\eta},\ \mathrm{Re} = \frac{\rho \mathrm{U}_0 a}{\mu}\ ,\ \mathrm{Re}.\varpi = \frac{\rho\sigma a^2}{\mu} \tag{10.7}$$

Substituting (10.5) in (10.1), we notice that stream function ψ can be introduced as

$$u = \frac{1}{r^2 Sin\theta}\frac{\partial\psi}{\partial\theta}\ \ and\ \ v = -\frac{1}{rSin\theta}\frac{\partial\psi}{\partial r}\quad \text{i.e}\qquad \boldsymbol{q} = \nabla\times\left(\frac{\psi}{h_3}\boldsymbol{e}_\emptyset\right) \tag{10.8}$$

Using (10.5), (10.6) and (10.7) in (10.2) we get

$$Re.S\frac{\partial \boldsymbol{q}}{\partial t} = -Re.S.\nabla p - S\,\nabla\times\ \nabla\times\boldsymbol{q} - \ \nabla\times\ \nabla\times\nabla\times\nabla\times\boldsymbol{q} \tag{10.9}$$

Substituting (10.8) in equation (10.9) and then eliminating pressure we get,

$$E^2(E^2 - \lambda_1^2)\,(E^2 - \lambda_2^2)\,\psi = 0 \tag{10.10}$$

where $E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{Cot\theta}{r^2}\frac{\partial}{\partial \theta}$ (10.11)

where $\lambda_1^2 + \lambda_2^2 = S$ and $\lambda_1^2\lambda_2^2 = Re.S.i\varpi$ (10.12)

The solution for ψ if $\lambda_1 \neq \lambda_2$ in (10.10) was given by Lakshmana Rao et al. (1971) for the case of Micro-polar fluids. The solution for ψ for the case, $\lambda_1 = \lambda_2$ cannot be obtained as a limiting case of $\lambda_1 \to \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation in non-dimensional form.

$$S = 4Re.\,i\varpi \tag{10.13}$$

In this chapter we are interested in the solution for ψ for the case of resonance $\lambda_1 = \lambda_2 = \lambda$. In this case, the equation for ψ is given by

$$E^2\left(E^2 - \lambda^2\right)^2 \psi = 0 \tag{10.14a}$$

For the case of non-resonance

$$E^2\left(E^2 - \lambda_1^2\right)\left(E^2 - \lambda_2^2\right)\psi = 0 \tag{10.14b}$$

10.3.1 Boundary Conditions

The sphere is oscillating in the direction of X axis. Hence the non-dimensional velocity of sphere Γ after removing $e^{i\varpi t}$ is given by

$\boldsymbol{q}_{\Gamma} = \boldsymbol{i} = cos\theta\boldsymbol{e}_r - sin\theta\boldsymbol{e}_{\theta}$ which implies by no-slip condition

$u = cos\theta$ and $v = - sin\theta$ on $r = 1$ (10.15)

By hyper-stick condition $\boldsymbol{\upsilon}_{\Gamma} = \frac{1}{2}(\text{curl}\,\boldsymbol{q})_{\Gamma} = 0$ on r=1 (10.16)

10.4 Solution of the Problem

To match with the boundary conditions, stream function ψ is assumed in the form

$$\psi = f(r)Sin^2\theta \tag{10.17}$$

Substituting (10.17) in (10.14a) and (10.14b) we get equation for *f* for Resonance and non-resonance cases as below

$$D_s^2\left(D_s^2 - \lambda^2\right)^2 f = 0 \tag{10.18a}$$

$$D_s^2\left(D_s^2 - \lambda_1^2\right)\left(D_s^2 - \lambda_2^2\right) f = 0 \tag{10.18b}$$

Where $D_s^2 = \frac{d^2}{dr^2} - \frac{2}{r^2}$

From the boundary conditions in (10.15) and (10.16), the conditions on *f* are obtained as:

$$f(1) = \frac{1}{2},\ f'(1) = 1 \text{ and } D_s^2 f = 0 \text{ on r=1} \tag{10.19}$$

Since the equation for *f* is linear, *f* is considered as

$$f = A_0 f_0 + A_1 f_1 + A_2 f_2$$

$$\text{with } D_s^2 f_0 = 0,\quad \left(D_s^2 - \lambda^2\right) f_1 = 0 \text{ and } \left(D_s^2 - \lambda^2\right)^2 f_2 = 0 \tag{10.20a}$$

for the case of resonance and

$$D_s^2 f_0 = 0,\quad \left(D_s^2 - \lambda_1^2\right) f_1 = 0 \text{ and } \left(D_s^2 - \lambda_2^2\right) f_2 = 0 \tag{10.20b}$$

for the case of non-resonance.

On solving (10.20a), the solution for *f* is obtained for resonance case as

$$f(\mathrm{r}) = \frac{\mathrm{A_0}}{\mathrm{r}} + \mathrm{A_1}\sqrt{\mathrm{r}}\mathrm{K}_{\frac{3}{2}}(\lambda \mathrm{r}) + \mathrm{A_2}\mathrm{r}^{\frac{3}{2}}\mathrm{K}'_{\frac{3}{2}}(\lambda \mathrm{r}) \tag{10.21a}$$

and for non-resonance case as

$$f(\mathrm{r}) = \frac{\mathrm{A_0}}{\mathrm{r}} + \mathrm{A_1}\sqrt{\mathrm{r}}\mathrm{K}_{\frac{3}{2}}(\lambda_1 \mathrm{r}) + \mathrm{A_2}\sqrt{\mathrm{r}}\mathrm{K}_{\frac{3}{2}}(\lambda_2 \mathrm{r}) \tag{10.21b}$$

The following results are useful to note in the case of resonance and non-resonance.

$$D_s^2 f_1 \; = \; \lambda^2 f_1 \quad \text{and} \quad D_s^2 f_2 \; = 2\lambda f_1 + \; \lambda^2 f_2 \qquad (10.22a)$$

$$D_s^2 f_1 \; = \; \lambda_1^2 f_1 \quad \text{and} \quad D_s^2 f_2 \; = \; \lambda_2^2 f_2 \qquad (10.22b)$$

The constants A_0 , A_1 , A_2 are obtained from the boundary conditions (10.19) as follows:

In case of resonance

$$\begin{bmatrix} 1 & K_{\frac{3}{2}}(\lambda) & K'_{\frac{3}{2}}(\lambda) \\ -1 & \frac{1}{2}K_{\frac{3}{2}}(\lambda) + \lambda K'_{\frac{3}{2}}(\lambda) & \frac{3}{2}K'_{\frac{3}{2}}(\lambda) + \lambda K''_{\frac{3}{2}}(\lambda) \\ 0 & K_{\frac{3}{2}}(\lambda) & 2K_{\frac{3}{2}}(\lambda) + K'_{\frac{3}{2}}(\lambda) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \qquad (10.23a)$$

And in case of non-resonance

$$\begin{bmatrix} 1 & K_{\frac{3}{2}}(\lambda_1) & K_{\frac{3}{2}}(\lambda_2) \\ -1 & \frac{1}{2}K_{\frac{3}{2}}(\lambda_1) + \lambda_1 K'_{\frac{3}{2}}(\lambda_1) & \frac{1}{2}K_{\frac{3}{2}}(\lambda_2) + \lambda_2 K'_{\frac{3}{2}}(\lambda_2) \\ 0 & \lambda_1^2 K_{\frac{3}{2}}(\lambda_1) & \lambda_2^2 K_{\frac{3}{2}}(\lambda_2) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \qquad (10.23b)$$

On solving the equation (10.23) for A_0, A_1, A_2 we get f completely and hence ψ is known.

10.4.1 Pressure

From equation (10.9) pressure is obtained as follows.

$$dp = \nabla p.\boldsymbol{dr} = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta \qquad (10.24)$$

By comparing components in (10.9), we get

$$Re.S \frac{\partial p}{\partial r} = -Re.Si\varpi \frac{1}{r^2 sin\theta} \frac{\partial \psi}{\partial \theta} + \frac{S}{r^2 sin\theta} \frac{\partial}{\partial \theta} (\mathrm{E}^2 \psi) - \frac{1}{r^2 sin\theta} \frac{\partial}{\partial \theta} (\mathrm{E}^4 \psi) \qquad (10.25)$$

$$Re.\frac{S}{r} \frac{\partial p}{\partial \theta} = \; Re.Si\varpi \frac{1}{rsin\theta} \frac{\partial \psi}{\partial r} - \frac{S}{rsin\theta} \frac{\partial}{\partial r} (\mathrm{E}^2 \psi) + \frac{1}{rsin\theta} \frac{\partial}{\partial r} (\mathrm{E}^4 \psi) \qquad (10.26)$$

By substituting (10.25) and (10.26) in (10.24), we get

$$Re.Sp = \frac{A_0\lambda_1^2\lambda_2^2}{r^2}cos\theta$$

Hence $p = \frac{i\omega A_0}{r^2}cos\theta$ (10.27)

10.4.2 Drag acting on the sphere of radius a

Drag $= D^* = 2\pi a^2 \int_0^\pi (T^*_{rr} cos\theta - T^*_{r\theta}\ sin\theta)sin\theta\,|_{R=a}\ d\theta$ (10.28)

$$T^*_{rr} = \frac{\mu U_0}{a}\left[-Re.p + \frac{4}{r^2}(f' - \frac{2f}{r^2})cos\theta\right] \quad (10.29)$$

$$T^*_{r\theta} = \frac{\mu U_0}{a}\left[\frac{1}{r}\left(\frac{1}{S}D_s^4 f - D_s^2 f + \frac{2}{r}(f' - \frac{2f}{r})\right)\right]sin\theta \quad (10.30)$$

Substitute (10.29) and (10.30) in (10.28), we get

$$D^* = 2\pi\mu U_0 a \int_0^\pi \{ (-Re.p\ cos\theta) - (\frac{1}{S}D_s^4 f - D_s^2 f)\sin^2\theta \}sin\theta|_{r=1}\ d\theta \quad (10.31)$$

Substitute (10.27) in (10.31), we get the Drag on the sphere (for resonance and non-resonance cases – without the factor $e^{i\omega t}$) after dividing it by $2\pi\mu U_0 a$ in the following non-dimensional form as

For resonance $Drag = D^* = Real\frac{2}{3}\{Re.i\varpi(1 - A_0) - 2\}$ (10.32a)

For non-resonance $D^* = \text{Real}\left\{\frac{2}{3}\left[\text{Re}.i\varpi A_0 - 2 - \frac{2}{S}\left(A_1\lambda_1^4 K_{\frac{3}{2}}(\lambda_1) + A_2\lambda_2^4 K_{\frac{3}{2}}(\lambda_2)\right)\right]\right\}$

(10.32b)

10.5 Results and Discussions

The values of λ are obtained from (10.16) by solving $x^2 - Sx + i\varpi ReS = 0$ for x. Then for resonance case

$$\lambda = \sqrt{x} = \begin{cases} \sqrt{\frac{S\pm\sqrt{S^2-4S.Re.i\omega}}{2}} & \text{for nonresonance} \\ \sqrt{\frac{S}{2}} & \text{for resonance} \end{cases} \quad (10.33)$$

In the case of resonance, $S = 4i.Re.\varpi$

Here ϖ *and* Re are choosen independently, with $Re \ll 1$ and $\varpi \gg 1$ such that $\varpi.Re$ is not negligibly small (say>1) then λ is obtained from (10.33). Then

A_0, A_1 *and* A_2 and hence stream function ψ and Drag are obtained. To get physical quantities, the corresponding real part of the quantities are taken.

$2\lambda_1^2 = S + \sqrt{\frac{\sqrt{S^2p}+S^2}{2}} - i\sqrt{\frac{\sqrt{S^2p}+S^2}{2}}$ where $p = S^2 + 16Rw^2$ and $Rw = Re.\varpi$

Taking modulus on bothsides we get, $4|\lambda_1|^4 = S^2 + S\sqrt{p} + S\sqrt{2(\sqrt{S^2p} + S^2)}$

This can be Rearranged as $N = \frac{4|\lambda_1|^4 - S^2}{S} = \sqrt{p} + \sqrt{2(S\sqrt{p} + S^2)}$

Squaring and reaaranging we get, $N^2 + p - 2S^2 = 2\sqrt{p}(S + f) = \frac{8\sqrt{p}|\lambda_1|^4}{S}$

Again squaring and rearranging we get, $S^2\left(N^2 + 16Rw^2 – S^2\right)^2 = 64p|\lambda_1|^8$

This equation can be rearranged as a polynomial in S as below:

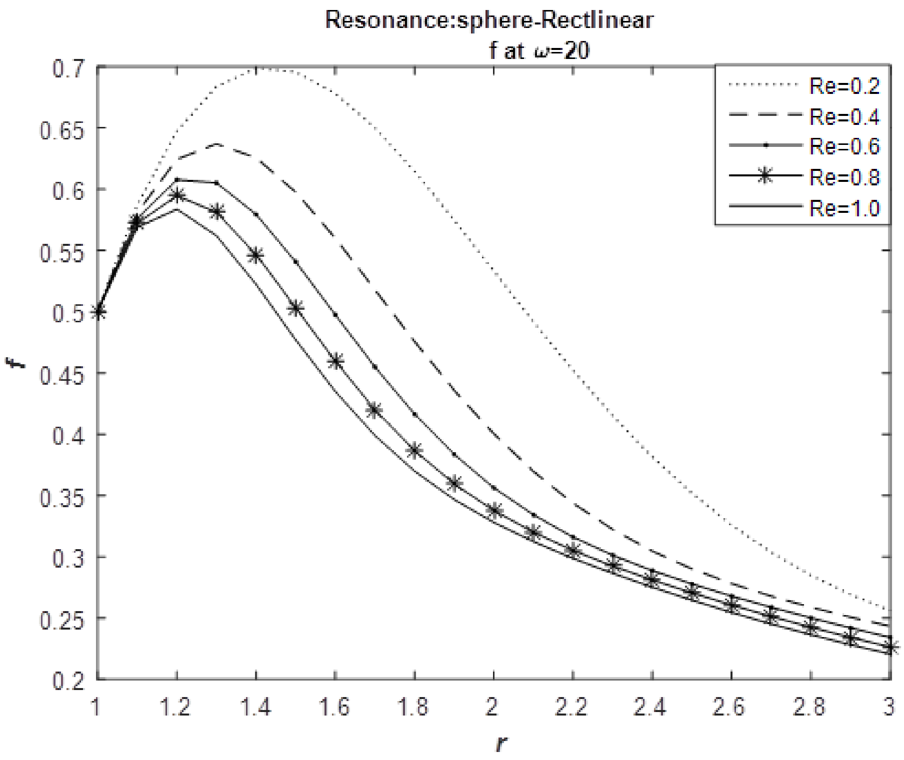
$$Rw^2(Rw^2 - L)S^4 - L^2(L + 2Rw^2)S^2 + L^4 = 0 \ \ where\ L = |\lambda_1|^4 \tag{10.34a}$$

From this by fixing L and Rw we can find S or by fixing S and L we can find Rw.

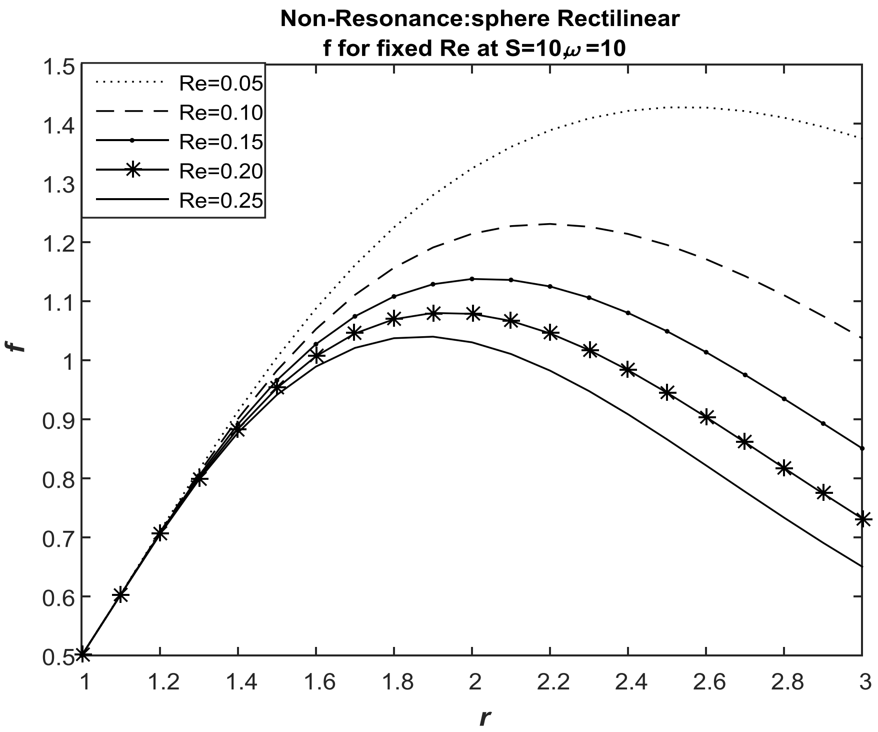
In the case of resonance (10.34) reduces to: $S = 2\lambda^2 = 4i\ Re.\varpi$ (10.34b)

10.5.1 Stream function

The stream function for the flow is obtained from (10.21). It is presented in terms of function f in the form of figures below in Fig 10.2 and Fig 10.3. It is observed that stream function raises near the sphere. But for resonance its peak is obtained at a hight less than the case of non-resonance. Again we observe that stream function vanishes at a longer distance from origin of the sphere for non-resonance. From stream line pattern we observe circulations at the cap (pole of the sphere). For resonance the values of the stream lines are less than 0.8 (entire region is in green or yellow in color indicating values less than 0.8). For non-resonance, the stream lines take values more than 1.5 also. In this case of non-resonance, we can find stream lines again in small circulations on left and right side ways also with values near to 2. This may be due to twisting effect of Couple stresses. This effect is reduced to minimum for the case of resonance.



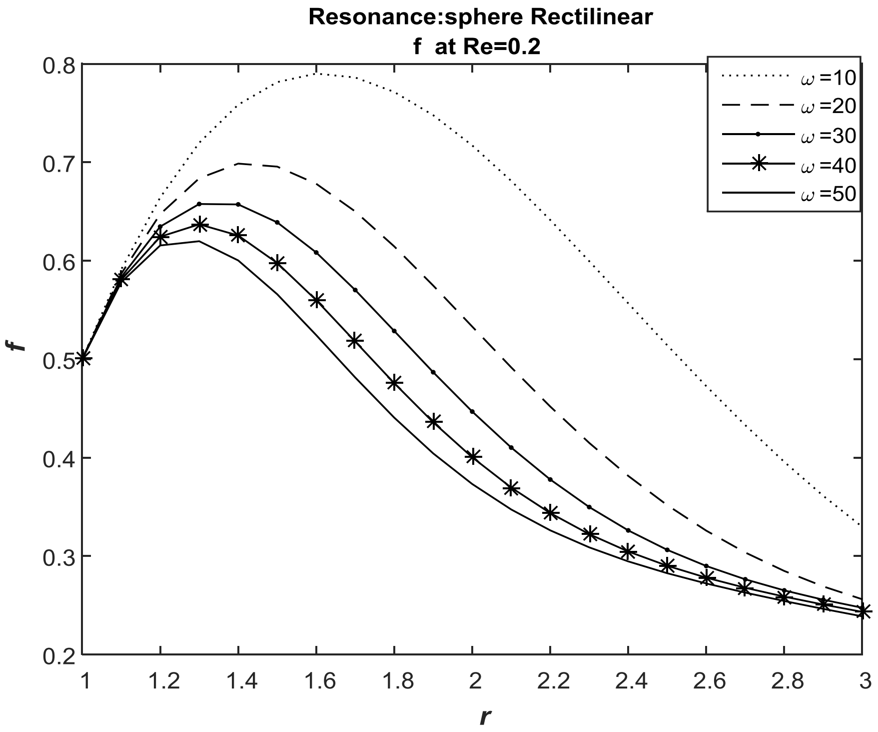
(a)



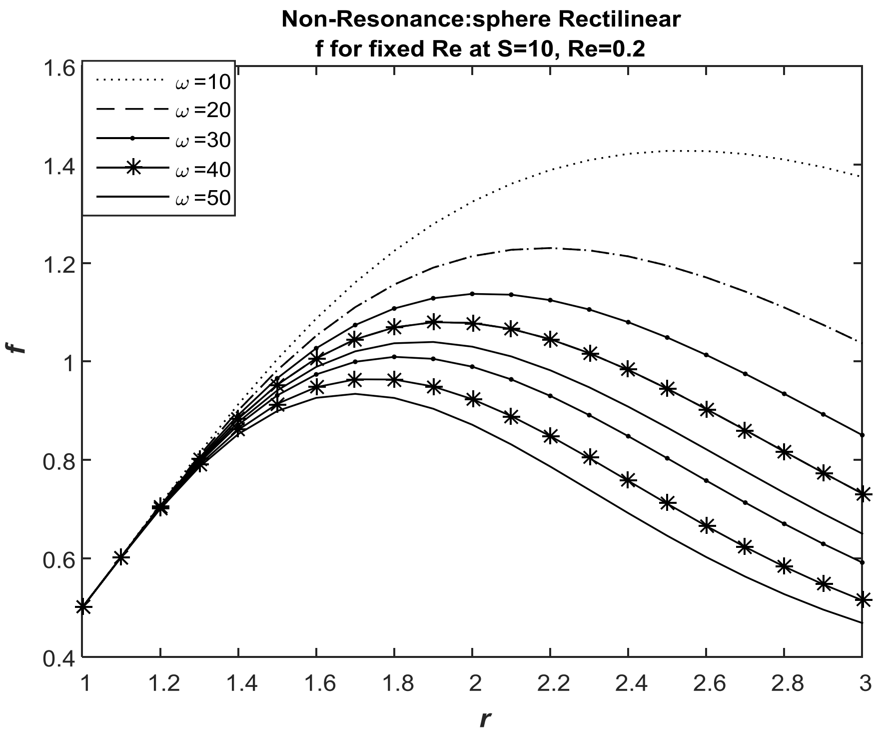
(b)

Fig 10.2 Stream function f at different Re for the case of

(a) resonance and (b) non-resonance



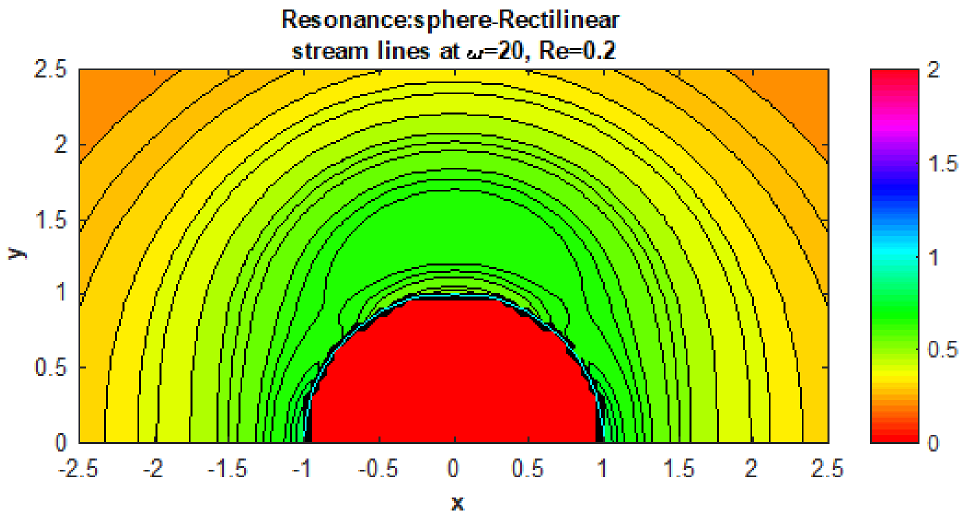
(a)



(b)

Fig 10.3 Stream function f at different ϖ for the case of

(a) resonance and (b) non-resonance



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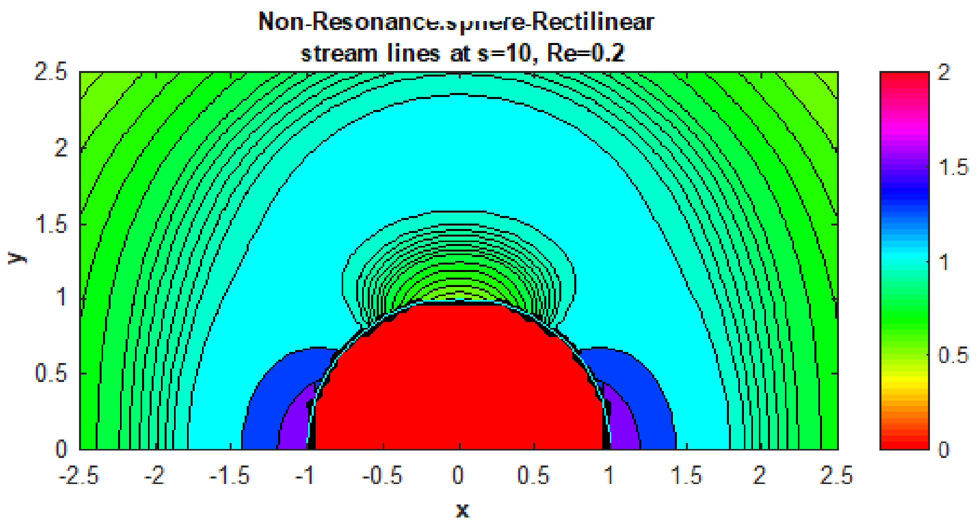
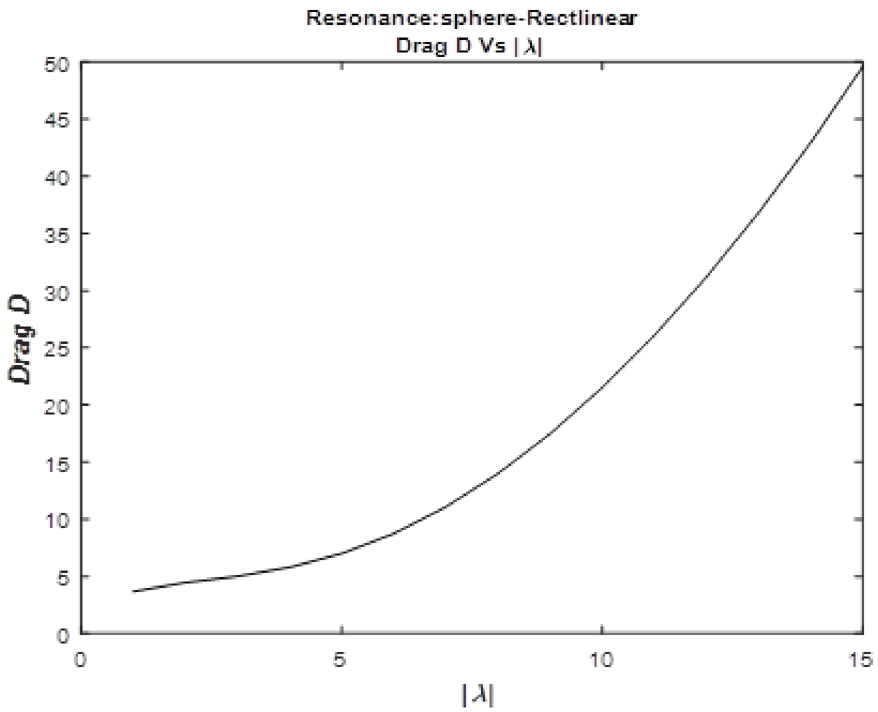


(b)

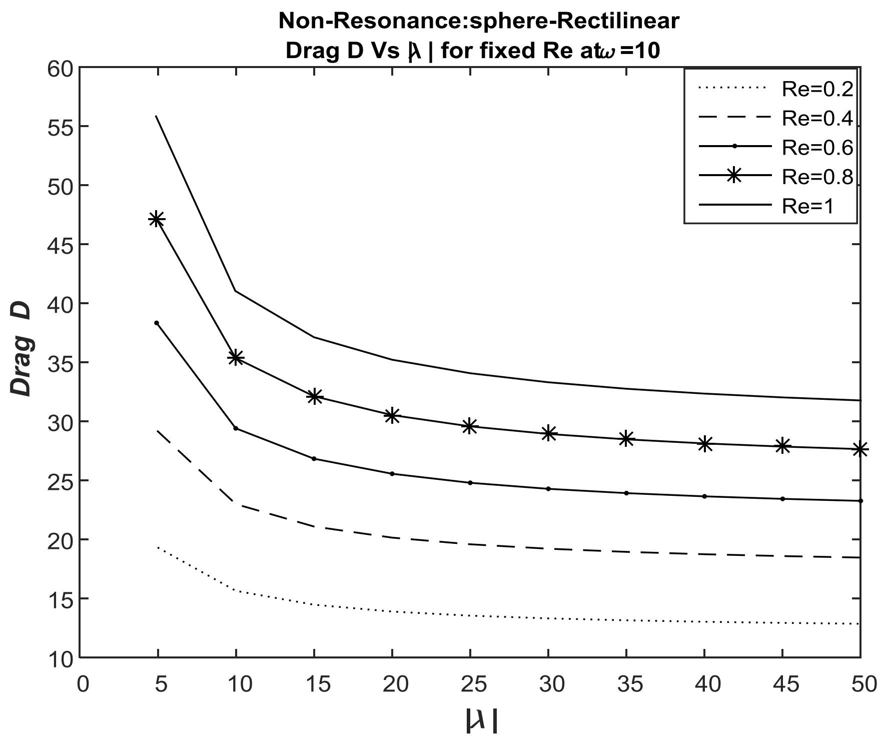
Fig 10.4 Stream line pattern for the case of (a) resonance and (b) non-resonance

10.5.2 Drag

When |λ| is fixed S is obtained as polynomial of degree 4 (eq. 10.34a). This will not contain Re for resonance case (see 10.34b). Hence for resonance we get only one curve as in Fig 10.5. When |λ| is fixed we get high values for Drag. When Re increases, Drag increases, in both the cases. But for the case of resonance, Drag will be reduced to minimum. (In non-resonance Drag is from 6 to 14, but for resonance it is in 4 to 7.5. Again in the case of non-resonance as |λ| increases, Drag decreases and reaches a constant value for fixed value of Reynolds number.

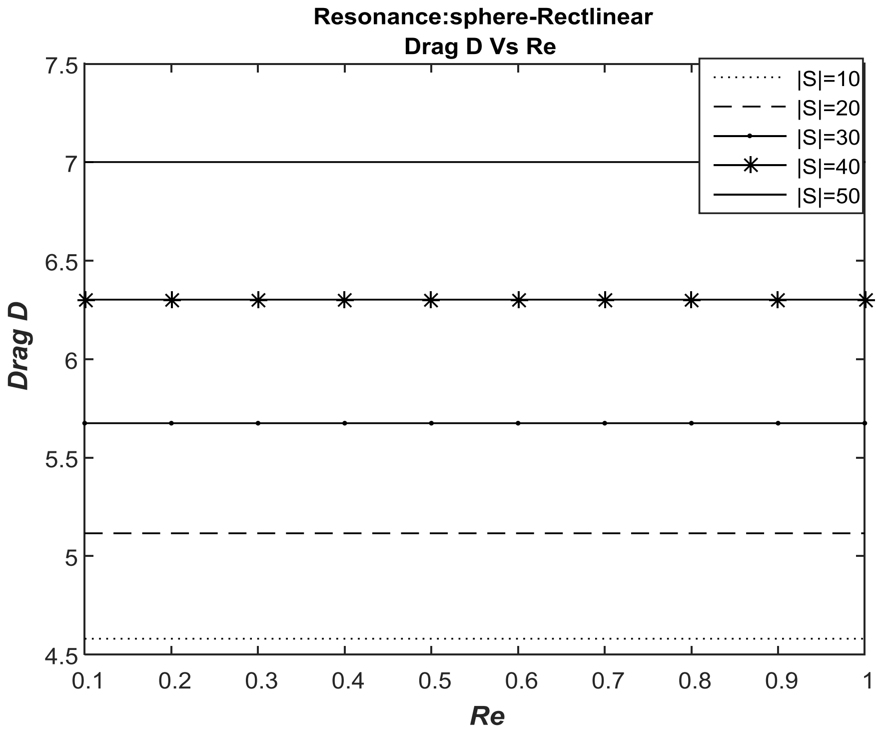


(a)

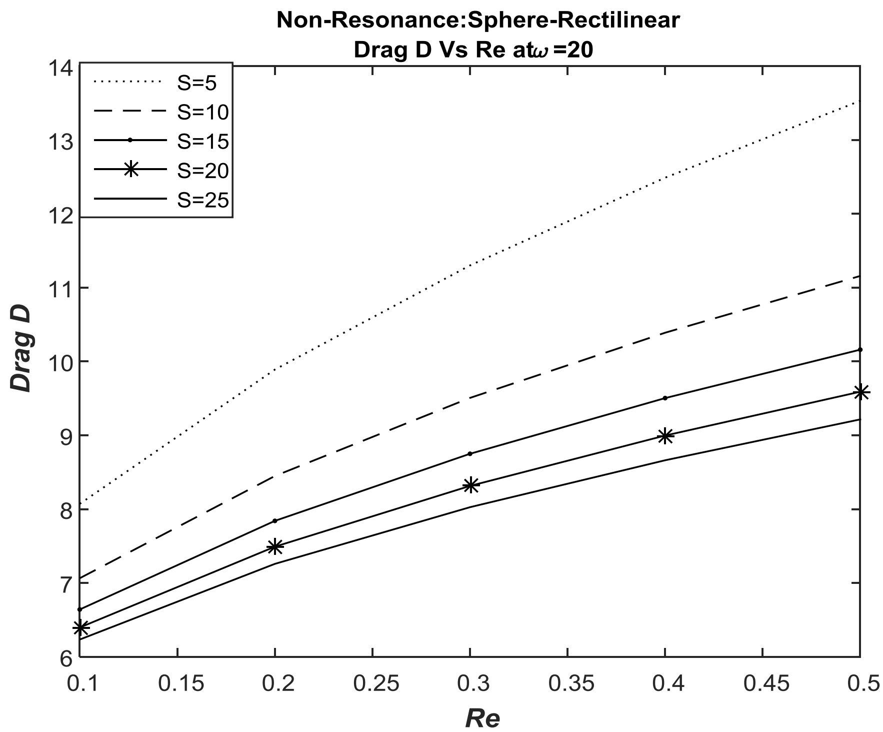


(b)

Fig 10.5 D Vs |λ| for the case of the case of (a) resonance and (b) non-resonance



(a)



(b)

Fig 10.6 D Vs Re for the case of (a) resonance and (b) non-resonance

10.6 Conclusions

Hence from above observations, we conclude that

1. For resonance, values of Stream function decrease and form small circulations on the cap of the sphere. For non-resonance stream function takes higher values and forms large circulations.

2. Drag is minimum for the case of resonance. This observation is very important to prepare fluids with minimum Drag or design machines to adjust for oscillations to meet resonance condition and then to get minimum Drag.

Chapter 11

Rotary oscillations of a Sphere in a Couple-stress fluid

This chapter concerns an analytic study of an incompressible Couple-stress fluid flow. The flow generated due to rotary oscillations of a sphere about its axis of symmetry. By taking Stokesian assumptions, nonlinear convective terms of motion are neglected and hence equations are made linear. In this situation, a special case, in which material constants satisfy a relation, is considered and the flow is technically termed as *resonance* flow. No-slip condition and hyper stick condition are used as boundary conditions. The mathematical expression for the velocity is obtained in terms of modified Bessel's functions. The Couple acting on the sphere is evaluated. The effect of physical parameters like Reynolds number and Couple stress parameter on the Couple due to oscillations is shown through graphs. *It is observed that Couple acting on the sphere is minimum for the resonance case*.

11.1 Introduction

The flow of non-Newtonian/Newtonian fluids to Sphere were studied by many authors. Frater (1967, 1968) studied the elastico-viscous fluid flows generated due to oscillations of sphere and evaluated Drag and damping force on the body. Ravindran (1972) studied simple oscillatory flow in polar fluids. Tekasakul et al. (1998, 2003) investigated axi-symmetric viscous flow generated due to rotary oscillations of arbitrary axi-symmetric bodies and obtained the solution by using green function technique. Ashmawy (2015, 2016) studied incompressible viscous fluid flow in which the author considered rotary oscillations of a composite sphere and Couple-stress fluid flow generated due to creeping motion of a rigid slip sphere wherein author obtained Drag acting on the slip sphere.

Lakshman Rao et al. (1971, 1981, 1983) have studied the Micro-polar fluid flows generated due to oscillations of different symmetric bodies like sphere and spheroid. These problems were attempted to obtained Drag or Couple on the symmetric body. In these studies, the effects of physical parameters on Drag or Couple were found. Lakshmana Rao et al. (1970, 1980) studied Micro-polar fluid flow past a sphere, Couple-stress fluid flow by analytically and computationally. In all these problems, authors found that a distinct flow exists which is technically termed as resonance. Till now this has not been investigated by many researchers. This case arises in Lakshmana Rao (1971, 1981, 1983), but resonance case was not attempted by the authors. Iyengar et al. (2001) studied rectilinear oscillations, rotary oscillations of approximate sphere in an incompressible viscous fluid and Micro-polar fluid respectively. Iyengar et al. (1993, 2004) studied Stokes flow of an incompressible Micro-polar fluid past an approximate sphere and oscillatory flow of a Micro-polar fluid generated due to rotary oscillations of two concentric spheres. Verma (1971) studied oscillatory fluid flow past a fixed porous sphere. Lai (1978) investigated elastic-viscous fluid flow generated due to rectilinear oscillations of sphere and evaluated Drag on the sphere. Anwar (2004) studied Micro-polar fluid flow of circular cylinder rotating and oscillating. Aparna (2012) examined oscillatory fluid flow of permeable sphere oscillating rotary oscillations in an incompressible Micro-polar fluid.

In all above problems, the case of resonance if exists was not studied. In this chapter we propose to investigate this case of resonance type flow, in Couple-stress fluids, due to rotary oscillations of a sphere about its axis of symmetry. The similar case investigated in Micro-polar fluid as chapter 6.

11.2 Basic Equations

The basic equations of an incompressible Couple stress fluid introduced by Stokes (1966) are given by:

$$div\, \bar{Q} = 0 \tag{11.1}$$

$$\rho\left(\frac{\partial \bar{Q}}{\partial \tau} + \bar{Q}.\nabla\bar{Q}\right) = -\nabla P - \mu\nabla\times\nabla\times\bar{Q} - \eta\nabla\times\nabla\times\nabla\times\nabla\times\bar{Q} \tag{11.2}$$

where $\bar{Q}$ is fluid velocity vector, ρ is density, P is pressure,τ is time and μ is viscosity coefficient.

By neglecting non linear convective terms from (11.2), we get

$$\rho\frac{\partial\bar{Q}}{\partial\tau} = -\nabla P - \mu\nabla\times\nabla\times\bar{Q} - \eta\nabla\times\nabla\times\nabla\times\nabla\times\bar{Q} \qquad (11.3)$$

For Couple stress fluids, the stress tensor T and Couple stress tensor M satisfy the following constitutive equations.

$$T = -PI + \lambda(\nabla_1 . Q)I + \mu(\nabla_1 Q + (\nabla_1 Q)^T) + \frac{1}{2}I\times(\nabla_1 . M) \qquad (11.4)$$

$$M = mI + 2\eta\nabla_1(\nabla_1\times Q) + 2\eta'[\nabla_1(\nabla_1\times Q)]^T \qquad (11.5)$$

11.3 Statement and Formulation of the Problem

A sphere of radius a is performing rotary oscillations with velocity $W_0 e^{i\sigma\tau}\boldsymbol{e_\phi}$ about its axis of symmetry in an infinite vat containing incompressible Couple stress fluid. A spherical coordinate system $(R,\theta,\emptyset)$ with base vectors $(\boldsymbol{e_R}, \boldsymbol{e}_\theta, \boldsymbol{e}_\emptyset)$ with origin at the center of the sphere and axis of symmetry along $\boldsymbol{e}_\emptyset$is considered. The flow is axially symmetric, hence it is independent of toroidal coordinate ϕ. Hence the fluid velocity will be in cross sectional plane of the sphere containing the base vectors $(\boldsymbol{e_R}, \boldsymbol{e}_\theta)$. The velocity is assumed in the form:

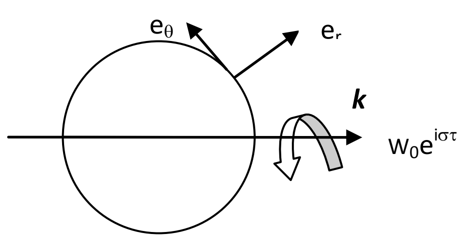


Fig11.1 Geometry of Rotary oscillations of a sphere

$$\boldsymbol{Q} = e^{i\sigma\tau}W(R,\theta)\boldsymbol{e}_\emptyset \qquad (11.6)$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$\mathrm{R} = ar, \mathrm{W} = \mathrm{W}_0 w, \mathbf{Q} = \mathbf{q}\mathrm{W}_0, \mathrm{P} = p\rho\mathrm{W}_0^2, \tau = \frac{\mathrm{at}}{W_0} \tag{11.7}$$

The following are non-dimensional parameters viz, ϖ is frequency parameter, S is Couple stress parameter and Re is Reynolds number for Couple-stress fluids.

$$\varpi = \frac{a\sigma}{W_0}, S = \frac{\mu a^2}{\eta},\ \mathrm{Re} = \frac{\rho \mathrm{W}_0 a}{\mu}, \mathrm{Re}.\varpi = \frac{\rho\sigma a^2}{\mu} \tag{11.8}$$

Using non dimensional scheme (11.7), (11.8) in (11.3) we get

$$Re.S\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{t}} = -Re.S.\nabla p - S\,\nabla \times\ \nabla \times \boldsymbol{q} -\ \nabla \times\ \nabla \times \nabla \times \nabla \times \boldsymbol{q} \tag{11.9}$$

Swirl ζ (moment of velocity) is defined as $\zeta = wh_3$ (11.10)

By the choice of velocity field (11.6) and swirl (11.10), the equations of motion (11.9) is reduced to

$$Re.S.i\varpi\left(\frac{\zeta}{h_3}e_\phi\right) = -Re.S.\nabla p + S\ \frac{1}{h_3}E^2\zeta\ \bar{e}_\emptyset - \frac{1}{h_3}E^4\zeta\ \bar{e}_\emptyset \tag{11.11}$$

Where $E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{cot\theta}{r^2}\frac{\partial}{\partial \theta}$ (11.12)

Let $\frac{\partial p}{\partial \emptyset} = 0$

By comparing coefficients of e_ϕ in (11.11), we get

$$(\mathrm{E}^4 - S\mathrm{E}^2 + Re.S.i\varpi)\zeta = 0 \tag{11.13}$$

This equation (11.13) can be written in the form as

$$(\mathrm{E}^2 - \lambda_1^2)\ (\mathrm{E}^2 - \lambda_2^2)\ \zeta = 0 \tag{11.14}$$

Where $\lambda_1^2 + \lambda_2^2 = S$ and $\lambda_1^2\lambda_2^2 = Re.S.i\varpi$ (11.15)

The solution for ζ if $\lambda_1 \neq \lambda_2$ in (11.14) is given Lakshmana Rao et al. (1971). The solution for ζ for the case, $\lambda_1 = \lambda_2 = \lambda$ cannot be obtained as a limiting case of

$\lambda_1 \to \lambda_2$. This case is referred to as "*Resonance*". This resonance occurs if the material coefficients follow the following relation in non-dimensional form:

$$2\lambda^2 = S = 4i\varpi Re \tag{11.16}$$

In this chapter we are interested in the solution for ζ for the case of resonance $\lambda_1 = \lambda_2 = \lambda$. In this case the equation for ζ is given by

$$\left(\mathrm{E}^2 - \lambda^2\right)^2 \zeta = 0 \tag{11.17a}$$

For the case of non-resonance

$$(\mathrm{E}^2 - \lambda_1^2)\,(\mathrm{E}^2 - \lambda_2^2)\,\zeta = 0 \tag{11.17b}$$

To match the boundary conditions,Swirl ζ is assumed in the form

$$\zeta = f(r)\sin^2\theta \tag{11.18}$$

Hence $\quad \mathrm{E}^2\zeta = D_s^2 f(r)\sin^2\theta \qquad$ (11.19)

Where $D_s^2 = \frac{d^2}{dr^2} - \frac{2}{r^2} \qquad$ (11.20)

Now we notice that the equations for ζ (11.17a) and (11.17b) are reduced to :

For Resonance case: $\qquad \left(D_s^2 - \lambda^2\right)^2 f(r) = 0 \qquad$ (11.21a)

For non-resonance case: $\left(D_s^2 - \lambda_1^2\right)\left(D_s^2 - \lambda_2^2\right) f(r) = 0 \qquad$ (11.21b)

11.3.1 Boundary Conditions

The non-dimensional swirl on the sphere Γ is given by

No-slip condition: $\qquad \zeta = \sin^2\theta \quad on\; r = 1$

Hyper-stick condition: $\qquad \frac{\partial\zeta}{\partial r} = 2\sin^2\theta \;\; on\; r = 1$

Hence the boundary conditions in terms of f at r=1 are obtained as:

By no-slip condition $\qquad f(1) = 1 \qquad$ (11.22a)

By hyper-stick condition $\quad f'(1) = 2 \qquad$ (11.22b)

11.4 Solution of the Problem

Since the equation for f is linear, the general solution for f is linear combination of individual solutions of factors in the differential operator. Hence f is considered as

$$f = A_1 f_1 + A_2 f_2 \tag{11.23}$$

where for the case of resonance:

$$\left(D_s^2 - \lambda^2\right) f_1 = 0 \text{ and } \left(D_s^2 - \lambda^2\right)^2 f_2 = 0 \tag{11.24a}$$

and for the case of non-resonance:

$$\left(D_s^2 - \lambda_1^2\right) f_1 = 0 \text{ and } \left(D_s^2 - \lambda_2^2\right) f_2 = 0 \tag{11.24b}$$

On solving (11.24a), the solution for f is obtained for resonance case as

$$f(\mathrm{r}) = \mathrm{A}_1 \sqrt{r} K_{\frac{3}{2}}(\lambda r) + \mathrm{A}_2 r^{\frac{3}{2}} K'_{\frac{3}{2}}(\lambda r) \tag{11.25a}$$

On solving (11.24b), the solution for f is obtained for non-resonance case as

$$f(\mathrm{r}) = \mathrm{A}_1 \sqrt{r} K_{\frac{3}{2}}(\lambda_1 r) + \mathrm{A}_2 \sqrt{r} K_{\frac{3}{2}}(\lambda_2 r) \tag{11.25b}$$

The following results are useful to note in the case of resonance and non-resonance.

$$D_s^2 f_1 = \lambda^2 f_1 \text{ and } D_s^2 f_2 = 2\lambda f_1 + \lambda^2 f_2 \tag{11.26a}$$

In case of non-resonance,

$$D_s^2 f_1 = \lambda_1^2 f_1 \text{ and } D_s^2 f_2 = \lambda_2^2 f_2 \tag{11.26b}$$

The constants A_1 and A_2 are obtained from the boundary conditions (11.22a) and (11.22b) as follows:

In the case of resonance

$$\begin{bmatrix} K_{\frac{3}{2}}(\lambda) & K'_{\frac{3}{2}}(\lambda) \\ K'_{\frac{3}{2}}(\lambda) & \left(1 + \frac{9}{4\lambda^2}\right) K_{\frac{3}{2}}(\lambda) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2\lambda} \end{bmatrix} \tag{11.27a}$$

In the case of non-resonance, the conditions for A_1 and A_2 are given by

$$\begin{bmatrix} K_{\frac{3}{2}}(\lambda_1) & K_{\frac{3}{2}}(\lambda_2) \\ \lambda_1 K'_{\frac{3}{2}}(\lambda_1) & \lambda_2 K'_{\frac{3}{2}}(\lambda_2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} \tag{11.27b}$$

Hence from (11.27a) and (11.27b) we can calculate A_1andA_2.And hence ζ is known.

11.4.1 Couple acting on the Sphere of radius a

Couple acting on sphere $C^* = 2\pi a^3 \int_0^\pi (T^*_{r\emptyset} \sin^2\theta)\,|_{R=a}\, d\theta$ (11.28)

For Couple-stress fluids, the constitutive equations for stress and Couple stresses are given by (11.4) and (11.5) as

$$T = -PI + \lambda(\nabla_1 . Q)I + 2\mu E + \frac{1}{2} I \times (\nabla_1 . M)$$

and $M = mI + 2\eta\nabla_1(\nabla_1 \times Q) + 2\eta'[\nabla_1(\nabla_1 \times Q)]^T$

$$E = \frac{1}{2}(\nabla_1 \bar{Q} + \nabla_1 \bar{Q}^T)$$

$$= \begin{bmatrix} \frac{\partial U}{\partial R} & \frac{1}{2R}\left[\frac{\partial U}{\partial \theta} + R\frac{\partial V}{\partial R} - V\right] & \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial U}{\partial \emptyset} + R\frac{\partial W}{\partial R} - W\right] \\ \frac{1}{2R}\left[\frac{\partial U}{\partial \theta} + R\frac{\partial V}{\partial R} - V\right] & \frac{1}{R}\left[\frac{\partial V}{\partial \theta} + U\right] & \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial V}{\partial \emptyset} + \frac{\partial W}{\partial \theta} - WCot\theta\right] \\ \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial U}{\partial \emptyset} + R\frac{\partial W}{\partial R} - W\right] & \frac{1}{2R}\left[\frac{1}{Sin\theta}\frac{\partial V}{\partial \emptyset} + \frac{\partial W}{\partial \theta} - WCot\theta\right] & \frac{1}{R}\left[\frac{1}{Sin\theta}\frac{\partial W}{\partial \emptyset} + VCot\theta + U\right] \end{bmatrix}$$

(11.29)

For this present problem, we get strain rate tensor as,

$$E = \begin{bmatrix} 0 & 0 & \frac{1}{2}\left[\frac{\partial W}{\partial R} - \frac{W}{R}\right] \\ 0 & 0 & \frac{1}{2R}\left[\frac{\partial W}{\partial \theta} - WCot\theta\right] \\ \frac{1}{2}\left[\frac{\partial W}{\partial R} - \frac{W}{R}\right] & \frac{1}{2R}\left[\frac{\partial W}{\partial \theta} - WCot\theta\right] & 0 \end{bmatrix} \tag{11.30}$$

And M is given by

$$M_{RR} = m + \frac{2\eta + 2\eta'}{sin\theta}\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial \zeta}{\partial \theta}\right) \tag{11.31}$$

$$M_{R\theta} = -\frac{2\eta}{sin\theta}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \zeta}{\partial R}\right) + \frac{2\eta'}{R^2}\left[\frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{1}{sin\theta}\frac{\partial \zeta}{\partial \theta}\right) + \frac{1}{sin\theta}\frac{\partial \zeta}{\partial R}\right] \tag{11.32}$$

$$M_{R\emptyset} = 0 \tag{11.33}$$

$$M_{\theta R} = 2\eta \frac{1}{R^2}\left[\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{1}{sin\theta}\frac{\partial\zeta}{\partial\theta}\right) + \frac{1}{sin\theta}\frac{\partial\zeta}{\partial R}\right] - \frac{2\eta^{'}}{sin\theta}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\zeta}{\partial R}\right) \tag{11.34}$$

$$M_{\theta\theta} = m + \frac{2\eta+2\eta^{'}}{R^2}\left[\frac{\partial}{\partial\theta}\left(-\frac{1}{sin\theta}\frac{\partial\zeta}{\partial R}\right) + \frac{1}{Rsin\theta}\frac{\partial\zeta}{\partial\theta}\right] \tag{11.35}$$

$$M_{\theta\emptyset} = 0 \tag{11.36}$$

$$M_{\emptyset R} = 0 \tag{11.37}$$

$$M_{\emptyset\theta} = 0 \tag{11.38}$$

$$M_{\emptyset\emptyset} = m + \frac{2\eta+2\eta^{'}}{Rsin\theta}\left[-\frac{cot\theta}{R}\frac{\partial\zeta}{\partial R} + \frac{1}{R^2}\frac{\partial\zeta}{\partial\theta}\right] \tag{11.39}$$

Hence we get

$$T_{R\emptyset} = \mu\left[\frac{\partial W}{\partial R} - \frac{W}{R}\right] + \eta\frac{1}{R^2 sin\theta}\left[2E^2\zeta - \frac{1}{R}\frac{\partial^3\zeta}{\partial R\partial\theta^2} + \frac{cot\theta}{R}\frac{\partial^2\zeta}{\partial R\partial\theta} - R\frac{\partial^3\zeta}{\partial R^3} - 2\frac{\partial^2\zeta}{\partial R^2}\right] \tag{11.40}$$

Using non dimensional scheme (11.7), (11.8) in (11.40) on the boundary r=1 we get

$$T_{r\emptyset} = -\frac{\mu W_0}{Sa^2} f^{'''} sin\theta \tag{11.41}$$

The Couple acting on the sphere (after deleting the factor $e^{i\varpi t}$) is obtained as:

$$C^* = -\frac{8\pi\mu a\, W_0}{3S} f^{'''} \tag{11.42}$$

Non dimensional Couple C is obtained by dividing C^* by $4\pi\mu a W_0$

$$C = -\frac{2}{3S} f^{'''}(1) \tag{11.43}$$

In the resonance case, the Couple is given by

$$C = -\frac{2}{3S}\left[2\lambda^2 + \lambda\left(\frac{1}{2}K_{\frac{3}{2}}(\lambda) + \lambda K_{\frac{3}{2}}^{'}(\lambda)\right)A_2\right] \tag{11.44a}$$

In the non-resonance case, the Couple is given by

$$C = -\frac{2}{3S}\left[\lambda_1^2\left(\frac{1}{2}K_{\frac{3}{2}}(\lambda_1) + \lambda_1 K_{\frac{3}{2}}^{'}(\lambda_1)\right)A_1 + \lambda_2^2\left(\frac{1}{2}K_{\frac{3}{2}}(\lambda_2) + \lambda_2 K_{\frac{3}{2}}^{'}(\lambda_2)\right)A_2\right] \tag{11.44b}$$

11.4.2 Pressure

By the choice of velocity field in (11.6), we get

$$\mathbf{q}.\nabla \boldsymbol{q} = -\frac{\zeta^2}{h_3^2}(sin\theta \bar{e}_r + cos\theta \bar{e}_\theta) \quad (11.45)$$

By comparing components along $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ in equation (11.45), pressure is obtained as follows.

$$Re.S\frac{\partial p}{\partial r} = \frac{\zeta^2}{h_3^2}sin\theta \quad (11.46)$$

$$Re.S\frac{\partial p}{\partial \theta} = r\frac{\zeta^2}{h_3^2}cos\theta \quad (11.47)$$

We know that $dp = \nabla p.\boldsymbol{dr} = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial \theta}d\theta$ (11.48)

Substituting (11.46) and (11.47) in (11.48) and integrating we get

Pressure $p = \frac{1}{3Re.S}\frac{f^2}{r}\sin^3\theta$ (11.49)

11.5 Results and Discussions

For resonance case, λ value cannot be taken randomly. In the case of resonance, the values of λ are obtained from (11.11) by solving the following equation for x.

$$x^2 - Sx + i\varpi ReS = 0$$

Then the roots of this equation gives the values of λ_1^2 and λ_2^2 and for resonance case, the values of λ are obtained if

$S = 4i\varpi.Re$ and $\lambda = \sqrt{x} = \sqrt{\frac{S}{2}}$ (11.50)

When $|\lambda_1|$ is fixed, S can be obtained from the equation:

$S = 4iR_0 = 2\lambda^2$ for resonance (11.51a)

$R_0^2(R_0^2 - L)S^4 - L^3(L + 2R_0^2)S^2 + L^4 = 0,$ for non-resonance (11.51b)

Here with $R_0 = Re.\varpi$ and $L = |\lambda_1^4|$

11.5.1 Couple

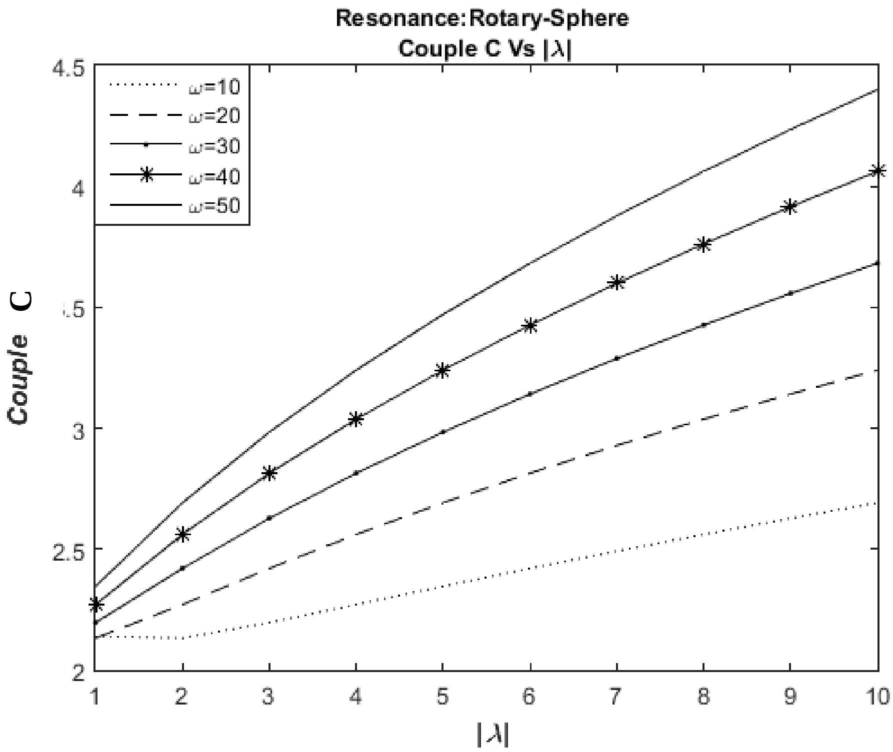
The coefficient of friction C_f is evaluated and presented in the form of graphs for various values of $Re, S\ and\ \varpi$. From the figures we can conclude that for the case of Resonance, the coefficient of friction is lesser for the case of Resonance than the case of non-resonance when |λ| is fixed. In this case, if $Re\ and\ \varpi$ are known then S and λ are known. But in the case of non-resonance all parameters can be chosen independently. From Fig 11.2, we observe the following. In the case of non-resonance, for small values of |λ|, Couple is high and as |λ| increases, Couple becomes constant for a particular value of ϖ. But the values of C_f are less than the corresponding C_f values of Resonance.

We can clearly observe from Fig 11.3 that coefficient of friction C_f is lower for the case of resonance than in the case of non-resonance. Again we see that effect of |λ| is dominating all other parameters. For fixed value of | λ |, C_f ranges upto 20 in non-resonance case while it ranges upto 10 in resonance case. Hence *we conclude that the case of resonance reduces the Couple on the body*. From Fig 11.4, we note that as Re increases, C_f also increases for non-resonance case. In Fig 11.4, |λ | is not fixed. Hence values of Re, S and ϖ are taken randomly and λ values are found.

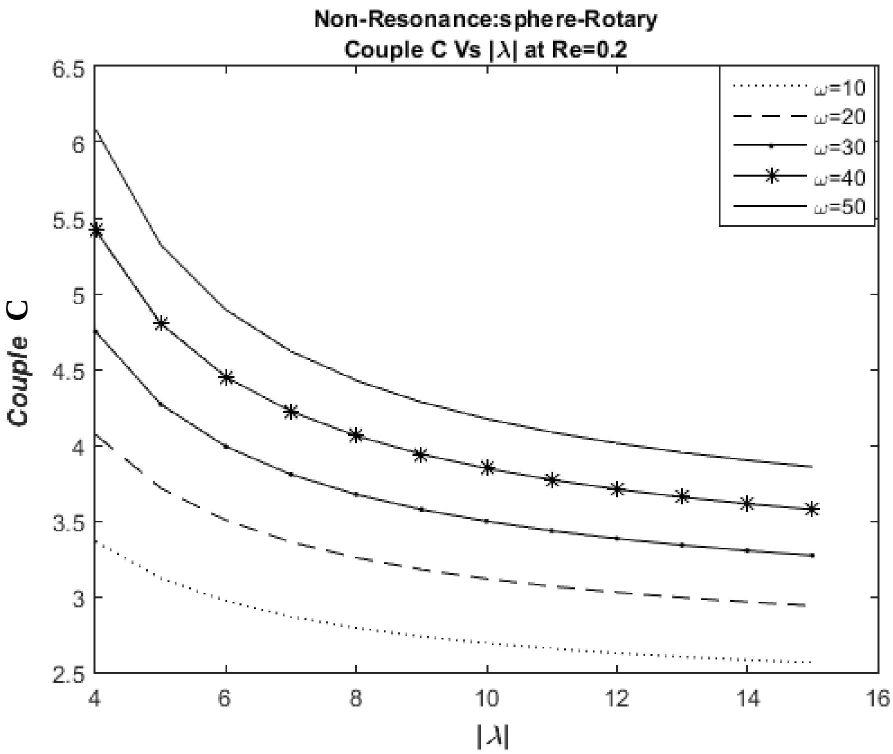
11.5.2 Velocity

From Fig 11.5 and 11.6, we observe that, in the case resonance, velocity vanishes near to r=5 and takes negative values near to sphere and then increase and then vanish after some distance. But in the case of non-resonance it f vanishes near to r=7 and variation in f from positive to negative and negative to positive values is not drastic. Hence we conclude that fluid will not be disturbed much far from the body only near to the body we can see high velocity for the case of resonance. In the case of non-resonance effect of oscillations will be present to more distances than the case of resonance.

From Fig 11.7, we see that for the case of resonance, circulations near to sphere at the pole are present with low values (colour does not change). In the case of non-resonance, circulations near to pole we can observe with more positive values (colour changes).

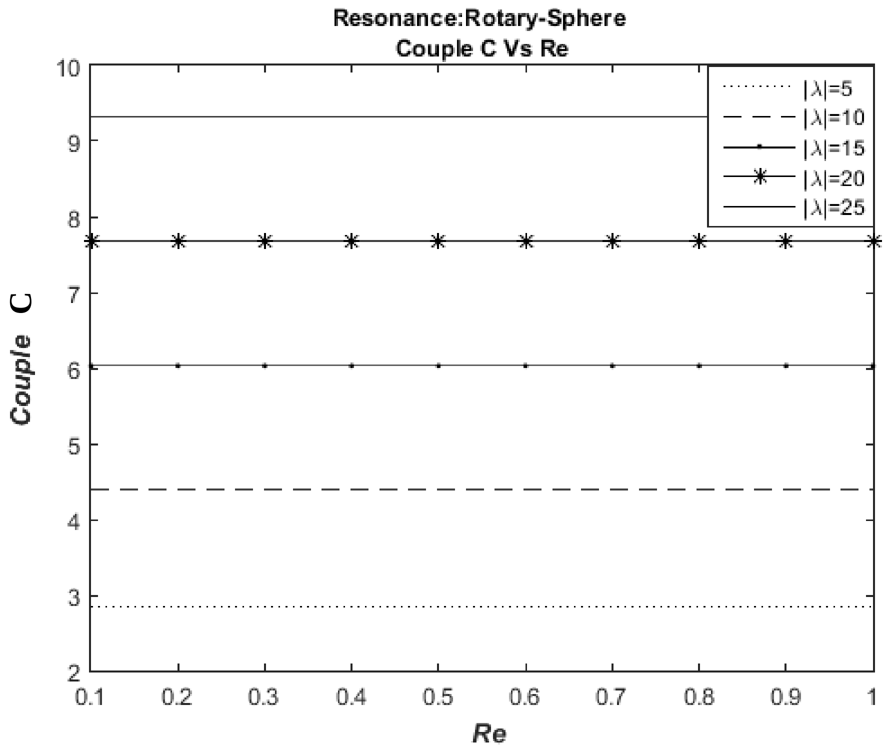


(a)

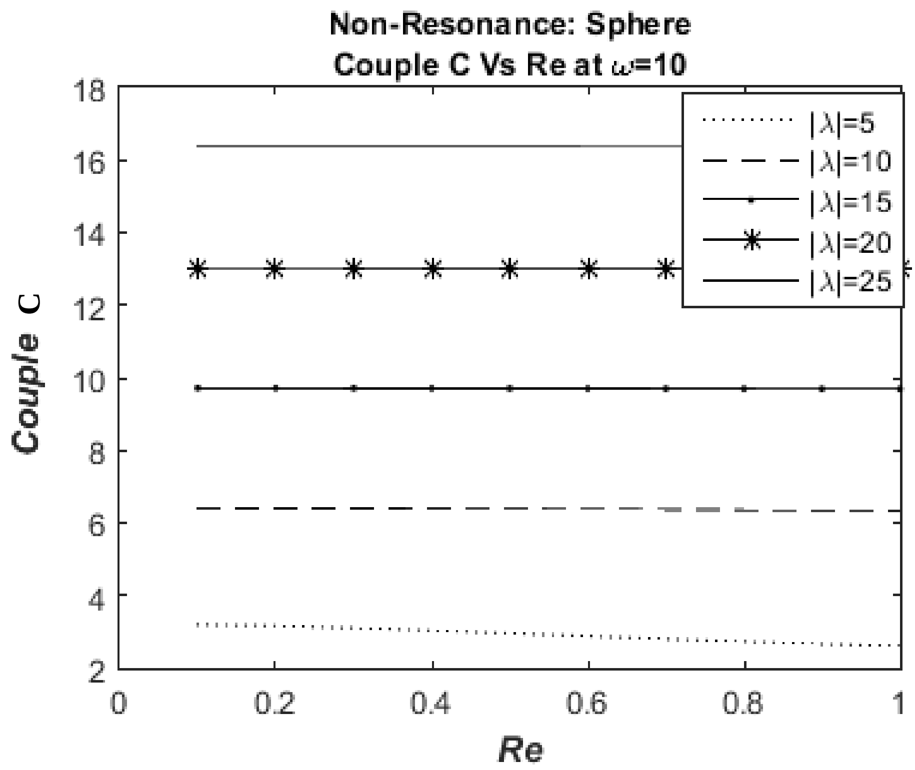


(b)

Fig 11.2 C_f Vs |λ| for the case of (a) resonance and (b) non-resonance

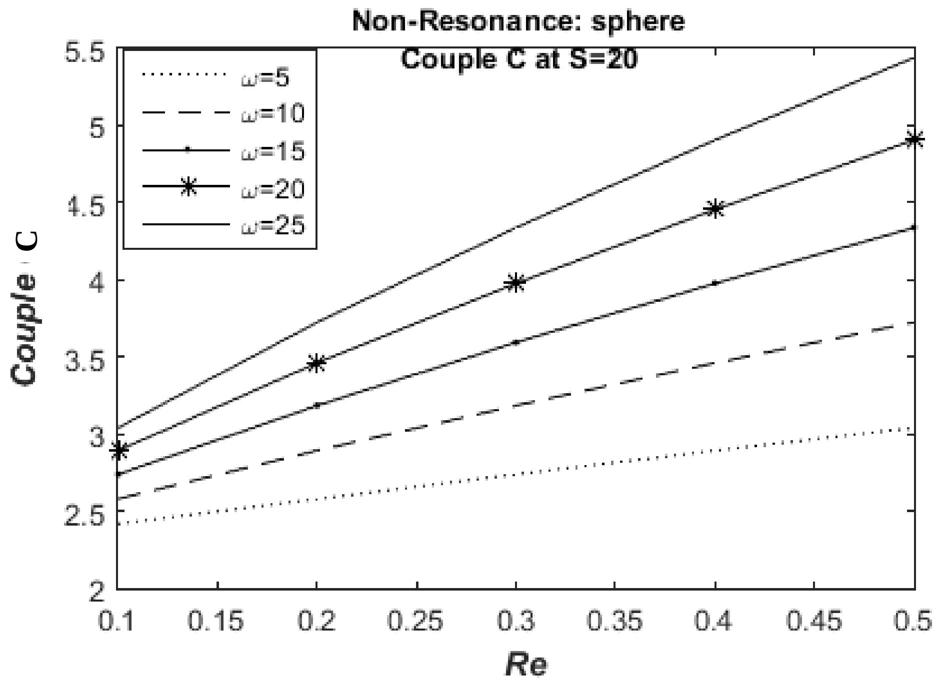


(a)

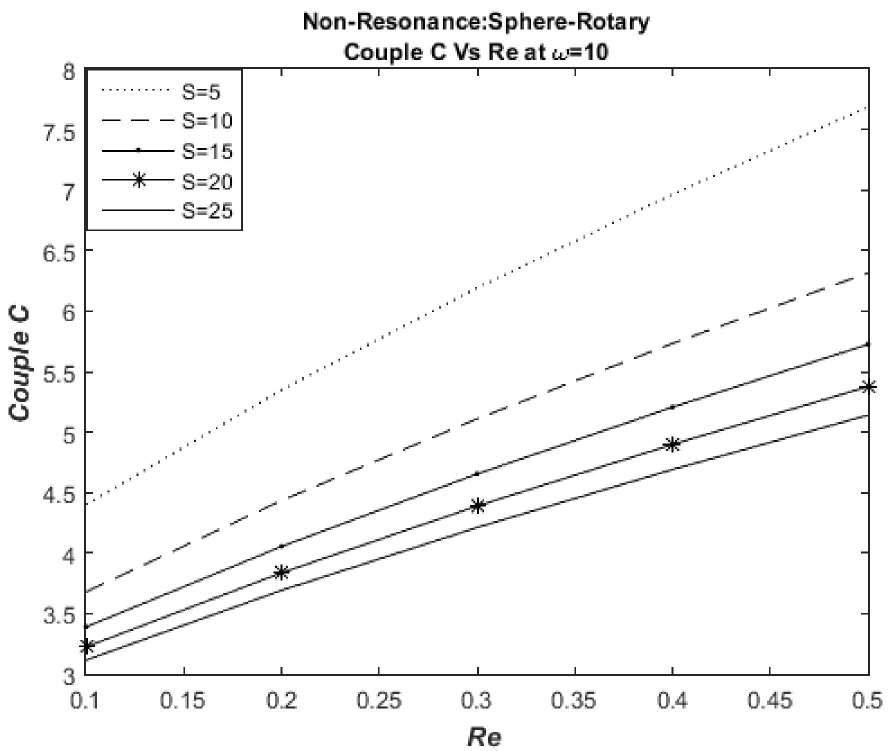


(b)

Fig 11.3 C_f Vs Re for the case of (a) resonance and (b) non-resonance

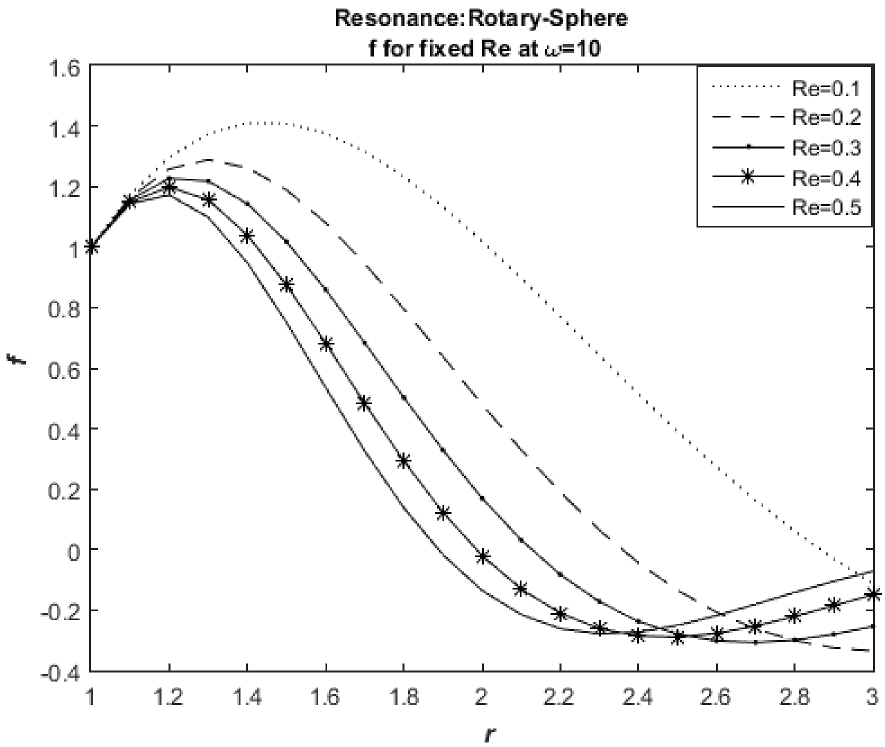


(a)

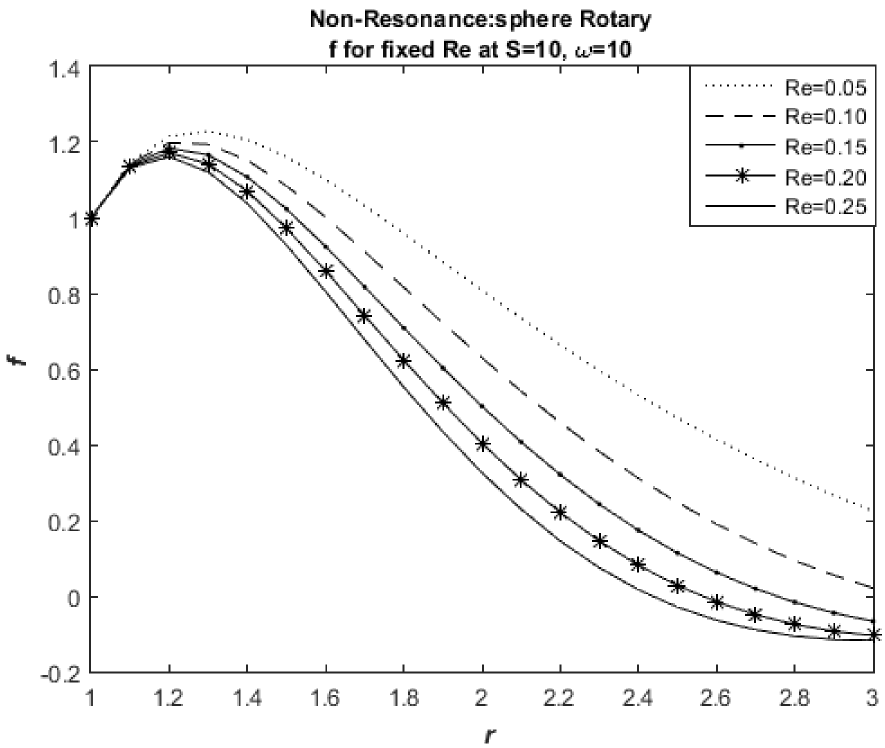


(b)

Fig 11.4 C_f Vs Re for the case of non-resonance (a) at different ϖ values and (b) at different S values

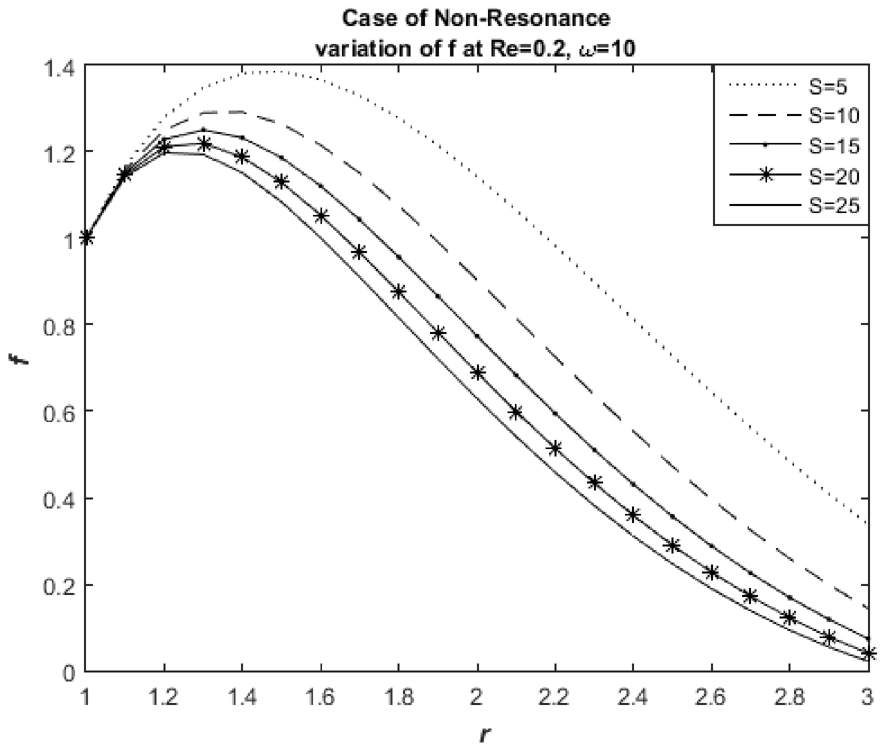


(a)

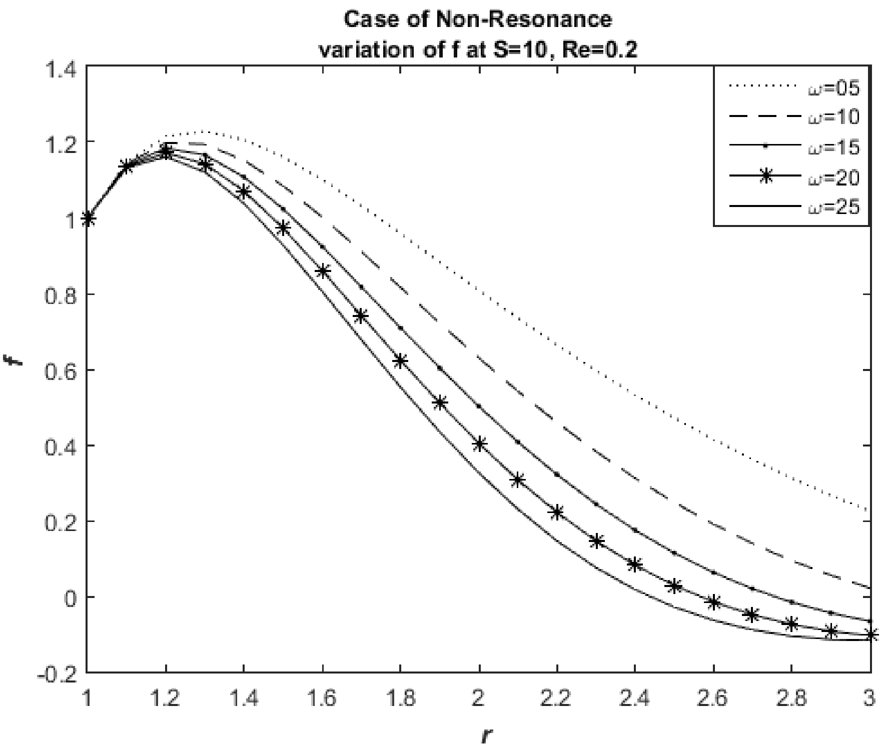


(b)

Fig 11.5 Velocity f at different Re for the case of (a) resonance and (b) non-resonance



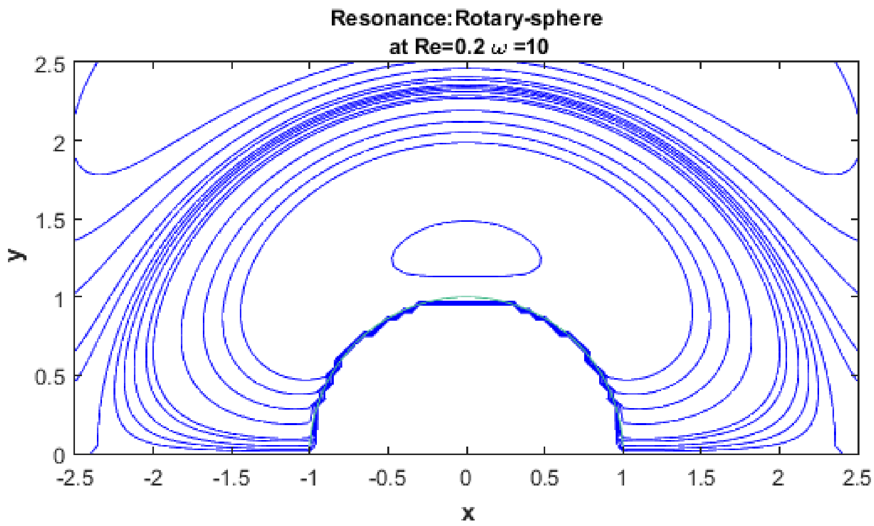
(a)



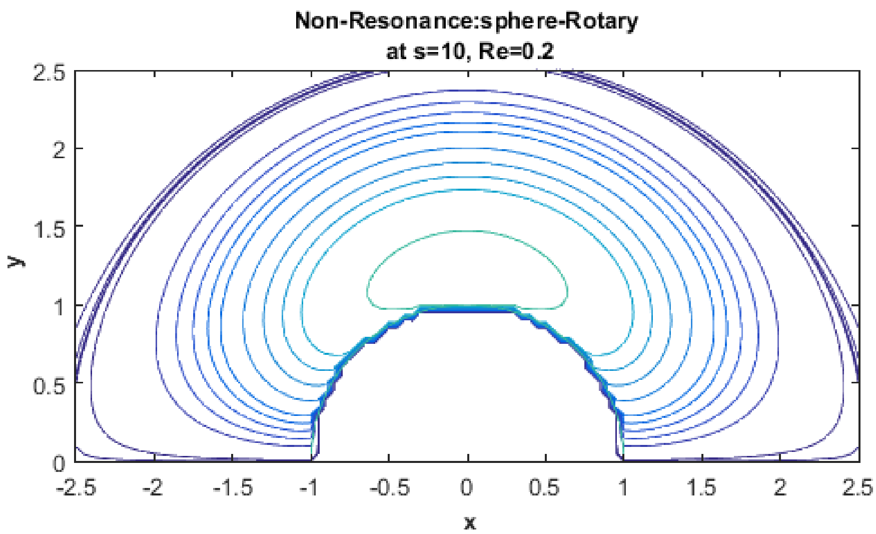
(b)

Fig 11.6 Velocity f for the case of non-resonance at different

(a) S values and (b) ϖ values



(a)



(b)

Fig 11.7 Velocity contours for the case of (a) resonance and (b) non-resonance

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11.6 Conclusions

In this chapter, we derived the velocity in term of swirl for the se of resonance and non-resonance. Our important observation is that Couple stresses offer less Couple on the body when resonance occurs.

Part – IV

Conclusions

Chapter 12

Conclusions

The thesis aims at analyzing the flows generated due to oscillations of symmetric bodies (cylinder or sphere) about a symmetric axis in fluids with Couple stresses, namely micro-polar fluids and Couple stress fluids. These two theories are developed independently. The theory of Couple stress fluid does not require independent rotation vector as in the case of micro-polar fluids. The flows generated in both the fluids appear same when we see the equations of motion. But when they are solved only, the differences and similarities can be known.

We observe the following similarities and differences in the flows of these fluids.

i) Rectilinear oscillations of cylinder resonance yields less Drag than non-resonance for micro-polar fluids. Similarly resonance offers less Drag in the case of Couple stress fluids.

ii) In the case of rotary oscillation of cylinder, resonance offers less Couple for Couple stress fluids. The same observations we find for micro-polar fluids also

iii) In the case of longitudinal oscillations of a cylinder, resonance offers less Skin friction for both the fluids.

iv) In the case of rectilinear oscillations of a sphere, for resonance micro-polar fluids offer less Drag. Similarly Couple fluids offer less Drag for non-resonance case.

v) In the case of rotary oscillations of a sphere, for resonance Couple stress fluids offer less Couple. Whereas micro-polar fluids offer less Couple for the case of non-resonance.

From the above observations, we note that the case of resonance has distinct behavior for problem to problem.

Main observation in the case of Couple stress fluids is that, in the case of resonance, the parameters are related by a simple equation given by

$$S = 4iR_0\varpi$$

This means Couple stress parameter takes imaginary values. If it is to take real values then either Re or ϖ must be imaginary, which is not correct. Here we are unable to give a physical meaning to this situation. This case of imaginary values to a physical parameter will not occur for micro-polar fluids. Hence we feel micro-polar fluid theory is more realistic, though some other believe that Couple stress fluids are more realistic.

The flows generated due to external applied magnetic field for similar situation of flows is one major area where we can pay very good attention. The problems related to heat transfer and mass transfer are also of interest for the case of resonance flows. These problems will have very high value for industrial applications.

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List of Publications

Considerable material presented in this thesis has been published / accepted / commnicated for publication in the form of following research papers:

PAPERS PUBLISHED / ACCEPTED

i. Resonance Flow Due to Rectilinear Oscillations of a Circular Cylinder In A Micro-polar Fluid, *59th proceedings of ISTAM*, 2014.

ii. Resonance Type Flow Due to Rectilinear Oscillations of a Sphere In A Micro-polar Fluid, *Journal of Physics: Conference Series* 662 (2015) 012015.(IOP)

iii. Resonance Type Flow Due to Rotary Oscillations of a Sphere In A Micro-polar Fluid, *Procedia Engineering*, 127 (2015), 1323-1329, (Elsevier).

iv. Couple-stress fluid flow due to rectilinear oscillations of a circular cylinder: case of resonance, *Lecture Notes in Mechanical Engineering* (2019), 978-981-13-1903-7, https://doi.org/10.1007/978-981-13-1903-7_65(Springer).

v. Longitudinal oscillations of a circular cylinder in a micro-polar fluid: case of resonance. (Accepted for Publication in 'Sadhana - Academy Proceedings in Engineering Sciences'), (Springer).

PAPERS COMMUNICATED

vi. Rotary oscillations of a circular cylinder in a micro-polar fluid-material resonance. (Communicated to a peer reviewed Journal 'Journal of Mathematical Analysis and Applications' Elsevier)

vii. Couple-stress fluid flow due to rotary oscillations of a sphere: case of resonance. (Communicated to a peer reviewed Journal 'Applied Mathematical Modelling', Elsevier).