CONE NONNEGATIVITY OF GENERALIZED INVERSES OF LINEAR OPERATORS

A

Thesis

submitted in partial fulfilment of the requirements

for the award of the degree of

DOCTOR OF PHILOSOPHY

by

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APPROVAL SHEET

This Dissertation Work entitled **"CONE NONNEGATIVITY OF GENERALIZED INVERSES OF LINEAR OPERATORS"**, by **Mrs. ARCHANA BHAT (Roll No. 718068)** is approved for the degree of **DOCTOR OF PHILOSOPHY**

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This is to certify that the work presented in the thesis entitled **"CONE NON-NEGATIVITY OF GENERALIZED INVERSES OF LINEAR OPERATORS"**, is a bonafide work done by me under the supervision of **Dr. R.S. Selvaraj** and **Dr. T. Kurmayya**. It has not been submitted elsewhere for the award of any degree.

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CERTIFICATE

This is to certify that the thesis entitled, **"CONE NONNEGATIVITY OF GENERALIZED INVERSES OF LINEAR OPERATORS"**, submitted in partial fulfilment of requirement for the award of degree of **DOCTOR OF PHILOSOPHY** to National Institute of Technology Warangal - 506 004, is a bonafide research work done by **Mrs. ARCHANA BHAT (Roll No. 718068)** under our supervision. The contents of the thesis have not been submitted elsewhere for the award of any degree.

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Archana Bhat

Abstract

Keywords: Linear operators, Closed operators, Bounded linear operators, Moore-Penrose inverse, Drazin inverse, Group inverse, Proper splitting, $B_\dagger$-splitting, Gram operator, Acute cone.

The main objective of this research work is to study the results related to characterization of Drazin monotonicity, various splitting methods and nonnegativity of Moore-Penrose inverse of unbounded Gram operators. The results of this thesis are primarily motivated by theoretical considerations. This thesis is divided into four chapters.

Chapter 1 is introductory in nature. It hits the bare essentials of the introduction part with historical development. The theory on linear operators is briefly outlined. The concepts on generalized inverses, ordering on a vector space by a cone are presented. The theory on splitting of operators and iterative schemes are explored with literature survey.

Chapter 2 deals with the Drazin monotonicity of bounded linear operators over ordered Banach space. The methods of index splitting are generalized to bounded linear operators over Banach space with index k and iterative method is established for computing the solution of non-invertible operator equation. The key aspect of this chapter is to characterize the cone nonnegativity conditions for Drazin inverses (that is, Drazin monotonicity) of operators over ordered Banach spaces.

The central theme of the Chapter 3 is to study $B_\dagger$-splitting of operators over ordered Hilbert space. At first, the discussion on different

splitting methods, particularly the proper splitting of non-invertible operators over Hilbert space is considered. This gives the foundation to the study on $B_\dagger$-splitting. The main result involves the existence of $B_\dagger$-splitting and construction methods to obtain $B_\dagger$-splitting for a singular operator over ordered Hilbert space.

Chapter 4 treats the unbounded Gram operators over ordered Hilbert space. In particular, closed and densely defined operators are considered. Characterization of the cone nonnegativity of unbounded Gram operators is provided and illustrated with examples. This characterization involves acuteness and obtuseness of the cone in real ordered Hilbert space.

Finally, the summary of the thesis is presented with a list of references.

Contents

List of Notations

Sets

$\in$	Element membership
$\subseteq$	Subset
$\emptyset$	Empty set
$\cap$	Set intersection

Spaces

$\mathbb{N}$	The space of all natural numbers
$\mathbb{R}$	The space of all real numbers
$\mathbb{R}^n$	The space of all real vectors with n co-ordinates
$\mathbb{R}^n_+$	The set of all nonnegative vectors in $\mathbb{R}^n$
$\mathbb{R}^{m\times n}$	The set of all real matrices of order $m \times n$
$\mathbb{R}^{n\times n}$	The set of all real square matrices of order n
$H, H_1, H_2, ...$	Hilbert spaces
$X, Y, ...$	Normed linear spaces
$X \otimes Y$	Cartesian product of X and Y
$\overline{M}$	Closure of a subset M
$M^\perp$	Orthogonal complement of M
C	A cone
C°	Interior of a cone C
C^*	The dual cone of C

Operations

$\langle .,. \rangle$	Euclidean inner product
$\oplus$	Direct sum
$\oplus^{\perp}$	Orthogonal direct sum

Operators and Matrices

I	The identity operator
$P_{M,L}$	Projection onto the space M along L
P_M	Orthogonal projection onto M
S, T, U, V	Operators on vector spaces
$\mathcal{L}(X, Y)$	Class of linear operators from X into Y
$\mathcal{L}(X)$	Class of linear operators from X into X
$\mathcal{B}(X, Y)$	Class of bounded linear operators from X into Y
$\mathcal{C}(X, Y)$	Class of closed linear operators from X into Y
δ_{nm}	Kronecker delta function
$rank(S)$	Rank of the matrix S
$ind(S)$	The index of an operator S
$S \geq 0$	Operator S is nonnegative
$D(S)$	Domain of a linear operator S
$Car(S)$	Carrier of S
$N(S)$	The null space of S
$R(S)$	The range space of S
$r(S)$	The spectral radius of S
S^*	The adjoint of S on Hilbert space
S^*S	Gram operator of S
S^{-1}	The inverse of S
$S^{\dagger}$	The Moore-Penrose inverse of S
$S^{\#}$	The group inverse of S
S^D	The Drazin inverse of S

Chapter 1

Introduction

"Life is a mathematical equation. In order to gain the most, you have to know how to convert negatives into positives"

- Anonymous

1.1 General Introduction

Abstract mathematical modelling can be formed for many real world problems which take the operator equation of the kind

$$Sx = b$$

where S is a bounded (or unbounded but closed) linear operator between certain vector spaces and b is the element in the co-domain of S. The important concepts in solving these system of operator equations are existence, uniqueness, stability and method of finding a solution.

Finding the positive solution makes sense in realm. To study the positive solution and it's existence, positively invertible operators are of interest. If $S : X \to Y$ be a positively invertible operator between the ordered normed spaces and b be a positive vector, then it is easy to obtain positive solution of the system $Sx = b$. But, it may turn out that the operator S may not be invertible, or the system is inconsistent, or even if the system is consistent, it may be difficult to find the exact solution of the equation. In such cases, one may look for the least squares solution with minimal norm. To find the solution of consistent system, the most appropriate computational method is iterative method which is obtained by splitting of operators.

The theory of nonnegative splitting is useful tool in convergence analysis of the iterative scheme and provides interesting comparison results. Nonnegative matrices and operators have applications in convex optimization, linear complementary problems and other mathematical problems. The convergence factor of the iterative scheme depends on the fact that spectral radius of iteration matrix is less than one. It is proved that, spectral radius of iteration matrix is less than one if and only if the matrix is monotone. Invertible matrices are monotone if and only the inverse of matrix is nonnegative. In addition, monotonicity of an operator is equivalent to cone nonnegativity of inverse of an operator. Hence, it is important to study the characterizations of cone nonnegativity of generalized inverses of operators.

This chapter gives a brief survey of the pertinent literature, some fundamental concepts, definitions and preliminary results that are used in the thesis. In section 1.2, we study cones and ordering on a vector space by a cone. In section 1.3, we collect the results on linear operators. The clear picture on bounded and unbounded linear operators are given in the subsections 1.3.2 and 1.3.1, respectively. Section 1.4 contains generalized inverses and their properties with geometrical interpretation. In particular, we discuss about Moore-Penrose inverses, Drazin inverses and group inverses with their historical development. In section 1.5, we provide a survey on iterative methods using splitting, monotonicity of matrices and monotonicity of operators. In section 1.6, we present an outline of the thesis.

1.2 Cones and Ordering

There are several ways of distinguishing elements by certain ordering properties. For instance, vectors by their length, functions by smoothness properties, numerical sequences by their convergence speed, and matrices by their rank. We consider the order relations of special type with special properties. These relations are similar to the properties of real numbers ordering relation denoted by $>$ and $\geq$ by their inherent properties. In this subsection, we discuss the concepts related to cone and partial ordering of a vector spaces.

Definition 1.2.1 *A binary relation "* $\leq$ *" on a set* X *is said to be partial order on* X *if the following assertions true.*

(i) For $x \in X$, $x \leq x$ *(reflexivity);*

(ii) For $x, y \in X$, $x \leq y$ *and* $y \leq x$ *imply* $x = y$ *(antisymmetry);*

(iii) For $x, y, z \in X$, $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ *(transitivity).*

A vector space X with a partial order " $\leq$ " is called partially ordered vector space if the order relation " $\leq$ " is compatible with the vector space X. This means, if $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$ and $x \leq y$ implies $\lambda x \leq \lambda y$ for $\lambda \geq 0$. Set X with order relation " $\leq$ " is denoted by $(X, \leq)$.

For many occasions, we can express the convergence of vectors by the notion of a norm, especially while discussing the solution of an algebraic system. A norm is the generalization to real vector space of the intuitive notion of "length" in the real world. Norm can be defined on a vector space in different ways. It is crucial to select most suitable norm for the practical situation. A vector space with a norm defined on it is called **normed linear space.** Some of the norms stem from an inner product. These inner products give the intuitive notion of orthogonality and angle in **inner product space.** An inner product space which is complete is called **Hilbert space.** A complete normed linear space is known to be **Banach space.** In this thesis, we denote the normed spaces and the Banach spaces with the notations X, Y and Hilbert spaces with H, H_1, H_2. The inner product of x and y is noted by $\langle \cdot, \cdot \rangle$. The norm function is noted by $||\cdot||$. If the norm is induced by an inner product then $||x||= \sqrt{\langle x, x\rangle}$ for every $x \in H$. Throughout the thesis, our results pertain to the spaces are defined on the field of real numbers.

An ordered vector space X together with a norm is called an ordered normed space. An ordered Banach space X is a complete ordered normed space. A complete ordered inner product space is known to be ordered Hilbert space H. Now we introduce the concept called cone, which is the backbone of this thesis.

Definition 1.2.2 ***Cone.*** *A nonempty subset C of a real vector space X is called a cone if*

(i) $x \in C,\ \lambda \in \mathbb{R},\ \lambda \geq 0 \implies \lambda x \in C$

(ii) $x,\ y \in C \implies x + y \in C.$

A cone C is called pointed cone if $C \cap (-C) = \emptyset$.

Example 1.2.1 *Let $\mathbb{R}^n$ be the Euclidean space and $\mathbb{R}^n_+$ be the nonnegative orthant (generalization of the first quadrant in n-dimension) in $\mathbb{R}^n$. Then $\mathbb{R}^n_+$ is a closed and pointed cone.*

Example 1.2.2 *Let $\mathbb{R}^n$ be the Euclidean space. The subset*

$$C = \{x \in \mathbb{R}^n : \sqrt{x_2^2 + x_3^2 + ... + x_n^2} \leq x_1, x_1 \geq 0\}$$

is a cone.

Example 1.2.3 *Let $X = l^2$ be the sequence space. Consider $C = l^2_+ = \{x \in l^2 : x_j \geq 0,\ \forall\ j \in \mathbb{N}\}$. The set C forms a cone in X.*

Example 1.2.4 *Consider* $X = (L^2[a,b], \langle \cdot, \cdot \rangle)$, *the space of square-integrable functions on* $[a,b]$ *with the inner product* $\langle f, g \rangle = \int_a^b f(u)g(u)du$. *Then* $C = \{f(u) : f(u) \geq 0, \ \forall \ u \in [a,b]\}$ *forms a cone in* $L^2[a,b]$.

Example 1.2.5 *Let* $\mathcal{S}^{n\times n}$ *be the space of* $n \times n$ *symmetric matrices. Then the set of positive semi-definite matrices will form a cone in* $\mathcal{S}^{n\times n}$.

Consider a real vector space X and cone C in X. Defined a relation by the notion " $\leq$ " as: $y - x \in C$ for every $x, \ y \in X$ if and only if $x \leq y$. It is simple to observe that " $\leq$ " is an order relation which forms X an ordered vector space. Conversely, suppose X is an ordered vector space with the order " $\leq$ ", then the subset $C = \{x \in X : x \geq 0\}$ forms a positive cone in X.

Consider a normed linear space X. Let X^* be the space of continuous linear functionals defined on X. For a cone C in X, the dual cone is defined by $C^* = \{f \in X^* : f(x) \geq 0, \ \forall \ x \in C\}$. Moreover, the dual cone C^* of the cone C in the Hilbert space H is given by $C^* = \{x \in H : \langle x, u \rangle \geq 0, \ \forall \ u \in H\}$. A cone C is known to be self-dual if $C^* = C$.

Definition 1.2.3 *A vector lattice* X *is called Dedekind complete (order complete) if every subsets which are nonempty with a lower bound and upper bound has an infimum and a supremum, respectively.*

Definition 1.2.4 *A cone* C *of an ordered vector space* X *is called generating (or reproducing), if any element* $x \in X$ *can be represented by* $x = u - v$, *for* $u, v \in C$. *Cone* C *is generating if* $X = C - C$, *analogously.*

Let the interior of C is denoted by C° in an ordered normed linear space X. In finite dimensional space X, a cone C is reproducing cone implies and implied by the condition $C^\circ \neq \emptyset$ [5]. But in case of an infinite dimensional ordered normed space, a cone C is reproducing if $C^\circ \neq \emptyset$, but not conversely.

Definition 1.2.5 *A pointed cone C in X is called solid if it contains some ball of positive radius. A cone C being solid means $C^{\circ} \neq \emptyset$.*

Definition 1.2.6 *The cone C is called normal if there exists the constant of normality δ such that $||x + y|| \geq \delta$ whenever $x,\ y \in C$ with $||x|| = ||y|| = 1$.*

Definition 1.2.7 *A cone C in X is called allow plastering if it can be imbedded into another cone C_1 of ordered space X in a way that each $x_0 \in C$ lies in C_1 together with a neighbourhood $\{x : ||x - x_0|| \leq b||x_0||\}$, where $b > 0$ is independent of point x_0. In such case, C_1 is called plastering of the cone C.*

Definition 1.2.8 *A linear functional $f : X \to \mathbb{R}$ is said to uniformly positive functional if there exists a $\lambda > 0$ such that $\lambda||x|| \leq f(x)$, for all $x \in C$.*

1.3 Linear Operators

The purpose of this section is to lay out the features of the theory of linear operators. The results of this section are basic and can be found in numerous classical textbooks ([1],[61] and [12]).

The linear operators theory is an increasingly important area in mathematics. Almost all operators encountered in quantum mechanics are linear operators. A linear operator is not just a map between two vector spaces, it is a map that respects the linear structure of the spaces and hence it has a nice properties compared to non-linear operator. In this section, we discuss linear operators which are either bounded or unbounded and it is mentioned specifically for each chapter. We begin this section with the linear operator definition on vector space and discuss its properties.

Definition 1.3.1 ***Linear operator.*** *A function $S : X \to Y$ is said to be a linear operator if*

$$S(\lambda x + \beta y) = \lambda Sx + \beta Sy$$

for every $x, y \in X$ and for scalars λ and β.

The space of all linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$ and if $Y = X$ then it is denoted by $\mathcal{L}(X)$. Suppose $Y = \mathbb{R}$ then $S : X \to \mathbb{R}$ is known to be **linear functional.** In some situations, the operator may be defined on the proper subspace of a vector space rather than defining on a entire vector space. In this scenario, the **domain of definition** (say, domain) is denoted by $D(S)$.

Every linear operator naturally forms two important subspaces: range (or range space) and null space. The definition of range and null space are given below.

Definition 1.3.2 *Let $S \in \mathcal{L}(X, Y)$. Then the* ***range*** *and* ***null space*** *of S are given by $R(S) = \{y \in Y : Sx = y \text{ for some } x \in X\}$ and $N(S) = \{x \in D(S) : Sx = 0\}$, respectively. In addition, the* ***carrier*** *of S is denoted with $Car(S)$ and is given by*

$$Car(S) = D(S) \cap N(S)^{\perp}.$$

Next, define the norm of an operator. Let X and Y be normed linear spaces with respective norms. Consider $S \in \mathcal{L}(X, Y)$. Then the quantity

$$||S||= \sup\{||Sx||: ||x||= 1\}$$

is called the **norm of the operator** S.

Let $S \in \mathcal{L}(X, Y)$ is called **bounded linear operator** if $||S||< \infty$ (finite norm). We denote the class of all bounded operators on X to Y by $\mathcal{B}(X, Y)$ which is the subclass of $\mathcal{L}(X, Y)$. If the norm of an operator is not finite, then the operator is called **unbounded linear operator.**

Example 1.3.1 ***(Differential operator):*** *Consider the normed linear spaces* $C[0,1]$ *and* $C^1[0,1]$ *with supremum norm. Define first order linear differential operator* $S : C^1[0,1] \to C[0,1]$ *by*

$$(Sx)(t) = x'(t), \ t \in [0,1].$$

Take a sequence $\{x_n(t)\} = \{t^n\}$ *in such a way that* $||Sx_n|| = n$ *and* $||x_n|| = 1$. *The norm of the operator* S *is not a finite quantity. Hence* S *is an unbounded operator.*

1.3.1 Unbounded Linear Operators

Unbounded linear operators appears in several applications, prominently in ordinary and partial differential equations, quantum mechanics. Unbounded operators which occur in the theory of partial differential equations, especially in potential theory, moment theory and other related areas, can be found in [26]. The theory of unbounded operators is more intricate than that of bounded operators. In the year 1920, to define the quantum mechanics on a strong mathematical foundation the unbounded operators theory was established. Marshall H. Stone and John von Neumann developed the unbounded operators theory systematically. The natural launguage of quantum mechanics is the self-adjoint operators on Hilbert space, it was an intuition of J. von Neumann. Modern physics is developed with this idea. We limit ourselves to Hilbert spaces for the discussion of unbounded linear operators since our focus is on the generalized inverse, specifically, the Moore-Penrose inverse of unbounded linear operator over Hilbert space.

Let H, H_1 and H_2 are Hilbert spaces over complex numbers field. Suppose M and L are subsets of H. The closure of a subset L is denoted by $\overline{L}$ and orthogonal complement is denoted by $L^{\perp}$. Moreover, if M, L are linear subspaces of H then $M + L$, $M \oplus L$ and $M \oplus^{\perp} L$ denotes *sum, direct sum* and *orthogonal direct sum,* respectively.

Next, we define the particular class of linear operators called densely defined operators.

Definition 1.3.3 *(Page no.333, [4]) Consider $S \in \mathcal{L}(H_1, H_2)$. If $\overline{D(S)} = H_1$ then S is called densely defined operator (simply, a dense operator).*

Every operator $S \in \mathcal{L}(H_1, H_2)$ can be considered to be a densely defined operator without loosing the generality since it can be seen as $S \in \mathcal{L}(\overline{D(S)}, H_2)$. By Riesz representation theorem, we can associate another operator S^* related to S in the following way.

Definition 1.3.4 *(Page no. 348, [61]) Consider $S \in \mathcal{L}(H_1, H_2)$ be dense operator. Then for all $x \in D(S)$ and $y \in D(S^*)$, there exists a unique operator S^* satisfies $\langle Sx, y\rangle = \langle x, S^*y\rangle$. This operator S^* is called **adjoint operator** of S. In such case, $D(S^*)$ contain all $y \in H_2$ for which the linear functional $x \to \langle Sx, y\rangle$ is continuous on $D(S)$.*

Example 1.3.2 *Let $S \in \mathcal{L}(l^2(\mathbb{N}))$ be forward shift operator. For a given $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ in $l^2(\mathbb{N})$ we can compute*

$$\begin{aligned}\langle Sx, y\rangle &= \langle (0, x_1, x_2, ...), (y_1, y_2, ...)\rangle \\ &= x_1\overline{y}_2 + x_2\overline{y}_3 + ... \\ &= \langle x, y^*\rangle.\end{aligned}$$

In this case, $S^*y = y^* = (y_2, y_3, ...)$. So, we can conclude that S^* is the backward shift operator.

Example 1.3.3 *Consider a Lebesgue measurable function $a(t)$ which is complex valued and bounded on $[a, b]$. Let $S \in \mathcal{B}(L^2([a, b]))$ defined by multiplication of functions,*

$$(Sg)(t) = a(t)g(t).$$

Let $f, g \in L^2([a,b])$,

$$\langle Sf, f\rangle = \int_a^b a(t)g(t)\overline{f}(t)dt = \langle g, \overline{a}f\rangle.$$

Thus $(S^*f)(t) = \overline{a}(t)f(t)$.

Next, the properties of an adjoint operator is collected in the following proposition. For more details on this, refer [25].

Proposition 1.3.1 *Consider* $S \in \mathcal{L}(H_1, H_2)$ *be a dense operator with adjoint* $S^* \in \mathcal{L}(H_2, H_1)$.

(i) Then, S^* *is closed and linear operator.*

(ii) Moreover, if S *is invertible, then* S^* *is invertible operator and* $(S^*)^{-1} = (S^{-1})^*$.

Remark 1.3.1

(i) An element S^*y *in* $\langle Sx, y\rangle = \langle x, S^*y\rangle$ *will be uniquely determined if and only if* $D(S)$ *is dense subspace of* H.

(ii) For an operator $S^* \in \mathcal{L}(H_2, H_1)$ *the domain* $D(S^*)$ *is a subspace of* H_2.

(iii) If $S \in \mathcal{B}(H_1, H_2)$, *then we have* $D(S^*) = H_2$ *and* $S^* \in \mathcal{B}(H_2, H_1)$.

John von Neumann introduced graphs to analyse unbounded operators in 1936. Graph of an operator provides information about the domain, null space, range space, inverse and adjoint.

Definition 1.3.5 *(Page no. 347, [61]) Let* $S \in \mathcal{L}(H_1, H_2)$. *The* ***graph*** $G(S)$ *of an operator* S *is the linear subspace of* $H_1 \times H_2$ *that includes ordered pairs given by* (x, Sx), *for* $x \in D(S)$.

Definition 1.3.6 *(Page no. 332, [4]) Let $S \in \mathcal{L}(H_1, H_2)$. If $G(S)$ is a closed linear subspace of $H_1 \times H_2$ then operator S is called* ***closed operator.*** *In other words, S is closed linear operator if a sequence $x_n \in D(S)$, such that $x_n \to x_0$, $Sx_n \to y_0$ implies $x_0 \in D(S)$ and $Sx_0 = y_0$, where $\to$ indicates strong convergence of a sequence.*

In $\mathcal{L}(H_1, H_2)$, the subclass of closed operators is represented by $\mathcal{C}(H_1, H_2)$. A bounded operator $S \in \mathcal{B}(H_1, H_2)$ is closed linear operator if and only if $D(S)$ is closed subspace. Hence, it can be written as $\mathcal{B}(H_1, H_2) \subset \mathcal{C}(H_1, H_2)$. Conversely, suppose $D(S)$ is entire space H_1 then $S \in \mathcal{C}(H_1, H_2)$ is bounded linear operator. This assertion gives the *closed graph theorem.* In addition, a closed operator has a closed null space.

Definition 1.3.7 *If an operator $S \in \mathcal{L}(H_1, H_2)$ be dense operator and $S \in \mathcal{C}(H_1, H_2)$ then S is referred as densely defined closed linear operator.*

Example 1.3.4 *Consider $H = l^2(\mathbb{N})$ and domain*

$$D(S) = \{x = (x_1, x_2, ..., x_n, ...) \in H : \sum_{j=1}^{\infty} |jx_j|^2 < \infty\}$$

Define

$$Sx = S(x_1, x_2, x_3, ..., nx_n, ...) = (x_1, 2x_2, 3x_3, ...) \ \ \forall \ x \in (x_1, x_2, ...) \in D(S)$$

Let $\{e_n : n \in \mathbb{N}\}$ be separable basis of H such that $e_n(m) = \delta_{nm}$ be the Kronecker delta function. Then $Se_n = ne_n$. So, S is unbounded operator. Since the space of sequences with at most finitely many non-zero terms c_{00}, contained in $D(S)$; we have $\overline{D(S)} = H$. Hence S is dense operator.

Let $(x_n) \in D(S)$ be a cauchy sequence. Hence there exists $x \in D(S)$ and $y \in H$ in such a way that $x_n \to x$ and $Sx_n \to y$. Completeness of $l^2(\mathbb{N})$ gives $Sx = y$. So, S is *closed linear operator.*

The adjoint of operator S is given below: For $x \in D(S)$ and $y \in D(S^*)$, we have $\langle Sx, y\rangle = \langle x, S^*y\rangle$

$$\begin{aligned}\langle x, S^*y\rangle &= \langle Sx, y\rangle \\ &= \langle S(x_1, x_2, x_3, ...), (y_1, y_2, ...)\rangle \\ &= \langle (x_1, 2x_2, 3x_3, ...), (y_1, y_2, ...)\rangle \\ &= x_1y_1 + 2x_2y_2 + 3x_3y_3 + ... \\ &= \langle (x_1, x_2, x_3, ...), (y_1, 2y_2, 3y_3, ...)\rangle.\end{aligned}$$

This implies $S^*y = S^*(y_1, y_2, ...) = (y_1, 2y_2, 3y_3, ...)$.

Among all unbounded operators, the class of closed densely defined operators have nice properties similar to bounded linear operators. So, it is our interest of study in upcoming chapters. The following proposition gives the basic properties of densely defined closed operators.

Proposition 1.3.2 *(Chapter 9, [4]) If $S \in \mathcal{C}(H_1, H_2)$ be a dense operator. Then*

(i) $S^* \in \mathcal{L}(H_2, H_1)$ *is densely defined and* $S^{**} = S$.

(ii) $N(S^*S) = N(S), \quad N(S) = R(S^*)^\perp$.

(iii) $N(SS^*) = N(S^*), \quad N(S^*) = R(S)^\perp$.

(iv) $\overline{R(S)} = \overline{R(SS^*)}, \quad \overline{R(S)} = N(S^*)^\perp$

(v) $\overline{R(S^*)} = \overline{R(S^*S)}, \quad \overline{R(S^*)} = N(S)^\perp$.

Definition 1.3.8 *(Page no. 332, [4]) Let $S, T \in \mathcal{L}(H_1, H_2)$ with the domains $D(S) \subset D(T)$. If $Tx = Sx$ for each $x \in D(S)$, then T is referred as an extension of S and S is referred as a restriction of T. These relations are shown by $S \subset T$ or by $S = T\big|_{D(S)}$.*

Next, we give the definition of some class of operators for which $H = H_1 = H_2$.

Definition 1.3.9 *(Page no. 334, [4]) A dense operator $S \in \mathcal{L}(H)$ is said to be symmetric if*

$$S \subset S^*$$

and self-adjoint if

$$S = S^*.$$

Example 1.3.5 *Let p, p', q, w are functions defined on $[a, b] \subset \mathbb{R}$ which are real valued and continuous. Let $w(t) > 0$, for all $t \in [a, b]$. Consider the Hilbert space*

$$H = \{v : \int_a^b |v(x)|^2 w(x) dx < \infty\}$$

with the inner product

$$\langle v, u \rangle = \int_a^b v(x) u(x) w(x) dx.$$

Let S be the ***Sturm-Liouville operator*** *given by*

$$Su = \frac{1}{w}[-(p'v)' + qv]$$

where

$$D(S) = \{v \in C^2[a, b] : \alpha_j v(t) + \beta_j v'(t) = 0, t \in \{a, b\}, |\alpha_j| + |\beta_j| > 0, j \in \{1, 2\}\}.$$

Then, S is symmetric and densely defined operator.

Fix a symmetric operator S and consider a symmetric extension T of S, that is $S \subset T$. We can notice that $T^* \subset S^*$. Since $T \subset T^*$, we get $S \subset T \subset T^* \subset S^*$. Thus every symmetric extension of S is a restriction of S^*.

Projection operators are the simplest non-scalar operators. Projection operators play an important role while defining the generalized inverses. These projections enable us to decompose the operator into sum of restriction operators which have the same property as the original operator and easy to work with. There is a close connection between the closed linear subspaces of a Hilbert space and the projection operators. The geometric properties of the subspaces can be obtained through the algebraic properties of the projection operators.

Next, we give the definition of projection operator and discuss its properties. More details on these results can be found in [30], [75] and [25].

Definition 1.3.10 *Consider an operator $S \in \mathcal{L}(H)$. Then S is said to be a* ***projection****(or idempotent, in general) if and only if $S^2 = S$.*

In the forthcoming chapters, the operator $P_{M,L}$ denotes projection operator onto subspace M along subspace L where $X = M \oplus L$. The following lemma gives the conditions on subspaces to have equal projections.

Lemma 1.3.1 *Let M and L be linear subspaces of the space X which allow the algebraic direct sum decomposition such that $X = M \oplus L$. Consider the associated projector P with $R(P) = M$ and $N(P) = R(I - P) = L$. Then for $S \in \mathcal{B}(X)$*

(i) $SP_{M,L} = S \iff N(S) \supset L$.

(ii) $P_{M,L}S = S \iff R(S) \subset M$.

The following proposition gives the algebraic operations involving linear operators:

Proposition 1.3.3 *Let $S, T \in \mathcal{L}(H_1, H_2)$ and $U \in \mathcal{L}(H_2, H_3)$. Then the following rules hold good:*

(i) ***Sum:*** *For an operator* $S+T \in \mathcal{L}(H_1, H_2)$ *the sum rule is given by* $(S+T)x = Sx+Tx$ for all $x \in D(S+T)$ *and defined domain is* $D(S+T) = D(S) \cap D(T)$.

(ii) ***Product:*** *For an operator* $US \in \mathcal{L}(H_1, H_3)$ *the product rule is given by* $(US)x = U(Sx)$ for all $x \in D(US)$ *and defined domain is* $D(US) = \{x \in D(S) : Sx \in D(U)\}$.

(iii) ***Scalar multiplication:*** *For* $\beta \in \mathbb{C} - \{0\}$, $\beta S \in \mathcal{L}(H_1, H_2)$ *with scalar multiplication* $(\beta S)x = \beta Sx$ for all $x \in D(S)$ *and domain* $D(\beta S) = D(S)$.

The following theorem gives the properties of adjoint operator.

Theorem 1.3.11 *(Theorem 13.2, [61])) Suppose* S, T *and* ST *are dense and linear operators defined over* H. *Then*

$$T^*S^* \subset (ST)^*$$

Additionally, if $S \in \mathcal{B}(H)$ *then*

$$T^*S^* = (ST)^*.$$

Definition 1.3.12 *Suppose* $S \in \mathcal{C}(H)$ *be a dense operator. Then* S *is called normal operator if* $S^*S = SS^*$ *and unitary operator if* $SS^* = I = S^*S$.

Definition 1.3.13 *Let* $S \in \mathcal{L}(H_1, H_2)$ *be one-to-one. Then the inverse of* S *is denoted by* S^{-1} *and* $S^{-1} : R(S) \to H_1$ *is defined by* $S^{-1}(Sx) = x$ *for all* $x \in D(S)$.

It is very important to know whether a bounded operator $S : X \to Y$ has a bounded inverse or not. For example, assume that the equation

$$Sx = b$$

has unique solution x for $b \in Y$. Suppose that the above equation is very difficult to solve numerically for a given b_0, but easy to solve for a value $\hat{b}$ near b_0. Then, if S^{-1} is continuous, i.e. bounded, then the corresponding solutions x_0 and $\hat{x}$ are also near. Since

$$||x_0 - \hat{x}|| = ||S^{-1}b_0 - S^{-1}\hat{b}|| \leq ||S^{-1}||||b_0 - \hat{b}||.$$

Therefore we can solve the equation for a near value $\hat{b}$ without obtaining a significant error. The next theorem states Banach's celebrated theorem on the inverse mapping.

Theorem 1.3.14 *Consider X, Y be complete spaces and $S \in \mathcal{B}(X, Y)$. If S is one-to-one and onto then S^{-1} is continuous (that is, S^{-1} is bounded).*

The necessary and sufficient conditions for a closed linear operator to have a bounded inverse is provided in the following lemma.

Lemma 1.3.2 *(Page no. 95, [26]) Consider X, Y be complete spaces and $S \in \mathcal{C}(X, Y)$. Then operator S has a bounded inverse if and only if S is one-one and has a closed range.*

The main idea of convergence of iterative scheme lies on the fact that the spectral radius of iterative operator/matrix is less than one. Given an operator S with domain and range in X, consider the operators of the form $\lambda I - S$, where λ is a scalar and I is an identity operator.

Definition 1.3.1 *([64]) Let $S \in \mathcal{L}(X)$. Then the set $\rho(S) \subset \mathbb{C}$ of all scalars λ such that the range of $\lambda I - S$ is dense subspace of X and $\lambda I - S$ has a continuous inverse is called the resolvent set of S. For $\lambda \in \rho(S)$, the operator $R_\lambda = (\lambda I - S)^{-1}$ is known to be the resolvent operator. The spectrum of S is the set $\sigma(S) = \{\lambda : \lambda \notin \rho(S)\}$.*

If S is defined on n dimensional space X, then S and $\lambda I - S$ can be represented by a $n \times n$ matrices. The spectrum of S is composed of scalars λ that are roots of the equation

$$det(\lambda I - S) = 0.$$

If $\lambda \in \sigma(S)$, then λ will be eigenvalue of S; that is, there exists a vector $x \neq 0$, called an eigenvector associated with λ, such that $Sx = \lambda x$, and hence $(\lambda I - S)x = 0$. In the infinite dimensional case, $\sigma(S)$ may contain scalars that are not eigenvalues of S.

The maximum absolute value of the spectrum $\sigma(S)$ is called **spectral radius** and it is denoted by $r(S)$.

Example 1.3.6 *Let (λ_n) be a bounded sequence of scalars. Consider $S \in \mathcal{L}(X)$ be a diagonal operator defined by*

$$(Sx)_j = \lambda_j x_j, \quad x \in X; \quad j \in \mathbb{N}$$

For $\lambda \in \mathbb{R}$, the equation $Sx = \lambda x$ is satisfied for a nonzero vector $x \in X$ whenever $\lambda = \lambda_j$ for some $j \in \mathbb{N}$. Hence,

$$\sigma(S) = \{\lambda_1, \lambda_2, ...\}.$$

For $n \in \mathbb{N}$, $e_n \in X$ be defined by Kronecker delta function $e_n(j) = \delta_{nj}$ and it is an eigenvector of the operator S for the eigenvalue λ_n. In such case, the spectral radius of operator S is $r(S) = \max\{(|\lambda_j|)\}$.

Next, we see some class of linear operators which can be either bounded or unbounded depending on the domain of definition.

Definition 1.3.15 *Consider an operator $S \in \mathcal{L}(X)$. It is said to be* ***nilpotent*** *if $S^k = 0$ for some k.*

Another essential point, we discuss below is ascent and descent of linear operators. This is helpful to define index of an operator.

Ascent and Descent of Linear Operators: To define ascent and descent of linear operator, we consider the case of operators in which domain and co-domain be the same vector space X. The considerations here are all algebraic in nature.

Consider $S \in \mathcal{L}(X)$. We define the iterates $S^2, S^3, ...$ of S. We follow the convention that $S^0 = I$ where I is an identity on X, $S^1 = S$ and $S^n x = S(S^{n-1}x)$. Then $D(S) = X$ and for $n \geq 1$, $D(S^n)$ be the set of all $x \in D(S^{n-1})$ such that $S^{n-1}x$ is in $D(S)$. If $D(S) \neq X$, then $D(S^n)$ is usually a proper subset of $D(S^{n-1})$. Operator S^n with null space $N(S^n) = \{x \in D(S^n) : S^n x = 0\}$. For this replayed operator, the relation among null spaces as below.

$$\{0\} = N(S^0) \subset N(S^1) \subset N(S^2) \subset ...$$

We observe that

$$N(S^{n+1}) = \{x \in D(S) : Sx \in N(S^n)\}.$$

If $N(S^n)$ coincides with $N(S^{n+1})$, then it coincides with all $N(S^k)$ for $k > n$.

Definition 1.3.16 *([64]) Let $S \in \mathcal{L}(X)$. The smallest natural number k which satisfies $N(S^k) = N(S^{k+1})$ is the **ascent** of S and denoted by $\alpha(S)$. If there is no such k exists, then we assume $\alpha(S) = \infty$. Note that $\alpha(S) = 0$ if and only if S is one-to-one.*

Similarly, for $n = 0, 1, 2, ...$ the range spaces $R(S^n)$ of the iterates of S form a nested chain of subspaces:

$$X = R(S^0) \supset R(S^1) \supset R(S^2) \supset ...$$

Evidently,

$$R(S^{n+1}) = S\{R(S^n) \cap D(S)\}.$$

If $R(S^n) = R(S^{n+1})$, then it follows that $R(S^n) = R(S^k)$ for $k > n$.

Definition 1.3.17 *([64]) Let $S \in \mathcal{L}(X)$. The smallest natural number k which satisfies $R(S^k) = R(S^{k+1})$ is the descent of S and denoted by $\delta(S)$. If $R(S^{n+1})$ is a proper subspace of $R(S^n)$ then we set $\delta(S) = \infty$. Note that $\delta(S) = 0$ if and only if $R(S) = X$.*

Lemma 1.3.3 *Suppose that $\delta(S) = 0$ and $\alpha(S) < \infty$. Then $\alpha(S) = 0$.*

The following lemma describes the case of equality of descent and ascent of an operator.

Lemma 1.3.4 *(Lemma 2.21, [1]) If $S \in \mathcal{L}(X)$ has finite descent and ascent, then they must coincide, that is $\alpha(S) = \delta(S) = p < \infty$. The space X has the direct sum decomposition*

$$X = R(S^p) \oplus N(S^p).$$

Moreover, $R(S^p)$ is a closed linear subspace of X.

Example 1.3.7 *Each $S \in \mathbb{R}^{m\times n}$ has finite descent and ascent.*

Example 1.3.8 *If a nilpotent operator $S : X \to X$ has a index of nilpotency k then $\delta(S) = \alpha(S) = k$.*

Example 1.3.9 *Let $S \in \mathcal{L}(H)$ be defined by*

$$S(e_j) = \begin{cases} 0, & \text{if } j = 0 \\ e_{j-1}, & j \geq 1. \end{cases}$$

where $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of separable Hilbert space H. The range space $R(S)$ is spanned by $(e_j)_{j\in\mathbb{N}}$, $\delta(S) = 0$. There is no k such that $N(S^k) = N(S^{k+1})$, hence $\alpha(S) = \infty$.

1.3.2 Bounded Linear Operators

As we have seen, the primary gap between bounded and unbounded operators is the domain of definition. In this subsection, we discuss the properties of bounded linear operators. In particular, we give the properties which hold good for bounded linear operators but not for densely defined closed operators.

Definition 1.3.18 *(Chapter 9, [4]) A projection operator $P \in \mathcal{B}(H)$ is an* ***orthogonal projector*** *if*

$$P = P^2 = P^*.$$

In this scenario, $R(P)$ is closed linear subspace and

$$H = R(P)^{\perp} \oplus N(P).$$

Example 1.3.10 *Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$P(x_1, x_2) = (x_1, x_1), \qquad \forall\ (x_1, x_2) \in \mathbb{R}^2.$$

The subspaces $R(P) = \{(x, x) : x \in \mathbb{R}\}$ and $N(P) = \{(0, x) : x \in \mathbb{R}\}$ are not orthogonal with each other. Thus P is not an orthogonal projection.

Example 1.3.11 *Consider S be a measurable subset of $[0, 1]$ and $H = L^2[0, 1]$. Let $P \in \mathcal{B}(H)$ given by*

$$(Px)(t) = x(t)\chi_S(t), \qquad \text{for all } x \in H$$

where χ_S is the characteristic function of S. Here P is the orthogonal projection because $R(P) = \{x \in H : x(t) = 0\ \ \forall\ \ t \notin S\}$ and $N(P) = \{x \in H : x(t) = 0,\ \ \forall\ \ t \in S\}$ are orthogonal to each other.

The below proposition states the properties of adjoint of a bounded operator defined on Hilbert space.

Proposition 1.3.4 *Consider $S, T \in \mathcal{B}(H)$. Then each of the following assertions are true:*

(i) $(\lambda S)^* = \overline{\lambda} S^*$

(ii) $||S^*|| = ||S||$

(iii) $||S^*S|| = ||SS^*|| = ||S||^2$

(iv) $(S+T)^* = S^* + T^*$

The next theorem establishes fundamental relationships between the range, null space and adjoint operator and these are helpful in upcoming chapters.

Theorem 1.3.19 *(Theorem 1.2.1, [28]) Consider $S \in \mathcal{B}(H_1, H_2)$. Then,*

$$R(S^*)^\perp = N(S);\;\; N(S^*)^\perp = \overline{R(S)};$$

$$R(S)^\perp = N(S^*); \quad N(S)^\perp = \overline{R(S^*)}$$

where the bar denotes the closure of the given subspace.

Definition 1.3.20 *(Page no.63, [25]) Let $S \in \mathcal{B}(H_1, H_2)$. If range of S is finite dimensional, then S is finite-rank operator. Specifically, if the dimension of the range space is one, then such operator is called rank-one operator.*

Next, we introduce the concept called cone nonnegativity of operators which plays a vital role in this thesis. Consider a real vector spaces X, Y with cones C_1 and C_2, respectively. Let $S \in \mathcal{L}(X, Y)$. Suppose a cone C_1 is mapped into the set SC_1, it can be easily seen as a cone because of the linearity property of the operator S. The cone nonnegativity of an operator S is defined below:

Definition 1.3.21 *Consider a real vector spaces X, Y with cones C_1 and C_2, respectively. An operator $S \in \mathcal{L}(X, Y)$ is **cone nonnegative** (or positive) if $SC_1 \subseteq C_2$ (or $SC_1 \subset C_2$) and is denoted by $S \geq 0$ (or $S > 0$.)*

The class of all positive bounded linear operators forms a cone in $\mathcal{B}(X, Y)$ and is denoted by $\mathcal{B}_+(X, Y)$. The following definition gives rise to a cone nonnegativity of generalized inverses of linear operators.

Definition 1.3.22 *Consider a real vector spaces X, Y with cone C_1 and C_2, respectively. An operator $S \in \mathcal{B}(X, Y)$ is said to be positive invertible if S^{-1} exist and $S^{-1}C_2 \subseteq C_1$.*

In the following example, we illustrate the application of positive invertibility.

Example 1.3.12 *In the study of differential equations [63], for the discrete approximations of the maximum principle of a matrix $S = (a_{ij})$ is given as follows: Let $Sx = b$ with $b = (b_1, b_2, ...) \in \mathbb{R}^n$. From the assertions $b \geq 0,\ b \neq 0$ it follows that $x \geq 0$. Also,*

$$\max_{i \in N} x_i = \max_{i \in N^+(y)} x_i,$$

for $N = \{1, 2, ..., n\}$ and $N^+(b) = \{i \in N : b_i > 0\}$. If S is a matrix with negative-off-diagonal and has a positive inverse (M-Matrix) and $Se \in \mathbb{R}^n_+$, where $e = (1, 1, ..., 1)^T$, then S satisfies above maximum condition. If S is positively invertible and satisfies maximum condition then S^{-1} is weakly diagonally dominant of its column entries.

1.4 Generalized Inverses

The linear system of algebraic equations $Sx = b$, where S is either a matrix or an operator, may be used to formulate many real-world problems. This system has a solution $x = S^{-1}b$, if the matrix S is square and invertible. The criteria for the matrix S to have an inverse is matrix should be non-singular, tantamount to matrix have non-zero determinant. However in practical problems,there has been a requirement of some kind of partial inverse of a matrix which is either non-invertible or rectangular. To fulfil this need, it was discovered that even if a matrix is non-invertible, there is still either a left or right sided inverse of that matrix. A matrix $S \in \mathbb{C}^{m\times n}$ is left invertible if there is a left inverse $L \in \mathbb{C}^{n\times m}$ such that

$$LS = I_n.$$

Similarly, matrix $S \in \mathbb{C}^{m\times n}$ is right invertible if there is a right inverse $R \in \mathbb{C}^{n\times m}$ such that

$$SR = I_m.$$

The property, where every matrix has some inverse-like matrix, shown way to define the generalized inverses. When matrix is invertible and possesses some properties of the ordinary inverse, these generalized inverses reduces to ordinary inverse. Moreover, the class of invertible matrices contained in the class of matrices having generalized inverses. Though the generalized inverse is frequently not used, as it is supplanted through various restrictions to create different generalized inverses for specific purposes. For instance, let X be the inner inverse of a matrix S satisfying $SXS = S$ alone used to analyze the solutions of the linear system of algebraic equations $Sx = b$. But inner inverse is not unique. If least squares properties are concerned, we have to look for Moore-Penrose inverse. If we are interested in spectral properties, we have to consider only square matrix which have eigenvalues and eigenvectors. With this strategy, we have to go for Drazin inverse. We give the definition with historical development of generalized inverses which we are used in this thesis.

1.4.1 Moore-Penrose Inverse of Operator

The literature on generalized inverses has enormous growth in the last several decades. Some authors used the phrase *"pseudo inverse"* instead of *"generalized inverse"*. In the year 1903, Fredholm [18] discussed the concept of a generalized inverse for the first time. He gave a generalized inverses of integral operators. In 1912, Hurwitz [34] characterized the class of all pseudo inverses by using the finite dimensionality of the null spaces of Fredholm operators.

The generalised inverses of matrices were preceded by the generalised inverses of integral and differential operators. The note on this is initially given by an American mathematician Eliakim Hastings Moore (1862–1932). In 1920, E.H. Moore published the paper on the reciprocal of the general algebraic matrix [52]. This was rediscovered by 91 years old English Mathematical Physicist and Nobel laureate, Sir Roger Penrose [57]. Hence this particular unique generalized inverse is called *Moore-Penrose inverse.* This can be considered as a new era in the development of generalized inverses and matrix analysis. The fact that Moore and Penrose defined actually the same notion was recognized by Richard Rado [60]. Ben-Israel [3] made an attempt to identify the reasons on why no noticeable reaction was kindled by Moore's paper [52] whereas the response caused by the article Penrose [57] was notable. The broad definition of generalized inverses of operators on Hilbert space was given by Tseng [65]. Later, it was found that the maximal Tseng inverse of an operator is the Moore-Penrose inverse. The monographs [4], [6], [36], [28] are for further readings on generalized inverse of matrices and operators with historical developments. Now, we define maximal Tseng inverse of linear operator on Hilbert space.

Definition 1.4.1 *(Page no. 339, [4]) Let* $S \in \mathcal{L}(H_1, H_2)$ *satisfy* $D(S) = N(S) \oplus^{\perp} Car(S)$. *Then* $S^g \in \mathcal{L}(H_2, H_1)$ *is a Tseng inverse of operator* S *if the following assertions hold true:*

(i) $R(S) \subset D(S^g)$

(ii) $R(S^g) \subset D(S)$

(iii) $S^g S x = P_{\overline{R(S^g)}} x, \quad \text{for all } x \in D(S)$

(iv) $S S^g y = P_{\overline{R(S)}} y, \quad \text{for all } y \in D(S^g)$

The maximal Tseng inverse of S, denoted by $S^\dagger$, is the Moore-Penrose inverse of S with domain and null space given by

$$D(S^\dagger) = R(S) \oplus^\perp R(S)^\perp$$

and $N(S^\dagger) = R(S)^\perp$, respectively.

The following theorems are due to operators with closed range.

Theorem 1.4.2 *(Page no. 343, [4]) Consider $S \in \mathcal{C}(H_1, H_2)$. Then $S^\dagger \in \mathcal{L}(H_2, H_1)$ is bounded operator if and only if $R(S)$ is closed linear subspace.*

Theorem 1.4.3 *(Page no. 343, [4]) Let $S \in \mathcal{B}(H_1, H_2)$ have closed range. Then*

$$S^\dagger = (S^*S)^\dagger S^* = S^*(SS^*)^\dagger$$

Next, discuss the geometrical meaning of the Moore-Penrose inverse of the bounded linear operators between Hilbert spaces. Consider the Hilbert spaces H_1, H_2 and operator $S \in \mathcal{B}(H_1, H_2)$ with closed range. For a given $b \in H_2$, consider the linear operator equation $Sx = b$. If $b \notin R(S)$ then the equation does not have a solution and if $N(S) \neq \emptyset$ then a solution may not be unique even though it exists. When $b \notin R(S)$, it is still desirable to investigate a solution as a generalized solution of the system $Sx = b$, in certain sense. Thus, in place of exact solution we look for the closet solution with geometric approach. Consider $P_{R(S)}$ be the orthogonal projection of H_2 on to $R(S)$ along $N(S)^\perp$. If the equation $Sx = b$ does not have a solution, an appropriate solution of the consistent system $Sx = P_{R(S)}b$ may be accepted as a generalized solution. Assume that a minimizer functional exists and $x_0 \in H_1$ is minimizer of the residual functional

$$\pi_b(x) = ||Sx - b||, \quad x \in H_1.$$

Then it makes sense to call x_0 as a generalized solution. These two definitions are equivalent and they are equivalent to another condition given in the next result.

Theorem 1.4.4 *(Theorem 2.1.1,[28]) Suppose $S \in \mathcal{B}(H_1, H_2)$ has closed range and $b \in H_2$. Then the following statements are equivalent.*

i) $Sx_0 = P_{R(S)}b$,

ii) $||Sx_0 - b|| \leq ||Sx - b||, \ \forall \ x \in H_1$,

iii) $S^*Sx_0 = S^*b$.

If a vector $x_0 \in H_1$ satisfies one of the above condition, then it is called least squares solution of the system $Sx = b$. The set of all least squares solutions of $Sx = b$ is denoted by

$$L_b = \{x_0 \in H_1 : S^*Sx_0 = S^*b\}.$$

The set L_b contains a unique vector of minimal norm and L_b is a closed convex set. Define a linear operator $S^\dagger : H_2 \to H_1$ by

$$S^\dagger b = x_0, \text{ for } b \in H_2,$$

where $S^\dagger$ is a continuous operator. Moreover, $S^\dagger$ is the unique solution of the operator equations mentioned below:

$$SXS = S;$$

$$XSX = X;$$

$$(SX)^* = SX;$$

$$(XS)^* = XS.$$

This operator $S^\dagger$ is called the Moore–Penrose inverse of operator S.

The further discussion on the properties of Moore-Penrose inverse of bounded and unbounded operators are considered in later chapters.

1.4.2 Drazin Inverse

In this subsection, we brief about Drazin inverse with its historical developments and applications.

A key feature of generalized inverse is to provide some type of solution or least squares solution linear system. In recent decades, there is a great deal of interest in generalized inverses of matrices, specifically Drazin inverse. Drazin inverse is the important tool in the study of ring theory [31]. However, the Drazin inverse will not provide solutions of linear algebraic equations, it will provide solutions for systems of linear differential equation and linear difference equations [7]. The Drazin inverse is useful tool in matrix theory and computations primarily because it has a very desirable spectral property: "The nonzero eigenvalues of the Drazin inverse are the reciprocals of the nonzero eigenvalues of the given matrix, and the corresponding generalized eigenvectors have the same multiplicity" [27].

The algebraic definition of Drazin inverse was first given by M.P. Drazin in 1958 [13] while setting of an abstract rings and semigroups, which does not have the reflexive property, but commutes with the element. For the bounded operators and elements of Banach algebra Ben-Israel [4], Caradus [8], King [42], Koliha [44] and several other researchers introduced and studied the Drazin inverse. Also, in [50], Marek and *Žitný* discused the Drazin inverse of operators as well as for elements of a Banach algebra in detail. The investigation on the Drazin inverse of closed linear operators and its application to singular evolution equations and partial differential equations carried by Nashed and Zhao [55]. In [14], Drazin studied the extremal definition of generalized inverses which gives a generalization of the original Drazin inverse. Studies shown that the Drazin inverse has application in analyzing Markov chains ([6], Chap.8), difference equation and differential equations([6], Chap.9), Cauchy problems, investigation of Cesaro-Neumann iterations [32], cryptography [33], and iterative procedures in numerical analysis.

In [8], Caradus proved $S \in \mathcal{B}(X)$, where X is a complex Banach space has a Drazin inverse S^D whenever 0 is a pole of the resolvent $(\lambda I - S)^{-1}$ of S; and the order of the pole is equal to the index of operator S. If 0 is a simple pole, then S^D is called the *Group Inverse* and denoted by $S^{\#}$.

The Drazin inverse definition is not restricted to Hilbert space, it is defined over Banach space. Similar to the matrix case, the Drazin inverse of bounded operators is defined for space X to X itself. To define the Drazin inverse, first we define index of an operator. In section 1.3.1, we have discussed the descent and ascent of an operator.

Descent and ascent of an operator are important to define the index of an operator. The following theorem gives the condition on subspace for a bounded linear operator to be invertible.

Theorem 1.4.5 *(Theorem 2.23, [1]) Consider $S \in \mathcal{B}(X)$. There exists a finite ascent and descent if and only if S has a reducing pair of closed linear subspaces (M, L) such that $S \in \mathcal{B}(M)$ is nilpotent and $S \in \mathcal{B}(L)$ is invertible. In addition, if $p = \alpha(S) = \delta(S) < \infty$, then the pair (M, L), where $M = N(S^p)$ and $L = R(S^p)$, is the only reducing pair for the operator S such that S is nilpotent on M and invertible on L.*

Next, we define the Drazin inverse of $S \in \mathcal{B}(X)$ with $R(S^k)$ is closed for some nonnegative integer k.

Definition 1.4.6 *(Definition 12.1.1, [68])* ***Drazin Inverse:*** *Let $S \in \mathcal{B}(X)$. If there exists $T \in \mathcal{B}(X)$ which satisfies the following conditions*

$$STS^k = S^k$$

$$TST = T$$

$$ST = TS$$

for some nonnegative integer k. Then T is said to be Drazin inverse of S and it is denoted by S^D. Also, such a smallest integer k is called index of S and denoted by $ind(S)$.

Lemma 1.4.1 *(Lemma 2.1, [10]) Consider $S \in \mathcal{B}(X)$. If S has a Drazin inverse S^D then it has finite descent as well as finite ascent, and vice versa. Also, k is the smallest nonnegative integer such that $k = ind(S) = \alpha(S) = \delta(S)$ which satisfies the conditions of Drazin inverse.*

Example 1.4.1 *([44]) Consider the Banach space $X = l^1(\mathbb{N})$. Let $S \in \mathcal{B}(l^1)$ defined by an infinite matrix*

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \cdots & & \cdots & & \cdots \end{pmatrix}$$

Observe that S is not a nilpotent element of $\mathcal{B}(l^1(\mathbb{N}))$. Thus, we can not find any non zero operator which satisfies the conditions of Drazin inverse. Hence, we may conclude that S^D, the Drazin inverse of S is zero operator.

Example 1.4.2 *([64]) Let $S \in \mathcal{B}(l^1(\mathbb{N}))$ be defined by a diagonal operator which can be represented by infinite matrix as shown below:*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta_2 & 0 & 0 & \cdots \\ 0 & 0 & \beta_3 & 0 & \cdots \\ 0 & 0 & 0 & \beta_4 & \cdots \\ \cdots & & \cdots & & \cdots \end{pmatrix}$$

where $0 < \epsilon \leq |\beta_n|$ for $n \geq 2$. Then the spectrum of S is the set $\sigma(S) = \{0\} \cup \mathrm{cl}\{\beta_n : n \geq 2\}$. Consider the set of all complex valued functions $f \in H(S)$ where f is *holomorphic in an open neighbourhood* $\Delta(f) \subset \sigma(S)$*. Thus, for* $f \in H(S)$*,*

$$f(S)(\zeta_n) = \sum_{n=2}^{\infty} \beta_n \zeta_n e_n$$

where $\{e_n : n \in \mathbb{N}\}$ *be the standard Schauder basis of* $l^1(\mathbb{N})$*. It can be verified that* 0 *is a simple pole of* S*. Thus,* $S^D = S^\#$ *is which is given by* $S^D = f(S)$*, where* $f(\zeta)$ *is* 0 *in a neighbourhood of 0 and* ζ^{-1} *in the neighbourhood of* $\sigma(S)\backslash\{0\}$*. Therefore,* S^D *is the matrix defined by*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta_2^{-1} & 0 & 0 & \cdots \\ 0 & 0 & \beta_3^{-1} & 0 & \cdots \\ 0 & 0 & 0 & \beta_4^{-1} & \cdots \\ \cdots & & \cdots & & \cdots \end{pmatrix}$$

Proposition 1.4.1 *(Page no. 166, [4]) Let* $S \in \mathcal{B}(X)$ *with the Drazin inverse* S^D*. Then, the following assertions are true for* S^D*:*

(i) $(S^D)^* = (S^*)^D$

(ii) $(S^D)^l = (S^l)^D$ *for* $l \in \mathbb{N}$

(iii) If S^l *with index 1 and* S *has index* k*, then* $(S^l)^\# = (S^D)^l$

(iv)) S^D *has index* 1*, and* $(S^D)^\# = S^2 S^D$*.*

(v) If ind$(S) = k$*, then* $R(S^l) = R(S^D)$ *and* $N(S^l) = N(S^D)$ *for* $l \geq k$*.*

1.4.3 Group Inverse

The name *group inverse* was opted by I. Erdélyi [17] because the powers of a given operator S, together with $S^\#$ as a inverse element of S and the projection operator

$S^{\#}S$ as a unit element, forms an Abelian group. Englefield [16] highlighted the spectral properties of the group inverse and named it as commuting reciprocal inverse. For the given operator S, if $N(S)$ and $R(S)$ are complementary subspaces then the group inverse exists. Also, it is equivalent to index of the operator S is one. If group inverse exists then it is unique. An element $S \in \mathcal{B}(H)$ is group invertible if and only if there is a projection (in general, idempotent) $P \in \mathcal{B}(H)$ such that $S + P$ is invertible, $SP = 0$ and $SP = PS$. If these conditions are satisfied, the group inverse $S^{\#}$ of S is given by $S^{\#} = (S+P)^{-1}(I - P)$, and the idempotent $P = S^{\pi} = I - SS^{\#}$. The general definition of group inverse is given below.

Definition 1.4.7 *Let $S \in \mathcal{B}(H)$ with $ind(S) = 1$. Then the group inverse $S^{\#}$ exists uniquely and that satisfies the operator equations mentioned below:*

$$SS^{\#}S = S;$$

$$S^{\#}SS^{\#} = S^{\#};$$

$$SS^{\#} = S^{\#}S.$$

Proposition 1.4.2 *Let $S \in \mathcal{B}(H)$ with the group inverse $S^{\#} \in \mathcal{B}(H)$. Then the following properties hold true for $S^{\#}$.*

(i) *If S is invertible then $S^{\#} = S^{-1}$.*

(ii) $S^{T\#} = S^{\#T}$.

(iii) $S^{*\#} = S^{\#*}$.

(iv) $(S^{\#})^{\#} = S$.

(v) *For $l \in \mathbb{N}$, $(S^{l})^{\#} = (S^{\#})^{l}$.*

Example 1.4.3 *Consider the subset of class of matrices $\mathbb{R}^{n\times n}$ as defined below.*

$$G = \{S \in \mathbb{R}^{n\times n} : S = \beta T,\ 0 \neq \beta \in \mathbb{R},\ \ T = (a_{ij}),\ a_{ij} = 1,\ i,j \in 1,2,...,n\}$$

In this case, G forms a multiplicative group. The multiplicative identity of G is $E = \frac{1}{n}T$. If $S = \beta T,\ \beta \neq 0$, then the group inverse of S is $S^{\#} = \frac{1}{\beta n^2}T$.

1.5 Iterative Methods using Splitting of Operators and Monotonicity

In the study of general system theory, mechanics, control theory, applied mathematics, economics et cetera, an important role is to solve linear systems of algebraic equations. Iterative methods are the most popular methods used to solve linear systems $Sx = b$. This idea was proposed by Gauss in 1823. The history of iterative techniques was given by Varga (see pages 1-2, [66]). For a matrix S, there are several iterative methods available, direct method like *Gaussian elimination;* iterative methods such as *Jacobi method* and *Gauss-Seidel method* which depend on decomposition of S into diagonal, lower triangular, upper triangular matrices to get iteration matrix. Initially, iterative methods are developed to solve integral operators. Gradually, it is extended to find the solution via generalized inverses of operator equations on Hilbert space and Banach space. To name few, steepest descent method [53], conjugate gradient method [28], and iterative methods by splitting of operator were discussed in [41]. A large and growing body of literature has investigated that for any iterative method, the standard convergence condition is the spectral radius of the iteration matrix/operator is less than one.

Suppose the matrix S is large in size and sparse, we prefer to apply iterative method rather than direct method. The iterative method has an advantage that during the computation, the initial matrix S is not altered. Hence the roundoff error is much less serious. Simplicity is another advantage of such method because in computation we have to do only vector addition and matrix-vector multiplication. The drawback with these methods are the rate of convergence may be slow or the method may even diverge, and in such case we need to find a stopping test. The theory of splittings is the major topic in the study of iterative approches. Therefore, it is essential to study the theory of splittings. Consider a real $n \times n$ matrix S which is decomposed to $S = U - V$. If U is invertible then a decomposition $S = U - V$ is called splitting. But, in this thesis, splitting means simply a decomposition of operator into difference of two operators. In analysis of iterative methods, splitting of a matrix is considered to study the convergence of iterative schemes to get the solutions of linear systems.

Let $Sx = b$ be the linear system of algebraic equation. Consider the splitting $S = U - V$, with U invertible. This splitting is used to associate S to an iterative method

$$Ux^{k+1} = Vx^k + b, \quad k \in \mathbb{N}_0$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This can be modified to iteration scheme

$$x^{k+1} = (U^{-1}V)x^k + U^{-1}b, \quad k \in \mathbb{N}_0.$$

In the above method, we call $U^{-1}V$ as iteration matrix/operator. The above iteration scheme converges to the solution if the spectral radius of iteration matrix $U^{-1}V$ is less than one and vice versa. Many results in literature shows that for certain kind of splittings of S, spectral radius of $U^{-1}V$ is less than one if and only if S is monotone. A real square matrix S is monotone if $Sx \geq 0$ implies $x \geq 0$. Here $x = (x_j) \geq 0$ means that $x_j \geq 0$ for all $j \in \{1, 2, ..., n\}$. If the matrix is rectangular or singular, the analogous iterative scheme is obtained for the system $Sx = b$ as follows: Let $S^\dagger$ be the generalized inverse of S, and with the splitting $S = U - V$. Then, we can write

$$Ux^{k+1} = Vx^k + b, \quad k \in \mathbb{N}_0$$

Since U is non-invertible, we can simplify as below:

$$x^{k+1} = (U^\dagger V)x^k + U^\dagger b, \quad k \in \mathbb{N}_0.$$

Above iterative methods are governed by convergence factor of the spectral radius of an iteration matrix $r(U^{\dagger}V) = \max\{|\lambda|, \lambda \in \sigma(U^{\dagger}V), \lambda \neq 1\}$. The history of above scheme goes back to Berman and Plemmons [5], who proved that the above iterative scheme converges to the solution if spectral radius of $r(U^{\dagger}V) < 1$ and vice versa. Also, it is shown that $r(U^{\dagger}V) < 1$ if and only if S is monotone operator. Monotonicity plays an important role in convex optimization, stochastic processes and numerical analysis (ref. [5],[66],[11]).

The first systematic study of monotone matrix was reported by Collatz in [11] while discussing the initial and boundary value problems. He points out that, a matrix is monotone if and only if matrix is invertible and the inverse of the matrix is nonnegative. Equivalently, matrix S is monotone if and only if S is positively invertible. Gil investigated the case of positive invertibility of integral operators over separable Hilbert lattices ([22], [20]). To obtain the necessary and/or sufficient conditions for the monotonicity of matrices much work has been made by several authors [21], [45], [19], [5]. Positive invertibility of matrices is characterized by Peris [58] which involve the splitting of matrices. Later it is called B-splitting. In addition to this result, Weber ([69], [70]) proved condition for positive invertibility of the operator with certain spectral property and positive splittings of an operator on ordered Banach space. This idea is used to obtain the cone nonnegativity of the Moore-Penrose inverses for operators on Hilbert spaces by authors Kurmayya and Sivakumar [49]. The topics discussed in the current thesis are motivated by the above mentioned problems.

1.6 Outline of the Thesis

The thesis is organized in the following way: In chapter 2, we discuss the Drazin inverse of bounded linear operators over Banach space. Nonnegative Drazin inverse with historical note are discussed in 2.1. Some basic results to define the index splitting and results on cone nonnegative operators are given in 2.2, which are used in the main part of the chapter. Next, in section 2.3 we introduce the index splitting of an operator and Drazin monotonicity. We categorize the index splitting of "first type ” and “second type .” Theorem 2.3.2 gives the convergence conditions for the solution of algebraic linear system of equations with the help of first type method of index splitting. In section 2.4, the main characterization of Drazin monotonicity is given. Further, in Theorem 2.4.1 we present the main results on characterization of Drazin monotonicity of bounded linear operator over ordered Banach space where order is being induced by a cone. We illustrate them with suitable examples in section 2.5.

In chapter 3, we present the different types of splittings of non-invertible bounded linear operator on Hilbert space with closed range. Section 3.1, deals with the introduction to the chapter. Section 3.2 contains the preliminary results of the chapter. In section 3.3, we give definition of proper splitting on Hilbert space, different conditions for splitting of non-invertible operator S and conditions to form $B_{\dagger}$-splitting. In section 3.4, we present the construction methods to obtain the $B_{\dagger}$-splitting of an operator with suitable examples. We provide the analogous theorem for group inverse of bounded operators.

In chapter 4, we discuss the unbounded Gram operators and cone nonnegativity characterizations of Moore–Penrose inverses. In particular, we consider the closed densely defined linear operator on ordered Hilbert space. This characterization is based on acuteness and obtuseness of a cone. Section 4.1 gives the introduction to the chapter with related literature review. In section 4.2 we present the preliminary results of the chapter. In section 4.3 we prove the main result with the help of sequence of lemmas. Illustrative examples are provided in section 4.4.

Finally, we summarize the contents of the thesis with conclusion, future work and present a list of references.

Chapter 2

A Characterization of Drazin Monotonicity of operators on Banach space

2.1 Introduction

In this chapter[1], we focus on the index splitting of a bounded linear operator on Banach space and characterize the cone nonnegativity of Drazin inverse of bounded linear operators defined on an ordered Banach space.

The theory of splitting of operators and matrices, apart from being a useful tool in convergence analysis of the iterative scheme, also provides interesting comparison results. Pye [59] investigated the class of nonnegative matrices which have a nonnegative Drazin inverse, obtained necessary and sufficient conditions for a real matrix to have nonnegative Drazin inverse. An interesting result derived by Eiermann et al. [15], for a square matrix S, where $(I - S)$ is possibly singular, the necessary and

[1]The contents of this chapter have been published as a research article under the same title, in *The Journal of Analysis*, 2023, https://doi.org/10.1007/s41478-023-00567-6

sufficient conditions implying that a semiiterative method tends to a solution vector which can be described in terms of the Drazin inverse of $(I - S)$. In [2], authors introduced a few matrix spliting arising from index-proper splittings and studied their convergence results as well as applications to multisplitting theory. Jena and Pani [38] introduced the notion of interval Drazin monotonicity. Then characterized interval Drazin monotonicity using the notion of interval boundedness. Also, proved that nonnegative decomposition satisfying an eigenvalue property implies that nonnegativity of Drazin inverse. In [43], one can see different applications of nonnegative Drazin inverse and Group inverse.

Wei [73] introduced the method of index splitting to solve the non-invertible linear system $Sx = b$ where $S \in \mathbb{R}^{n\times n}$. He continued the same study to characterize the nonnegativity conditions for Drazin inverse based on the index splitting of matrices [74].

In [10], the method of index splitting was generalized to bounded linear operator over Banach space where the index of an operator is assumed to be k. For this type of splitting, iterative method can be established for computing the solution of non-invertible operator equation $Sx = b, \;\; b \in R(S^k)$ and $ind(S) = k$. In [10], index splittings with different conditions on range and null space of an operator with index k are observed. Since this type of splittings occur frequently, these are called "First type" method of index splittings. In this chapter, we consider both types of method of index splittings which are defined on Banach space. These findings further support the idea of characterization of the cone nonnegativity condition for Drazin inverse (that is, Drazin monotonicity) over ordered Banach space. The motivation for this characterization comes from the classical results of [5]. The results from the monograph [45], helped to understand the ordered Banach space and prove the results of this chapter.

2.2 Preliminary Results

In this chapter, we consider the ordered Banach space $X = (X, C)$ with order being induced by the cone C. An operator $S \in \mathcal{B}(X)$ be bounded linear operator from X

into itself with closed range $R(S^k)$ and $ind(S) = k$. We recall the definition of Drazin inverse of bounded operator over Banach space.

Definition 2.2.1 ***Drazin Inverse:*** *Let $S \in \mathcal{B}(X)$. For some nonnegative integer k, If there exists $X \in \mathcal{B}(X)$ which satisfies the conditions: $SXS^k = S^k$; $XSX = X$; $SX = XS$; then such operator B is said to be Drazin inverse of S and denoted by S^D. The smallest such integer k is called index of S and denoted by $ind(S) = k$. If $ind(S) = 1$ then Drazin inverse is called group inverse and denoted by $S^{\#}$.*

If $S \in \mathcal{B}(X)$ has a Drazin inverse, then it is unique. The following lemma gives the conditions for the existence of the Drazin inverse of bounded operator over Banach space.

Lemma 2.2.1 *(Lemma 2.1, [10]) An operator $S \in \mathcal{B}(X)$ has a Drazin inverse S^D if it has finite ascent and finite descent, vice versa. In this case, $\alpha(S) = \delta(S) = k = ind(S)$ is the smallest nonnegative integer which satisfies the conditions of Drazin inverse.*

The following proposition gives the conditions for Drazin inverse to be continuous operator(that is, bounded linear operator).

Proposition 2.2.1 *(Chapter 12, [68]) Let $S \in \mathcal{B}(X)$ with $ind(S) = k$ which is finite. Then there exists Drazin inverse $S^D \in \mathcal{L}(X)$. If $R(S^k)$ is closed subspace then $S^D \in \mathcal{B}(X)$. Moreover, $N(S^k) = N(S^D)$ and $R(S^k) = R(S^D)$.*

In the rest of this section, we collect some results that will be used in the main results of this chapter.

Lemma 2.2.2 *Suppose $S \in \mathcal{B}(X)$. Then for nonnegative integers i, j there exists an algebraic isomorphism $\cong$ between subspaces such that*

$$\frac{R(S^i)}{R(S^{i+j})} \cong \frac{X}{R(S^j) + N(S^i)}$$

Theorem 2.2.2 *(Theorem 2.1, [10]) Let* $S \in \mathcal{B}(X)$*. For operator* S*,* $ind(S) = k$ *if* $k = \min\{l : R(S^l) \oplus N(S^l) = X\}$ *and vice versa.*

Proof If $ind(S) = k$, then from Lemma 1.3.4 and Lemma 2.2.1 we have

$$R(S^k) \oplus N(S^k) = X.$$

Then there is a nonnegative integer $p \leq k$ such that

$$R(S^p) \oplus N(S^p) = X.$$

From this, $N(S^p) = N(S^{p+1})$. Suppose a contrary that $N(S^p) \neq N(S^{p+1})$. We have $N(S^p) \subset N(S^{p+1})$, so there exists a $x \in X$ such that $x \in N(S^{p+1})$ but $x \notin N(S^p)$, that is $S^{p+1}x = 0$ but $S^p x \neq 0$. Let $b = S^p x \in R(S^p)$, then $S^p b = S^{2p} b = 0$. That is $b \in N(S^p)$. Thus $0 \neq b = S^p x \in R(S^p) \cap N(S^p) = \{0\}$ is a contradiction. From $N(S^p) = N(S^{p+1})$ we have $N(S^p) = N(S^{p+l})$, for all $l \in \mathbb{N}$. Consequently, $\alpha(S) \leq p \leq k$. Meanwhile, from the Lemma 2.2.1, $\alpha(S) = ind(S) = k \geq p$. Hence, either k is the minimum non-negative integer or $p = k$ which satisfies the decomposition $R(S^k) \oplus N(S^k) = X$.

Conversely, assume $k = \min\{l : R(S^l) \oplus N(S^l) = X\}$. By reiterating the above, we get $\alpha(S) \leq k$. In Lemma 2.2.2, consider the equality of $k = i = j$, we obtain

$$\frac{R(S^k)}{R(S^{2k})} \cong \frac{X}{R(S^k) + N(S^k)}$$

Since $N(S^k) + R(S^k) = X$, we get $R(S^k) = R(S^{2k})$. Note that $R(S^k) \supset R(S^{k+1}) \supset R(S^{k+2}) \supset ... \supset R(S^{2k})$. So, $R(S^k) = R(S^{k+1}) = R(S^{k+2}) = ... = R(S^{2k})$. Hence $R(S^k) = R(S^{k+l})$ for $l \in \mathbb{N}$. So, $\delta(S) \leq k$. From Lemma 2.2.1, we obtain $k = ind(S) = \alpha(S) = \delta(S)$. If $ind(S) = p < k$, then $R(S^p) \oplus N(S^p) = X$, which is a contradiction to the given definition of k. Hence $ind(S) = k$.

The following lemma describes the Drazin inverse solution to the singular operator equation.

Lemma 2.2.3 *Consider* $S \in \mathcal{B}(X)$ *be non-invertible operator with* $ind(S) = k$. *Then* $x = S^D b$ *is the unique solution of the linear system*

$$Sx = b, \quad x \in R(S^k).$$

Proof Consider a solution x given by an equation $x = S^k y$ for some y. We can write $S^{k+1}y = b$, and

$$\begin{aligned} x &= S^k y = S^{k+1} S^D y \\ &= S^D S^{k+1} y \\ &= S^D b. \end{aligned}$$

Uniqueness of the solution is follows from $R(S^k) \cap N(S^k) = \{0\}$.

We discussed the positive invertible operator over ordered vector space in Definition 1.3.22. Suppose the operator $S \in \mathcal{B}(X)$ is singular and there exists a Drazin inverse $S^D \in \mathcal{B}(X)$ as a generalized inverse of S. Then similar to the positive invertibility, the definition of Drazin monotonicity will be given as follows:

Definition 2.2.3 ***Drazin Monotonicity.*** *Consider a Banach space* $X = (X, C)$ *be ordered by a cone* C. *An operator* $S \in \mathcal{B}(X)$ *with* $ind(S) = k$ *and* S^D *exists. Operator* S *is said to be Drazin monotone with respect to the cone* C *in* H *if* $S^D C \subseteq C$.

Taylor and Lay [64] proved that $r(S) = \lim_{n\to\infty} |S^n|^{\frac{1}{n}}$, where $S \in \mathcal{B}(X)$. By using this fact, the following lemma can be derived.

Lemma 2.2.4 *Suppose* $S \in \mathcal{B}(X)$. *Then* $\underset{n\to\infty}{lim} S^n = 0$ *if and only if* $r(S) < 1$.

Next, we state two theorems from [45], which will be helpful to prove the main results of this chapter.

Theorem 2.2.4 *Consider $X = (X, C)$ be a Banach space ordered by a cone C. Defined an operator $S \in \mathcal{B}(X)$ such that $S \geq 0$. Then $r(S) < 1$ implies $(I - S)^{-1} \geq 0$. Moreover, if the cone C is normal and generating, the existence of $(I - S)^{-1}$ and $(I - S)^{-1} \geq 0$ implies that $r(S) < 1$.*

Theorem 2.2.5 *Consider a Banach space $X = (X, C)$ ordered by a cone C which is normal and reproducing. Let S, $T \in \mathcal{B}(X)$ such that*

$$-Tx \leq Sx \leq Tx \quad \text{for} \quad x \in C.$$

Then $r(S) \leq r(T)$.

2.3 Index Splitting of Operators over Banach Space

In this section, we define the two types of index splittings of bounded linear operators and the relevant results. Let $S \in \mathcal{B}(X)$ be non-invertible operator with $ind(S) = k$ and $R(S^k)$ be closed subspace of X. Consider the splitting $S = U - V$ of S such that $U, V \in \mathcal{B}(X)$. This splitting can be classified into two types depending on $ind(U)$. The following is the definition of first type.

Definition 2.3.1 ***First type method of Index splitting:*** *Let $S \in \mathcal{B}(X)$ with $ind(S) = k$. The splitting $S = U - V$ is called first type method of Index splitting, if it satisfies the conditions $R(U^k) = R(S^k)$ and $N(U^k) = N(S^k)$.*

The following theorem [10] gives the Drazin inverse formula for first type method of index splitting. This is used to prove the main result.

Theorem 2.3.2 *Consider $S \in \mathcal{B}(X)$ with $ind(S) = k$. The splitting $S = U - V$ being an index splitting of S satisfies $R(U^k) = R(S^k)$, $N(U^k) = N(S^k)$. Then* *(a)* $ind(U) = k$.

(b) $S^D = (I - U^D V)^{-1} U^D$.

(c) *There is an iteration scheme*

$$x^{(n+1)} = U^D V x^{(n)} + U^D b, \text{ for } n \geq 0$$

which converges to $S^D b$ *for all initial vector* $x^{(0)} \in X$ *if and only if* $r(U^D V) < 1$.

Proof (a) Given that $R(U^k) = R(S^k)$ and $ind(S) = k$. From Theorem 2.2.2, we get $ind(U) = ind(S) = k$.
(b) First we show that $I - U^D V$ is invertible. Assume that for $p \in X$, $(I - U^D V)p = 0$. Then $p = U^D V p \in R(U^D) = R(U^k) = R(S^k)$. Meanwhile, we get

$$\begin{aligned} 0 &= U^D U (I - U^D V) p \\ &= (U^D U - U^D U U^D V) p \\ &= U^D (U - V) p \\ &= U^D S p \end{aligned}$$

which gives $Sp \in N(U^D) = N(U^k) = N(S^k)$ and gives $p \in N(S^{k+1})$. Note that $\alpha(S) = ind(S) = k$, which gives $N(S^k) = N(S^{k+1})$. Hence $p \in N(S^k)$. Therefore $p \in N(S^k) \cap R(S^k) = \{0\}$, that is, $p = 0$. So $N(I - U^D V) = \{0\}$. Hence $I - U^D V$ is invertible. Now consider

$$\begin{aligned} (I - U^D V) S^D =& S^D - U^D (U - S) S^D \\ =& S^D - U^D U S^D + U^D S S^D \\ =& S^D - S^D + U^D \\ =& U^D \end{aligned}$$

Hence we obtain the expression for Drazin inverse $S^D = (I - U^D V)^{-1} U^D$.

(c) The iteration scheme

$$x^{(n+1)} = U^D V x^{(n)} + U^D b, \text{ for } n \geq 0$$

will converge to the solution $S^D b$ for every initial vector $x^{(0)} \in X$. The solution is

$$S^D b = U^D V S^D b + U^D b.$$

The error term is given by

$$\begin{aligned} x^{(n+1)} - S^D b =& U^D V x^{(n)} + U^D b - (U^D V S^D b + U^D b) \\ =& U^D V (x^{(n)} - S^D b) \\ =& (U^D V)^{n+1} (x^{(0)} - S^D b). \end{aligned}$$

By assumption, we have for each initial vector $x^{(0)}$, the n^{th} iteration term $x^{(n)}$ converges to $S^D b$. Hence we obtain $\lim\limits_{n \to \infty} (U^D V)^n = 0$. Then by Lemma 2.2.4, we get $r(U^D V) < 1$.

Conversely, by reiterating the scheme, we get

$$x^{(n+1)} = U^D V x^{(n)} + U^D b = ... = (U^D V)^{n+1} x^{(0)} + \sum_{i=0}^{n} (U^D V)^i U^D b.$$

By Lemma 2.2.4, if $r(U^D V) < 1$, then $\lim\limits_{n+1 \to \infty} (U^D V)^{n+1} = 0$ and $\sum_{i=0}^{\infty} (U^D V)^i = (I - U^D V)^{-1}$. So $x^{(n+1)}$ converges to $(I - U^D V)^{-1} U^D b = S^D b$ for each $x^{(0)} \in X$.

The special case S is invertible, that is for $ind(S) = \alpha(S) = \delta(S) = 0$ is given in the corollary.

Corollary 2.3.1 *Let $S \in \mathcal{B}(X)$ be invertible. Consider the splitting $S = U - V$ be such that U is invertible. Then,*

$$S^{-1} = (I - U^{-1} V)^{-1} U^{-1}$$

and the iteration

$$x^{(n+1)} = U^{-1}Vx^{(n)} + U^{D}b, \text{ for } n \geq 0$$

converges to $S^{-1}b$ *for each initial vector* $x^{(0)} \in X$ *if and only if* $r(U^{-1}V) < 1$.

Remark 2.3.1 *The case of above Theorem 2.3.2 reduces to well known proper splitting when* $ind(S) = \alpha(S) = \delta(S) = 1$ *and the operator* $S \in \mathcal{B}(H)$.

Next, we give the definition of second type method of index splitting. In this type of splitting, operator S has index k but the operator U has the index 1.

Definition 2.3.3 ***Second type method of index splitting:*** *Let* $S \in \mathcal{B}(X)$ *with* $ind(S) = k$. *The splitting* $S = U - V$ *is called second type method of Index splitting, if it satisfies the conditions* $R(U) = R(S^k)$ *and* $N(U) = N(S^k)$.

The following theorem gives the convergence criteria of the iterative scheme and explicit formula for Drazin inverse of an operator S having a second type method of index splitting. Since, $ind(U) = 1$, operator U involves with group inverse.

Theorem 2.3.4 *Let* $S \in \mathcal{B}(X)$ *with* $ind(S) = k$. *A splitting*

$$S = U - V$$

be second type index splitting such that $R(U) = R(S^k)$, $N(U) = N(S^k)$. *Then the following conditions are true.*

(a) $ind(U) = 1$;

(b) $S^D = (I - U^{\#}V)^{-1}U^{\#}$;

(c) The iteration scheme

$$x^{(n+1)} = U^{\#}Vx^{(n)} + U^{\#}b, \quad \text{for} \quad n \geq 0$$

converges to the solution $S^D b$ *for every initial vector* $x^{(0)} \in X$ *if and only if* $r(U^{\#}V) < 1$.

Proof *The proof of the theorem is analogous to proof of Theorem 2.3.2.*

In the following example, we observe the case of second type of index splitting for operator on finite dimensional case.

Example 2.3.1 *Consider a linear operator* $S \in \mathcal{B}(\mathbb{R}^4)$, *which can be represented by a matrix*

$$S = \begin{pmatrix} 2 & 4 & 6 & 5 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -2 & -3 & -3 \end{pmatrix}$$

where $ind(S) = 2$, $rank(S^2) = 2$. *Choose*

$$U = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{2}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{2}{5} & -\frac{2}{5} \end{pmatrix}$$

and $V = U - S$. *Since the* $ind(U) = 1$, *we get the group inverse of* U *as*

$$U^{\#} = \begin{pmatrix} 13 & 12 & 25 & 25 \\ 12 & 13 & 25 & 25 \\ -5 & -5 & -10 & -10 \\ -5 & -5 & -10 & -10 \end{pmatrix}$$

and hence $S^D = (I - U^{\#}V)^{-1}U^{\#}$. *So,*

$$S^D = \begin{pmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}$$

2.4 A Characterization of Drazin Monotonicity

In this section, we state and prove the equivalent conditions for characterization of the cone nonnegativity of Drazin inverse of bounded linear operators over ordered Banach space, where order is induced by a cone. The details on the concept of a cone was discussed in section 1.2.

The Drazin monotonicity seen in the Definition 2.2.3 is characterized for first type method of index splitting in the following theorem:

Theorem 2.4.1 *Consider an ordered Banach space* $X = (X, C)$ *with normal and reproducing cone* C. *Let* $S \in \mathcal{B}(X)$ *with* $ind(S) = k$. *Let* $S = U - V$ *be first type method of Index splitting such that* $R(U^k) = R(S^k)$ *and* $N(U^k) = N(S^k)$. *If* $U^D(C) \subseteq C$ *and* $U^D V \geq 0$ *then the following statements are equivalent:*

(i) $S^D(C) \subseteq C$.

(ii) $S^D V \geq 0$.

(iii) $r(U^D V) < 1$.

Proof We prove the above equivalent conditions as below:

$$(iii) \implies (ii) \implies (i) \implies (iii).$$

$(iii) \implies (ii)$: Let $r(U^D V) < 1$. From Theorem 2.3.2, we have $S^D = (I - U^D V)^{-1} U^D$. Consider,

$$\begin{aligned} S^D V &= (I - U^D V)^{-1} U^D V \\ &= \sum_{i=0}^{\infty} (U^D V)^i U^D V \\ &= \sum_{i=1}^{\infty} (U^D V)^i \end{aligned}$$

Since $U^D V \geq 0$ which gives $(U^D V)^i \geq 0$ for all $i = 1, 2, 3, \ldots.$ Hence $S^D V = \sum_{i=1}^{\infty} (U^D V)^i \geq 0$.

$(ii) \implies (i)$: Let $S^D V \geq 0$. i.e. $S^D V(C) \subseteq C$. Using the given $U^D(C) \subseteq C$ and $U^D V(C) \subseteq C$ we obtain

$$\begin{aligned} S^D(C) &= (I - U^D V)^{-1} U^D(C) \\ &= \sum_{i=0}^{\infty} (U^D V)^i U^D(C) \\ &\subseteq \sum_{i=0}^{\infty} (U^D V)^i (C) \\ &\subseteq (C). \end{aligned}$$

Hence $S^D(C) \subseteq (C)$ which is merely $S^D \geq 0$.

$(i) \implies (iii)$: Given that $S^D \geq 0$. To prove $r(U^D V) < 1$, first we show that $(I - U^D V)^{-1} \geq 0$. Then by the Theorem 2.2.4 we arrive at conclusion. From Theorem

2.3.2, we have $S^D = (I - U^D V)^{-1} U^D$. Using the given fact, $(I - U^D V)^{-1} U^D \geq 0$. Since $U^D \geq 0$ we get $(I - U^D V)^{-1} \geq 0$. The identity

$$(1 - \epsilon)I - U^D V = (I - U^D V)[I - \epsilon(I - U^D V)^{-1}]$$

implies that the operator $(1 - \epsilon)I - U^D V$ is positively invertible for a small $\epsilon > 0$. So,

$$[(1 - \epsilon)I - U^D V]^{-1} = \sum_{i=0}^{\infty} (1 - \epsilon)^{-1} (U^D V)^i$$

Moreover, the relation

$$\sum_{i=0}^{n} (1 - \epsilon)^{-i} (U^D V)^i = (1 - \epsilon)[(1 - \epsilon)I - U^D V]^{-1}[I - (1 - \epsilon)^{-n-1}(U^D V)^{n+1}]$$

implies that

$$0 \leq (1 - \epsilon)^{-n} (U^D V)^n \leq (1 - \epsilon)[(1 - \epsilon)I - U^D V]^{-1}$$

for $n = 1, 2, ...$ and small $\epsilon > 0$. From these estimates and Theorem 2.2.5, it follows that

$$r[(1 - \epsilon)^{-n} (U^D V)^n] \leq (1 - \epsilon) r\{[(1 - \epsilon)I - U^D V]^{-1}\} = a(\epsilon)$$

Consequently, $r(U^D V) \leq (1 - \epsilon)[a(\epsilon)]^{\frac{1}{n}}$ for $n = 1, 2, ...$ Therefore $r(U^D V) \leq 1 - \epsilon < 1$. Hence all the conditions are equivalent.

Remark 2.4.1 *In article [10], we observed second type method of index splitting. We discuss the results of second type method of index splitting below.*

Next, we state the theorem that characterize Drazin monotonicity of bounded linear operator over ordered Banach space with second type method of Index splitting.

Theorem 2.4.2 *Consider an ordered Banach space $X = (X, C)$ with normal and reproducing cone C. Let $S \in \mathcal{B}(X)$ with ind$(S) = k$. Let $S = U - V$ be second type method of index splitting such that $R(U) = R(S^k)$ and $N(U) = N(S^k)$. If $U^{\#}(C) \subseteq C$ and $U^{\#} V \geq 0$ then the following conditions are equivalent.*

(i) $S^D(C) \subseteq C$

(ii) $S^D V \geq 0$

(iii) $r(U^{\#}V) < 1.$

Proof *We omit the proof since it is similar to the proof of Theorem 2.4.1.*

2.5 Illustrations

In the following example, we illustrate Theorem 2.4.1 for an operator defined on infinite dimensional ordered Banach space.

Example 2.5.1 *Let* $X = l^1(\mathbb{N}) = \{(x_i) : x_i \in \mathbb{R}, \sum_{i=1}^{\infty}|x_i| < \infty\}$ *be real Banach space with* $||\cdot||_1$. *Define operator* $S : l^1(\mathbb{N}) \to l^1(\mathbb{N})$ *by*

$$S(x_1, x_2, x_3, ...) = (x_1, 0, x_3, 0, x_5, ...)$$

i.e. $(Sx)_j = \begin{cases} x_j & \text{if } j \text{ odd} \\ 0 & \text{otherwise} \end{cases}$

We see that null space $N(S) = \{(0, x_2, 0, x_4, ...) \in X : x_i \in \mathbb{R}\} \neq \emptyset$ *and range space* $R(S) = \{(x_1, 0, x_3, 0, x_5, ...) \in X : x_i \in \mathbb{R}\}$. *We observe that* S *is a non-invertible operator and range space is proper closed subspace. From range space and null space of iterative operator* S^n, *we have* $R(S) = R(S^2) = R(S^3) = ...$ *and* $N(S) = N(S^2) = N(S^3) = ...$, *respectively. Hence* $\alpha(S) = \delta(S) = ind(S) = 1$. *By direct computation, we have* $S^D = S^{\#} = S$. *Consider the cone* $C = l^1_+(\mathbb{N}) = \{x = (x_1, x_2, ...) \in X : x_n \geq 0, \ \forall \ n \in \mathbb{N} \}$ *and* C *is normal and reproducing cone. With respect to this cone,* $X = (l^1(\mathbb{N}), l^1_+(\mathbb{N}), ||\cdot||_1)$ *is ordered Banach space.*

Next, consider the index splitting $S = U - V$ *with* $Ux = (3x_1, 0, x_3, 0, x_5, 0, ...)$ *and* $V = U - S$. *Then,* $U^D x = U^D(x_1, x_2, x_3, ...) = (\frac{1}{3}x_1, 0, x_3, 0, x_5, ...)$ *and*

$$\begin{aligned} U^D V x &= U^D V(x_1, x_2, x_3, ...) \\ &= (\frac{2}{3}x_1, 0, 0, 0, ...). \end{aligned}$$

Hence we can verify that $U^D V(C) \subseteq C$ *and* $U^D(C) \subseteq C$. *Also by computation, we get* $r(U^D V) = \frac{2}{3} < 1$. *Hence by the equivalent conditions of Theorem 2.4.1, we conclude that* $S^D(C) \subseteq C$ *that is* $S^D \geq 0$, *Drazin monotonicity of* S.

Illustration of Theorem 2.4.1 is comparatively easy in the finite dimensional case. The following example gives the illustration of Theorem 2.4.1 for an operator defined on finite dimensional ordered Banach space.

Example 2.5.2 *Consider the finite dimensional Banach space* $(\mathbb{R}^4, \mathbb{R}^4_+, ||\cdot||_2)$ *where* $\mathbb{R}^4_+$ *be standard positive cone in* $\mathbb{R}^4$. *Define* $S : \mathbb{R}^4 \to \mathbb{R}^4$ *which have a matrix representation as shown below.*

$$S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $ind(S) = 2$. *Consider first type method of Index splitting, by taking*

$$U = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $V = U - S$. *Here* $R(U^2) = R(S^2)$ *and* $N(U^2) = N(S^2)$. *Computing* U^D *and* $U^D V$

$$we\ get,\ U^D = \begin{bmatrix} 0 & 0 & \frac{1}{50} & 0 \\ 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\ and\ \ U^DV = \begin{bmatrix} 0 & 0 & \frac{1}{10} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ respectively.$$

We observe that $U^D \geq 0$ *and* $U^DV \geq 0$. *Moreover,* $r(U^DV) = \frac{1}{2} < 1$. *Hence by equivalent conditions of Theorem 2.4.1, we conclude that* $S^D(\mathbb{R}^4_+) \subseteq \mathbb{R}^4_+$ *i.e.* $S^D \geq 0$.

Chapter 3

On $B_{\dagger}-$ Splitting and Nonnegativity of Moore-Penrose Inverses of operators

3.1 Introduction

The problem of characterizing positively invertible matrices has been abundantly studied in the literature (referred here and there in [5]). For instance, Johnson et al., ([39],[40]) characterized the positive invertibility and inverse nonnegative matrices in terms of sign patterns of the matrices. The positive invertibility of the matrix S can be characterized in terms of all positive splittings of S which satisfies eigenvalue property. Peris [58] investigated the particular positive splitting (which is called B-splitting) to characterize positively invertible matrix S. Also, he shown that a matrix is positively invertible if and only if it has a B-splitting satisfying the certain eigenvalue property.

Theorem 3.1.1 *(Theorem 5, [58]) Let $S \in \mathbb{R}^{n\times n}$. Then the following assertions are equivalent:*

(i) S is inverse-positive.

(ii) $S = U - V$ is the B-splitting such that $r(U^{-1}V) < 1$.

Note that, the term "inverse-positive" is used in the literature by many authors. This term is equivalent to "positive invertibility". The later term is used in the present thesis.

Weber [71] considered the more general case of [58]. He considered the positive splitting of an invertible operator acting on ordered normed space with certain kind of spectral property; shown the sufficient condition for the operator to be positively invertible and with additional requirements on the operator, it is necessary. Kurmayya and Sivakumar [49] extended the spectral property result of positive invertible operators [71] to non-invertible operators in the light of Moore-Penrose inverse as a generalized inverse. Later, over ordered Banach space, Weber [72] proved that positively invertible operator S possesses a B-decomposition if there exists a uniformly positive functional. Mishra and Sivakumar [51] defined the $B_\dagger$-splitting as a generalization of B-splitting for the non-invertible operators on finite dimensional space ordered by a standard cone and proved the analogous results [72]. In this article, $B-$splitting is generalized from finite dimensional to infinite dimensional and from classical inverses to generalized inverses. The generalized splitting is called $B_\dagger-$splitting. We discuss the existence of $B_\dagger-$splitting by characterizing the nonnegativity of Moore-Penrose inverse of bounded operator over ordered Hilbert space, where order is being induced by a cone.

3.2 Preliminaries

In this section, we recall the relevant results that will be used in the later sections. First, we discuss the finite-rank operators.

Definition 3.2.1 *(Page no.63,[24]) Let $S \in \mathcal{B}(H_1, H_2)$. If range of S is finite dimensional, then S is called finite-rank operator. Specifically, if the dimension of the range space is one, then such operator is called rank-one operator.*

Now, we recall the concepts on non-invertible bounded linear operators over Hilbert space. For more details on these results refer [4]. For an operator $S \in \mathcal{B}(H_1, H_2)$ with closed range, there exists a unique generalized inverse in $\mathcal{B}(H_2, H_1)$, called the Moore-Penrose inverse and it is denoted by $S^\dagger$.

Definition 3.2.2 *Let $S \in \mathcal{B}(H_1, H_2)$ with closed range. Then the Moore-Penrose inverse $S^\dagger \in \mathcal{B}(H_2, H_1)$ satisfies the following four equations.*

$$SS^\dagger S = S$$

$$S^\dagger SS^\dagger = S^\dagger$$

$$(SS^\dagger)^* = SS^\dagger$$

$$(S^\dagger S)^* = S^\dagger S$$

Equivalently, $SS^\dagger = P_{R(S)}$, $S^\dagger S = P_{R(S^*)}$ where P is the projection operator.

The following result is fundamental in the study of operator equation and the proof follows from the existing results in [4].

Lemma 3.2.1 *Let $S \in \mathcal{B}(H_1, H_2)$ and $b \in H_2$. The system $Sx = b$ has a solution if and only if $SS^\dagger b = b$. In that case, the general solution is given by $x = S^\dagger b + z$ for some $z \in N(S)$.*

In the following lemma, the formula for Moore-Penrose inverse of S is given, which follows the reverse order law with certain conditions.

Lemma 3.2.2 *(Page no. 223, [35]) Consider $S_1 \in \mathcal{B}(H_3, H_2)$ and $S_2 \in \mathcal{B}(H_1, H_3)$ with $R(S_2) = H_3 = R(S_1^*)$. Define $S \in \mathcal{B}(H_1, H_2)$ by $S = S_1 S_2$. Then*

$$\begin{aligned} S^\dagger &= S_2^*(S_2 S_2^*)^{-1}(S_1^* S_1)^{-1} S_1^* \\ &= S_2^\dagger S_1^\dagger \end{aligned}$$

A sufficient condition for $\mathcal{B}_+(X,Y)$ to be a generating cone is given in the following theorem.

Theorem 3.2.3 *([67]) Consider ordered normed spaces $X,\ Y$ with closed and normal cones $C_1,\ C_2$ respectively. Suppose Y is a Dedekind complete vector lattice and C_2 is solid. Then the pointed cone $\mathcal{B}_+(X,Y)$ is generating in $\mathcal{B}(X,Y)$.*

The following theorem gives the condition for the pointed cone $\mathcal{B}_+(X,Y)$ to be solid.

Theorem 3.2.4 *([67]) Consider ordered normed spaces $X,\ Y$. The pointed cone $\mathcal{B}_+(X,Y)$ to have a nonempty interior if and only if the cone C_2 has interior points and the cone C_1 allow plastering.*

Theorem 3.2.5 *(Theorem 25.1, [45]) Let X be an ordered Banach space. Consider $S \in \mathcal{B}(X)$ be such that $S \geq 0$. Then $r(S) < 1$ implies $(I-S)^{-1} \geq 0$, and conversely, if cone C is normal and reproducing then the existence of $(I-S)^{-1}$ and $(I-S)^{-1} \geq 0$ guarantee that $r(S) < 1$.*

Theorem 3.2.6 *(Theorem 25.4, [45]) Consider an ordered Banach space X with a normal cone C which satisfies $C^\circ \neq \emptyset$. Let $S,T : X \to X$ be two bounded operators such that T is positively invertible and $S \leq T$. Then S is positively invertible if and only if $S(C) \cap C^\circ \neq \emptyset$.*

Remark 3.2.1 *Let f is a uniformly positive functional on H_1 with $\delta > 0$. That is $f(x) \geq \delta ||x||$ for all $x \in C_1$ and $u \in C_2$. For some $\epsilon > 0$ consider the ball $B(u,\epsilon) \subset C_2$. Then the ball $B(f \otimes u, \delta\epsilon)$ belongs to $\mathcal{B}_+(H_1,H_2)$. As a result, the rank one operator $f \otimes u$ is an interior point of $\mathcal{B}_+(H_1,H_2)$. Additionally, for any operator $T \in \mathcal{B}(H_1,H_2)$ there is a number $\zeta > 0$ such that $\pm T \leq \zeta(f \otimes u)$ because any interior point of a cone is order unit.*

3.3 Splittings of Non-invertible Bounded Operators over Hilbert Spaces

In this section we introduce $B_{\dagger}$-splittings for non-invertible bounded linear operator S defined on ordered real Hilbert space, and prove some results related to such splittings. First, we recall the definition of proper splitting.

Definition 3.3.1 *(Page no. 278, [29]) An operator $S \in \mathcal{B}(H_1, H_2)$ with closed range. A splitting $S = U - V$ is said to be proper splitting if $R(S) = R(U)$ and $N(S) = N(U)$.*

Theorem 3.3.2 *Let $S \in \mathcal{B}(H_1, H_2)$ with closed range. Consider a proper splitting $S = U - V$. Then*

(i) $SS^{\dagger} = UU^{\dagger}$; $S^{\dagger}S = U^{\dagger}U$

(ii) $S = (I - VU^{\dagger})U$

(iii) $(I - VU^{\dagger})$ *is invertible.*

(iv) $S^{\dagger} = U^{\dagger}(I - VU^{\dagger})^{-1}$.

Proof (i) Since $R(S) = R(U)$, we get

$$SS^{\dagger} = P_{R(S)} = P_{R(U)} = UU^{\dagger}$$

and

$$S^{\dagger}S = P_{R(S^*)} = P_{N(S)^{\perp}} = P_{N(U)^{\perp}} = U^{\dagger}U$$

(ii) Since $N(S) = N(U)$ we have $N(S) \subseteq N(V)$. i.e. $R(S^*)^{\perp} \subseteq R(V^*)^{\perp}$. From this we obtain $R(V^*) \subseteq R(S^*) = N(S)^{\perp} = N(U)^{\perp} = R(U^*)$. Hence $R(V^*) \subseteq R(U^*)$

implies $U^\dagger UV^* = V^*$. Taking adjoint on both side gives $VU^\dagger U = V$. This deduces

$$S = U - V = U - VU^\dagger U = (I - VU^\dagger)U$$

The proof of result (iii) is available in [29]. However, for the completeness, the proof is given here.

(iii) To show $(I - VU^\dagger)$ is invertible, it is sufficient to show that '1' is not an eigenvalue of $VU^\dagger$. If $(I - VU^\dagger)x = 0$ for some nonzero vector x, then $x = VU^\dagger x \in R(V) \subseteq R(S) = R(U)$. Also, $x \in R(U)$ implies that $x = UU^\dagger x$. Therefore, $x = VU^\dagger x = (U - S)U^\dagger x = UU^\dagger x - SU^\dagger x = x - SU^\dagger x$ which gives $x = x - SU^\dagger x$. Hence $SU^\dagger x = 0$. This implies $U^\dagger x \in N(S) = N(U)$. So, $UU^\dagger x = x = 0$ which is a contradiction for x is an eigenvector with eigenvalue 1. Hence $(I - VU^\dagger)$ is invertible.

(iv) Let $S = S_1 S_2$ where $S_1 = (I - VU^\dagger)$ and $S_2 = U$. Since $R((I - VU^\dagger)^*) = H_2 = R(U) = R(S)$ and from Lemma 3.2.2 we get

$$\begin{aligned} S^\dagger &= S_2^\dagger S_1^\dagger \\ &= U^\dagger (I - VU^\dagger)^{-1} \end{aligned}$$

We begin with the definition of $B_\dagger$-splitting.

Definition 3.3.3 *Let $S \in \mathcal{B}(H_1, H_2)$ with closed range and $H_1 = (H_1, C_1)$ and $H_2 = (H_2, C_2)$. A proper splitting $S = U - V$ is called a $B_\dagger$-splitting if it satisfies the following conditions.*

(i) $U \geq 0$ *i.e.* $UC_1 \subseteq C_2$

(ii) $V \geq 0$ *i.e.* $VC_1 \subseteq C_2$

(iii) $VU^\dagger \geq 0$ *i.e.* $VU^\dagger C_2 \subseteq C_2$

(iv) $Sx, Ux \in C_2 + N(S^*)$ *and* $x \in R(S^*) \implies x \in C_1$.

Theorem 3.3.4 *Let $S \in \mathcal{B}(H_1, H_2)$ with closed range and $H_1 = (H_1, C_1)$, $H_2 = (H_2, C_2)$ with cone C_1 be solid and normal cone. Consider the following conditions.*

(i) $S^\dagger C_2 \subseteq C_1$

(ii) $Sx \in C_2 + N(S^*),\ x \in R(S^*) \implies x \in C_1$.

(iii) $C_2 \subseteq SC_1 + N(S^*)$

(iv) $\exists\ x_0 \in C_1 \cap R(S^*)$ *and* $z_0 \in N(S^*)$ *such that* $Sx_0 + z_0 \in C_2^\circ$.

Then we have (i) $\iff$ *(ii)* $\implies$ *(iii)* $\implies$ *(iv).*
Suppose S has a $B_\dagger$-splitting, then each of the above is equivalent to the following.

(v) $r(VU^\dagger) < 1$.

Proof (i) $\implies$ (ii): Let $Sx \in C_2 + N(S^*),\ x \in R(S^*)$. Given that $S^\dagger C_2 \subseteq C_1$. $x = S^\dagger Sx = S^\dagger(u+v) = S^\dagger u + S^\dagger v$ where $u \in C_2,\ v \in N(S^*)$. Since $N(S^*) = N(S^\dagger)$ we get $x = S^\dagger Sx = S^\dagger u \in C_1$. Hence $S^\dagger Sx = x \in C_1$.
(ii) $\implies$ (i): Let $Sx = u + v,\ u \in C_2,\ v \in N(S^*)$. Take $u = Sx - v$ and operate $S^\dagger$ on both side, we get $S^\dagger u = S^\dagger(Sx - v) = S^\dagger Sx - S^\dagger v = x - 0 = x \in C_1$. Hence $S^\dagger C_2 \in C_1$.
(ii) $\implies$ (iii): Let $y \in C_2$ and set $x \in S^\dagger y$. Then $x \in R(S^*)$. By Lemma 3.2.1, the general solution of $S^\dagger y = x$ is $y = Sx + r,\ r \in N(S^*)$. So, $Sx = y - r \in C_2 + N(S^*)$. From (ii), $x \in R(S^*)$ and $x \in C_1$. Hence, $y \in SC_1 + N(S^*)$. Therefore, $C_2 \subseteq SC_1 + N(S^*)$.
(iii) $\implies$ (iv): Let $u_0 \in C_2^\circ$ implies $u_0 \in C_2$. By hypothesis (iii), $\exists\ x_0 \in C_1$ and $z_0 \in N(S^*)$ such that $Sx_0 + z_0 = u_0$. Hence, $Sx_0 + z_0 \in C_2^\circ$.
(iv) $\implies$ (v): Let $S = U - V$ be a $B_\dagger$-splitting of S. Then, $R(S) = R(U),\ N(S) = N(U)$ and $U \geq 0,\ V \geq 0,\ VU^\dagger \geq 0$ and $Sx, Ux \in C_2 + N(S^*)$ and $x \in R(S^*)$ implies $x \in C_1$. We have $I \in \mathcal{B}(H_2)$ and it is positively invertible and $I - VU^\dagger \leq I$. Now, if we show $(I - VU^\dagger)C_2 \cap C_2^\circ \neq \emptyset$ then by Theorem 3.2.6, we get $I - VU^\dagger$ is

positively invertible. From given (iv), $\exists$ some $x_0 \in C_1$ and $z_0 \in N(S^*)$ such that $Sx_0 + z_0 \in C_2^\circ \subseteq C_2$. Set $w_0 = Ux_0 + z_0$ where $z_0 \in N(S^*) = N(U^*)$. Then

$$\begin{aligned} w_0 &= Ux_0 + z_0 \\ &= (S+V)x_0 + z_0 \\ &= Sx_0 + z_0 + Vx_0 \end{aligned}$$

Since $V \geq 0$, $x_0 \in C_1$ implies $Vx_0 \in C_2$ and $Sx_0 + z_0 \in C_2$ gives $w_0 \in C_2$. Further, from $z_0 \in N(S^*) = N(U^*) = N(U^\dagger)$ we have $(I - VU^\dagger)z_0 = z_0$. Hence,

$$\begin{aligned} (I - VU^\dagger)w_0 &= (I - VU^\dagger)(Ux_0 + z_0) \\ &= (I - VU^\dagger)Ux_0 + (I - VU^\dagger)z_0 \\ &= Sx_0 + z_0 \in C_2^\circ. \end{aligned}$$

Thus, $(I - VU^\dagger)C_2 \cap C_2^\circ \neq \emptyset$. This shows operator $(I - VU^\dagger)$ is positively invertible. From $VU^\dagger \geq 0$, $(I - VU^\dagger)^{-1} \geq 0$ and by Theorem 3.2.5, we conclude that $r(VU^\dagger) < 1$.

(v) $\implies$ (ii): Given that $r(VU^\dagger) < 1$. Suppose $Sx \in C_2 + N(S^*)$ and $x \in R(S^*)$. If we show $Ux \in C_2 + N(S^*)$, then from condition (iv) of $B_\dagger$-splitting we get $x \in C_1$. Let $Sx = p + q,\ p \in C_2, q \in N(S^*)$. From $S = (I - VU^\dagger)U$ and $(I - VU^\dagger)$ invertible, we have $U = (I - VU^\dagger)^{-1}S$.

$$\begin{aligned} Ux &= (I - VU^\dagger)^{-1}Sx \\ &= (I - VU^\dagger)^{-1}(p + q) \\ &= (I - VU^\dagger)^{-1}p + (I - VU^\dagger)^{-1}q \\ &= r + s \end{aligned}$$

Since $(I - VU^\dagger)^{-1} \geq 0$ and $p \in C_2$ we get $r = (I - VU^\dagger)^{-1}p \in C_2$. Also, $q \in N(S^*) = N(U^*) = N(U^\dagger)$ yields

$$0 = S^\dagger q = U^\dagger (I - VU^\dagger)^{-1} q = U^\dagger s$$

which implies $s \in N(U^\dagger) = N(U^*)$. Thus, we conclude $Ux = r + s \in C_2 + N(U^*) \in C_2 + N(S^*)$ Therefore, $Sx, Ux \in C_2 + N(S^*)$, $x \in R(S^*)$ implies $x \in C_1$.

3.4 Construction Method of $B_\dagger$-Splitting with Examples

The following theorem provide the conditions for an operator to have a $B_\dagger$-splitting and in proof we can see the construction of $B_\dagger$-splitting.

Theorem 3.4.1 *Let $S \in \mathcal{B}(H_1, H_2)$ with closed range and $H_1 = (H_1, C_1)$, $H_2 = (H_2, C_2)$ where cone C_1 be closed, normal, allows plastering and cone C_2 be normal, reproducing, $C_2^\circ \cap R(S) \neq \emptyset$. Further, $S = U - V$ be a proper splitting with $S^\dagger \geq 0$ and $S^\dagger S \geq 0$. Then S possesses a $B_\dagger$-splitting such that $r(VU^\dagger) < 1$.*

Proof Let $u \in C_2^\circ \cap R(S)$ be fixed interior point and $f \in H_1^{'}$ be fixed uniformly positive functional on H_1. Define an operator $f \otimes u : H_1 \to H_2$ by $f \otimes u = u^T \cdot f$ which is a rank-one operator. The conditions on the cone in ordered Hilbert space gives that a positive operator $f \otimes u$ is an interior point of $\mathcal{B}(H_1, H_2)$. For some real scalar $\alpha > 0$, define an arbitrary operator $T : H_2 \to H_2$ by

$$T = \frac{1}{\alpha + f(S^\dagger u^T)}(f \otimes u)S^\dagger$$

Since $\alpha > 0$, $f(S^\dagger u^T) > 0$ and $S^\dagger \geq 0$ we get $TC_2 \subseteq C_2$. Now, to show $r(T) < 1$, for eigenvalue $\lambda \neq 0$, consider $Tx = \lambda x$.

$$\frac{1}{\alpha + f(S^\dagger u^T)}(f \otimes u)S^\dagger x = \lambda x$$

which implies

$$\frac{u^T \cdot f(S^\dagger x)}{\alpha + f(S^\dagger u^T)} = \lambda x.$$

Hence $\frac{f(S^\dagger x)}{\alpha+f(S^\dagger u^T)}u^T = \lambda x$. i.e. $\beta u^T = \lambda x$ where $\beta = \frac{f(S^\dagger x)}{\alpha+f(S^\dagger u^T)}$. Substitute $x = \frac{\beta}{\lambda}u^T$ in β expression to deduce eigenvalue λ, we get $\lambda = \frac{f(S^\dagger u^T)}{\alpha+f(S^\dagger u^T)}$ and $0 < \lambda < 1$. Hence all the eigenvalues less than one and so $r(T) < 1$.

From $T \geq 0$, $r(T) < 1$, we can apply Theorem 3.2.5 and obtain $(I-T)^{-1} \geq 0$. Also, $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$. Using the method of induction, we have $T^{k+1} = \lambda^k T$ for $k \geq 0$. So,

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T = I + \sum_{k=0}^{\infty} \lambda^k T.$$

Since $\lambda < 1$, using the sum of geometric series we get $\sum_{k=0}^{\infty} \lambda^k = \frac{\alpha+f(S^\dagger u^T)}{\alpha}$.

$$(I-T)^{-1} = I + \frac{\alpha + f(S^\dagger u^T)}{\alpha} T$$

$$\begin{aligned}(I-T)^{-1}S &= (I + \frac{\alpha + f(S^\dagger u^T)}{\alpha} T)S \\ &= S + \frac{1}{\alpha}(f \otimes u)S^\dagger S\end{aligned}$$

Choose α such that $\alpha\zeta \leq 1$ and from Remark 3.2.1, $\pm S \leq \zeta(f \otimes u)$ and this implies $0 \leq S + \zeta(f \otimes u)S^\dagger S$. Hence, $(I-T)^{-1}S = S + \zeta(f \otimes u)S^\dagger S \geq 0$. Set $U = (I-T)^{-1}S = S + \frac{1}{\alpha}(f \otimes u)S^\dagger S \geq 0$ and $V = TU \geq 0$. With the above setting of U and V, the splitting $S = U - V$ is a positive splitting. Also, $R(T) \subseteq R(S)$. Let $x_0 \in R(U), Uy_0 = x_0$. Hence $(I-T)x_0 = (I-T)Uy_0 = Sy_0$ implies $x_0 \in R(S)$. i.e. $R(U) \subseteq R(S)$. From $U = (I-T)^{-1}S$ we get $R(S) \subseteq R(U)$ which follows $R(S) = R(U)$. To verify the condition $N(S) = N(U)$, consider $x_0 \in N(S)$ i.e. $Sx_0 = (U-V)x_0 = (I-T)Ux_0 = 0$. implies $x_0 \in N(U)$ which gives $N(S) \subseteq N(U)$. Similarly, $N(U) \subseteq N(S)$. Hence $R(U) = R(S)$ and $N(U) = N(S)$, the splitting $S = U - V$ is a proper splitting with this setting.

We have $R(T^*) \subseteq R(S) = R(U)$. This implies $UU^\dagger T^* = T^*$. Taking adjoint of operators on both side, we get $TUU^\dagger = T$. i.e. $VU^\dagger = T \geq 0$. Therefore $VU^\dagger C_2 \subseteq C_2$. This proves $VU^\dagger \geq 0$.

From Theorem 3.4.1, $S^\dagger \geq 0$ gives $Sx \in C_2 + N(S^*)$, $x \in R(S^*)$ implies $x \in C_1$. Since $R(S) = R(U)$, $Ux \in C_2 + N(S^*)$. Hence the last condition of $B_\dagger$-splitting $Sx, Ux \in$

$C_2 + N(S^*)$, $x \in R(S^*)$ implies $x \in C_1$ is satisfied. Moreover, $r(VU^\dagger) = r(T) < 1$. Hence with the above splitting operator S possesses a $B_\dagger$-splitting with $r(VU^\dagger) < 1$.

Next, we give examples for $B_\dagger$-splitting based on above Theorem 3.4.1, which is constructive in nature.

Example 3.4.1 *Consider the Hilbert spaces $H_1 = l^2$, $H_2 = \mathbb{R}^n$, respectively. The cone on H_1 and H_2 be $C_1 = l^2_+ = \{x \in l^2 : x_i \geq 0, \forall i\}$ and $C_2 = \mathbb{R}^n_+$, respectively. An operator $S : l^2 \to \mathbb{R}^n$ be defined by*

$$S(x_1, x_2, ..., x_n, ...) = \sum_{i=1}^{n} x_i(1, 1, ..., 1)$$

The range is finite dimensional, so it is closed subspace and S is non-invertible operator. The adjoint $S^* : \mathbb{R}^n \to l^2$ defined by

$$S^*(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i(1, 1, ..., 1, 0, 0, ...)$$

By computation, the Moore-Penrose inverse of operator S is $S^\dagger = \frac{1}{n^2}S^*$. Also, $S^\dagger \mathbb{R}^n_+ \subseteq l^2_+$ and $S^\dagger S \geq 0$. Cone C_1 is closed, normal, allows plastering and C_2 is normal, reproducing with $C_2^\circ \cap R(S) \neq \emptyset$.
Since $\mathbb{R}^n_+$ is solid cone, we have $u = (1, 1, ..., 1)$ is a point in interior of $(\mathbb{R}^n_+)$ and $R(S)$.
Define uniformly positive functional $f : l^2 \to \mathbb{R}$ by

$$f(x_1, x_2, ..., x_n, ...) = x_1 + x_2 + ... + x_n.$$

Hence we can represent f as $f = (1, 1, ..., 1)$.

$$\begin{aligned} S^\dagger u^T &= \frac{1}{n^2} S^* u^T \\ &= \frac{1}{n^2} S^* (1, 1, ..., 1)^T \\ &= \frac{1}{n}(1, 1, ..., 1, 0, 0, ...)^T \end{aligned}$$

$f(S^\dagger u^T) = f(\frac{1}{n}(1, 1, ..., 1, 0, 0, ...)^T) = \frac{n}{n} = 1.$

The rank-one operator $f \otimes u : l^2 \to \mathbb{R}^n$ is given by

$$f \otimes u = u^T \cdot f = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots \end{bmatrix}$$

Choose the value of ζ such that $-S \leq \zeta(f \otimes u)$ is satisfied.

We decompose operator S satisfies the $B_\dagger$-splitting using the following construction method. Define $T_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_\alpha = \frac{1}{f(S^\dagger u) + \alpha}(f \otimes u) S^\dagger$$

where the constant α satisfies the condition $\alpha\zeta \leq 1$. In particular, let $\zeta = \frac{1}{2}$ and hence $\alpha = 2$. The representation of $T_{\alpha=2}$ with respect to standard basis of $\mathbb{R}^n$ is

$$T_{\alpha=2} = \begin{bmatrix} \frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \\ \frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \end{bmatrix}$$

Then

$$I - T_{\alpha=2} = \begin{bmatrix} 1-\frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \\ \frac{1}{3n} & 1-\frac{1}{3n} & \cdots & \frac{1}{3n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{3n} & \frac{1}{3n} & \cdots & 1-\frac{1}{3n} \end{bmatrix}$$

$(I - T_{\alpha=2})$ *is invertible and its inverse is given by*

$$(I - T_{\alpha=2})^{-1} = \begin{bmatrix} \frac{2n+1}{2n} & \frac{2n+1}{2n} & \cdots & \frac{2n+1}{2n} \\ \frac{2n+1}{2n} & \frac{2n+1}{2n} & \cdots & \frac{2n+1}{2n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{2n+1}{2n} & \frac{2n+1}{2n} & \cdots & \frac{2n+1}{2n} \end{bmatrix}$$

Let $U = (I - T_{\alpha=2})^{-1}S$ *and* $V = T_{\alpha=2}U$

$$\begin{aligned} U &= (I - T_{\alpha=2})^{-1}S \\ &= \begin{bmatrix} \frac{2n+1}{2n} & \frac{2n+1}{2n} & \cdots & \frac{2n+1}{2n} \\ \frac{2n+1}{2n} & \frac{2n+1}{2n} & \cdots & \frac{2n+1}{2n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{2n+1}{2n} & \frac{2n+1}{2n} & \cdots & \frac{2n+1}{2n} \end{bmatrix} (\sum_{i=1}^{n} x_i) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \frac{3}{2}\sum_{i=1}^{n} x_i \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

$$V = T_{\alpha=2}U = \begin{bmatrix} \frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \\ \frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{3n} & \frac{1}{3n} & \cdots & \frac{1}{3n} \end{bmatrix} (\frac{3}{2}\sum_{i=1}^{n} x_i) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{3}{2}(\sum_{i=1}^{n} x_i) \begin{pmatrix} \frac{1}{3n} + \frac{1}{3n} + \cdots + \frac{1}{3n} \\ \frac{1}{3n} + \frac{1}{3n} + \cdots + \frac{1}{3n} \\ \vdots \\ \frac{1}{3n} + \frac{1}{3n} + \cdots + \frac{1}{3n} \end{pmatrix}$$

$$= \frac{1}{2}(\sum_{i=1}^{n} x_i) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

So, $S = U - V$ *with* $U \geq 0$, $V \geq 0$, *and* $VU^\dagger \geq 0$ *and* $Sx, Ux \in C_2 + N(S^*)$ $x \in C_1$. *Therefore* S *possesses a* $B_\dagger$*-splitting with* $r(VU^\dagger) < 1$.

Remark 3.4.1 *It is noted that, we can obtain different* U *and* V *which satisfies the* $B_\dagger$*-splitting of* S *with different choices of* α *satisfies the condition* $\alpha\zeta \leq 1$.

Example 3.4.2 *Let* $H = l^2(\mathbb{N})$. *Define operator* $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *by*

$$S(x_1, x_2, x_3, ...) = (x_1, 0, x_3, 0, x_5, ...)$$

i.e. $(Sx)_j = \begin{cases} x_j & \text{if } j \text{ odd} \\ 0 & \text{otherwise} \end{cases}$

The null space and range are $N(S) = \{(0, x_2, 0, x_4, ...) \in H : Sx = 0\} \neq \emptyset$ *and*

$R(S) = \{(x_1, 0, x_3, 0, x_5, ...) \in H\}$, *respectively. Operator* S *is non-invertible operator and range space is closed subspace. Also, by computation* $S = S^\dagger$. *Consider a cone* $C = \{x \in H : x_1 \geq x_n,\ x_n \geq 0\ \forall\ n \geq 2,\ \}$. *Cone* C *is solid, normal and allow plastering. Also,* $S^\dagger C \subseteq C$, $S^\dagger S \geq 0$. *Hence the conditions for* S *to possess* $B_\dagger$-*splitting are satisfied. Let the point* $u = (1, 0, 0, 0, ...) \in C^\circ$. *Define uniformly positive functional* $f : l^2(\mathbb{N}) \to \mathbb{R}$ *by*

$$f(x_1, x_2, .., x_n, ...) = x_1 + x_2 + ... + x_n.$$

It can be represented by $f = (1, 1, 1, ..., 1, 0, 0, ...)$. *Hence the rank-one operator*

$$\begin{aligned} f \otimes u &= u^T \cdot f \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots \end{bmatrix} \end{aligned}$$

We represent rank-one operator by orthonormal basis.

$$f \otimes u = \begin{bmatrix} e_1 & e_1 & e_1 & \cdots & e_1 & 0 & 0 & \cdots \end{bmatrix}$$

Consider α *such that* $-S \leq \zeta(f \otimes u)$ *and* $\alpha\zeta \leq 1$. *Take* $\zeta = 2$ *and so* $\alpha = \frac{1}{2}$. *For* $\alpha = \frac{1}{2}$, *define operator* $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *by*

$$T_\alpha = T_{\frac{1}{2}} = \frac{1}{f(S^\dagger u^T) + \alpha}(f \otimes u)S^\dagger$$

Computing $S^\dagger u^T = S^\dagger(1, 0, 0, ...)^T = (1, 0, 0, ...)^T$ *and*

$$f(S^\dagger u^T) = f(1, 0, 0, ...) = 1.$$

After substitution of values in $T_{\frac{1}{2}}$ we get

$$\begin{aligned} T_{\frac{1}{2}} &= \frac{1}{1+\frac{1}{2}}\begin{bmatrix} e_1 & e_1 & e_1 & \cdots & e_1 & 0 & 0 & \cdots \end{bmatrix}\begin{bmatrix} e_1 & 0 & e_3 & 0 & e_5 & 0 & \cdots \end{bmatrix} \\ &= \frac{2}{3}\begin{bmatrix} e_1 & 0 & 0 & \cdots \end{bmatrix} \end{aligned}$$

$$\begin{aligned} I - T_{\frac{1}{2}} &= \begin{bmatrix} e_1 & e_2 & e_3 & \cdots \end{bmatrix} - \begin{bmatrix} \frac{2}{3}e_1 & 0 & 0 & \cdots \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3}e_1 & e_2 & e_3 & \cdots \end{bmatrix} \end{aligned}$$

Hence $(I - T_{\frac{1}{2}})^{-1} = \begin{bmatrix} 3e_1 & e_2 & e_3 & \cdots \end{bmatrix}$
Computing U and V with orthonormal basis representation we get,

$$\begin{aligned} U &= (I - T_{\frac{1}{2}})^{-1}S \\ &= \begin{bmatrix} 3e_1 & e_2 & e_3 & \cdots \end{bmatrix}\begin{bmatrix} e_1 & 0 & e_3 & 0 & e_5 & \cdots \end{bmatrix} \\ &= \begin{bmatrix} 3e_1 & 0 & e_3 & 0 & e_5 & \cdots \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} V &= T_{\frac{1}{2}}U \\ &= \begin{bmatrix} \frac{2}{3}e_1 & 0 & 0 & \cdots \end{bmatrix}\begin{bmatrix} 3e_1 & 0 & e_3 & 0 & e_5 & \cdots \end{bmatrix} \end{aligned}$$

Hence $U - V = S$ with this construction. We see that $U \geq 0$, $V \geq 0$, and $VU^\dagger \geq 0$. The condition $Sx, Ux \in C_2 + N(S^*)$ implies $x \in C_1$ is satisfied. Hence $S = U - V$ is a $B_\dagger$-splitting with $r(VU^\dagger) = \frac{2}{3} \leq 1$.

Chapter 4

Cone Nonnegativity of Moore–Penrose Inverses of Unbounded Gram Operators

4.1 Introduction

In this chapter[1], we characterize the cone nonnegativity of the Moore-Penrose inverse of Gram operators which are linear and unbounded. This characterization extends the results of [62] from bounded linear operators to unbounded linear operators.

Let us review the results which provide a motivation to the results of this chapter. The obtuseness of the standard cone in $\mathbb{R}^n$ was first provided by Goffin [23]. Novikoff proposed the notion of acute and obtuse cone in [56]. The terminology is motivated by the fact that, in two dimension vector space, a cone is acute if and only if its central angle is at most $\frac{\pi}{2}$. In two dimensions, a cone is obtuse if and only if its central angle is at least $\frac{\pi}{2}$.

[1]The contents of this chapter have been published as a research article under the same title, in *Positivity*, Vol.26, Article no.67, 2022, https://doi.org/10.1007/s11117-022-00908-y

Recently, the concept of the cone positivity (or nonnegativity) of the generalized inverse of Gram operators and Gram matrices has received a high attention. It is due to the applications in convex optimization theory. More specifically, when the operator equation $Sx = b$ is inconsistent, the normal equation $S^*Sx = S^*b$ can be employed to find the least squares solution of minimal norm. In solving the normal equation, the nature of the Gram operator S^*S plays a vital role.

In 2001, Cegielski [9] characterized the monotonicity of Gram matrices in terms of acuteness (or obtuseness) of certain cones which are polyhedral. This results have been generalized to characterize the nonnegativity of Moore-Penrose inverse of Gram matrices in [48]. This characterization was extended to Gram operators which are bounded and defined on infinite dimensional real Hilbert spaces in [47]. This characterization is stated in the following theorem:

Theorem 4.1.1 *(Theorem 3.6,[47]) Consider an operator $S \in \mathcal{B}(H_1, H_2)$ with closed linear subspace $R(S)$. Suppose a convex closed cone C of H_1 with $S^\dagger SC \subseteq C$ with cones $C_1 = SC$ and $C_2 = (S^\dagger)^* C'$, respectively. Then the assertions below are equivalent:*

(i) $(S^*S)^\dagger(-C') \subseteq C$.

(ii) $C_1' \cap R(S) \subseteq -C_1$.

(iii) C_2 *is acute.*

(iv) C_1 *is obtuse.*

(v) $S^*Sx \in -C'$, $x \in R(S^*) \implies x \in C$.

(vi) $S^*Sx \in P_{R(S^*)}(-C'),\ x \in R(S^*) \implies x \in C$.

where $C' = \{x \in H_1 : \langle x, u\rangle \leq 0,\ \forall\ u \in C\}$ *called polar of a cone* C.

Kurmayya and Ramesh [46] generalized the existing results due to Sivakumar and Kurmayya [47] for densely defined closed operators. On the other hand, Sivakumar [62] has characterized the nonnegativity of Moore-Penrose inverses of Gram operators over infinite dimensional real Hilbert spaces, based on the results of Novikoff [56]. In this chapter, the results of [62] are generalized to the unbounded Gram operators, in particular to the closed and densely defined operators over real Hilbert spaces. It is relevant to mention that several concepts from the theory of unbounded operators and its Moore-Penrose inverses are being used to prove the main results of this chapter.

This chapter is organized as follows. Section 4.2 contains preliminary results on Moore-Penrose inverses of closed dense operators and introduction to acute, obtuse cone which are used in this chapter. Section 4.3 contains a series of lemmas and the main theorem on the characterization of the cone nonnegativity of the Moore-Penrose invese of unbounded Gram operators. The final section contains some examples.

4.2 Preliminaries

As we mentioned in the Introduction 1, $\mathcal{L}(H_1, H_2)$ denotes the space of linear operators on H_1 to H_2. For any $S \in \mathcal{L}(H_1, H_2)$, the notions *domain* of S by $D(S)$, the *range* of S by $R(S)$, closure of $R(S)$ by $\overline{R(S)}$, the *null space* of S by $N(S)$, and the *carrier* of S by $Car(S)$, where $Car(S) = D(S) \cap N(S)^{\perp}$ are followed. Next, we move on to some basic definitions which will be used in further results. For more details of these concepts, one can refer [4] and [26].

The following result collects some basic properties of densely defined closed operators.

Proposition 4.2.1 *(Chapter 9, [4]) If $S \in \mathcal{C}(H_1, H_2)$ is densely defined linear operator then*

(i) $S^* \in \mathcal{L}(H_2, H_1)$ is densely defined and $S^{**} = S$.

(ii) $N(S^*) = R(S)^\perp, \; N(SS^*) = N(S^*)$.

(iii) $N(S^*S) = N(S), \; N(S) = R(S^*)^\perp$.

(iv) $\overline{R(S^*)} = N(S)^\perp, \; \overline{R(S^*)} = \overline{R(S^*S)}$.

(v) $\overline{R(S)} = N(S^*)^\perp, \; \overline{R(S)} = \overline{R(SS^*)}$.

Next, we recall the most important definition namely the Moore-Penrose inverse, and we collect its basic properties in the subsequent propositions.

Definition 4.2.1 ***Moore-Penrose Inverse*** *(Chapter 9, [4]) Let $S \in \mathcal{C}(H_1, H_2)$. The Moore-Penrose inverse of S is the map $S^\dagger : R(S) \oplus^\perp R(S)^\perp \to H_1$ defined as follows:*

$$S^\dagger y = \begin{cases} S_0^{-1}y, & \text{if } y \in R(S) \\ 0, & \text{if } y \in R(S)^\perp, \end{cases}$$

where $S_0 = S|_{Car(S)}$. i.e. S_0 is the restriction of S on $Car(S)$.

Proposition 4.2.2 *(Theorem 2, Chapter 9, [4]) Let $S \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

(i) $D(S^\dagger) = R(S) \oplus^\perp R(S)^\perp, \; N(S^\dagger) = N(S^*)$.

(ii) $R(S^\dagger) = Car(S)$.

(iii) *$S^\dagger$ is densely defined and $S^\dagger \in \mathcal{C}(H_2, H_1)$.*

(iv) *$S^\dagger$ is bounded if and only if $R(S)$ is closed.*

(v) *$(S^*S)^\dagger = S^\dagger (S^*)^\dagger$ and $(SS^*)^\dagger = (S^*)^\dagger S^\dagger$.*

Proposition 4.2.3 *(Theorem 5.7, [54]) Let $S \in \mathcal{C}(H_1, H_2)$ be dense operator. Then each of the following set of conditions characterize the Moore-Penrose inverse:*

(i) *(a)* $S^\dagger S x = P_{\overline{(R(S^\dagger))}} x$ for all $x \in D(S)$.

(b) $S^\dagger S S^\dagger y = S^\dagger y$ for all $y \in D(S^\dagger)$.

(c) $S S^\dagger y = P_{\overline{(R(S))}} y$ for all $y \in D(S^\dagger)$.

(ii) *(a)* $S S^\dagger S x = S x$ for all $x \in D(S)$.

(b) $S^\dagger S S^\dagger y = S^\dagger y$ for all $y \in D(S^\dagger)$.

(c) $S^\dagger S$ *and* $S S^\dagger$ *are symmetric operators. That is* $S^\dagger S \subset (S^\dagger S)^*$ *and* $S S^\dagger \subset (S S^\dagger)^*$.

Note:

(1) For bounded operators over Hilbert space $S^\dagger S$ and $SS^\dagger$ are self-adjoint operators. But for closed and densely defined operators over Hilbert space $S^\dagger S$ and $SS^\dagger$ are symmetric operators.

(2) Suppose $S : H_1 \to H_2$ is a closed dense operator with closed range then $(S^*)^\dagger = (S^\dagger)^*$.

(3) We can obtain the inclusion $R((S^\dagger)^*) \subseteq D(S^*)$ from the following:

$$\begin{aligned} R((S^\dagger)^*) = R((S^*)^\dagger) = Car(S^*) &= D(S^*) \cap N(S^*)^\perp \\ &= D(S^*) \cap \overline{R(S)} \\ &= D(S^*) \cap R(S). \end{aligned}$$

Next, we give a result which is fundamental in the study of linear equations and it can be proved easily from the existing results in ([54], page 63).

Lemma 4.2.1 *Let $S \in \mathcal{C}(H_1, H_2)$ be dense operator with range closed and $y \in H_2$. Then for all $x \in D(S)$, the linear system of equations $Sx = b$ has a solution whenever $b \in R(S)$. In this case the general solution is given by $x = Tb+z$ for some T satisfying $STSx = Sx$ for all $x \in D(S)$ and for any $z \in N(S)$.*

Remark 4.2.1 *Specifically, suppose $Sx = b$ is consistent, then $x = S^{\dagger}b + z$ for some $z \in N(S)$.*

According to the Defined 1.2.2, consider C be a closed convex cone on real Hilbert space H and C^* be the dual cone of C. In addition to this, the definition of acute cone and obtuse cone are stated below:

Definition 4.2.2 *Consider a cone C on H. If $\langle x, y\rangle \geq 0$, for $x,\ y \in C$ then C is said to be acute cone. If $C^* \cap \{\overline{spanC}\}$ is acute, then C said to be obtuse where span C is the linear subspace spanned by C.*

According to [56], on real Hilbert space H the acuteness C is given by the inclusion $C \subseteq C^*$ and the obtuseness of C is given by the inclusion $C^* \subseteq C$. For more details on a cone in infinite dimensional Banach space and its properties, one may refer to [45].

4.3 Cone Nonnegativity of Moore-Penrose Inverses

This section deals with the main results of the article. As we mentioned in the Chapter 1, let $S \in \mathcal{L}(H_1, H_2)$. The operator S^*S is called the Gram operator of S. Here, we derive some results which characterize the cone nonnegativity of the Moore-Penrose inverse of unbounded Gram operators. We reiterate that the results of this section are extensions of [62] from bounded operators to unbounded operators. However, these results are new and the proof techniques are also very interesting. We now begin this section with a series of lemmas which lead to the main theorem.

Throughout the section, we assume that $S \in \mathcal{C}(H_1, H_2)$ is densely defined linear operator with closed range and C is a closed cone in $D(S^*S)$ such that $C^* \subset D(S^*S)$.

Lemma 4.3.1 *Consider $S \in \mathcal{C}(H_1, H_2)$ be dense operator with closed range and C be a closed cone in H_1. Then the following conditions hold:*

(i) $u \in (SC)^* \cap D(S^*) \iff S^*u \in C^*$.

(ii) $S^\dagger SC \subseteq C \iff S^\dagger SC^* \subseteq C^*$.

Proof (i) Let $u \in (SC)^* \cap D(S^*)$ and $r \in C$. Then $0 \leq \langle u, Sr\rangle = \langle S^*u, r\rangle$. Thus $S^*u \in C^*$. Conversely, let $S^*u \in C^*$ and $r \in C$. Then $0 \leq \langle S^*u, r\rangle = \langle u, Sr\rangle$. Thus $u \in (SC)^* \cap D(S^*)$.

(ii) Let $y = S^\dagger Sx$ with $x \in C^*$. By proposition 2.9, we have $S^\dagger S \subset (S^\dagger S)^*$. So, $\forall u \in C$, $S^\dagger Su = (S^\dagger S)^*u$ and since $S^\dagger SC \subseteq C$, $u' = S^\dagger Su \in C$. Now $\langle y, u\rangle = \langle S^\dagger Sx, u\rangle = \langle x, (S^\dagger S)^*u\rangle = \langle x, S^\dagger Su\rangle = \langle x, u'\rangle \geq 0$. i.e. $\langle y, u\rangle \geq 0, \forall\ u \in S$. This implies that $y \in C^*$. Converse part can be proved similarly using the fact that $C^{**} = C$.

Theorem 4.3.1 *Consider $S \in \mathcal{C}(H_1, H_2)$ be dense operator with range closed and C be a closed cone in H_1 such that $S^\dagger SC \subseteq C$. Then $(SC)^* \cap D(S^*) = (S^\dagger)^*C^* + N(S^*)$.*

Proof Let $y \in (SC)^* \cap D(S^*)$. Then by Lemma 4.3.1, $z = S^*y \in C^*$. Now, by Lemma 4.2.1, $y = (S^\dagger)^*z + w$ for some $w \in N(S^*)$. i.e. $y \in (S^\dagger)^*C^* + N(S^*)$. Therefore, $(SC)^* \cap D(S^*) \subseteq (S^\dagger)^*C^* + N(S^*)$.

Conversely, let $u = u_1 + u_2$ where $u_1 = (S^\dagger)^*l$ with $l \in C^*$ and $u_2 \in N(S^*)$. Let $v = St$ for some $t \in C$. Set $t' = S^\dagger St \in S^\dagger SC \subseteq C$. Then $\langle u, v\rangle = \langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle = \langle (S^\dagger)^*l, St\rangle + \langle S^*u_2, t\rangle = \langle l, (S^\dagger)^{**}St\rangle + 0 = \langle l, S^\dagger St\rangle = \langle l, t'\rangle \geq 0$. Thus $\forall\ v = St,\ t \in C,\ \langle u, v\rangle \geq 0$ which gives $u \in (SC)^*$. So, $(S^\dagger)^*C^* + N(S^*) \subseteq (SC)^* \cap D(S^*)$.

Lemma 4.3.2 *Consider $S \in \mathcal{C}(H_1, H_2)$ be densely operator with range closed and let C be a closed cone in H_1 such that $S^\dagger SC \subseteq C$. Then the following conditions are true.*

(i) $((S^\dagger)^* C^*)^* \cap D(S^\dagger) = SC + N(S^*)$.

(ii) $(SC)^* \cap D(S^*) \cap R(S) = (S^\dagger)^* C^*$.

Proof (i) In Theorem 4.3.1, replacing S by $(S^\dagger)^*$ and C by C^*. We obtain

$$((S^\dagger)^* C^*)^* \cap D(S^\dagger) = SC + N(S^\dagger)$$

which gives

$$((S^\dagger)^* C^*)^* \cap D(S^\dagger) = SC + N(S^*).$$

(ii) Let $y = Sx \in (SC)^* \cap D(S^*) \cap R(S)$. By Lemma 4.3.1, $S^* y \in C^*$. Since $y = Sx \in R(S)$ we can write $y = P_{R(S)} y = P_{\overline{R(S)}} y = SS^\dagger y = (SS^\dagger)^* y = (S^\dagger)^* S^* y \in (S^\dagger)^* C^*$. Thus $y \in (S^\dagger)^* C^*$. Therefore, $(SC)^* \cap D(S^*) \cap R(S) \subseteq (S^\dagger)^* C^*$.

Conversely, let $x = (S^\dagger)^* u$ for $u \in C^*$. Since $R((S^\dagger)^*) = D(S^*) \cap R(S)$ we have $x \in R((S^\dagger)^*) = D(S^*) \cap R(S)$. Now, for $w \in C, Sw = v \in SC$ and set $w' = S^\dagger Sw$. Then $w' \in C$ by $S^\dagger SC \subseteq C$. So, $\langle x, v \rangle = \langle (S^\dagger)^* u, Sw \rangle = \langle u, S^\dagger Sw \rangle = \langle u, w' \rangle \geq 0$. Hence $x \in (SC)^* \cap D(S^*) \cap R(S)$. So, $(S^\dagger)^* C^* \subseteq (SC)^* \cap D(S^*) \cap R(S)$.

The next lemma gives necessary and sufficient conditions for the cone nonnegativity of the Moore-Penrose inverse of Gram operator $S^* S$ with respect to the cones C^* and C in terms of acuteness of the cone $(S^\dagger)^* C^*$.

Lemma 4.3.3 *Consider $S \in \mathcal{C}(H_1, H_2)$ be densely operator with range closed and C be a cone in H_1 such that $S^\dagger SC \subseteq C$. Then the following conditions are equivalent.*

(i) $(S^\dagger)^* C^* \subseteq SC + N(S^*)$

(ii) $(S^*S)^\dagger C^* \subseteq C + N(S)$

(iii) $(S^*S)^\dagger C^* \subseteq C$

Proof $(i) \implies (ii)$: By proposition 2.8, we have $(S^*S)^\dagger = S^\dagger(S^\dagger)^* = S^\dagger(S^*)^\dagger$. Let $x \in C^* \cap D(S),$ and $y = (S^*S)^\dagger x = S^\dagger(S^\dagger)^* x$. Then

$$\begin{aligned} Sy &= SS^\dagger(S^\dagger)^* x \\ &= P_{\overline{R(S)}}(S^\dagger)^* x \\ &= (S^\dagger)^* x \in (S^\dagger)^* C^* \subseteq SC + N(S^*) \end{aligned}$$

Since $Sy \in SC + N(S^*)$ we write $Sy = Sv + w$ for $w \in N(S^*)$ and $v \in C$. So $S(y - v) \in \overline{R(S)} \cap N(S^*) = \{0\}$. i.e. $S(y - v) = 0$ which implies $y - v \in N(S)$. One can take $y = v + u$, for some $u \in N(S)$ which gives $y \in C + N(S)$. Hence $(S^*S)^\dagger C^* \subseteq C + N(S)$.

$(ii) \implies (i)$: Let $y = (S^\dagger)^* x,\ x \in C^*$. Then $S^\dagger y = S^\dagger(S^\dagger)^* x = (S^*S)^\dagger x \in C + N(S)$. Set $S^\dagger y = u + v$ where $u \in C, v \in N(S)$. So $S^\dagger y - u \in N(S)$. i.e. $S(S^\dagger y - u) = 0$. which gives $SS^\dagger y - Su = 0$. Since $y \in R((S^\dagger)^*) = D(S^*) \cap R(S)$, by proposition 2.9, we get $SS^\dagger y = y$. Thus, $y = Su$. Then it follows that $y = Su + w$ for $w \in N(S^*)$. Hence $y \in (S^\dagger)^* C^* \subseteq SC + N(S^*)$.

$(ii) \implies (iii)$: Let $x \in C^* \cap D(S)$ and $y = (S^*S)^\dagger x$. By hypothesis, we can write $y = u + v$ for some $u \in C$ and $v \in N(S)$. The linear equation $y = (S^*S)^\dagger x$ gives

$x = S^*Su$ by Lemma 4.2.1. Then

$$\begin{aligned} y &= (S^*S)^\dagger x \\ &= (S^*S)^\dagger (S^*S)u \\ &= P_{\overline{R(S^*S)^*}} u \\ &= P_{\overline{R(S^*S)}} u \\ &= P_{\overline{R(S^*)}} u \\ &= P_{R(S^*)} u \\ &= S^\dagger S u \in C \end{aligned}$$

Similarly, we can prove $(iii) \implies (ii)$.

Lemma 4.3.4 *Consider $S \in \mathcal{C}(H_1, H_2)$ be dense operator with range closed and a closed cone C be in H_1. Then the following implication holds.*

$$(S^\dagger)^* C^* \subseteq SC \implies (S^*S)^\dagger C^* \subseteq C + N(S).$$

Proof Let $y \in (S^*S)^\dagger C^*$. By proposition 2.8, we have $(S^*S)^\dagger = S^\dagger (S^\dagger)^* = S^\dagger (S^*)^\dagger$. For $x \in C^*,\ y = (S^*S)^\dagger x = S^\dagger (S^*)^\dagger x$. Then,

$$\begin{aligned} Sy &= SS^\dagger (S^*)^\dagger x \\ &= SS^\dagger (S^\dagger)^* x \\ &= (S^\dagger)^* x \in (S^\dagger)^* C^* \subseteq SC \end{aligned}$$

Writing $Sy = Sv + w$ for $v \in C$ and $w \in R(S)^\perp = N(S^*)$. Thus $Sy - Sv \in N(S^*)$. $S^*(Sy - Sv) = 0$ implies that $S^*S(y - v) = 0$. Thus $y - v \in N(S^*S) = N(S)$ which gives $y = v + u$ for $u \in N(S), v \in C$. Hence the proof.

Lemma 4.3.5 *Consider $S \in \mathcal{C}(H_1, H_2)$ be densely operator with range closed and a closed cone C be in H_1 such that $S^\dagger SC \subseteq C$. Then*

$$(S^*S)^\dagger C^* \subseteq C + N(S) \iff C^* \cap R(S^*) \subseteq S^*SC + N(S).$$

Proof To prove the necessary condition, consider $y = S^*x \in C^* \cap R(S^*)$.
Then $(S^*S)^\dagger y \in (S^*S)^\dagger C^* \subseteq C + N(S)$. So, $(S^*S)^\dagger y \in C + N(S)$. Write $(S^*S)^\dagger y = u + v$ for $u \in C,\ v \in N(S)$.

$$\begin{aligned} y &= (S^*S)^{\dagger\dagger}(u+v) \\ &= (S^*S)u + (S^*S)v \end{aligned}$$

for $u \in C, v \in N(S^*S) = N(S)$. This implies, $y \in S^*SC + N(S)$. Hence $C^* \cap R(S^*) \subseteq S^*SC + N(S)$.

To prove the converse part, consider $y = (S^*S)^\dagger x,$ for $x \in C^*$. Applying S^*S on both side.

$$\begin{aligned} S^*Sy &= (S^*S)(S^*S)^\dagger x \\ &= P_{\overline{R(S^*S)}}x \\ &= P_{\overline{R(S^*)}}x \\ &= S^\dagger Sx \in S^\dagger SC^* \subseteq C^* \text{ (From given } S^\dagger SC \subseteq C) \end{aligned}$$

So $z = S^*Sy \in C^* \cap R(S^*) \subseteq S^*SC + N(S)$. Writing $z = (S^*S)u + w$ for some $u \in C,\ w \in N(S^*S) = N(S)$.

$$\begin{aligned} y &= (S^*S)^\dagger z \\ &= (S^*S)^\dagger (S^*S)u + (S^*S)^\dagger w \\ &= P_{\overline{R(S^*S)}}u + (S^*S)^\dagger w \\ &= P_{\overline{R(S^*)}}u + v \\ &= u + v \text{ for } u \in C \text{ and } v \in N((S^*S)^\dagger) = N(S). \end{aligned}$$

Hence $C^* \cap R(S^*) \subseteq S^*SC + N(S)$ if and only if $(S^*S)^\dagger C^* \subseteq C + N(S)$, whenever $S^\dagger SC \subseteq C$.

So far, we have proved a series of lemmas which will be used in proving the main result of this chapter. The main result follows next.

Theorem 4.3.2 ***(Main Theorem)***

Consider $S \in \mathcal{C}(H_1, H_2)$ be densely operator with range closed and a closed cone C be in $D(S^*S)$ such that $C^* \subset D(S^*S)$ and $S^\dagger SC \subseteq C$. Then the following conditions are equivalent:

(i) $(S^\dagger)^* C^*$ *is acute.*

(ii) $(S^*S)^\dagger C^* \subseteq C + N(S)$

(iii) $(SC)^* \cap D(S^*) \cap R(S) \subseteq SC$

Proof $(i) \iff (ii)$: Let $(S^\dagger)^* C^*$ is acute. Then $(S^\dagger)^* C^* \subseteq ((S^\dagger)^* C^*)^*$. By Lemma 4.3.2, $(S^\dagger)^* C^* \subseteq ((S^\dagger)^* C^*)^* \cap D(S^\dagger) = SC + N(S^*)$. So, $(S^\dagger)^* C^* \subseteq SC + N(S^*)$. From Lemma 4.3.3, it follows that $(S^*S)^\dagger C^* \subseteq C + N(S)$. Similarly, converse also can be proved.

$(ii) \implies (iii)$: Let $y \in (SC)^* \cap D(S^*) \cap R(S)$. For $y = Sx$, by Lemma 4.3.1, $S^*y \in C^*$. Also $S^*y \in R(S^*)$. So $S^*y \in C^* \cap R(S^*)$. By Lemma 4.3.4, it can be written $S^*y = S^*Su + z$ for $u \in C,\ z \in N(S)$. Since $S^*y,\ S^*Su$ are in $R(S^*)$, it follows that $z = 0$. So $S^*y = S^*Su,\ u \in C$. Then we have solution $y = (S^*)^\dagger S^*Su + w$ for some $w \in N(S^*)$. Then $y = (SS^\dagger)^*Su + w = SS^\dagger Su + w = Su + w$. Since $y,\ Su$ both belong to $R(S)$, it follows that $w = 0$. That is $y \in SC$.

$(iii) \implies (ii)$: Let $(SC)^* \cap D(S^*) \cap R(S) \subseteq SC$. Then by Lemma 4.3.2, we get $(S^\dagger)^* C^* \subseteq SC$. Now, by Lemma 4.3.4, it follows that $(S^*S)^\dagger C^* \subseteq C + N(S)$.

4.4 Illustrations

In this section, we discuss two examples which illustrate the main results.

Example 4.4.1 *Consider* $H = l^2(\mathbb{N})$ *with the domain of operator* S *be*

$$D(S) = \{(x_1, x_2, ..., x_n, ...) : \sum_{j=1}^{\infty} |jx_j|^2 < \infty\}.$$

Define the operator $S : D(S) \to H$ *by* $S(x_1, x_2, ..., x_n, ...) = (0, 2x_2, ..., nx_n, ...)$ *for all* $(x_1, x_2, ..., x_n, ...) \in D(S)$. *Note that* S *is dense and closed operator. Also,* $R(S)$ *is closed and* $N(S) = \{(x_1, 0, 0, ...) : x_1 \in \mathbb{R}\}$.

$$S^{\dagger}(y_1, y_2, y_3, ...) = (0, \frac{y_2}{2}, \frac{y_3}{3}, ...)$$

where $(y_n) \in l^2(\mathbb{N})$. *Observe that* $S = S^*$ *and* $D(S^*S) = D(S^2) = \{(x_n) \in H : \sum_{n=1}^{\infty} n^4 |x_n|^4 < \infty\}$. *Let* $C = \{(x_n) \in D(S^2) : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$ *One can verify that* $C^* = C$ *and* $S^{\dagger}SC \subseteq C$. *Consider* $D = (S^{\dagger})^*(C^*) = S^{\dagger}(C)$. *Let* $x,\ y \in D$. *Then* $x = S^{\dagger}u,\ y = S^{\dagger}v$ *for some* $u,\ v \in H$. *Consider the standard orthonormal basis* $\{e_n : n \in \mathbb{N}\}$ *of* $l^2(\mathbb{N})$. *Then, we can have* $u = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n$ *and* $v = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n$. *Since* $\langle u, e_n \rangle \geq 0, \langle v, e_n \rangle \geq 0$, *we get*

$$\begin{aligned} \langle x, y \rangle &= \langle S^{\dagger}u, S^{\dagger}v \rangle \\ &= \sum_{n=2}^{\infty} \frac{1}{n^2} \langle u, e_n \rangle \langle v, e_n \rangle \\ &\geq 0 \end{aligned}$$

Hence $D = (S^{\dagger})^*(C^*)$ *is acute. Therefore, by Theorem 4.3.2, it follows that* $(S^*S)^{\dagger}$ *is cone nonnegative.*

Example 4.4.2 *Consider* $H = L^2[0, \pi]$ *be the space of real valued functions.* $H' =$

$\{f \in \mathcal{AC}[0,\pi] : f' \in H\}$ where $\mathcal{AC}[0,\pi]$ be the space of real functions on $[0,\pi]$ which are absolutely continuous. $H'' = \{f \in H' : f' \in H'\}$ be the subspace of H'. Define the differential operator S by $\frac{d}{dt}$ with $D(S) = \{x \in H' : f(0) = f(\pi) = 0\}$. By the fundamental theorem of integral calculus, it is observed that $S \in \mathcal{C}(H)$. Take $f_n = \sin(nt),\ n \in \mathbb{N}$. Then $\{f_n : n \in \mathbb{N}\}$ is an orthonormal basis of the space H and functions $\{f_n : n \in \mathbb{N}\}$ contained in $D(S)$. Hence S is dense operator. Moreover, $Car(S) = D(S)$. So, S is one-one operator. Note that $R(S) = \{y \in H : \int_0^\pi y(t)dt = 0\} = span\{1\}^\perp$. Thus, $D(S^\dagger) = H$. Suppose $g_n = \sqrt{\frac{2}{\pi}}\cos(nt),\ t \in [0,\pi],\ n \in \mathbb{N}$. Then an orthonormal basis for $R(S)$ is given by $\{g_n : n \in \mathbb{N}\}$. So, $S^*S = -\frac{d^2}{dt^2}$ with $D(S^*S) = \{f \in H'' : f(0) = 0 = f(\pi)\}$ (refer [46]). By using projection method, it can be shown that $S^\dagger(y) = \sum_{n=1}^{\infty}\frac{1}{n}\langle y, g_n\rangle f_n$. Let $C = \{f \in D(S^*S) : \langle f, f_n\rangle \geq 0,\ \forall\ n \in \mathbb{N}\}$. Then C forms a cone and $C^* = C$. To check the second condition of the Theorem 4.3.2, let $\phi \in C$ and consider

$$(S^*S)^\dagger(\phi) = S^\dagger(S^\dagger)^*(\phi) = \sum_{n=1}^{\infty}\frac{1}{n^2}\langle \phi, f_n\rangle f_n.$$

Since $\phi \in C$, we have $\langle \phi, f_n\rangle \geq 0$ for all $n \in \mathbb{N}$ and hence $\frac{1}{n^2}\langle \phi, f_n\rangle \geq 0$ for $n \in \mathbb{N}$. Thus, $(S^*S)^\dagger$ is cone nonnegative.

Chapter 5

Conclusions

In this chapter, we summarize the key findings of the investigation carried out in this thesis and provide concluding remarks from our work. Moreover, we also mention few problems that give scope for further research.

5.1 Concluding Remarks

The thesis comprises of results on cone nonnegativity of generalized inverses of linear operators. The following is its summary:

Methods of index splitting are discussed in Chapter 2. The iteration schemes to solve the operator equation of the type $Sx = b,\ b \in R(S^k),\ k = ind(S)$ are discussed based on these methods of index splitting and convergence results are given. The Drazin monotonicity of bounded linear operators over ordered Banach space is characterized in the main results Theorem 2.4.1 and Theorem 2.4.2 with examples.

The splitting of non-invertible bounded linear operators into difference of operators over ordered Hilbert space is discussed in Chapter 3.4. The special class of operators which have $B_{\dagger}$-splitting are represented with their existence conditions and constructions methods in Theorem 3.3.4 and Theorem 3.4 respectively. Examples are given to illustrate the $B_\dagger$-splitting.

For unbounded Gram operators, the cone nonnegativity of Moore-Penrose inverse is characterized in Chapter 4. Specifically, a subclass of closed densely defined linear operators is considered in this chapter. This characterization involves acuteness of the cone. The equivalent conditions for the characterization of cone nonnegativity of Moore-Penrose inverse of unbounded Gram operators are given in Theorem 4.3.2 and these are proved with the help of sequence of lemmas given in the section 4.3. Examples are provided to show the cone nonnegativity of Moore-Penrose inverse of unbounded Gram operators.

5.2 Future Scope of Work

The scope of the present study lies in the applicability of results in many problems of operator theory as well as applied linear algebra. Specifically, we would like to highlight the following:

- The defintion of $B_\dagger$-splitting is given in Chapter 3. In [37], B_D-splitting of matrix is introduced by assuming the existence of Drazin inverse of a marix. This B_D-splitting can be studied on the class of operators for which Drazin inverse exists.
- The study on $B_\dagger$-splitting can be extended to the class of closed densely defined operators. For this one has to investigate whether the class of positive closed densely defined operators forms a cone or not; in the class of densely defined closed linear operators.
- To analyse the solution of the linear system, several comparison results can be studied for $B_\dagger$-splitting and index splitting methods.
- The characterization given in Chapter 4 can be discussed for an arbitrary operator in place of Gram operator.

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List of Publications

Published Articles

1 Archana Bhat and T. Kurmayya, “A characterization of cone nonnegativity of Moore–Penrose inverses of unbounded Gram operators”, *Positivity*, Vol.26, Article No. 67, 2022, https://doi.org/10.1007/s11117-022-00908-y.

2 Archana Bhat, T. Kurmayya and R.S. Selvaraj, “A characterization of Drazin monotonicity of operators over ordered Banach space”, *The Journal of Analysis,* 2023, https://doi.org/10.1007/s41478-023-00567-6 .

Communicated Articles

1 Archana Bhat and T. Kurmayya, “On $B_{\dagger}-$ splitting and Nonnegativity of Moore-Penrose inverses of bounded linear operators”, Communicated to *Operators and Matrices.*