Algorithmic Aspects of Variants of Roman Domination in Graphs

Submitted in partial fulfillment of the requirements

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DOCTOR OF PHILOSOPHY

Submitted by

Chakradhar Padamutham

(Roll No. 701633)

Under the guidance of

Dr. P. Venkata Subba Reddy

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

NATIONAL INSTITUTE OF TECHNOLOGY WARANGAL

TELANGANA - 506004, INDIA

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DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY WARANGAL
TELANGANA - 506004, INDIA

THESIS APPROVAL FOR Ph.D.

This is to certify that the thesis entitled, Algorithmic Aspects of Variants of Roman Domination in Graphs, submitted by Mr. Chakradhar Padamutham [Roll No. 701633] is approved for the degree of DOCTOR OF PHILOSOPHY at National Institute of Technology Warangal.

Examiner

Research Supervisor
Dr. P. Venkata Subba Reddy
Dept. of Computer Science and Engg.
NIT Warangal
India

Chairman
Prof. P. Radha Krishna
Head, Dept. of Computer Science and Engg.
NIT Warangal
India

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY WARANGAL
TELANGANA - 506004, INDIA

CERTIFICATE

This is to certify that the thesis entitled, Algorithmic Aspects of Variants of Roman Domination in Graphs, submitted in partial fulfillment of requirement for the award of degree of DOCTOR OF PHILOSOPHY to National Institute of Technology Warangal, is a bonafide research work done by Mr. Chakradhar Padamutham (Roll No. 701633) under my supervision. The contents of the thesis have not been submitted elsewhere for the award of any degree.

Research Supervisor
Dr. P. Venkata Subba Reddy
Assistant Professor
Dept. of CSE
NIT Warangal
India

Place: NIT Warangal
Date: 14 January, 2021

DECLARATION

This is to certify that the work presented in the thesis entitled "*Algorithmic Aspects of Variants of Roman Domination in Graphs*" is a bonafide work done by me under the supervision of Dr. P. Venkata Subba Reddy and was not submitted elsewhere for the award of any degree.

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Chakradhar Padamutham

(Roll No. 701633)

Date: 14-01-2021

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ABSTRACT

Let $G = (V, E)$ be a simple, undirected and connected graph. In this thesis, we study the algorithmic aspects of Roman domination and its variants, namely (i) Roman $\{2\}$-domination, (ii) double Roman domination, (iii) perfect Roman domination, (iv) perfect double Roman domination, (v) independent Roman domination, (vi) independent Roman $\{2\}$-domination, (vii) independent double Roman domination, (viii) total Roman domination, (ix) total double Roman domination, (x) weakly connected Roman domination, (xi) Roman $\{3\}$-domination, (xii) total Roman $\{2\}$-domination and (xiii) total Roman $\{3\}$-domination.

For a simple, undirected graph $G = (V, E)$, a Roman dominating function (RDF) $f : V \rightarrow \{0, 1, 2\}$ has the property that, every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A Roman $\{2\}$-dominating function (R2DF) $f : V \rightarrow \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N_G(v)$, with $f(u) = 2$, or at least two vertices $x, y \in N_G(v)$ with $f(x) = f(y) = 1$, where $N_G(v)$ is the set of vertices adjacent to v in G. A double Roman dominating function (DRDF) on G is a function $f : V \rightarrow \{0, 1, 2, 3\}$ such that for every vertex $v \in V$ if $f(v) = 0$, then v has at least two neighbors $x, y \in N_G(v)$ with $f(x) = f(y) = 2$ or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then v must have at least one neighbor w with $f(w) \geq 2$ A perfect Roman dominating function (PRDF) $f : V \rightarrow \{0, 1, 2\}$ has the property that, every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. A function $h : V(G) \rightarrow \{0, 1, 2, 3\}$ which satisfies the following two conditions is called a perfect double Roman dominating function (PDRDF).

C1). For all $q \in V$ with $h(q) = 0$, either there exist exactly two vertices r_1, r_2 such that $(q, r_1) \in E$, $(q, r_2) \in E$, $h(r_1) = 2$, $h(r_2) = 2$ and $\forall s$, if $h(s) = 3$ then $(q, s) \notin E$, or there exists exactly one vertex t such that $h(t) = 3$, $(q, t) \in E$ and $\forall u$, if $h(u) = 2$ then $(q, u) \notin E$.

C2). For all $q \in V$ with $h(q) = 1$, there exists exactly one vertex t such that $h(t) = 2$, $(q, t) \in E$ and $\forall u$, if $h(u) = 3$ then $(q, u) \notin E$.

A *Roman $\{3\}$-dominating function* (R3DF) is a function $g : V(G) \rightarrow \{0, 1, 2, 3\}$

having the property that $\sum_{v \in N_G(u)} g(v) \geq 3$, if $g(u) = 0$, and $\sum_{v \in N_G(u)} g(v) \geq 2$, if $g(u) = 1$ for any vertex $u \in G$. An independent Roman dominating function (IRDF), independent Roman $\{2\}$-dominating function (IR2DF) and independent double Roman dominating function (IDRDF), respectively, is a RDF, R2DF and DRDF with additional constraint that no two vertices assigned positive values are adjacent. A total Roman dominating function (TRDF), total double Roman dominating function (TDRDF), total Roman $\{2\}$-dominating function (TR2DF) and total Roman $\{3\}$-dominating function (TR3DF), respectively, is a RDF, DRDF, R2DF and R3DF with an additional property that the subgraph induced by the set of vertices labeled positive weight has no isolated vertices. A function $h : V(G) \rightarrow \{0, 1, 2\}$ which satisfies the following two conditions is called a weakly connected Roman dominating function (WCRDF) of G. C1). for all $q \in V$ with $h(q) = 0$ there exists a vertex r such that $(q, r) \in E$ and $h(r) = 2$ and
C2). the graph with vertex set $V(G)$ and edge set $\{(p, z) : h(p) \geq 1$ or $h(z) \geq 1$ or both $\}$ is connected.

The weight of a RDF (R2DF, DRDF, PRDF, PDRDF, IRDF, IR2DF, IDRDF, TRDF, TR3DF, WCRDF, TR2DF, R3DF, TDRDF) is the sum $f(V) = \sum_{v \in V} f(v)$. Given a graph G and a positive integer k, the Roman domination problem (RDP), Roman $\{2\}$-domination problem (R2DP), double Roman domination problem (DRDP), perfect Roman domination problem (PRDP), perfect double Roman domination problem (PDRDP), independent Roman domination problem (IRDP), independent Roman $\{2\}$-dominatin problem (IR2DP), independent double Roman domination problem (IDRDP), total Roman domination problem (TRDP), total double Roman domination problem (TDRDP), Roman $\{3\}$-domination problem (R3DP), total Roman $\{2\}$-domination problem (TR2DP) and total Roman $\{3\}$-domination problem (TR3DP), respectively, is to check whether G has a RDF, R2DF, DRDF, PRDF, PDRDF, IRDF, IR2DF, IDRDF, TRDF, TDRDF, R3DF, TR2DF and TR3DF of weight at most k.

The minimum RDP (MRDP), minimum R2DP (MR2DP), minimum DRDP (MDRDP), minimum PRDP (MPRDP), minimum PDRDP (MPDRDP), minimum IRDP (MIRDP), minimum IR2DP (MIR2DP) and minimum IDRDP (MIDRDP), minimum TRDP (MTRDP), minimum TDRDP (MTDRDP), minimum R3DP (MR3DP), minimum TR2DP

(MTR2DP), minimum TR3DP (MTR3DP), minimum WCRDP (MWCRDP), respectively, is to find an RDF, R2DF, DRDF, PRDF, PDRDF, IRDF, IR2DF, IDRDF, TRDF, TDRDF, R3DF, TR2DF, TR3DF and WCRDF of minimum weight in the input graph.

In this thesis, we show that RDP, R2DP, DRDP, PRDP, IRDP, IR2DP, IDRDP and R3DP are NP-complete for star convex and comb convex bipartite graphs, R2DP is NP-complete for bisplit graphs, and PDRDP is NP-complete for bipartite graphs, and IR2DP, IDRDP, R3DP, PDRDP and TR3DP are NP-complete for chordal graphs, and IRDP, IR2DP and IDRDP are NP-complete for dually chordal graphs, and R3DP is NP-complete for planar graphs. We show that MRDP, MR2DP, MDRDP, MPRDP, MPDRDP, MIRDP, MIR2DP, MIDRDP, MTRDP, MTDRDP, MR3DP, MTR2DP, MTR3DP and MWCRDP are linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs.

We show that the MRDP for star convex bipartite graphs and comb convex bipartite graphs cannot be approximated within $(1-\epsilon)\ln|V|$ for any $\epsilon > 0$ unless $P = NP$ and show that MTRDP (MTDRDP, MR3DP, MTR2DP, MTR3DP, MWCRDP) cannot have $(1-\delta)\ln|V|$ ratio approximation algorithm for any $\delta > 0$ unless $P = NP$.

We propose a, $2(1+\ln(\Delta+1))$-approximation algorithm (APX-AL) for the MRDP, $2(1+\ln(\Delta+1))$-APX-AL for the MR2DP and $3(1+\ln(\Delta+1))$-APX-AL for the MDRDP, $2(\ln(\Delta-0.5)+1.5)$-APX-AL for the MTRDP and the MTR2DP, $3(\ln(\Delta-0.5)+1.5)$-APX-AL for the MTDRDP and the MTR3DP, $3(1+\ln(\Delta-1))$-APX-AL for the MR3DP and $2(1+\epsilon)(1+\ln(\Delta-1))$-APX-AL for the MWCRDP, where Δ is the maximum degree of G and $\epsilon > 0$.

We show that the, MIRDP, MIR2DP and MIDRDP are APX-hard for graphs with $\Delta = 4$, and MRDP, MTRDP and MTDRDP are APX-complete for graphs with $\Delta = 5$, and MR3DP, MTR2DP, MTR3DP and MWCRDP are APX-complete for graphs with $\Delta = 4$.

We show that domination problem and IRDP (PDRDP, IR2DP, IDRDP, R3DP, TR2DP, TR3DP, WCRDP) are not equivalent in computational complexity aspects.

We adopt an Integer Linear Programming (ILP) approach towards computing the solutions of MR3DP and MTR3DP.

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$x \in S$	x is a *member* of a set S
$\|S\|$	*size* or *cardinality* of the set S
$S \subseteq V$	S is a *subset* of V
$X \cup Y$	*union* of two sets X and Y
$X \cap Y$	*intersection* of two sets X and Y
$X \setminus Y$	*set difference* of two sets X and Y
$\emptyset$	*empty set*
$:$	*such that*
$\mathbb{N}$	set of *natural numbers*
$\mathbb{R}$	set of *real numbers*
$\mathbb{Z}^+$	set of *positive integers*
$V(G)$ or V	set of *vertices*
$E(G)$ or E	set of *edges*
$N[v]$	*closed neighborhood* of a vertex v
$N(v)$	*neighborhood* or *open neighborhood* of a vertex v
Δ or $\Delta(G)$	*maximum degree* of a graph G
δ or $\delta(G)$	*minimum degree* of a graph G
$G[S]$ or $\{S\}$	*induced subgraph* of G with a set S
$\gamma(G)$	*domination number* of a graph G
$\gamma_c(G)$	*connected domination number* of a graph G

$i(G)$	*independent domination number* of a graph G
$\gamma_R(G)$	*Roman domination number* of a graph G
$\gamma_{\{R2\}}(G)$	*Roman* $\{2\}$*-domination number* of a graph G
$\gamma_{dR}(G)$	*double Roman domination number* of a graph G
$\gamma_R^P(G)$	*perfect Roman domination number* of a graph G
$\gamma_{dR}^p(G)$	*perfect double Roman domination number* of a graph G
$i_R(G)$	*independent Roman domination number* of a graph G
$i_{\{R2\}}(G)$	*independent Roman* $\{2\}$*-domination number* of a graph G
$i_{dR}(G)$	*independent double Roman domination number* of a graph G
$\gamma_{tR}(G)$	*total Roman domination number* of a graph G
$\gamma_{tdR}(G)$	*total double Roman domination number* of a graph G
$\gamma_R^{wc}(G)$	*weakly connected Roman domination number* of a graph G
$\gamma_{\{R3\}}(G)$	*Roman* $\{3\}$*-domination number* of a graph G
$\gamma_{tR2}(G)$	*total Roman* $\{2\}$*-domination number* of a graph G
$\gamma_{t\{R3\}}(G)$	*total Roman* $\{3\}$*-domination number* of a graph G
$\square$	*end of a proof*
$\gamma_R(G)$	*Roman domination number* of a graph G
$H\square G$	*Cartesian product* of graphs H and G
P_n	*path graph* of order n
C_n	*cycle graph* of order n
K_n	*complete graph* of order n
$deg(v)$	*degree* of a vertex v

Chapter 1

Introduction

In this thesis, we mainly focus on determining algorithmic complexity aspects of Roman domination and its variants. The theory of Roman domination has several real life applications, namely, facility location problems, planning of defence strategies, surveillance related problems, ad hoc wireless networks, etc. We present results related to Roman domination and its variants namely, Roman $\{2\}$-domination, double Roman domination, perfect Roman domination, perfect double Roman domination, independent Roman domination, independent Roman $\{2\}$-domination, independent double Roman domination, total Roman domination, total double Roman domination, weakly connected Roman domination, Roman $\{3\}$-domination, total Roman $\{2\}$-domination and total Roman $\{3\}$-domination.

1.1 Preliminaries

Here, we present few definitions and results pertaining to graph theory and algorithms which will be used throughout the thesis.

1.1.1 Graph Theoretic Terminology

We consider $G(V, E)$ as an undirected, simple and connected graph. For a vertex u of G, the (*open*) *neighborhood* denoted $N_G(u)$ is the set $\{v : (v, u) \in E(G)\}$ and its *degree* is $|N_G(u)|$. $N_G[u] = \{u\} \cup N_G(u)$ is the *closed neighborhood* of u. *Maximum degree* of G

denoted Δ (or clearly $\Delta(G)$) is $max_{u \in V(G)}|N_G(u)|$. A vertex v is called *isolated vertex* if $|N_G(v)| = 0$. A vertex v of G is called *universal vertex* if $|N_G(v)| = |V(G)| - 1$. If each pair of vertices is connected by an edge then the graph is *complete*. An *induced subgraph* of H denoted $H[S]$ (or $\langle S \rangle$) is the graph formed with the vertex set $S \subseteq V(H)$ and the edge set $\{(u, v) : u, v \in S\}$. A set $S \subseteq V$ forms a *clique* if the induced subgraph $\langle S \rangle$ is complete. If no two vertices of a set $S \subseteq V$ are adjacent then, S is said to be an *independent set*. We refer to [20], for undefined notations and terminology.

1.1.2 Algorithmic Preliminaries

Here, we present few concepts and notations of complexity theory, used in this thesis. Throughout this thesis, the O (Big 'Oh') notation is used to bound the running time of an algorithm. Let $g : \mathbb{N} \rightarrow \mathbb{R}^+$ and $f : \mathbb{N} \rightarrow \mathbb{R}^+$. We say that $f(n) = O(g(n))$ if there exists two constants $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ satisfying $f(n) \leq c.g(n)$, $\forall n \geq n_0$.

1.1.2.1 Complexity Classes

The most common complexity classes are *P*, *NP*, *NP-hard*, and *NP-complete*.

> "The decision problems that can be solvable in polynomial time in worst-case belongs to the class *P*. In detail, a problem belongs to class *P* if there exists an algorithm that solves the problem for any input size in $O(n^k)$, where n is input size and k is a constant. The decision problems whose 'yes' instances can be verifiable in polynomial time belongs to the class *NP*."

A polynomial time solvable problem can be verified in polynomial time. Therefore, any problem in class *P* belongs to *NP*.

> "A problem is said to be *NP-hard* if every problem in *NP* class is polynomially reducible to it. If a problem is in NP and NP-hard then it is said to be in *NP-complete*."

The problems in *NP-complete* are considered as hardest problems in *NP*.

Let I_π denote the set of all instances of a decision problem π. An instance x of π is called an yes (no) instance if the answer to the problem π for the instance x is yes (no). A decision problem π_1 is said to be *polynomially reducible* to another decision problem π_2 if there exists a function $f : I_{\pi_1} \rightarrow I_{\pi_2}$ such that (i) f is computable deterministically in polynomial time and (ii) x is an yes instance of π_1 iff $f(x)$ is an yes instance of π_2. A decision problem π is said to be *NP-complete* if (i) $\pi \in NP$, and (ii) for any problem $\pi' \in NP$, π' is polynomially reducible to π. An optimization problem π is *NP-hard* if a polynomial time algorithm for π would imply a polynomial time algorithm for every problem in NP. In general, an optimization problem is NP-hard if its corresponding decision problem is NP-complete.

1.1.2.2 Approximation Hardness

One of the ways to deal with problems which are NP-hard is to give approximation algorithms for giving a feasible solution to an optimization version of the problem. An algorithm A for a minimization problem π is called a $\rho(n)$*-approximation algorithm*, if the cost C of the solution given by the algorithm A is within $\rho(n)$ times the cost C^* of an optimal solution, for any input of size n, that is, $\frac{C}{C^*} \leq \rho(n)$. If $\rho(n) = c$ for some constant $c > 1$, then the algorithm A is called a *constant approximation algorithm* or a c*-approximation algorithm*. The class of all NP optimization problems that have constant approximation algorithms which run in polynomial time is called *APX*.

"An optimization problem π is *APX-complete* if:

1. $\pi \in$ APX, and
2. $\pi \in$ APX-hard, that is, there exists an L-reduction from known APX-complete problem to π."

L-reduction is one of the several approximation-preserving reductions available in the literature. This can be formally defined as follows.

Definition 1.1.1. *"(L-reduction) [16] An optimization problem* π *is L-reducible to optimization problem* π' *if there exists a function* $f : \pi \rightarrow \pi'$ *and two constants* $\alpha, \beta \in \mathbb{Z}^+$*, which satisfy the following for every instance* x *of* π*:*

1. $opt_{\pi'}(f(x)) \leq \alpha.opt_{\pi}(x)$

2. *for every feasible solution* y *of* $f(x)$ *with objective value* $m_{\pi'}(f(x), y) = c_2$ *in polynomial time one can find a solution* y' *of* x *with* $m_{\pi}(x, y') = c_1$ *such that* $|opt_{\pi}(x) - c_1| \leq \beta |opt_{\pi'}(f(x)) - c_2|$

Here, $opt_{\pi}(x)$ *represents the size of an optimal solution for an instance* x *of* π*."*

1.2 Graph Classes Studied in the Thesis

Although most of the graph optimization problems are algorithmically hard to solve (*i.e.* NP-hard) in arbitrary graphs, these problems are proved to have polynomial time algorithms when the input restricted to specific graph class. Here, we define the graph classes considered in the rest of the thesis.

1.2.1 Bipartite Graphs

Definition 1.2.1. *"A graph is bipartite if its vertex set* V *can be split into disjoint independent sets* P *and* Q *such that every edge* (p, q) *is incident on a vertex in* P *and other in* Q*."*

A bipartite graph with partition P and Q, we denote as $G = (P, Q, E)$. Further, if each vertex $p \in P$ is adjacent to every $q \in Q$ then G is a *bi-clique* or *complete bipartite* and denoted as $K_{m,n}$, here $m = |P|$ and $n = |Q|$. Clearly, bipartite graphs can be properly colorable with two colors (i.e., *2-colorable*). Since sets X and Y are independent, bipartite graph does not contain odd cycle. Bipartition is determined in polynomial time using Breadth First Search algorithm [14].

1.2.2 Bisplit Graphs

Definition 1.2.2. *"A graph $G = (V, E)$ is a bisplit graph if vertex set V can be split into three disjoint independent sets X, Y, and Z such that $X \cup Y$ forms a complete bipartite graph (or bi-clique) [2]."*

In a bisplit graph, if one partition of bi-clique is empty then, it is a bipartite graph. Hence, bipartite graphs is a proper subclass of bisplit graphs. Therefore, all bisplit graphs are not *2-colorable*.

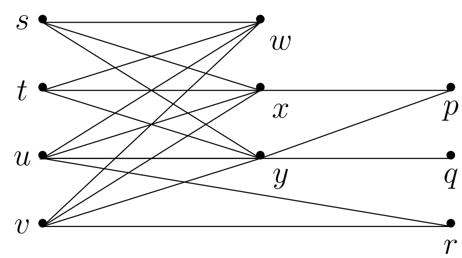

Figure 1.1: Bisplit graph

The bisplit graph with partition $X = \{s, t, u, v\}$, $Y = \{w, x, y\}$, and $Z = \{p, q, r\}$ is illustrated in Figure 1.1. We can recognize in polynomial time that whether the graph is a bisplit graph [2].

1.2.3 Star Convex Bipartite Graphs

Definition 1.2.3. "([70]) *For a bipartite graph $G = (P, Q, E)$ if there exists an associated tree $T = (P, F)$, such that for each vertex $q \in Q$, its neighborhood $N(q)$ induces a subtree of T then, it is a tree convex bipartite graph."*

G is called a *star convex bipartite graph* if T is a *star* in the Definition 1.2.1. Figure 1.2 depicts a star convex bipartite graph with associated star.

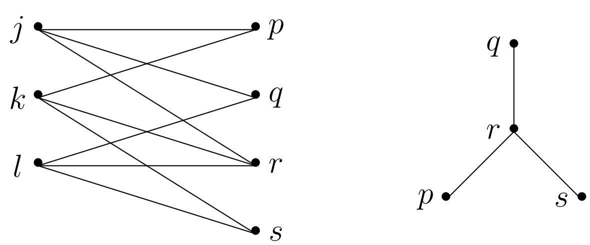

Figure 1.2: Star convex bipartite graph

1.2.4 Comb Convex Bipartite Graphs

A bipartite graph is *comb convex* if T is a *comb* in the Definition 1.2.1.

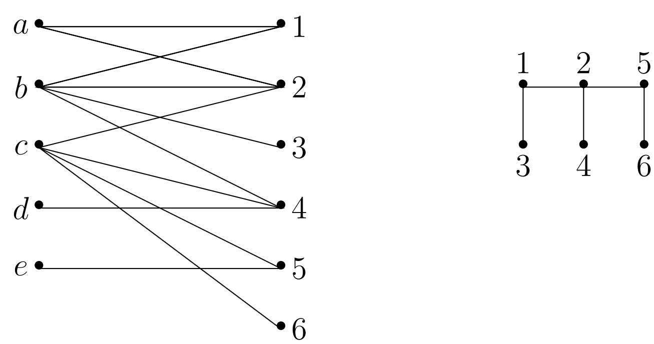

Figure 1.3: Comb convex bipartite graph

An example comb convex bipartite graph along with its corresponding comb is depicted in Figure 1.3. Given a bipartite graph, we can construct comb convex bipartite graph as follows.

Theorem 1.2.1 (*Canonical transformation*)**.** *([29]) "Given a bipartite graph* $G = (L, M, E)$*, where* $L = \{l_1, l_2, \ldots, l_{|L|}\}$*,* $M = \{m_1, m_2, \ldots, m_{|M|}\}$*, we can construct a comb convex bipartite graph* $G' = (L', M, E')$*, where* $L' = L \cup \{l_{|L|+1}, l_{|L|+2}, \ldots, l_{2|L|}\}$ *and* $E' = E \cup \{(l_i, m') : |L| + 1 \leq i \leq 2|L|, m' \in M\}$*."*

1.2.5 Chordal Graphs

In a graph, if any cycle with length at least 3 has a chord, then it is *chordal*. Alternately, if a graph does not contain a simple cycle having length greater than 3 then it is *chordal* (or *triangulated graph*). Since chordal graphs is a proper subclass of perfect graphs, where the chromatic and clique numbers of a graph are equal, the study of the graph problems in this graph class and in its subclasses is quite interesting. The important subclasses of chordal graphs considered in this thesis include split graphs, block graphs, threshold graphs, undirected path graphs, and doubly chordal graphs.

A vertex v is *simplicial* if its closed neighborhood $N[v]$ induces a complete subgraph of G. It is known that there exists at least one simplicial vertex in a chordal graph and if it is not complete then it has at least two simplicial vertices which are not adjacent [25]. The *perfect elimination ordering* (PEO) is an ordering of vertices say, $(v_1, \ldots, v_n)$, if v_i is the simplicial vertex of $G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for every i, $1 \leq i \leq n$. From hereditary nature of chordality, all the vertices of the graph can be removed successively removing simplicial vertices one after the other. This leads to the following characterization for chordal graphs due to Fulkerson et al [21].

Theorem 1.2.2. ([21]) *"A graph G is chordal iff G admits a PEO."*

For example, the graph depicted in Figure 1.4 admits a PEO $(p_1, p_2, p_3, p_5, p_4, p_6, p_7)$, hence it is a chordal graph. A PEO of a chordal graph can be obtained in linear time [11].

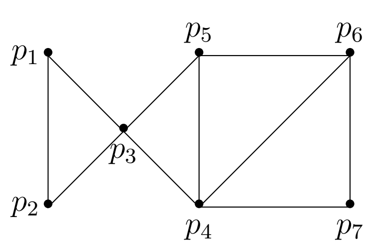

Figure 1.4: Chordal graph

1.2.6 Chain Graphs

A *chain graph* $G = (X, Y, E)$ is a bipartite graph such that a *chain* can be formed with sets of open neighborhood of X, i.e., $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$. If a bipartite graph $G = (X, Y, E)$ is a chain graph, then there is a chain with open neighborhoods of the vertices of Y as well. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is referred as *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \ldots \supseteq N_G(y_q)$. Chain ordering exists for every chain graph [37]. Figure 1.5 illustrates some examples of chain graphs.

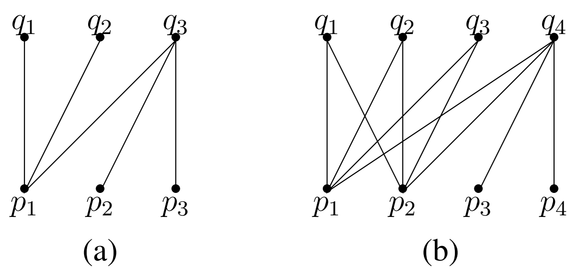

Figure 1.5: Chain graphs

1.2.7 Dually Chordal Graphs

A vertex $u \in N_G[v]$ is a *maximum neighbour* of v in G if $N_G[w] \subseteq N_G[u]$ holds for each $w \in N_G[v]$. A vertex ordering $(v_1, v_2, \ldots, v_n)$ is a maximum neighbourhood ordering (MNO) if for each $i < n$, v_i has a maximum neighbour in $\langle \{v_i, v_{i+1}, \ldots, v_n\} \rangle$.

> **Theorem 1.2.3.** *([67]) "A graph G is dually chordal iff G admits a MNO."*

A MNO of a dually chordal graph is determined in linear time [3]. Figure 1.6 depicts a dually chordal graph with a MNO $(v_4, v_8, v_3, v_7, v_6, v_2, v_1, v_5)$.

1.2.8 Split Graphs

A graph is a *split graph* if the vertex set can be split into an independent set I and a clique C. It can be observed that this partition may not be unique. Since every split graph admits a

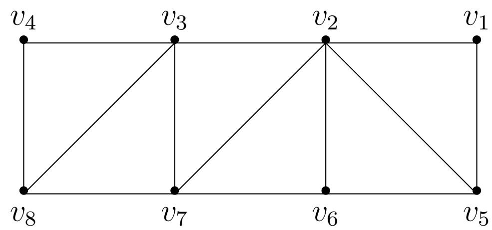

Figure 1.6: Dually chordal graph

perfect elimination ordering with independent set vertices followed by clique vertices, split graphs is one of the chordal subclasses. It can be noted that complement of a split graph is also a split graph. In a split graph $G = (C, I)$, if every vertex in I is adjacent to all vertices of C then G is called *complete split graph*.

1.2.9 Threshold Graphs

A graph is *threshold graph* if there exists a real number $w(v)$, $\forall v \in V$ and a real number t such that a set $S \subseteq V$ is independent iff $\sum_{v \in S} w(v) \leq t$.

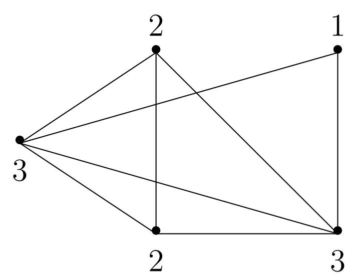

Figure 1.7: Threshold graph

The graph illustrated in Figure 1.7 is a threshold graph with threshold $t = 3$. The following characterizations are defined for threshold graphs in [56].

- Threshold graph G can be generated by adding an isolated vertex or a dominating vertex repeatedly, starting with single vertex.
- A graph is a *threshold graph* iff for a split partition (C, I) of V, there is an ordering $(x_1, x_2, \ldots, x_p)$ of C satisfying $N_G[x_1] \subseteq \ldots \subseteq N_G[x_p]$, and there is an ordering $(y_1, y_2, \ldots, y_q)$ of I satisfying $N_G(y_1) \supseteq \ldots \supseteq N_G(y_q)$.

1.3 Organization of the Thesis

The rest of the thesis is organized as follows.

In chapter 2, we discuss the Roman domination problem and its thirteen variants namely (i) Roman $\{2\}$-domination, (ii) double Roman domination, (iii) perfect Roman domination, (iv) perfect double Roman domination, (v) independent Roman domination, (vi) independent Roman $\{2\}$-domination, (vii) independent double Roman domination, (viii) total Roman domination, (ix) total double Roman domination, (x) weakly connected Roman domination, (xi) Roman $\{3\}$-domination, (xii) total Roman $\{2\}$-domination and (xiii) total Roman $\{3\}$-domination. We give the importance of studying these variants of Roman domination. We also present the survey of the results related to some well studied variants of domination problems like connected domination, total domination, that are present in the literature. The algorithmic complexity of Roman domination, Roman $\{2\}$-domination and double Roman domination problems are investigated in chapter 3. In chapter 4, we investigate the perfect Roman domination, perfect double Roman domination, total Roman domination and total double Roman domination problems complexity. Finally, we study the complexity difference between domination and perfect double Roman domination problems. In chapter 5, we investigate the independent Roman domination, independent Roman $\{2\}$-domination and independent double Roman domination and investigate its complexity in threshold graphs, chain graphs and bounded-tree width graphs. We study the complexity difference between domination and these variants of independent Roman domination problems. In chapter 6, we investigate the weakly connected Roman domination, Roman $\{3\}$-domination, total Roman $\{2\}$-domination and total Roman $\{3\}$-domination problems complexity. In chapter 7, we present approximation results related to all the above said Roman domination problems. Finally, chapter 8 summarizes the work presented in this thesis and mentions future directions of research related to these problems.

Chapter 2

Problems Studied in the Thesis and Related Work

2.1 Problems Studied in the Thesis

Due to wide variety of applications in several fields, many varieties of Roman domination problem have been emerged and studied in terms of exact values, lower and upper bounds, computational complexity, and approximation point of views. By identifying the importance of variants of Roman domination problem, we studied the following Roman domination and its variants in the thesis. Throughout the thesis P refers to polynomial time solvable and NPC refers to NP-complete.

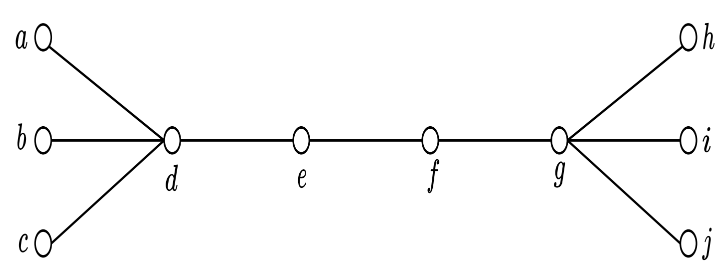

Figure 2.1: Example graph G

2.1.1 Roman Domination

In 2004, Cockayne et al. in [23] introduced the concept Roman domination (RDOM).

Definition 2.1.1. *"A Roman Dominating Function (RDF) $f : V \to \{0, 1, 2\}$ on G is a function such that $\forall u \in V$ where $f(u) = 0$ has a neighbour v such that $f(v) = 2$. The weight of a RDF is the value $f(V) = \sum_{v \in V} f(v)$. The Roman domination number is the minimum weight of a RDF on G and is denoted by $\gamma_R(G)$."*

Let $p : V(G) \to \{0, 1, 2\}$ be a function defined, on the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 2, & \text{if } v \in \{d, g\} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

Clearly, p is a RDF of G and $p(V) = \sum_{u \in V} p(u) = 4$. We refer to [17, 23, 31, 42, 43, 44, 57, 58, 60] for the liturature on RDOM in graphs.

2.1.2 Roman {2}-domination

In 2016, Roman $\{2\}$-domination (R2DOM) was introduced by Chellali et al. in [36].

Definition 2.1.2. *"A Roman $\{2\}$-dominating function (R2DF) is a RDF with an additional property that every vertex with weight zero may also adjacent to at least two vertices such that each with weight one. The weight of a R2DF is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a R2DF is called the Roman $\{2\}$-domination number and is denoted by $\gamma_{\{R2\}}(G)$."*

Let $q : V(G) \to \{0, 1, 2\}$ be a function defined, on the graph G depicted in Figure 2.1, as follows.

$$q(v) = \begin{cases} 2, & \text{if } v = g \\ 1, & \text{if } v \in \{a, b, c, e\} \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Clearly, q and the function p defined in section 2.1.1 are R2DFs.

2.1.3 Double Roman domination

Double Roman domination (DRDOM) was initiated in 2016 by Robert et al. in [7].

Definition 2.1.3. *"A double Roman dominating function (DRDF) on G is a function $f : V \rightarrow \{0, 1, 2, 3\}$ such that every vertex with weight zero should have at least either two neighbors such that each with weight two or one neighbor with weight three, and every vertex with weight one should have at least one neighbor with weight one. The weight of a DRDF is the value $f(V) = \sum_{v \in V} f(v)$. The double Roman domination number equals the minimum weight of a DRDF on G, denoted by $\gamma_{dR}(G)$."*

Let $p : V(G) \rightarrow \{0, 1, 2, 3\}$ and $q : V(G) \rightarrow \{0, 1, 2, 3\}$ are functions defined, on the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 3, & \text{if } v \in \{d, g\} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

$$q(v) = \begin{cases} 3, & \text{if } v = g \\ 2, & \text{if } v \in \{a, b, c, e\} \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

Clearly, p and q are DRDFs.

2.1.4 Perfect Roman Domination

In 2018, the concept of perfect Roman domination (PRDOM) was introduced by Henning et al. in [38].

Definition 2.1.4. *"A function $f : V \rightarrow \{0,1,2\}$ is a perfect Roman Dominating Function (PRDF) on G, if every vertex with weight zero is adjacent to exactly one vertex with weight two. The weight of a PRDF is the value $f(V) = \sum_{u \in V} f(u)$. The perfect Roman domination number is the minimum weight of a PRDF on G and is denoted by $\gamma_R^P(G)$."*

Clearly, the function p defined in section 2.1.1 is a PRDF. The PRDOM has been studied in [40, 65].

2.1.5 Perfect Double Roman Domination

In 2020, the concept of perfect double Roman domination (PDRDOM) was introduced by Egunjobi et al. in [9].

Definition 2.1.5. *"A function $h : V(G) \rightarrow \{0,1,2,3\}$ which satisfies the following conditions is called a perfect double Roman dominating function (PDRDF).*
C1). For all $q \in V$ with $h(q) = 0$, either there exist exactly two vertices r_1, r_2 such that $(q, r_1) \in E$, $(q, r_2) \in E$, $h(r_1) = 2$, $h(r_2) = 2$ and $\forall s$, if $h(s) = 3$ then $(q, s) \notin E$, or there exists exactly one vertex t such that $h(t) = 3$, $(q, t) \in E$ and $\forall u$, if $h(u) = 2$ then $(q, u) \notin E$.
C2). For all $q \in V$ with $h(q) = 1$, there exists exactly one vertex t such that $h(t) = 2$, $(q, t) \in E$ and $\forall u$, if $h(u) = 3$ then $(q, u) \notin E$.
The weight of a PDRDF is the value $f(V) = \sum_{u \in V} f(u)$. The perfect double Roman domination number is the minimum weight of a PDRDF on G and is denoted by $\gamma_{dR}^P(G)$."

Clearly, the function p defined in section 2.1.3 is a PDRDF.

2.1.6 Independent Roman Domination

In 2004, Independent Roman domination (IRDOM) was introduced by Cockayne et al. in [23].

Definition 2.1.6. *"An independent Roman dominating function (IRDF) on G is a RDF f with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \geq 1\}$ contains only isolated vertices. The weight of an IRDF f is the value $f(V) = \sum_{v \in V} f(v)$. The independent Roman domination number equals the minimum weight of an IRDF on G, denoted by $i_R(G)$."*

Clearly, the function p defined in section 2.1.1 is a IRDF. The concept of IRDOM has been studied in [48, 49].

2.1.7 Independent Roman $\{2\}$-domination

In 2018, Independent Roman $\{2\}$-domination (IR2DOM) was introduced by Rahmouni et al. in [8].

Definition 2.1.7. *"An independent Roman $\{2\}$-dominating function (IR2DF) on G is a R2DF f with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \geq 1\}$ contains only isolated vertices. The weight of an IR2DF f is the value $f(V) = \sum_{v \in V} f(v)$. The independent Roman $\{2\}$-domination number equals the minimum weight of an IR2DF on G, denoted by $i_{\{R2\}}(G)$."*

Let $q : V(G) \to \{0, 1, 2\}$ be a function defined, on the graph G depicted in Figure 2.1, as follows.

$$q(v) = \begin{cases} 2, & \text{if } v = e \\ 1, & \text{if } v \in \{a, b, c, h, i, j\} \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

Clearly, q and the function p defined in section 2.1.1 are IR2DFs. The literature on IR2DOM in graphs has been surveyed in [59].

2.1.8 Independent Double Roman Domination

In 2019, Independent double Roman domination (IDRDOM) was introduced by Maimani et al. in [30].

Definition 2.1.8. *"An independent double Roman dominating function (IDRDF) is a DRDF g with the additional property that the subgraph of G induced by the set $\{v \in V : g(v) \geq 1\}$ contains only isolated vertices. The weight of an IDRDF f is the value $f(V) = \sum_{v \in V} f(v)$. The independent double Roman domination number equals the minimum weight of an IDRDF on G, denoted by $i_{dR}(G)$."*

Clearly, the functions p and q defined in section 2.1.3, for the graph G depicted in Figure 2.1, are IDRDFs.

2.1.9 Weakly Connected Roman Domination

In 2019, Raczek et al. in [34] initiated the study of weakly connected Roman domination (WCRDOM).

Definition 2.1.9. *"A function $h : V(G) \to \{0, 1, 2\}$ which satisfies the following conditions is called a weakly connected Roman dominating function (WCRDF) of G with weight $w(h) = h(V) = \sum_{p \in V} h(p)$.*
C1). for all $z \in V$ with $h(z) = 0$ there exists a vertex y such that $(y, z) \in E$ and $h(y) = 2$ and
C2). the graph with vertex set $V(G)$ and edge set $\{(w, z) : h(w) \geq 1 \text{ or } h(z) \geq 1$ or both $\}$ is connected.
The weakly connected Roman domination number equals the minimum weight of WCRDF on G, denoted by $\gamma_R^{wc}(G)$."

Let $p : V(G) \to \{0, 1, 2\}$ be a function defined, on the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 2, & \text{if } v \in \{d, g\} \\ 1, & \text{if } v = e \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

Clearly, p is a WCRDF.

2.1.10 Roman $\{3\}$-domination

In 2020, Mojdeh et al. in [18] initiated the study of Roman $\{3\}$-domination (R3DOM).

Definition 2.1.10. *"A function $g : V \rightarrow \{0, 1, 2, 3\}$ having the property that $\sum_{v \in N_G(u)} g(v) \geq 3$, if $g(u) = 0$, and $\sum_{v \in N_G(u)} g(v) \geq 2$, if $g(u) = 1$ for any vertex $u \in G$ is called a Roman $\{3\}$-Dominating Function (R3DF) of G. The weight of a R3DF g is the sum $g(V) = \sum_{v \in V} g(v)$. The minimum weight of a R3DF is called the Roman $\{3\}$-domination number, denoted by $\gamma_{\{R3\}}(G)$."*

Let $p : V(G) \rightarrow \{0, 1, 2, 3\}$ be a function defined, on the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 3, & \text{if } v \in \{d, g\} \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

Clearly, p is a R3DF.

2.1.11 Total Roman Domination

In 2013, The notion of total Roman domination (TRDOM) was introduced by Liu et al. in [15].

Definition 2.1.11. *"A total Roman dominating function (TRDF) is a RDF f with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \geq 1\}$ is isolated-free. The weight of a TRDF g is the value $g(V) = \sum_{v \in V} g(v)$. The total Roman domination number equals the minimum weight of a TRDF, denoted by $\gamma_{tR}(G)$."*

Let $p : V(G) \to \{0, 1, 2\}$ and $q : V(G) \to \{0, 1, 2\}$ are two functions defined, on the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 2, & \text{if } v \in \{d, g\} \\ 1, & \text{if } v \in \{a, h\} \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

$$q(v) = \begin{cases} 2, & \text{if } v \in \{f, g\} \\ 1, & \text{if } v \in \{a, b, c, d\} \\ 0, & \text{otherwise} \end{cases} \tag{2.9}$$

Clearly, p and q are TRDFs. The concept of TRDOM has been studied in [6, 26, 33, 55].

2.1.12 Total Double Roman Domination

In 2019, Total double Roman domination (TDRDOM) was introduced by Shao et al. in [71], which is a variant of double Roman domination.

Definition 2.1.12. *"A total double Roman dominating function (TDRDF) is a DRDF g with the additional property that the subgraph of G induced by the set $\{v \in V : g(v) \geq 1\}$ is isolated-free. The weight of a TDRDF g is the value $g(V) = \sum_{v \in V} g(v)$. The total double Roman domination number equals the minimum weight of a TDRDF, denoted by $\gamma_{tdR}(G)$."*

let $p : V(G) \to \{0, 1, 2\}$ and $q : V(G) \to \{0, 1, 2\}$ are two functions defined, for the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 3, & \text{if } v \in \{d, g\} \\ 1, & \text{if } v \in \{e, f\} \\ 0, & \text{otherwise} \end{cases} \tag{2.10}$$

$$q(v) = \begin{cases} 3, & \text{if } v = g \\ 2, & \text{if } v = d \\ 1, & \text{if } v \in \{a, b, c, e, f\} \\ 0, & \text{otherwise} \end{cases} \tag{2.11}$$

Clearly, p and q are TDRDFs. The concept TDRDOM has been studied in [24, 71].

2.1.13 Total Roman $\{2\}$-domination

Recently, Ahangar et al. in [28] initiated the study of total Roman $\{2\}$-domination (TR2DOM).

Definition 2.1.13. *"A total Roman $\{2\}$-dominating function (TR2DF) g is a R2DF on G with the added property that $\langle\{v \in V : g(v) \geq 1\}\rangle$ is isolated-free. The weight of a TR2DF g is the value $g(V) = \sum_{v \in V} f(v)$. The total Roman $\{2\}$-domination number is the minimum weight of a TR2DF, denoted by $\gamma_{tR2}(G)$."*

Let $p : V(G) \rightarrow \{0, 1, 2\}$ be a function defined, for the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 2, & \text{if } v \in \{d, g\} \\ 1, & \text{if } v \in \{a, h\} \\ 0, & \text{otherwise} \end{cases} \tag{2.12}$$

Clearly, p is a TR2DF.

2.1.14 Total Roman $\{3\}$-domination

Recently, Shao et al. in [72] initiated the study of total Roman $\{3\}$-domination (TR3DOM).

Definition 2.1.14. *"A total Roman $\{3\}$-dominating function (TR3DF) g is a R3DF on G with the added property that $\langle\{v \in V : g(v) \geq 1\}\rangle$ is isolated-free. The weight of a TR3DF g is the value $g(V) = \sum_{v \in V} f(v)$. The total Roman $\{3\}$-domination number is the minimum weight of a TR3DF, denoted by $\gamma_{t\{R3\}}(G)$."*

Let $p : V(G) \rightarrow \{0, 1, 2, 3\}$ be a function defined, for the graph G depicted in Figure 2.1, as follows.

$$p(v) = \begin{cases} 3, & \text{if } v \in \{d, g\} \\ 1, & \text{if } v \in \{a, h\} \\ 0, & \text{otherwise} \end{cases} \tag{2.13}$$

Clearly, p is a TR3DF.

Given a graph G and a positive integer k, the RDOM problem (RDP), R2DOM problem (R2DP), DRDOM problem (DRDP), PRDOM problem (PRDP), PDRDOM problem (PDRDP), IR2DOM problem (IR2DP), IRDOM problem (IRDP), IDRDOM problem (IDRDP), TRDOM problem (TRDP), TDRDOM problem (TDRDP), R3DOM problem (R3DP), TR2DOM problem (TR2DP) and TR3DOM problem (TR3DP), respectively, is to check whether G has a RDF, R2DF, DRDF, PRDF, PDRDF, IRDF, IR2DF, IDRDF, TRDF, TDRDF, R3DF, TR2DF and TR3DF of weight at most k. It is known that the RDP is NPC for bipartite graphs, planar graphs and split graphs [42], R2DP is NPC for bipartite graphs [36], DRDP is NPC for bipartite and chordal graphs [27], PRDP is NPC for bipartite graphs, planar graphs and chordal graphs [65], IR2DP is NPC for bipartite graphs [8], IDRDP is NPC for bipartite graphs [30], WCRDP is NPC for split and bipartite graphs [34], R3DP is NPC for bipartite graphs [18], TRDP is NPC for bipartite and chordal graphs [15], TDRDP is NPC for bipartite and chordal graphs [71], TR2DP is NPC for bipartite and chordal graphs [28] and TR3DP is NPC for bipartite graphs [72]. In this thesis, we study the complexity of these problems in other graph classes.

The minimum RDP (MRDP), minimum R2DP (MR2DP), minimum DRDP (MDRDP), minimum PRDP (MPRDP), minimum PDRDP (MPDRDP), minimum IRDP (MIRDP), minimum IR2DP (MIR2DP), minimum IDRDP (MIDRDP), minimum TRDP (MTRDP), minimum TDRDP (MTDRDP), minimum R3DP (MR3DP), minimum TR2DP (MTR2DP), minimum TR3DP (MTR3DP), minimum WCRDP (MWCRDP), respectively, is to find an RDF, R2DF, DRDF, PRDF, PDRDF, IRDF, IR2DF, IDRDF, TRDF, TDRDF, R3DF, TR2DF, TR3DF and WCRDF of minimum weight in the input graph.

2.2 Related Work

In a graph $G = (V, E)$, a *dominating set* (DS) of a graph G is a set D such that $D \subseteq V(G)$ and $\cup_{w \in D} N_G[w] = V(G)$. The *domination number* of G denoted by $\gamma(G)$ is $min\{|T| : T$ is a DS of $G\}$. The problem of finding a DS of smallest cardinality in a graph is called the MINIMUM DOMINATION problem or minimum dominating set (MDS) problem. Given a graph H and a positive integer l, the domination decision problem (DDP) is to check whether H has a DS of size at most l. We refer to [66, 67] for the literature on the concept of domination. The DOMINATION problem has been proved as NPC for general graphs [54]. This problem has also been proved as NPC for bipartite graphs [1]. Further, this problem complexity has been investigated in bipartite subclasses. Consequently, it has been proved as NPC for comb convex and star convex bipartite graphs.

"The optimization version of DOMINATION problem is given below.

MINIMUM DOMINATION Problem

Instance: A simple, undirected graph $G = (V, E)$.

Solution: Minimum cardinality dominating set D of G.

Measure: Cardinality of D."

In approximation perspective, it has been proved that for the MINIMUM DOMINATION problem with ratio $1+\ln(\Delta(G)+1)$ [14]. The MINIMUM DOMINATION problem cannot be approximated within $(1-\epsilon)\ln n$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME($n^{O(\log \log n)}$), where

$n = |V|$ [39]. The APX-completeness of this problem in graphs with $\Delta = 3$ has been proved in [14].

A *connected dominating set* (CDS) S is a DS and $G[S]$ is connected. The minimum size of a connected dominating set of G denoted by $\gamma_c(G)$, is called the *connected domination number* of G. The Connected Domination Decision (CDM) problem is to check if an input graph G has a CDS S, with $|S| \leq k$, where $k \in \mathbb{Z}^+$. The MINIMUM CONNECTED DOMINATING SET (MCDS) problem is to find a CDS of minimum size in the input graph. It has been proved that one can give an approximation algorithm for the MINIMUM CONNECTED DOMINATING SET problem with ratio of $(1 + \epsilon)(1 + \ln(\Delta - 1))$ [19]. This problem cannot be approximated within $(1 - \epsilon) \ln n$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME$(n^{O(\log \log n)})$, where $n = |V|$ [39]. The APX-completeness of this problem in graphs with $\Delta = 3$ has been proved in [14].

A DS S is called an *independent dominating set* (IDS) of G if S is an independent set. The *independent domination number* is the minimum cardinality of an IDS in G and is denoted by $i(G)$. The MINIMUM INDEPENDENT DOMINATING SET problem is to find an IDS of minimum cardinality [66]. The Independent Domination Decision (IDOM) problem is to check if an input graph G has an IDS S, with $|S| \leq k$, where $k \in \mathbb{Z}^+$. The complexity of IDOM problem has been studied in several graph classes including bipartite graphs, split graphs, etc. in [22, 41]. The literature on various domination parameters in several graph classes has been surveyed in [66, 67].

A DS S is called a *total dominating set* (TDS) of G if $G[S]$ is isolated free. The (*total*) *domination number* of G denoted by $\gamma_t(G)$ is $min\{|T| : T$ is a TDS of $G\}$. The problem of finding a TDS of smallest cardinality in a graph is called the MINIMUM TOTAL DOMINATING SET (MTDS) problem. It has been proved that one can give an approximation algorithm for the MINIMUM TOTAL DOMINATING SET problem with ratio of $\ln(\Delta - 0.5) + 1.5$ [32]. Literature on various results in several graph classes has been surveyed in [15, 35, 63].

"Linear programming (LP) is a technique for optimizing a linear objective function, subject to a set of linear equality and linear inequality constraints. Mathematically, it is represented as below

maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

.

.

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

$x_i \geq 0$ for all $i = 1, 2, \ldots, n$."

Each variable in the feasible region of the LP model is restricted to over a continuous interval. It becomes an integer LP (ILP) model if variables are further restricted to integer values. For the Roman domination problem, ILP formulations have been proposed in [17, 50]. For the weak Roman domination problem, ILP formulations have been proposed in [51], and for the double Roman domination problem, ILP formulations have been proposed in [62].

Chapter 3

Algorithmic Complexity of Roman Domination, Roman $\{2\}$-domination and Double Roman Domination

In this chapter, we show that RDP, R2DP and DRDP are NP-complete for star convex bipartite graphs and comb convex bipartite graphs, and R2DP is NP-complete for bisplit graphs, by proposing a polynomial reduction from a well-known NP-complete problem, Exact Three Set Cover $(X3SC)$[54], which is defined as follows.

" **Exact Three Set Cover** $(X3SC)$

INSTANCE : A set $X = \{x_1, x_2, . . ., x_{3q}\}$, where $q \geq 1$ and another set $C = \{c_1, c_2, . . ., c_t\}$, where c_i is a subset of X with $\mid c_i \mid = 3$.

QUESTION : Does C have a subset C' such that $\cup_{c_i \in C'} c_i = X$ and $1 \leq i, j(\neq i) \leq t$, $c_i \cap c_j = \emptyset$?"

Next, we show that MRDP, MR2DP and MDRDP are linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs.

3.1 Algorithmic Complexity of Roman Domination

In this section, we present complexity results for Roman domination.

3.1.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for RDP in subclasses of bipartite is proved.

3.1.1.1 Star Convex Bipartite Graphs

Here, NP-completeness of RDP in star convex bipartite graphs is proved.

Theorem 3.1.1. *"RDP is NP-complete for star convex bipartite graphs."*

Proof. Given a graph G and a function f, whether f is a RDF of size at most k can be checked in polynomial time. Hence RDP is a member of NP. Now we show that RDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{ x_1, x_2, \ldots, x_{3q} \}$ and $C = \{ c_1, c_2, \ldots, c_t \}$, to an instance $\langle G, k\rangle$ of RDP as follows.

Create vertices x_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. The graph constructed is shown in the Figure 3.2. Let $A = \{a\} \cup \{x_i : 1 \leq i \leq 3q\}$, $B = \{c_i : 1 \leq i \leq t\} \cup \{a_1, a_2, a_3\}$. The set A induces a star with vertex a as central vertex, as shown in the Figure 3.1, and the neighbors of each element in B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. Next, we show that $X3SC$ has a solution if and only if G has a RDF with weight at most $2q + 2$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in C' \text{ or } v = a \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

Clearly, f is a RDF and $f(V) = 2q + 2$. Let $k = 2q + 2$.

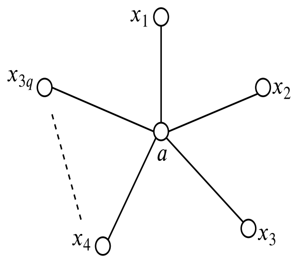

Figure 3.1: Star graph

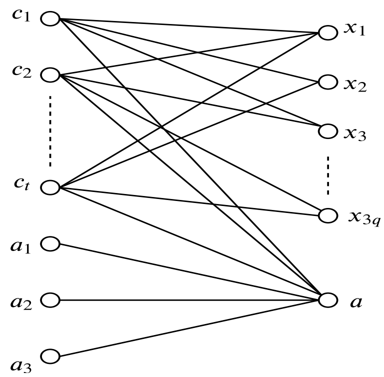

Figure 3.2: Construction of a star convex bipartite graph from an instance of X3SC

Conversely, suppose that G has a Roman dominating function g with weight at most k. Clearly, $g(a) + g(a_1) + g(a_2) + g(a_3) \geq 2$. Without loss of generality, let $g(a) = 2$ and $g(a_1) = g(a_2) = g(a_3) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0. We have the following claim.

Claim 3.1.1. *"If $g(V) \leq k$ then for each $x_i \in X$, $g(x_i) = 0$."*

Proof. (Proof by contradiction) Assume $g(V) \leq k$ and there exist some x_i's such that $g(x_i) \neq 0$. Let $m = |\{x_i : g(x_i) \neq 0\}|$. The number of x_i's with $g(x_i) = 0$ is $3q - m$. Since g is a RDF, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 2 + m + 2\lceil \frac{3q-m}{3} \rceil$, which is greater than k, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. □

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N_G(c_i) \right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C. □

3.1.1.2 Comb Convex Bipartite Graphs

Here NP-completeness of RDP in comb convex bipartite graphs is proved.

Theorem 3.1.2. *"RDP is NP-complete for comb convex bipartite graphs."*

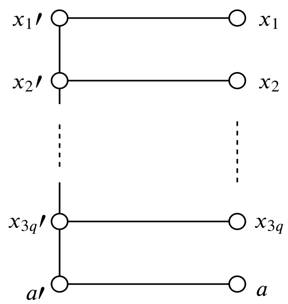

Figure 3.3: Comb graph

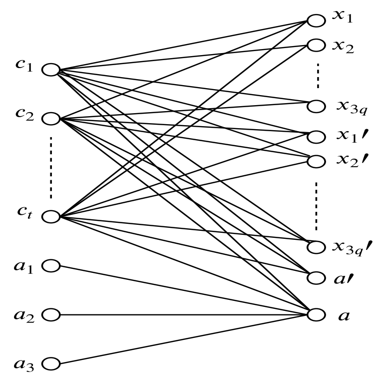

Figure 3.4: Construction of a comb convex bipartite graph from an instance of X3SC

Proof. Clearly, RDP is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3SC$, where $X = \{\ x_1, x_2, \ldots, x_{3q}\ \}$ and $C = \{\ c_1, c_2, \ldots, c_t\ \}$, to an instance $\langle G, k\rangle$ of RDP as follows.

Create vertices x_i, x'_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a', a_1, a_2 and a_3. Add edges (a_i, a) for each a_i and (c_j, x_i) if $x_i \in c_j$. Next add edges (c_j, a) and (c_j, a') for each c_j. Also add edges by joining each c_j to every x'_i. The graph constructed is shown in the Figure 4.2. Let $A = \{a, a'\} \cup \{x_i, x'_i : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. The set A induces a comb with elements $\{x'_i : 1 \leq i \leq 3q\ \} \cup \{a'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\ \} \cup \{a\}$ as teeth, as shown in Figure 4.1, and the neighbors of each element in B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. Next we show that $X3SC$ has a solution if and only if G has a RDF with weight at most $2q + 2$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a RDF f, on G, same as in equation 3.1. Clearly, $f(V) = 2q + 2 = k$.

Conversely, suppose that G has a RDF g with weight k. This proof is obtained with similar arguments as in the converse proof of Theorem 3.1.1 and by using the assignment $g(v) = 0$ if $v \in \{x'_i : 1 \leq i \leq 3q\} \cup \{a'\}$. □

From Theorems 3.1.1 and 3.1.2, the result below follows.

Theorem 3.1.3. *"RDP is NP-complete for tree convex bipartite graphs."*

3.1.1.3 Chain Graphs

Here, MRDP is proved to be linear time solvable for chain graphs. The following proposition is stated in [23].

Proposition 3.1.1. *"Let $G = K_{m_1,\ldots,m_n}$ be the complete n-partite graph with $m_1 \leq m_2 \leq \ldots \leq m_n$.*
(a) If $m_1 \geq 3$ then $\gamma_R(G) = 4$.
(b) If $m_1 = 2$ then $\gamma_R(G) = 3$.
(c) If $m_1 = 1$ then $\gamma_R(G) = 2$."

If $G(X, Y, E)$ is a complete bipartite graph then $\gamma_R(G)$ is obtained directly from Proposition 3.1.1. Otherwise, the following theorem holds.

Theorem 3.1.4. *"Let $G(X, Y, E)(\ncong K_{r,s})$ be a chain graph. Then,*

$$\gamma_R(G) = \begin{cases} 3, & \textit{if } |X| = 2 \textit{ or } |Y| = 2 \\ 4, & \textit{otherwise"} \end{cases} \tag{3.2}$$

Proof. If $G \cong K_1$ then $\gamma_R(G) = 1$. Otherwise, let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

Case (1) : $|X| \geq 2$ and $|Y| = 2$ then $f(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$

Case (2) : $|X| = 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v = x_2 \\ 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is a RDF and $\gamma_R(G) \leq 3$. From the definition of RDF, it follows that $\gamma_R(G) \geq 3$. Therefore $\gamma_R(G) = 3$.

Case (3) : $|X| \neq 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is a RDF and $\gamma_R(G) \leq 4$. Since $p \geq 2$ and $q \geq 2$, in any RDF of G, $f(X) \geq 2$ and $f(Y) \geq 2$. Therefore $\gamma_R(G) \geq 4$. Hence $\gamma_R(G) = 4$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $\gamma_R(G) = \sum_{i=1}^{k} \gamma_R(G_i)$. Now, the following result is immediate from Theorem 3.1.4.

Theorem 3.1.5. *"MRDP can be solvable in linear time for chain graphs."*

Proof. Since the chain ordering and the connected components can be computed in linear time [14, 64], the result follows. □

3.1.2 Complexity in Threshold Graphs

Here, MRDP is proved to be linear time solvable for threshold graphs.

Theorem 3.1.6. *"Let G be a threshold graph. Then $\gamma_R(G) = k + 1$, where k is the number of connected components in G."*

Proof. Let G be a threshold graph with n clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Now, define a function $f : V \rightarrow \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } deg(v) = 0 \\ 2, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

Clearly, f is a RDF and $\gamma_R(G) \leq k + 1$. From the definition of RDF, it follows that $\gamma_R(G) \geq k + 1$. Therefore $\gamma_R(G) = k + 1$. □

From Theorem 3.1.6, the result below follows.

Theorem 3.1.7. *"MRDP can be solvable in linear time for threshold graphs."*

Proof. Since the ordering of the vertices of the clique and the number of connected components in a threshold graph can be determined in linear time [14, 56], the result follows. □

3.1.3 Complexity in Bounded Tree-width Graphs

A graph problem for bounded tree-width graphs, is linear time solvable if there exists a counting monadic second-order logic (CMSOL) formula for it [10]. We show that RDP can be expressed in CMSOL.

Theorem 3.1.8 (*Courcelle's Theorem*)**.** *([10]) "Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P."*

Theorem 3.1.9. *"Given a graph G and a positive integer k, RDP can be expressed in CMSOL."*

Proof. Let $f : V \to \{0, 1, 2\}$ be a function on a graph G, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the RDP is expressed as follows.

$Rom\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p(p \in V_1 \vee p \in V_2 \vee (p \in V_0 \wedge \exists q \in V_2 \wedge adj(p, q))),$

where $adj(p, q)$ is the binary adjacency relation which holds iff, p, q are two adjacent vertices of G. □

From Theorems 3.1.8 and 3.1.9, the following result is immediate.

Theorem 3.1.10. *"MRDP can be solvable in linear time for bounded tree-width graphs."*

3.2 Algorithmic Complexity of Roman $\{2\}$-Domination and the Double Roman Domination in Graphs

In this section, we present complexity results for Roman $\{2\}$-domination and the double Roman domination in graphs.

3.2.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for R2DP and DRDP in subclasses of bipartite is proved.

3.2.1.1 Star Convex Bipartite Graphs

Here NP-completeness of R2DP and DRDP in star convex bipartite graphs is proved.

Theorem 3.2.1. *"R2DP is NP-complete for star convex bipartite graphs."*

Proof. Given a graph G and a function f, whether f is a R2DF of size at most k can be checked in polynomial time. Hence R2DP is a member of NP. Now we show that R2DP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{ x_1, x_2, \ldots, x_{3q} \}$ and $C = \{ c_1, c_2, \ldots, c_t \}$, to an instance $\langle G, k\rangle$ of R2DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. Let $A = \{a\} \cup \{x_i : 1 \leq i \leq 3q\}$ and $B = \{y_i : 1 \leq i \leq 3q\} \cup \{c_i : 1 \leq i \leq t\} \cup \{a_1, a_2, a_3\}$. The subgraph induced by A is a star with vertex a as central vertex and the neighbors of each element of B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time.

Next we show that, $X3SC$ has a solution if and only if G has a R2DF with weight at most $4q+2$. Let $k = 4q+2$. Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } v \in \{y_i : 1 \leq i \leq 3q\} \cup \{c_i : c_i \in C'\} \\ 2, & \text{if } v = a \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

It can be easily verified that f is a R2DF of G and $f(V) = 4q + 2 = k$.

Conversely, suppose that G has a R2DF g with weight k. Let $M = \{a, a_1, a_2, a_3\}$. Clearly, $\sum_{u \in M} g(u) \geq 2$, and so we may assume, without loss of generality, $g(a) = 2$ and $g(a_1) = g(a_2) = g(a_3) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0. Clearly, $g(x_i) = 0$ and $g(y_i) = 0$ case doesn't occur.

> **Claim 3.2.1.** *"If $g(V) = k$ then for each pair of vertices (x_i, y_i), $g(x_i) = 0$ and $g(y_i) = 1$."*

Proof. (Proof by contradiction) Assume $g(V) = k$ and there exist some pairs (x_i, y_i) such that $g(x_i) + g(y_i) > 1$. Let m be the number of pairs of (x_i, y_i) with $g(x_i) + g(y_i) = 2$. The number of pairs of (x_i, y_i) with $g(x_i) = 0$ and $g(y_i) = 1$ is $3q - m$. Since g is R2DF, each x_i with $g(x_i) = 0$, where $g(y_i) = 1$, should have neighbor c_j with $g(c_j) = 1$. Then minimum number of c_j's required with $g(c_j) = 1$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 3q + 2 + m + \lceil \frac{3q-m}{3} \rceil$, which is greater than k. Our assumption leads to a contradiction. Therefore for each pair (x_i, y_i), $g(x_i) = 0$ and $g(y_i) = 1$. Hence the claim.

Since each c_i has exactly three neighbors in X, clearly, there exist at least q number of c_i's with weight at least 1 such that $\left(\bigcup_{g(c_i) \geq 1} N(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. □

> **Theorem 3.2.2.** *"DRDP is NP-complete for star convex bipartite graphs."*

Proof. The proof is obtained with similar arguments as in Theorem 3.2.1, in which replace the assigned values 1 with 2 and 2 with 3. □

3.2.1.2 Comb Convex Bipartite Graphs

Here NP-completeness of R2DP and DRDP in comb convex bipartite graphs is proved.

Theorem 3.2.3. *"R2DP is NP-complete for comb convex bipartite graphs."*

Proof. Clearly, R2DP for comb convex bipartite graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3SC$, where $X = \{ x_1, x_2, \ldots, x_{3q} \}$ and $C = \{ c_1, c_2, \ldots, c_t \}$, to an instance $\langle G, k\rangle$ of R2DP as follows.

Create vertices x_i, x'_i and y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a', a_1, a_2 and a_3. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i and (c_j, x_i) if $x_i \in c_j$. Next add edges (c_j, a) and (c_j, a') for each c_j. Also add edges by joining each c_j to every x'_i. Let $A = \{a, a'\} \cup \{x_i, x'_i : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. The subgraph induced by A is a comb with the elements $\{x'_i : 1 \leq i \leq 3q \} \cup \{a'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q \} \cup \{a\}$ as teeth and the neighbors of each element of B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. Next, we show that, $X3SC$ has a solution if and only if G has a R2DF with weight at most $4q + 2$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We construct a R2DF f, on G, same as in Theorem 3.2.1. Clearly, $f(V) = 4q + 2 = k$.

The proof of the converse is similar to the proof given in Theorem 3.2.1. □

Theorem 3.2.4. *"DRDP is NP-complete for comb convex bipartite graphs."*

Proof. The proof is obtained by replacing the values 1 with 2 and 2 with 3 in Theorem 3.2.3. □

From Theorems 3.2.1 and 3.2.3, the result below follows.

Theorem 3.2.5. *"RD2P and DRDP are NP-complete for tree convex bipartite graphs."*

3.2.1.3 Chain Graphs

In this subsection, we show that R2DOM and DRDOM problems can be solvable in linear time for chain graphs. The following is a proposition without proof.

Proposition 3.2.1. *"Let $G = K_{r,s}$ be a complete bipartite graph with $r \leq s$.*

(a) If $r = 1$ then $\gamma_{\{R2\}}(G) = 2$.

(b) If $r = 2$ then $\gamma_{\{R2\}}(G) = 3$.

(c) If $r \geq 3$ then $\gamma_{\{R2\}}(G) = 4$."

If chain graph G is a complete bipartite graph then $\gamma_{\{R2\}}(G)$ is obtained directly from Proposition 3.2.1. Otherwise, the following theorem holds.

Theorem 3.2.6. *"Let $G\ (\neq K_{r,s})$ be a connected chain graph. Then,*

$$\gamma_{\{R2\}}(G) = \begin{cases} 3, & \textit{if } |X| = 2 \textit{ or } |Y| = 2 \\ 4, & \textit{otherwise"} \end{cases} \tag{3.5}$$

Proof. Let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$ where $p, q \geq 2$. Now, define a function $f : V \rightarrow \{0, 1, 2\}$ as follows.

Case (1) : $|X| = 2$ and $|Y| = 2$ then $f(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$

Case (2) : $|X| = 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v = x_2 \\ 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

Case (3) : $|X| \neq 2$ and $|Y| = 2$ then same condition holds as in case (1).

Clearly, f is a R2DF and $\gamma_{\{R2\}}(G) \leq 3$. From the definition of R2DF, it follows that $\gamma_{\{R2\}}(G) \geq 3$. Therefore $\gamma_{\{R2\}}(G) = 3$.

Case (4) : $|X| \neq 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is a R2DF and $\gamma_{\{R2\}}(G) \leq 4$. By contradiction, it can be easily verified that $\gamma_{\{R2\}}(G) \geq 4$. Therefore $\gamma_{\{R2\}}(G) = 4$. □

If the chain graph G is disconnected then weight of the R2DF is increased by k, where k is the number of isolated vertices in G.

The following propositions are proved in [69] and [7].

Proposition 3.2.2. *[69] "For any complete bipartite graph $K_{p,q}$, with $p, q \geq 3$, $\gamma_{dR}(K_{p,q}) = 6$."*

Proposition 3.2.3. *[7] "For any complete bipartite graph $K_{p,q}$ with $p \leq q$, $\gamma_{dR}(K_{1,q}) = 3$ and $\gamma_{dR}(K_{2,q}) = 4$."*

If G is a complete bipartite graph then $\gamma_{dR}(G)$ is obtained directly from Propositions 3.2.2 and 3.2.3. Otherwise, the following theorem holds.

Theorem 3.2.7. *"Let G $(\neq K_{r,s})$ be a connected chain graph. Then,*

$$\gamma_{dR}(G) = \begin{cases} 5, & \textit{if } |X| = 2 \textit{ or } |Y| = 2 \\ 6, & otherwise\text{"} \end{cases} \tag{3.6}$$

Proof. The proof is obtained by replacing the values 1 with 2 and 2 with 3 in Theorem 3.2.6. □

If the chain graph G is disconnected then weight of the DRDF is increased by $2k$, where k is the number of isolated vertices in G. From Theorems 3.2.6 and 3.2.7, the result below follows.

Theorem 3.2.8. *"MR2DP and MDRDP can be solvable in linear time for chain graphs."*

3.2.2 Complexity in Bisplit Graphs

Here NP-completeness of R2DP in bisplit graphs is proved.

Theorem 3.2.9. *"R2DP is NP-complete for bisplit graphs."*

Proof. It is clear that R2DP for bisplit graphs is in NP. We transform an instance of $X3SC$, where $X = \{ x_1, x_2, \ldots, x_{3q} \}$ and $C = \{ c_1, c_2, \ldots, c_t \}$, to an instance $\langle G, k \rangle$ of R2DP as follows.

Create vertices x_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. Let $P = \{x_i : 1 \leq i \leq 3q\}$, $Q = \{c_i : 1 \leq i \leq t\} \cup \{a_1, a_2, a_3\}$ and $R = \{a\}$. In the constructed graph G, P forms an independent set and $Q \cup R$ is a complete bipartite graph. Hence, making G a bisplit graph can be constructed from the given instance $\langle X, C \rangle$ of X3SC in polynomial time. Next we show that, $X3SC$ has a solution if and only if G has a R2DF with weight at most $2q + 2$. Let $k = 2q + 2$.

Suppose C' is a solution for $X3SC$ with $\mid C' \mid = q$. We define a function $f : V \rightarrow \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a\} \cup \{c_i : c_i \in C'\} \\ 0, & \text{otherwise} \end{cases} \tag{3.7}$$

It can be easily verified that f is a R2DF of G and $f(V) = 2q + 2 = k$.

Conversely, suppose that G has a R2DF g with weight k. Clearly, as in Theorem 3.2.1, $g(a) = 2$ and $\forall a_i$, $g(a_i) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0.

Claim 3.2.2. *"If $g(V) = k$ then for each $x_i \in X$, $g(x_i) = 0$."*

Proof. (Proof by contradiction) Assume $g(V) = k$ and there exist some x_i's such that $g(x_i) \neq 0$. Let $m = |\{x_i : g(x_i) \neq 0\}|$. The number of x_i's with $g(x_i) = 0$ is $3q - m$. Since g is a R2DF, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 2 + m + 2\lceil \frac{3q-m}{3} \rceil$, which is greater than k. Our assumption leads to a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. Hence the claim.

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C. □

3.2.3 Complexity in Threshold Graphs

In this subsection, we determine the Roman $\{2\}$-domination number and double Roman domination number of threshold graphs. For a threshold graph $G(V, E)$, if $|V| = 1$ then, clearly, $\gamma_{\{R2\}}(G) = 1$ and $\gamma_{dR}(G) = 2$. Otherwise, the following theorem holds.

Theorem 3.2.10. *"Let G be a threshold graph. Then $\gamma_{\{R2\}}(G) = k + 1$ and $\gamma_{dR}(G) = 2k + 1$, where k is the number of connected components in G."*

Proof. Let G be a threshold graph with n clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } deg(v) = 0 \\ 2, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

Clearly, f is a R2DF and $\gamma_{\{R2\}}(G) \leq k + 1$. From the definition of R2DF, it follows that $\gamma_{\{R2\}}(G) \geq k + 1$. Therefore $\gamma_{\{R2\}}(G) = k + 1$.

Similarly, let $g : V \to \{0, 1, 2, 3\}$ be a function on G as follows.

$$g(v) = \begin{cases} 2, & \text{if } deg(v) = 0 \\ 3, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

Clearly, g is a DRDF and $\gamma_{dR}(G) \leq 2k+1$. From the definition of DRDF, it follows that $\gamma_{dR}(G) \geq 2k+1$. Therefore $\gamma_{dR}(G) = 2k+1$. □

Now, the following result is immediate from Theorem 3.2.10.

Theorem 3.2.11. *"MR2DP and MDRDP can be solvable in linear time for threshold graphs."*

3.2.4 Complexity in Bounded Tree-width Graphs

Here, we show that for bounded tree-width graphs, MR2DP and MDRDP can be solvable in linear time.

Theorem 3.2.12. *"Given a graph G and a positive integer k, MR2DP can be expressed in CMSOL."*

Proof. Let $f = (V_0, V_1, V_2)$ be a function $f : V \to \{0,1,2\}$ on a graph G, where $V_i = \{v | f(v) = i\}$ for $i \in \{0,1,2\}$. The CMSOL formula for the R2DP is expressed as follows.

$Rom\_\{2\}\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p((p \in V_0 \wedge ((\exists q \in V_2 \wedge adj(p,q)) \vee (\exists r, s \in V_1 \wedge adj(p,r) \wedge adj(p,s)))) \vee (p \in V_1) \vee (p \in V_2))$ □

From Theorems 3.1.8 and 3.2.12, the following result is immediate.

Theorem 3.2.13. *"MR2DP problem can be solvable in linear time for bounded tree-width graphs."*

Theorem 3.2.14. *"Given a graph G and a positive integer k, MDRDP can be expressed in CMSOL."*

Proof. Let $g = (V_0, V_1, V_2, V_3)$ be a function $g : V \to \{0,1,2,3\}$ on a graph G, where $V_i = \{v | g(v) = i\}$ for $i \in \{0,1,2,3\}$. The CMSOL formula for the DRDP is expressed as follows.

$Double\_Rom\_Dom(V) = (g(V) \leq k) \wedge \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r \in V_2 \wedge adj(p,q) \wedge adj(p,r)) \vee (\exists s \in V_3 \wedge adj(p,s))) \vee (p \in V_1 \wedge (\exists t \in V_2 \wedge adj(p,t) \vee (\exists u \in V_3 \wedge adj(p,u))))) \vee (p \in V_2) \vee (p \in V_3))$ □

From Theorems 3.1.8 and 3.2.14, the following result is immediate.

Theorem 3.2.15. *"MDRDP problem can be solvable in linear time for bounded tree-width graphs."*

3.3 Summary

In this chapter, the RDP, R2DP and DRDP complexity has been investigated in various graph classes and the obtained results are tabulated below.

Graph Class	**RDP**	**R2DP**	**DRDP**
Bisplit graphs	NPC	-	-
Star convex bipartite graphs	NPC	NPC	NPC
Comb convex bipartite graphs	NPC	NPC	NPC
Chain graphs	P	P	P
Threshold graphs	P	P	P
Bounded tree-width graphs	P	P	P

Table 3.1: Complexity status of RDP, R2DP and DRDP

Chapter 4

Algorithmic Complexity of Perfect Roman Domination, Perfect Double Roman Domination, Total Roman Domination and Total Double Roman Domination

In this chapter, first we show that PRDP is NP-complete for star convex and comb convex bipartite graphs and PDRDP is NP-complete for chordal and bipartite graphs by proposing a polynomial reduction from NP-complete problem, Exact Three Set Cover $(X3SC)$[54]. Next, we show that MPRDP, MPDRDP, MTRDP and MTDRDP are linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs. Finally, we study the complexity difference of PDRDP with DOMINATION problem.

4.1 Algorithmic Complexity of Perfect Roman Domination

In this section, we present complexity results for perfect Roman domination.

4.1.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for PRDP in subclasses of bipartite is proved.

4.1.1.1 Star Convex Bipartite Graphs

In this section, NP-completeness of PRDP in star convex bipartite graphs is proved.

Theorem 4.1.1. *"PRDP is NP-complete for star convex bipartite graphs."*

Proof. The proof is obtained with similar arguments as in the proof of the Theorem 3.1.1. □

4.1.1.2 Comb Convex Bipartite Graphs

In this section, NP-completeness of PRDP in comb convex bipartite graphs is proved.

Theorem 4.1.2. *"PRDP is NP-complete for comb convex bipartite graphs."*

Proof. Clearly, PRDP is a member of *NP*. We transform an instance of $X3SC$, where $X = \{ x_1, x_2, . . ., x_{3q} \}$ and $C = \{ c_1, c_2, . . ., c_t \}$, to an instance of PRDP as follows.

Create vertices x_i for each $x_i \in X$, c_i, a_i, c'_i for each $c_i \in C$ and also create vertices a, a' and b. Add edges (a_i, c_i) for each c_i and (c_j, x_i) if $x_i \in c_j$. Next add edges (c'_j, b) for each c'_j, (b, a) and (b, a'). Also add edges by joining each c'_j to every x_i. Let $A = \{a, a'\} \cup \{c_i, c'_i : 1 \leq i \leq t\}$ and $B = V \setminus A$. The set A induces a comb with elements $\{c'_i : 1 \leq i \leq t \} \cup \{a'\}$ as backbone and $\{c_i : 1 \leq i \leq t \} \cup \{a\}$ as teeth as shown in the Figure 4.1. From the Figure 4.2, it is clear that the graph constructed is a comb convex bipartite graph since the neighbors of each element in B induce a subtree of the comb, where $|V| = 3t + 3q + 3$ and $|E| = 3qt + 5t + 2$. Next we show that, $X3SC$ has a solution iff G has a PRDF with weight at most $2t + 2$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We construct a PRDF f on G as

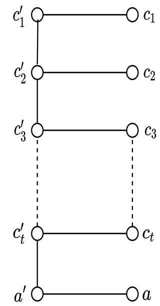

Figure 4.1: Comb graph

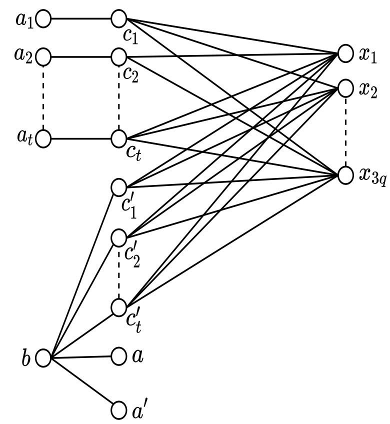

Figure 4.2: Construction of a comb convex bipartite graph from an instance of X3SC

follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{c_i : c_i \in C'\} \cup \{a_i : c_i \notin C'\} \text{ or } v = b \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

Clearly, $f(V) \leq 2t + 2$.

Conversely, suppose that G has a PRDF g with weight at most $2t + k$. Clearly, for each i, $g(a_i) + g(c_i) \geq 2$, these make the size at least $2t$, and $g(b) + g(a) + g(a') \geq 2$. Without loss of generality, $g(b) = 2$, $g(a) = 0$, $g(a') = 0$, $g(x_i) = 0$ where $1 \leq i \leq 3q$ and $g(c'_j) = 0$ where $1 \leq j \leq t$. Since g is a PRDF with weight $2t + k$ or less, the c_i vertices with $g(c_i) = 2$ should be Roman dominating over all the x_j vertices in G. Then $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C; because if some vertex x_i is not covered exactly once in C', the vertex x_i would not be Roman dominated exactly once in G and g would not be a PRDF. □

The result below follows from Theorems 4.1.1 and 4.1.2.

Theorem 4.1.3. *"PRDP is NP-complete for tree convex bipartite graphs."*

4.1.1.3 Chain Graphs

Here, we show that MPRDP can be solvable in linear time for chain graphs. If $G(X, Y, E)$ is a complete bipartite graph then $\gamma_R^P(G)$ is obtained directly from Proposition 6.4.1. Otherwise, the following theorem holds.

Theorem 4.1.4. *"MPRDP can be solvable in linear time for chain graphs."*

Proof. The proof is obtained with similar arguments from the Theorems 3.1.4 and 3.1.5. □

4.1.2 Complexity in Threshold Graphs

Here, we determine the perfect Roman domination number of threshold graph.

Theorem 4.1.5. *"MPRDP can be solvable in linear time for chain graphs."*

Proof. The proof is obtained with similar arguments from the theorems in Section 3.1.2. □

4.1.3 Complexity in Bounded Tree-width Graphs

Here, we prove that MPRDP for bounded tree-width graphs is linear time solvable.

Theorem 4.1.6. *"Given a graph G and a positive integer k, MPRDP can be expressed in CMSOL."*

Proof. Let $f = (V_0, V_1, V_2)$ be a function $f : V \to \{0, 1, 2\}$ on a graph G, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the PRDP is expressed as follows.

$Perfect\_Rom\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p((p \in V_1) \vee (p \in V_2) \vee (p \in V_0 \wedge \exists r(r \in V_2 \wedge adj(p, r)) \wedge \neg(\exists s, s \in V_2 \wedge s \neq r \wedge adj(p, s)))$. □

The result below follows from Theorems 3.1.8 and 4.1.6.

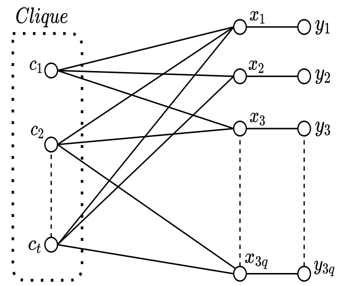

Figure 4.3: Chordal graph construction from an instance of X3SC

Theorem 4.1.7. *"MPRDP problem can be solvable in linear time for bounded tree-width graphs."*

4.2 Algorithmic Complexity of Perfect Double Roman Domination

In this section, we present complexity results for perfect double Roman domination in graphs.

4.2.1 Complexity in Chordal Graphs

Here, NP-completeness of PDRDP in chordal graphs is proved.

Theorem 4.2.1. *"PDRDP is NP-complete for chordal graphs."*

Proof. Given a function f of a graph G and an integer l, if f is a PDRDF of G with $\sum_{v\in V(G)} f(v) \leq l$ can be determined in $O(|V(G)|^c)$ time, where $c > 0$. Hence $PDRDP \in NP$. Now we show that PDRDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC to an instance $\langle G, l\rangle$ of PDRDP as follows.

Create vertices x_i, y_i for each $x_i \in X$, c_i for each $c_i \in C$. Add edges (x_i, y_i) for each x_i and (c_j, x_i) if $x_i \in c_j$. Also add edges (c_i, c_j), $\forall c_i, c_j \in C$, where $i \neq j$. The graph constructed is shown in the Figure 4.3. Next we show that, X3SC has a solution iff G has a PDRDF f with $f(V) \leq 8q$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. A function f on G defined below is clearly a PDRDF of G with $f(V) = 8q$.

$$f(v) = \begin{cases} 2, & \text{if } v \in C' \cup \{y_i : 1 \leq i \leq 3q\} \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

Conversely, let g be a PDRDF of G with $g(V) = 8q$.

Claim 4.2.1. *"If $g(V) = 8q$ then $\forall x_i, g(x_i) = 0$."*

Proof. (Proof by contradiction) Assume $g(V) = l$ and $A = \{x_i : g(x_i) > 0\}$. Clearly, $\forall x_a \in A, g(x_a) + g(y_a) \geq 3$. This makes the weight at least $3|A|$. The number of x_i's with $g(x_i) = 0$ is $3q - |A|$. Clearly, for each $x_i \notin A$, $g(y_i) \geq 2$. Since g is a PDRDF, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is at least $\lceil \frac{3q-|A|}{3} \rceil$. Therefore $g(V) \geq 3|A| + 2(3q - |A|) + 2\lceil \frac{3q-|A|}{3} \rceil$, which is greater than $8q$, a contradiction. Hence the claim. □

Therefore, the set $\{c_i : g(c_i) = 2\}$ is a solution for $X3SC$. □

4.2.2 Complexity in Bipartite Graphs

Here, NP-completeness of PDRDP in bipartite graphs is proved.

Theorem 4.2.2. *"PDRDP is NP-complete for bipartite graphs."*

Proof. The proof is similar to the proof given in Theorem 4.2.1, in which make the set $\{c_1, c_2, \ldots, c_t\}$ as an independent set. □

4.2.3 Complexity in Threshold Graphs

Here, we prove that MPDRDP is linear time solvable for threshold graphs. For a threshold graph G with split partition (C, I), where C is clique and I is independent set, if $|C| = 0$

then weight 2 is assigned for each vertex. then $\gamma^p_{dR}(G) = 2|V|$. Otherwise, the following theorem holds.

Theorem 4.2.3. *"Let G be a threshold graph with m connected components. Then $\gamma^p_{dR}(G) = 2m + 1$."*

Proof. The proof is similar to the proof given for double Roman domination number in Theorem 3.2.10. □

The result below follows from Theorem 4.2.3.

Theorem 4.2.4. *"MPDRDP is linear time solvable for threshold graphs."*

4.2.4 Complexity in Chain Graphs

Here, we determine the perfect double Roman number for chain graphs in linear time. The proposition below is from [9].

Proposition 4.2.1. *"For the paths P_n,*

$$\gamma^p_{dR}(G) = \begin{cases} n, & \text{if } n \equiv 0 \text{ mod } 3 \\ n+1, & \text{otherwise"} \end{cases} \tag{4.3}$$

Theorem 4.2.5. *"For a chain graph $G(X, Y, E)$,*

$$\gamma^p_{dR}(G) = \begin{cases} 3, & \text{if } G \cong K_{1,s}, \text{ where } s \geq 1 \\ 4, & \text{if } G \cong K_{r,s}, \text{ where } r = 2 \text{ or } s = 2 \\ 5, & \text{if } G \cong P_4 \\ 6, & \text{otherwise"} \end{cases} \tag{4.4}$$

Proof. If $G \cong K_1$ then $\gamma^p_{dR}(G) = 2$. Otherwise, let $G(X, Y, E)$ be a chain graph with

$|X| = p$ and $|Y| = q$. Now, define a function $h : V \to \{0, 1, 2, 3\}$ as follows.

Case (*1*) : If $G \cong K_{1,s(\geq 1)}$ i.e., G has a universal vertex then, clearly, $\gamma^p_{dR}(G) = 3$.

Case (*2*) : If $G \cong K_{r,s(r=2 \text{ or } s=2)}$, the following are possible subcases. Let $r = |X|$ and $s = |Y|$.

Case (*2.1*) : If $r = s = 2$ then $h(v) = \begin{cases} 2, & \text{if } v \in \{x_1, x_2\} \\ 0, & \text{otherwise} \end{cases}$

Case (*2.2*) : If $r \neq 2$, $s = 2$ then $h(v) = \begin{cases} 2, & \text{if } v \in \{y_1, y_2\} \\ 0, & \text{otherwise} \end{cases}$

Case (*2.3*) : If $r = 2$, $s \neq 2$ then $h(v) = \begin{cases} 2, & \text{if } v \in \{x_1, x_2\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, h is a PDRDF and $\gamma^p_{dR}(G) \leq 4$. Since G has no universal vertex, from the PDRDF definition, it follows that $\gamma^p_{dR}(G) \geq 4$. Hence $\gamma^p_{dR}(G) = 4$.

Case (*3*) : If $G \cong P_4$ then, from Proposition 4.2.1, $\gamma^p_{dR}(G) = 5$.

Case (*4*) : Otherwise, $h(v) = \begin{cases} 3, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, h is a PDRDF and $\gamma^p_{dR}(G) \leq 6$. By contradiction, it is easy to show that $\gamma^p_{dR}(G) \geq 6$. Therefore $\gamma^p_{dR}(G) = 6$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then $\gamma^p_{dR}(G) = \sum_{i=1}^{k} \gamma^p_{dR}(G_i)$. The result below follows from Theorem 4.2.5.

Theorem 4.2.6. *"MPDRDP for chain graphs is solvable in linear time."*

4.2.5 Complexity in Bounded Tree-width Graphs

Here, we show that MPDRDP for bounded tree-width graphs can be solvable in linear time.

Theorem 4.2.7. *"Let $H(V, E)$ be a graph and k be a positive integer. Then MPDRDP is expressible in CMSOL."*

Proof. Let $g : V(H) \to \{0, 1, 2, 3\}$ defined on H. Also, let $V_i = \{v|g(v) = i\}$ for

$i \in \{0, 1, 2, 3\}$. We make use of the following two conditions in order to express PDRDP in CMSOL.

$Condition\_0 = \exists V_0, V_1, V_2, V_3, \forall p((p \in V_2) \vee (p \in V_3) \vee (p \in V_0 \wedge ((\exists r_1, r_2(r_1, r_2 \in V_2 \wedge r_1 \neq r_2 \wedge adj(p, r_1) \wedge adj(p, r_2)) \wedge \neg(\exists s, s \in V_3 \wedge adj(p, s))) \vee (\exists r(r \in V_3 \wedge adj(p, r)) \wedge \neg(\exists s, s \in V_2 \wedge adj(p, s))))))$.

$Condition\_1 = \exists V_0, V_1, V_2, V_3, \forall p((p \in V_1 \wedge (\exists r(r \in V_2 \wedge adj(p, r)) \wedge \neg(\exists s, s \in V_3 \wedge adj(p, s)))))$.

Here, the binary adjacency relation $adj(p, q)$ holds iff $(p, q) \in E(H)$. Now, the CMSOL formula for the MPDRDP is given below.

$PDRDOM(V) = (g(V) \leq k) \wedge Condition\_0 \wedge Condition\_1$ □

Now, from Theorem 4.2.7 and the Courcelle's result in [10], the theorem below follows.

Theorem 4.2.8. *"MPDRDP for graphs with treewidth at most a constant is solvable in linear time."*

4.2.6 Complexity Difference between Perfect Double Roman Domination and Domination Problems

Perfect double Roman domination and domination problems vary in computational complexity aspects. Specifically, there exist graphs for which the domination problem is NP-complete whereas PDRDP is polynomial time solvable and vice versa. We refer to [12, 13, 52] for similar kind of study. We design a new graph class called GI graph, in which DDP $\in$ NPC, whereas the PDRDP $\in$ P.

Let $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ be a connected graph. Then GI *graph* is constructed from G by adding additional vertices and edges to G in the following way :

1. Create four copies of P_2 graphs such as $b_i - c_i$, $d_i - e_i$, $g_i - h_i$ and $i_i - j_i$, for each i.
2. Consider $2n$ additional vertices $\{a_1, a_2, \ldots, a_n, f_1, f_2, \ldots, f_n\}$.
3. Add edges $\{(v_i, a_i), (a_i, b_i), (a_i, d_i), (v_i, f_i), (f_i, g_i), (f_i, i_i) : 1 \leq i \leq n\}$.

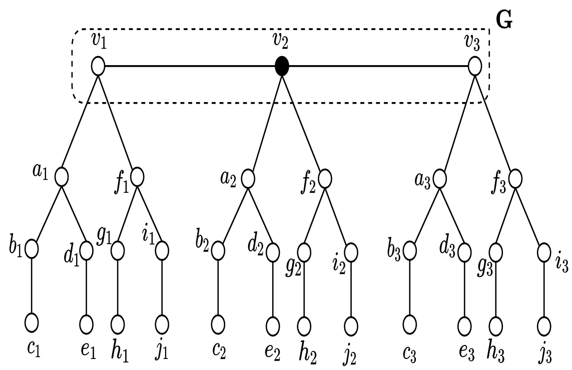

Figure 4.4: Construction of GI graph from G

General GI graph construction is shown in Figure 4.4.

Theorem 4.2.9. *"If G' is a GI graph constructed from G, then $\gamma^p_{dR}(G') = 12n$."*

Proof. Let $G' = (V', E')$ is a GI graph constructed from G. Let $f : V'(G') \to \{0, 1, 2, 3\}$ be a function defined as below

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a_i, c_i, e_i, f_i, h_i, j_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{4.5}$$

Clearly, f is a PDRDF and $\gamma^p_{dR}(G') \leq 12n$.

Next, we show that $\gamma^p_{dR}(G') \geq 12n$. Let g be a PDRDF on graph G'. Then following claim holds.

Claim 4.2.2. *"If $g(V) \geq 12n$ then $\forall v_i \in V$, $g(v_i) = 0$."*

Proof. (Proof by contradiction) Assume $g(V) < 12n$ and there exist m (≥ 1) v_i's such that $g(v_i) \neq 0$. Clearly, $g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) \geq 6$ and $g(f_i) + g(g_i) + g(h_i) + g(i_i) + g(j_i) \geq 6$, where $1 \leq i \leq n$. Therefore $g(V) \geq m + 12n > 12n$, a contradiction. Hence the claim. □

Clearly, each $\langle\{v_i, a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i\}\rangle$ requires a weight of at lease 12. Hence $g(V) \geq 12n$. Therefore $\gamma^p_{dR}(G) = 12n$. □

Lemma 4.2.9.1. *"G has a DS of size at most k iff GI has a DS of size at most $k + 4n$."*

Proof. Suppose D be a DS of G such that $|D| \leq k$. Then, clearly, $D' = D \cup \{b_i, d_i, g_i, i_i : 1 \leq i \leq n\}$ is a DS of GI such that $|D'| \leq k + 4n$.

Conversely, suppose D' is a DS of GI such that $|D'| \leq k + 4n$. Then, D' should contain at least one vertex from each pair $\{b_i, c_i\}$, $\{d_i, e_i\}$, $\{g_i, h_i\}$ and $\{i_i, j_i\}$. Let D'' be a set formed from D' by replacing all a_i's (f_i's) with the corresponding v_i's. Clearly, D'' is a DS of G such that $|D''| \leq k$. Hence the lemma. □

From the fact DDP is NP-complete [54] and above lemma, the following theorem is immediate.

Theorem 4.2.10. *"The DDP for GI graphs is NP-complete."*

4.3 Algorithmic Complexity of Total Roman Domination and Total Double Roman Domination in Graphs

In this section, we present complexity results for TRDOM and the TDRDOM in graphs.

4.3.1 Complexity in Threshold Graphs

Here, we solve MTRDP and MTDRDP for connected threshold graphs in linear time.

Theorem 4.3.1. *"Let G be a connected threshold graph. Then,*

$$\gamma_{tR}(G) = \begin{cases} 2, & \textit{if } \boldsymbol{G} \cong K_2 \\ 3, & otherwise \end{cases} \tag{4.6}$$

and

$$\gamma_{tdR}(G) = \begin{cases} 3, & \textit{if } \boldsymbol{G} \cong K_2 \\ 4, & otherwise\text{"} \end{cases} \tag{4.7}$$

Proof. Let G be a connected threshold graph with p clique vertices and q independent vertices as described above. Since, x_p is a universal vertex of G, clearly, this implies that $\gamma_{tR}(G) = 3$ and $\gamma_{tdR}(G) = 4$, except when $G \cong K_2$ where $\gamma_{tR}(G) = 2$ and $\gamma_{tdR}(G) = 3$. □

Now, the following result is immediate from Theorem 4.3.1.

Theorem 4.3.2. *"MTRDP and MTDRDP for threshold graphs are linear time solvable."*

If threshold graph G is disconnected i.e., G contains isolated vertices, then TRDF and TDRDF can not be defined on G.

4.3.2 Complexity in Chain Graphs

Here, we solve MTRDP and MTDRDP for chain graphs in linear time.

Theorem 4.3.3. *"Let $G(X, Y, E)$ be a connected chain graph. Then,*

$$\gamma_{tR}(G) = \begin{cases} 2, & \textit{if G is } K_2 \\ 3, & \textit{if G is } K_{1,s}, \textit{ where } s \geq 2 \\ 4, & \textit{otherwise} \end{cases} \tag{4.8}$$

and

$$\gamma_{tdR}(G) = \begin{cases} 3, & \textit{if G is } K_2 \\ 4, & \textit{if G is } K_{1,s}, \textit{ where } s \geq 2 \\ 6, & \textit{otherwise"} \end{cases} \tag{4.9}$$

Proof. Let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$ where $p, q \geq 1$. If $G \cong K_2$ or $G \cong K_{1,s}$, where $s \geq 2$, then $\gamma_{tR}(G)$ and $\gamma_{tdR}(G)$ can be determined directly from Theorem 4.3.1. Otherwise, define functions $f : V \to \{0, 1, 2\}$ and $g : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

$$g(v) = \begin{cases} 3, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases} \tag{4.11}$$

Clearly, $f(g)$ is a TRDF (TDRDF) and $\gamma_{tR}(G) \leq 4$ ($\gamma_{tdR}(G) \leq 6$). By contradiction, it can be easily verified that $\gamma_{tR}(G) \geq 4$ ($\gamma_{tdR}(G) \geq 6$). Therefore $\gamma_{tR}(G) = 4$ ($\gamma_{tdR}(G) = 6$). □

Now, from Theorem 4.3.3, the theorem below follows.

Theorem 4.3.4. *"MTRDP and MTDRDP for chain graphs are solvable in linear time."*

If chain graph G is disconnected i.e., G contains isolated vertices, then TRDF and TDRDF can not be defined on G.

4.3.3 Complexity in Bounded Tree-width Graphs

Here, we show that MTRDP and MTDRDP can be solvable in linear time for bounded tree-width graphs.

Theorem 4.3.5. *"Given a graph G and a positive integer k, MTRDP can be expressed in CMSOL."*

Proof. Let $f : V \rightarrow \{0, 1, 2\}$ be a function on a graph G, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2\}$. We make use of a property called, $Total\_Rom(V)$, to express TRDP in CMSOL. The CMSOL formula for the $Total\_Rom(V)$, which says that every vertex $p \in V_1 \cup V_2$ is adjacent to some vertex q in $V_1 \cup V_2$, is expressed as follows.

$Total\_Rom(V) = \exists V_0, V_1, V_2, \forall p, \exists q(p \in (V_1 \cup V_2) \wedge q \in (V_1 \cup V_2) \wedge adj(p, q))$.

Now, The CMSOL formula for the TRDP is expressed, by using $Rom\_Dom(V)$ from Section 3.1.3, as follows.

$Total\_Rom\_Dom(V) = (f(V) \leq k) \wedge Rom\_Dom(V) \wedge Total\_Rom(V)$. □

Now, from Theorem 4.3.5 and Courcelle's result in [10], the theorem below follows.

Theorem 4.3.6. *"MTRDP for graphs with treewidth at most a constant is solvable in linear time."*

Theorem 4.3.7. *"Given a graph G and a positive integer k, MTDRDP can be expressed in CMSOL."*

Proof. Let $g : V \rightarrow \{0, 1, 2, 3\}$ be a function on a graph G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. We make use of a property called, $Total\_Double\_Rom(V)$, to express TDRDP in CMSOL. The CMSOL formula for the $Total\_Double\_Rom(V)$, which says that every vertex $p \in V_1 \cup V_2 \cup V_3$ is adjacent to some vertex q in $V_1 \cup V_2 \cup V_3$, is expressed as follows.

$Total\_Double\_Rom(V) = \exists V_0, V_1, V_2, V_3, \forall p, \exists q(p \in (V_1 \cup V_2 \cup V_3) \wedge q \in (V_1 \cup V_2 \cup V_3) \wedge adj(p, q))$.

Now, the CMSOL formula for the TDRDP is expressed, by using $Double\_Rom\_Dom(V)$ from Section 3.2.4, as follows.

$Total\_Double\_Rom\_Dom(V) = (g(V) \leq k) \wedge Double\_Rom\_Dom(V) \wedge Total\_Double\_Rom(V)$. □

Now, from Theorem 4.3.7 and Courcelle's result in [10], the theorem below follows.

Theorem 4.3.8. *"MTDRDP for graphs with treewidth at most a constant is solvable in linear time."*

4.4 Summary

In this chapter, the PRDP, PDRDP, TRDP and TDRDP complexity has been investigated in various graph classes.

Graph Class	**PRDP**	**PDRDP**	**TRDP**	**TDRDP**
Bipartite graphs	NPC [65]	NPC	NPC [15]	NPC [71]
Chordal graphs	NPC [65]	NPC	NPC [15]	NPC [71]
Star convex bipartite graphs	NPC	-	-	-
Comb convex bipartite graphs	NPC	-	-	-
Chain graphs	P	P	P	P
Threshold graphs	P	P	P	P
Bounded tree-width graphs	P	P	P	P

Table 4.1: Complexity status of PRDP, PDRDP, TRDP and TDRDP

Chapter 5

Algorithmic Complexity of Independent Roman Domination, Independent Roman $\{2\}$-domination and Independent Double Roman Domination

In this chapter, we show that IRDP, IR2DP and IDRDP are NP-complete for, dually chordal graphs, star convex bipartite graphs and comb convex bipartite graphs, and IR2DP and IDRDP are NP-complete for chordal graphs. Next, we show that MIRDP, MIR2DP and MIDRDP are linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs. Finally, we study the complexity difference of IRDP (IR2DP, IDRDP) with DOMINATION problem.

5.1 Algorithmic Complexity of Independent Roman Domination

In this section, we present complexity results for independent Roman domination.

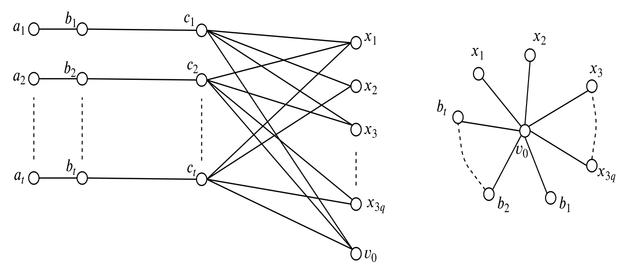

Figure 5.1: An illustration to the construction of star graph from an instance of X3SC

5.1.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for IRDP in subclasses of bipartite is proved.

5.1.1.1 Star Convex Bipartite Graphs

Here, NP-completeness of IRDP in star convex bipartite graphs is proved.

Theorem 5.1.1. *"IRDP is NP-complete for star convex bipartite graphs."*

Proof. Given a graph G and a function f, whether f is an IRDF of size at most k can be checked in polynomial time. Hence IRDP is a member of NP. Now we show that IRDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of IRDP as follows.

Create vertices x_i for each $x_i \in X$, a_i, b_i, c_i for each $c_i \in C$ and v_0. Add edges $(c_i, b_i), (a_i, b_i), (c_i, v_0)$ for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. Let $A = \{v_0\} \cup \{x_i : 1 \leq i \leq 3q\} \cup \{b_i : 1 \leq i \leq t\}$, $B = V \setminus A$. The subgraph induced by A is a star with vertex v_0 as central vertex and the neighbors of each element of B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. The graph constructed and associated star is shown in the Figure 5.1. Next we show that, $X3SC$ has a solution if and only if G has an IRDF with weight at most $2t + q$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \rightarrow$

$\{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in C' \cup \{b_i : c_i \notin C'\} \\ 1, & \text{if } v \in \{a_i : c_i \in C'\} \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

It can be easily verified that f is an IRDF of G and $f(V) = 2t + q$.

Conversely, suppose that G has an IRDF g with weight $2t + q$. Clearly, each path $a_i - b_i - c_i$ requires a weight of at least 2. This makes the weight at least $2t$.

Claim 5.1.1. *"If $g(V) = 2t + q$, then $g(v_0) = 0$."*

Proof. Assume $g(V) = 2t + q$ and $g(v_0) > 0$. Then, $\forall c_i, g(c_i) = 0$. Clearly, each path $a_i - b_i$ requires a weight of at least 2 and $\forall x_i, g(x_i) > 0$. This leads to the total weight at least $2t+g(v_0)+3q$, which is greater than $2t+q$, a contradiction. Therefore $g(v_0) = 0$. □

Claim 5.1.2. *"If $g(V) = 2t + q$, then for each $x_i \in X$, $g(x_i) = 0$."*

Proof. Assume $g(V) = 2t+q$ and there exist m ($\geq$1) x_i's such that $g(x_i) \neq 0$. The number of x_i's with $g(x_i) = 0$ is $3q - m$. Since g is an IRDF, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. Also each $a_i - b_i$ path requires a weight of at least 2. Hence $g(V) \geq 2t+m+2\lceil \frac{3q-m}{3} \rceil > 2t+q$, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. □

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C. □

5.1.1.2 Comb Convex Bipartite Graphs

Here NP-completeness of IRDP in comb convex bipartite graphs is proved.

Theorem 5.1.2. *"IRDP is NP-complete for comb convex bipartite graphs."*

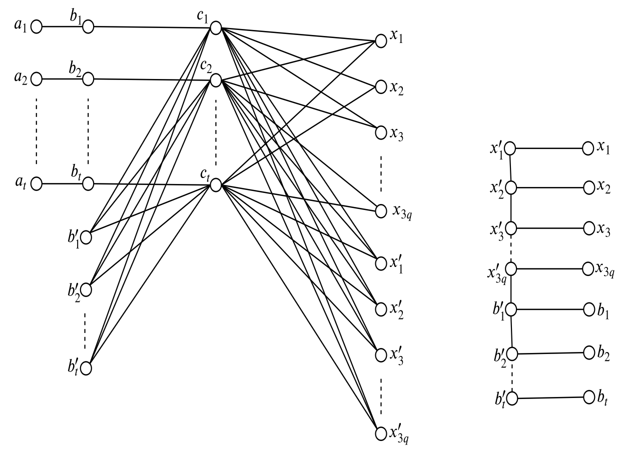

Figure 5.2: An illustration to the construction of comb graph from an instance of X3SC

Proof. Clearly, IRDP for comb convex bipartite graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3SC$, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of IRDP as follows.

Create vertices x_i, x'_i for each $x_i \in X$ and a_i, b_i, c_i, b'_i for each $c_i \in C$. Add edges $(c_i, b_i), (a_i, b_i)$ for each c_i and (c_j, x_i) if $x_i \in c_j$. Next add edges by joining each c_j to every b'_i and x'_i. Let $A = \{x_i, x'_i : 1 \leq i \leq 3q\} \cup \{b_i, b'_i : 1 \leq i \leq t\}$ and $B = V \setminus A$. The subgraph induced by A is a comb with the elements $\{x'_i : 1 \leq i \leq 3q\} \cup \{b'_i : 1 \leq i \leq t\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\} \cup \{b_i : 1 \leq i \leq t\}$ as teeth and the neighbors of each element of B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. The graph constructed and associated comb is shown in the Figure 5.2. Next we show that, $X3SC$ has a solution if and only if G has an IRDF with weight at most $2t + q$.

The forward and converse proofs are similar to the proof given in Theorem 5.1.1. □

From Theorems 5.1.1 and 5.1.2, the result below follows.

Theorem 5.1.3. *"IRDP is NP-complete for tree convex bipartite graphs."*

5.1.1.3 Chain Graphs

Here, we propose a method to compute the independent Roman domination number of chain graphs.

Theorem 5.1.4. *"Let $G(X, Y, E)$ be a connected chain graph. Then,*

$$i_R(G) = \begin{cases} 2, & \text{if } G \cong K_{1,s}, \text{ where } s \geq 1 \\ 1 + |X|, & \text{if } |X| \leq |Y| \\ 1 + |Y|, & \text{otherwise"} \end{cases} \tag{5.2}$$

Proof. If $G \cong K_1$ then $i_R(G) = 1$ and if $G \cong K_{1,s}$, where $s \geq 1$ i.e., G has a universal vertex then, clearly, $i_R(G) = 2$. Otherwise, let $G(X, Y, E)$ be a chain graph with $|X| = p$ and $|Y| = q$, where $p, q \geq 1$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

Case 1 : If $|X| \leq |Y|$, then

$$f(v) = \begin{cases} 2, & \text{if } v = x_p \\ 1, & \text{if } v \in \{x_i : 1 \leq i < p\} \\ 0, & \text{otherwise} \end{cases} \tag{5.3}$$

Clearly, f is an IRDF and $i_R(G) \leq 1 + |X|$. From the definition of independent Roman domination, it follows that $i_R(G) \geq 1 + |X|$. Therefore $i_R(G) = 1 + |X|$.

Case 2 : Otherwise,

$$f(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v \in \{y_i : 2 \leq i \leq q\} \\ 0, & \text{otherwise} \end{cases} \tag{5.4}$$

Clearly, f is an IRDF and $i_R(G) \leq 1 + |Y|$. From the definition of independent Roman domination, it follows that $i_R(G) \geq 1 + |Y|$. Therefore $i_R(G) = 1 + |Y|$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $i_R(G) = \sum_{i=1}^{k} i_R(G_i)$. Now, from Theorem 5.1.4, the theorem below follows.

Theorem 5.1.5. *"MIRDP can be solvable in linear time for chain graphs."*

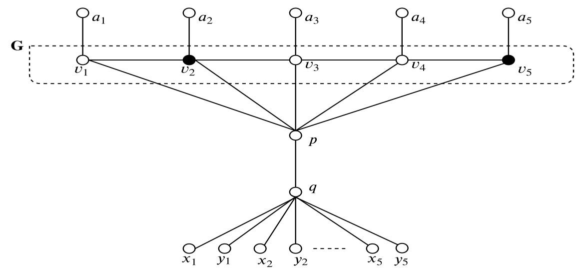

Figure 5.3: An example construction of G' from G

5.1.2 Complexity in Dually Chordal Graphs

In this section, we show that IRDP is NP-complete for dually chordal graphs by giving a polynomial time reduction from the independent domination problem, which has been proved as NP-hard for dually chordal graphs [4]. The decision version of independent domination problem is defined as follows.

"INDEPENDENT DOMINATION (IDOM)

INSTANCE : A simple, undirected graph G and a positive integer p.

QUESTION : Does G have an IDS of size at most p?"

Theorem 5.1.6. *"IRDP is NP-complete for dually chordal graphs."*

Proof. Clearly, IRDP is a member of NP. Given an instance $G = (V, E)$ of IDOM, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V', E')$ of IRDP such that $V' = V \cup \{a_i, x_i, y_i : 1 \leq i \leq n\} \cup \{p, q\}$ and $E' = E \cup \{(v_i, a_i), (v_i, p) : 1 \leq i \leq n\} \cup \{(p, q)\} \cup \{(q, x_i), (q, y_i) : 1 \leq i \leq n\}$. An example construction of G', when G is a path on five vertices, is shown in Figure 5.3. Since G' admits a maximum neighbourhood ordering $(a_1, a_2, \ldots, a_n, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, v_1, v_2, \ldots, v_n, p, q)$, it is a dually chordal graph. Next we show that, G has an IDS of size k iff G' has an IRDF with weight at most $n + k + 2$.

Suppose G has an IDS D of size k. We define a function $f : V' \to \{0, 1, 2\}$ on G' as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in D \cup \{q\} \\ 1, & \text{if } v \in \{a_i : v_i \notin D\} \\ 0, & \text{otherwise} \end{cases} \tag{5.5}$$

Clearly, f is an IRDF of G' and $f(V') \leq n + k + 2$.

Conversely, suppose that G' has an IRDF g with weight $n+k+2$, where $V_i' = \{v | g(v) = i\}$ for $i \in \{0, 1, 2\}$. Clearly, $g(p) + g(q) + g(x_i) + g(y_i) \geq 2$. Then, $D = \{v_i : g(v_i) = 2$ or $g(a_i) = 2\}$ is an IDS of G. The following claim holds.

Claim 5.1.3. *"$|D| \leq k$."*

Proof. Assume $g(V) = n + k + 2$ and $|D| = l$, where $l > k$. Clearly, there exist $n - l$ number of a_i's, where $g(v_i) = 0$, such that each such a_i needs a weight of at least 1. Therefore $g(V) \geq n + l + 2 > n + k + 2$, a contradiction. Hence the claim. □

D is an IDS of G; because if D is not an IDS, a vertex $v_i \in V_2'$ would be adjacent to $v_j \in V_2'$ and g would not be an IRDF. □

5.1.3 Complexity in Threshold Graphs

Here, we show that MIRDP for threshold graph can be solvable in linear time.

Theorem 5.1.7. *"Let $G(V, E)$ be a threshold graph with split partition (C, I). Then $i_R(G) = k + 1$, where k is the number of connected components in G."*

Proof. Let G be a threshold graph with n clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } deg(v) = 0 \\ 2, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

Clearly, f is an IRDF and $i_R(G) \leq k+1$. From the definition of IRDF, it follows that $i_R(G) \geq k+1$. Therefore $i_R(G) = k+1$. □

The following result is immediate from Theorem 5.1.7.

Theorem 5.1.8. *"MIRDP can be solvable in linear time for threshold graphs."*

5.1.4 Complexity in Bounded Tree-width Graphs

Here, we show that MIRDP for bounded tree-width graphs can be solvable in linear time.

Theorem 5.1.9. *"Given a graph G and a positive integer k, IRDP can be expressed in CMSOL."*

Proof. Let $f : V \rightarrow \{0,1,2\}$ be a function on a graph G, where $V_i = \{v | f(v) = i\}$ for $i \in \{0,1,2\}$. The CMSOL formula for the IRDP is expressed, by using $Rom\_Dom(V)$ defined in Theorem 3.1.9, as follows.

$Independent\_Rom\_Dom(V) = (f(V) \leq k) \wedge Rom\_Dom(V) \wedge Independent(V_1 \cup V_2)$. □

The result below follows from Theorems 3.1.8 and 5.1.9.

Theorem 5.1.10. *"MIRDP can be solvable in linear time for bounded tree-width graphs."*

5.1.5 Complexity Difference with Domination

Here, we show that domination problem and IRDP differ in computational complexity aspects by creating a new class of graphs. We build a new class of graphs in which the DDP is NP-complete, whereas the MIRDP can be solved trivially.

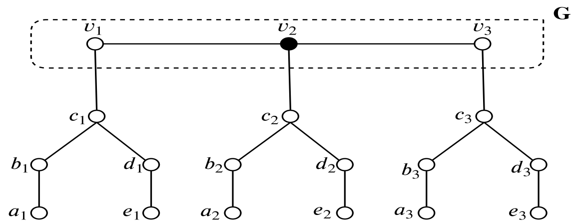

Figure 5.4: An illustration to the construction of GP graph from G

"**Definition 1**. (GP graph). A graph is *GP graph* if it can be constructed from a connected graph $G = (V, E)$ where $|V| = n$ and $V = \{v_1, v_2, \ldots, v_n\}$, in the following way :

1. Create n copies of P_5 graphs with vertices a_i, b_i, c_i, d_i, e_i in the order a_i, b_i, c_i, d_i and e_i.

2. Add edges $\{(c_i, v_i) : 1 \leq i \leq n\}$."

General GP graph construction is shown in Figure 5.4.

Theorem 5.1.11. *"If G' is a GP graph obtained from a graph $G = (V, E)$ $(|V| = n)$, then $i_R(G') = 4n$."*

Proof. Let $G' = (V', E')$ is a GP graph constructed from G. Let $f : V' \to \{0, 1, 2\}$ be a function on graph G', which is defined as follows

$$f(v) = \begin{cases} 2, & \text{if } v \in \{c_i : 1 \leq i \leq n\} \\ 1, & \text{if } v \in \{a_i, e_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{5.7}$$

Clearly, f is an IRDF and $i_R(G') \leq 4n$.

Next, we show that $i_R(G') \geq 4n$. Let g be an IRDF on graph G'. Clearly, $\forall i, 1 \leq i \leq n$, $g(a_i)+g(b_i)+g(c_i)+g(d_i)+g(e_i) \geq 4$. Therefore $i_R(G') \geq 4n$. Hence $i_R(G') = 4n$. □

Lemma 5.1.11.1. *"Let G' be a GP graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + 2n$."*

Proof. Suppose D be DS of G of size at most k, then it is clear that $D \cup \{b_i, d_i : 1 \leq i \leq n\}$ is a DS of G' of size at most $k + 2n$.

Conversely, let D' be a DS of G' of size at most $k + 2n$. Then, from each pair of the vertices $\{a_i, b_i\}$, $\{e_i, d_i\}$, at least one vertex must be included in D'. Let D'' be the set formed by replacing all c_i's in D' by the corresponding v_i's. Clearly, D'' is a DS of G of size at most k. Hence the lemma. □

From the fact DDP is NP-complete [54] and Lemma 5.1.11.1, the following theorem is immediate.

Theorem 5.1.12. *"The DDP is NP-complete for GP graphs."*

5.2 Algorithmic Complexity of Independent Roman {2}-Domination

In this section, we present complexity results for independent Roman $\{2\}$-domination.

5.2.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for IR2DP in subclasses of bipartite is proved.

5.2.1.1 Star Convex Bipartite Graphs

Here, NP-completeness of IR2DP in star convex bipartite graphs is proved.

Theorem 5.2.1. *"IR2DP is NP-complete for star convex bipartite graphs."*

Proof. Given a graph G and a function f, whether f is an IR2DF of size at most k can be checked in polynomial time. Hence IR2DP is a member of NP. Now we show that IR2DP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of IR2DP given as in Theorem 3.1.1. Next we show that, $X3SC$ has a solution if and only if G has an IR2DF with weight at most $2t + q$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in C' \\ 1, & \text{if } v \in \{a_i : 1 \le i \le t\} \cup \{c_i : c_i \notin C'\} \\ 0, & \text{otherwise} \end{cases} \tag{5.8}$$

It can be easily verified that f is an IR2DF of G and $f(V) = 2t + q$.

The proof of the converse is similar to the proof given in Theorem 5.1.1. □

5.2.1.2 Comb Convex Bipartite Graphs

Here NP-completeness of IR2DP in comb convex bipartite graphs is proved.

Theorem 5.2.2. *"IR2DP is NP-complete for comb convex bipartite graphs."*

Proof. Clearly, IR2DP for comb convex bipartite graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3SC$, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of IR2DP given as in Theorem 3.1.2. Next we show that, $X3SC$ has a solution if and only if G has an IR2DF with weight at most $2t + q$.

The forward and converse proofs are similar to the proof given in Theorem 5.2.1. □

From Theorems 5.2.1 and 5.2.2, the result below follows.

Theorem 5.2.3. *"IR2DP is NP-complete for tree convex bipartite graphs."*

5.2.1.3 Chain Graphs

Theorem 5.1.4 also produces independent Roman $\{2\}$-domination number of chain graph. Therefore the following result follows.

Theorem 5.2.4. *"MIR2DP can be solvable in linear time for chain graphs."*

5.2.2 Complexity in Chordal Graphs

Here NP-completeness of IR2DP in chordal graphs is proved.

Theorem 5.2.5. *"IR2DP is NP-complete for chordal graphs."*

Proof. Clearly, IR2DP for chordal graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3SC$, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of IR2DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, p_i, r_i, s_i, c_i for each $c_i \in C$ and $y_{i_{kl}}$, where $1 \leq i, k \leq 3q$ and $1 \leq l \leq 2$. Add edges $(c_i, s_i), (s_i, p_i), (s_i, r_i), (p_i, r_i)$ for each c_i and (c_j, x_i) if $x_i \in c_j$. Next add edges $(x_i, y_i), (y_i, y_{i_{kl}})$, where $1 \leq i, k \leq 3q$ and $1 \leq l \leq 2$. Also add edges $(x_i, x_j), \forall x_i, x_j \in X$, where $i \neq j$. The graph constructed is shown in the Figure 6.6. Since G admits a PEO $\{y_{i_{k2}} : 1 \leq i, k \leq 3q\} \cup \{y_{i_{k1}} : 1 \leq i, k \leq 3q\} \cup \{y_i : 1 \leq i \leq 3q\} \cup \{p_i : 1 \leq i \leq t\} \cup \{r_i : 1 \leq i \leq t\} \cup \{s_i : 1 \leq i \leq t\} \cup \{c_i : 1 \leq i \leq t\} \cup \{x_i : 1 \leq i \leq 3q\}$, it is a chordal graph. Next we show that, $X3SC$ has a solution if and only if G has an IR2DF with weight at most $2t + 9q^2 + 4q$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{s_i : c_i \notin C'\} \cup \{r_i : c_i \in C'\} \\ 1, & \text{if } v \in C' \cup \{y_i, y_{i_{k2}} : 1 \leq i, k \leq 3q\} \\ 0, & \text{otherwise} \end{cases} \tag{5.9}$$

It can be easily verified that f is an IR2DF of G and $f(V) = 2t + 9q^2 + 4q$.

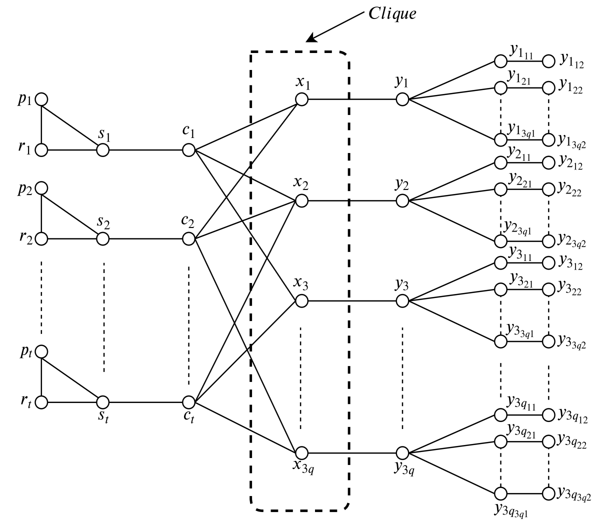

Figure 5.5: An illustration to the construction of chordal graph from an instance of X3SC

Conversely, suppose that G has an IR2DF g with weight $2t + 9q^2 + 4q$. The following claim holds.

Claim 5.2.1. *"If $g(V) = 2t + 9q^2 + 4q$ then for each $x_i \in X$, $g(x_i) = 0$."*

Proof. Assume $g(V) = 2t + 9q^2 + 4q$ and there exists an x_i such that $g(x_i) \neq 0$. Let x_a be a vertex such that $g(x_a) \geq 1$. Then $g(y_a) = 0$, each $y_{a_{k1}} - y_{a_{k2}}$, where $1 \leq k \leq 3q$, path requires a weight of at least 2 and each $\langle\{y_i, y_{i_{k1}}, y_{i_{k2}}\}\rangle$, where $1 \leq i, k \leq 3q$ and $y_i \neq y_a$, requires a weight of at least $3q + 1$. And also each $\langle\{p_i, r_i, s_i\}\rangle$, where $1 \leq i \leq t$, requires a weight of at least 2. Hence $g(V) \geq 2t + 9q^2 + 6q - 1 + g(x_a) > 2t + 9q^2 + 4q$, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. □

Clearly, $g(y_i) + g(y_{i_{k1}}) + g(y_{i_{k2}}) \geq 3q + 1$, where $1 \leq i, k \leq 3q$ and $g(p_i) + g(r_i) + g(s_i) \geq 2$, where $1 \leq i \leq t$. Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 1 such that $\left(\bigcup_{g(c_i)=1} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. □

5.2.3 Complexity in Dually Chordal Graphs

Here NP-completeness of IR2DP in dually chordal graphs is proved.

Theorem 5.2.6. *"IR2DP is NP-complete for dually chordal graphs."*

Proof. The proof is similar to the proof given in Theorem 5.1.6. □

5.2.4 Complexity in Threshold Graphs

Theorem 5.1.7 also determines the independent Roman $\{2\}$-domination number of threshold graph. Hence, the following result is immediate.

Theorem 5.2.7. *"MIR2DP can be solvable in linear time for threshold graphs."*

5.2.5 Complexity in Bounded Tree-width Graphs

Here, we show that for bounded tree-width graphs, MIR2DP can be solvable in linear time.

Theorem 5.2.8. *"Given a graph G and a positive integer k, IR2DP can be expressed in CMSOL."*

Proof. Let $f : V \rightarrow \{0, 1, 2\}$ be a function on a graph G, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the IR2DP is expressed, by using $Rom\_\{2\}\_Dom(V)$ from Theorem 3.2.12, as follows.

$Independent\_Rom\_Dom(V) = (f(V) \leq k) \wedge Rom\_\{2\}\_Dom(V) \wedge Independent(V_1 \cup V_2)$. □

Now, the following result is immediate from Theorems 3.1.8 and 5.2.8.

Theorem 5.2.9. *"MIR2DP can be solvable in linear time for bounded tree-width graphs."*

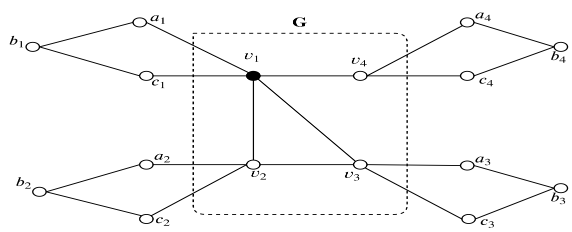

Figure 5.6: An illustration to the construction of GS graph from G

5.2.6 Complexity Difference in Domination and Independent Roman {2}-Domination

Here, we show the complexity difference between domination problem and IR2DP by constructing a new class of graphs.

> "**Definition 2**. (GS graph). A graph is *GS graph* if it can be constructed from a connected graph $G = (V, E)$ where $|V| = n$ and $V = \{v_1, v_2, \ldots, v_n\}$, in the following way :
>
> 1. Create n copies of P_3 graphs with vertices a_i, b_i, c_i with b_i as the middle vertex.
> 2. Add edges $\{(a_i, v_i), (c_i, v_i) : 1 \leq i \leq n\}$."

General GS graph construction is shown in Figure 5.6.

> **Theorem 5.2.10.** *"If G' is a GS graph obtained from a graph $G = (V, E)$ ($|V| = n$), then $i_{\{R2\}}(G') = 2n$."*

Proof. Let $G' = (V', E')$ is a GS graph constructed from G. Let $f : V' \to \{0, 1, 2\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 1, & \text{if } v \in \{a_i, c_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{5.10}$$

Clearly, f is an IR2DF and $i_{\{R2\}}(G') \leq 2n$.

Next, we show that $i_{\{R2\}}(G') \geq 2n$. Let g be an IR2DF of G'. Clearly, $\forall i, 1 \leq i \leq n$,

$g(a_i) + g(b_i) + g(c_i) \geq 2$. Hence $g(V') \geq 2n$. Therefore $i_{\{R2\}}(G') = 2n$. □

Lemma 5.2.10.1. *"Let G' be a GS graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + n$."*

Proof. Suppose D be DS of G, where $|D| \leq k$, then it is clear that $D \cup \{b_i : 1 \leq i \leq n\}$ is a DS of G' of size at most $k + n$.

Conversely, suppose D' is a DS of G' of size at most $k + n$. Then, from each of the vertices a_i, b_i, c_i, at least one vertex must be included in D'. Let D'' be the set formed by replacing all a_i's (c_i's) in D' by the corresponding v_i's. Clearly, D'' is a DS of G such that $|D''| \leq k$. Hence the lemma. □

From the fact DDP is NP-complete [54] and Lemma 5.2.10.1, the following theorem is immediate.

Theorem 5.2.11. *"The DDP is NP-complete for GS graphs."*

5.3 Algorithmic Complexity of Independent Double Roman Domination

Here, we present complexity results for independent double Roman domination.

5.3.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for IRDP in subclasses of bipartite is proved.

5.3.1.1 Star Convex Bipartite Graphs

Here, NP-completeness of IDRDP in star convex bipartite graphs is proved.

Theorem 5.3.1. *"IDRDP is NP-complete for star convex bipartite graphs."*

Proof. The proof is similar to the proof given in Theorem 5.1.1, in which replace the assigned values, for the vertices, 1 with 2 and 2 with 3. □

5.3.1.2 Comb Convex Bipartite Graphs

Here, NP-completeness of IDRDP in comb convex bipartite graphs is proved.

Theorem 5.3.2. *"IDRDP is NP-complete for comb convex bipartite graphs."*

Proof. The proof is similar to the proof given in Theorem 5.1.2, in which replace the assigned values, for the vertices, 1 with 2 and 2 with 3. □

From Theorems 5.3.1 and 5.3.2, the result below follows.

Theorem 5.3.3. *"IDRDP is NP-complete for tree convex bipartite graphs."*

5.3.1.3 Chain Graphs

Here, we propose a method to compute the independent double Roman domination number of chain graphs.

Theorem 5.3.4. *"Let $G(X,Y,E)$ be a chain graph. Then,*

$$i_{dR}(G) = \begin{cases} 3, & \text{if } G \cong K_{1,s},\ \text{where } s \geq 1 \\ 1 + 2|X|, & \text{if } |X| \leq |Y| \\ 1 + 2|Y|, & \text{otherwise"} \end{cases} \tag{5.11}$$

Proof. The proof is obtained with similar arguments as in Theorem 5.1.4, in which replace the assigned values, for the vertices, 1 with 2 and 2 with 3. □

The following result is immediate from Theorem 5.3.4.

Theorem 5.3.5. *"MIDRDP can be solvable in linear time for chain graphs."*

5.3.2 Complexity in Chordal Graphs

Here NP-completeness of IDRDP in chordal graphs is proved.

Theorem 5.3.6. *"IDRDP is NP-complete for chordal graphs."*

Proof. The proof is similar to the proof given in Theorem 5.2.5, in which replace the assigned values, for the vertices, 1 with 2 and 2 with 3. □

5.3.3 Complexity in Dually Chordal Graphs

Here NP-completeness of IDRDP in dually chordal graphs is proved.

Theorem 5.3.7. *"IDRDP is NP-complete for dually chordal graphs."*

Proof. The proof is similar to the proof given in Theorem 5.1.6, in which replace the assigned values, for the vertices, 1 with 2 and 2 with 3. □

5.3.4 Complexity in Threshold Graphs

Here, we solve the MIDRDP of threshold graph.

Theorem 5.3.8. *"Let $G(V, E)$ be a threshold graph with split partition (C, I). Then $i_{dR}(G) = 2k + 1$, where k is the number of connected components in G."*

Proof. Let G be a threshold graph with n clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Let $g : V \to \{0, 1, 2, 3\}$ be a function defined on G as follows.

$$g(v) = \begin{cases} 2, & \text{if } deg(v) = 0 \\ 3, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{5.12}$$

Clearly, g is an IDRDF and $i_{dR}(G) \leq 2k + 1$. From the definition of IDRDF, it follows that

$i_{dR}(G) \geq 2k+1$. Therefore $i_{dR}(G) = 2k+1$. □

The following result is immediate from Theorem 5.3.8.

Theorem 5.3.9. *"MIDRDP can be solvable in linear time for threshold graphs."*

5.3.5 Complexity in Bounded Tree-width Graphs

Here, we show that MIDRDP can be solvable in linear time for bounded tree-width graphs.

Theorem 5.3.10. *"Given a graph G and a positive integer k, IDRDP can be expressed in CMSOL."*

Proof. Let $g : V \rightarrow \{0, 1, 2, 3\}$ be a function on a graph G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. The CMSOL formula for the IDRDP is expressed, by using $Double\_Rom\_Dom(V)$ defined in Theorem 3.2.14, as follows.

$Independent\_Double\_Rom\_Dom(V)$ = $(g(V) \leq k) \wedge Double\_Rom\_Dom(V) \wedge Independent(V_1 \cup V_2 \cup V_3)$. □

Now, the following result is immediate from Theorems 3.1.8 and 5.3.10.

Theorem 5.3.11. *"MIDRDP can be solvable in linear time for bounded tree-width graphs."*

5.3.6 Complexity Difference in Domination and Independent Double Roman Domination

Here, we show the complexity difference between domination problem and IDRDP by constructing a new class of graphs.

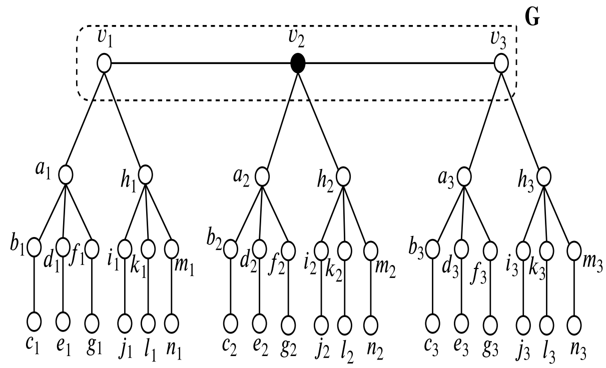

Figure 5.7: An illustration to the construction of GI graph from G

"**Definition 3**. (GI graph). A graph is *GI graph* if it can be constructed from a connected graph $G = (V, E)$ where $|V| = n$ and $V = \{v_1, v_2, \ldots, v_n\}$, in the following way :

1. Create six copies of P_2 graphs such as $b_i - c_i, d_i - e_i, f_i - g_i, i_i - j_i, k_i - l_i$ and $m_i - n_i$, for each i.
2. Consider $2n$ additional vertices $\{a_1, a_2, \ldots, a_n, h_1, h_2, \ldots, h_n\}$.
3. Add edges $\{(v_i, a_i), (a_i, b_i), (a_i, d_i), (a_i, f_i), (v_i, h_i), (h_i, i_i), (h_i, k_i), (h_i, m_i) : 1 \leq i \leq n\}$."

General GI graph construction is shown in Figure 6.5.

Theorem 5.3.12. *"If G' is a GI graph obtained from a graph $G = (V, E)$ ($|V| = n$), then $i_{dR}(G') = 16n$."*

Proof. Let $G' = (V', E')$ is a GI graph constructed from G. Let $f : V' \to \{0, 1, 2\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a_i, h_i, c_i, e_i, g_i, j_i, l_i, n_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{5.13}$$

Clearly, f is an IDRDF and $i_{dR}(G') \leq 16n$.

Next, we show that $i_{dR}(G') \geq 16n$. Let g be an IDRDF on graph G'. Then following claim holds.

Claim 5.3.1. *"If $g(V) = 16n$ then for each $v_i \in V$, $g(v_i) = 0$."*

Proof. Assume $g(V) = 16n$ and there exist some v_i's such that $g(v_i) \neq 0$. Let v_p be a vertex such that $g(v_p) \geq 1$. Then $g(a_p) = g(h_p) = 0$, each of the six p_2's $b_p - c_p, d_p - e_p, f_p - g_p, i_p - j_p, k_p - l_p, m_p - n_p$ requires a weight of at least 3 and each $\langle\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, k_i, l_i, m_i, n_i : 1 \leq i \leq n, i \neq p\}\rangle$, requires a weight of at least 16. Hence $g(V) \geq 16n+2+g(v_p) > 16n$, a contradiction. Therefore for each $v_i \in V$, $g(v_i) = 0$. □

Clearly, $g(a_i)+g(b_i)+g(c_i)+g(d_i)+g(e_i)+g(f_i)+g(g_i) \geq 8$, $g(h_i)+g(i_i)+g(j_i)+g(k_i) + g(l_i) + g(m_i) + g(n_i) \geq 8$, where $1 \leq i \leq n$. Hence $g(V) \geq 16n$. Therefore $g(V) = 16n$. □

Lemma 5.3.12.1. *"Let G' be a GI graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + 6n$."*

Proof. Suppose D be DS of G, where $|D| \leq k$, then it is clear that $D \cup \{b_i, d_i, f_i, i_i, k_i, m_i : 1 \leq i \leq n\}$ is a DS of G' of size at most $k + 6n$.

Conversely, suppose D' is a DS of G' such that $|D'| \leq (k + 6n)$. Then, from each pair of the vertices $\{b_i, c_i\}$, $\{d_i, e_i\}$, $\{f_i, g_i\}$, $\{i_i, j_i\}$, $\{k_i, l_i\}$, $\{m_i, n_i\}$, at least one vertex must be included in D'. Let D'' be the set formed by replacing all a_i's (h_i's) in D' by the corresponding v_i's. Clearly, D'' is a DS of G such that $|D''| \leq k$. Hence the lemma. □

From the fact DDP is NP-complete [54] and Lemma 5.3.12.1, the following theorem is immediate.

Theorem 5.3.13. *"The DDP is NP-complete for GI graphs."*

5.4 Summary

In this chapter, the IRDP, IR2DP and IDRDP complexity has been investigated in various graph classes.

Graph Class	**IRDP**	**IR2DP**	**IDRDP**
Dually chordal graphs	NPC	NPC	NPC
Chordal graphs	-	NPC	NPC
Star convex bipartite graphs	NPC	NPC	NPC
Comb convex bipartite graphs	NPC	NPC	NPC
Chain graphs	P	P	P
Threshold graphs	P	P	P
Bounded tree-width graphs	P	P	P

Table 5.1: Complexity status of IRDP, IR2DP and IDRDP

Chapter 6

Algorithmic Complexity of Weakly Connected Roman Domination, Roman {3}-domination, Total Roman {2}-domination and Total Roman {3}-domination

In this chapter, we show that the R3DP is NP-complete for chordal graphs, planar graphs and for two subclasses of bipartite graphs namely, star convex bipartite graphs and comb convex bipartite graphs and the TR3DP is NP-complete for chordal graphs by giving a polynomial time reduction from Exact-3-SET-Cover $(X3SC)$ [54]. Next, we show that MWCRDP, MR3DP, MTR2DP and MTR3DP are linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs. Next, we study the complexity difference of WCRDP (R3DP, TR3DP, TR3DP) with DOMINATION problem. Finally we propose ILP formulations for M(T)R3DP.

6.1 Algorithmic Complexity of Weakly Connected Roman Domination

In this section, we present complexity results for weakly connected Roman domination.

6.1.1 Complexity in Threshold Graphs

Here, we solve MWCRDP for connected threshold graphs in linear time.

Theorem 6.1.1. *"If G is a connected threshold graph then $\gamma_R^{wc}(G) = 2$."*

Proof. Let G be a connected threshold graph with p (≥ 1) clique vertices and q (≥ 1) independent vertices as described above. Clearly, $\gamma_R^{wc}(G) \geq 2$.

Next, a labeling f on G is defined as: $f(v) = 2$, if $v = x_p$ and 0 otherwise. It is easy to verify that f is a WCRDF and $\gamma_R^{wc}(G) \leq 2$. Therefore, $\gamma_R^{wc}(G) = 2$. □

The result below follows from 6.1.1.

Theorem 6.1.2. *"MWCRDP for threshold graphs is linear time solvable."*

If threshold graph G is disconnected i.e., G contains isolated vertices, then WCRDF cannot be defined on G.

6.1.2 Complexity in Chain Graphs

Here, we solve MWCRDP for chain graphs in linear time.

Theorem 6.1.3. *"For a chain graph $G(X, Y, E)$,*

$$\gamma_R^{wc}(G) = \begin{cases} 2, & \text{if } G \text{ is a star} \\ 3, & \text{if } |X| = 2 \text{ or } |Y| = 2 \\ 4, & \text{otherwise"} \end{cases} \tag{6.1}$$

Proof. Let $G(Y, Z, E)$ be a chain graph with $|X| = p\ (\geq 1)$ and $|Y| = q\ (\geq 1)$. Next, a labeling g on G is defined as below.

Case (1) : If G is a star i.e., G has a universal vertex then $\gamma_R^{wc}(G)$ can be determined same as in Theorem 3.1.6 .

Case (2) : If $|X| = 2$ or $|Y| = 2$. We consider the following subcases.

Case (2.1) : If $|X| = 2$ and $|Y| \geq 2$ then $g(v) = \begin{cases} 2, & \text{if } v = x_2 \\ 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

Case (2.2) : Otherwise, $g(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$

Clearly, g is a WCRDF and $\gamma_R^{wc}(G) \leq 3$. Since G has no universal vertex, from the definition of WCRDF, it follows that $\gamma_R^{wc}(G) \geq 3$. Therefore $\gamma_R^{wc}(G) = 3$.

Case (3) : Otherwise, $g(v) = \begin{cases} 2, & \text{if } v \in \{y_1, x_p\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, g is a WCRDF and $\gamma_R^{wc}(G) \leq 4$. By contradiction, it is easy to show that $\gamma_R^{wc}(G) \geq 4$. Therefore $\gamma_R^{wc}(G) = 4$. □

Now, the result below follows from Theorem 6.1.3.

Theorem 6.1.4. *"MWCRDP for chain graphs is solvable in linear time."*

If chain graph G is disconnected i.e., G contains isolated vertices, then WCRDF cannot be defined on G.

6.1.3 Complexity in Bounded Treewidth Graphs

Here, we show that MWCRDP for bounded tree-width graphs can be solvable in linear time.

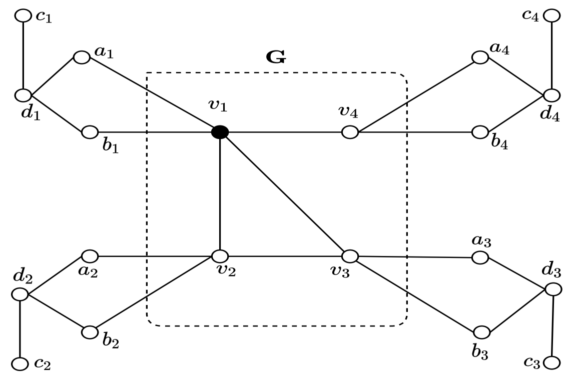

Figure 6.1: Construction of GS graph from G

Theorem 6.1.5. *"Let H be a graph and k a positive integer. Then WCRDP is expressible in CMSOL."*

Proof. Let $g : V(G) \to \{0,1,2\}$ defined on G. Also, let $V_i = \{v|g(v) = i\}$ for $i \in \{0,1,2\}$. We make use of property called, $Weakly\_Connected$ and property $Rom\_Dom(V)$ defined in Theorem 3.1.9 to express WCRDP in CMSOL. Let $V_{12} = V_1 \cup V_2$.

$Weakly\_Connected(V) = (\forall(t,q) \in E(t \in V_{12} \vee q \in V_{12}),$

which guarantees that for each edge $(t,q) \in E(G)$, either $t \in V_{12}$ or $q \in V_{12}$.

Now, the CMSOL formula for the WCRDP is given below.

$$Weakly\_Connected\_Rom\_Dom(V) = (f(V) \leq k) \wedge Rom\_Dom(V) \wedge Weakly\_Connected(V).$$ □

Now, from Theorem 6.1.5 and Courcelle's result in [10], the theorem below follows.

Theorem 6.1.6. *"MWCRDP for graphs with treewidth at most a constant is solvable in linear time."*

6.1.4 Computational Complexity Contrast between WCRDOM and Domination Problems

Here, we show that WCRDOM and Domination problems vary in computational complexity aspects.

We design a new graph class in which DDP ∈ NP-complete, whereas the WCRDP ∈ P.

"**Definition 1**. (GS graph). A graph is a *GS graph* if it is obtained from an n vertex labeled graph with vertices $\{v_1, v_2, \ldots, v_n\}$, in the following way:

1. Create n copies of $K_{1,3}$ star graphs whose four vertices are labeled as a_i, b_i, c_i, d_i with d_i being the central vertex.
2. Add edges $\{(a_i, v_i), (b_i, v_i) : 1 \leq i \leq n\}$."

General GS graph construction is shown in Figure 6.1.

Theorem 6.1.7. *"If H is a GS graph obtained from a graph $G(V, E)$ ($|V| = n$), then $\gamma_R^{wc}(H) = 3n$."*

Proof. Let H be a GS graph constructed from G and f be a function defined on H as $f(v) = 2$, if $v \in \{d_1, d_2, \ldots, d_n\}$; $f(v) = 1$, if $v \in \{v_1, v_2, \ldots, v_n\}$ and 0 otherwise. Clearly, f is a WCRDF and $\gamma_R^{wc}(H) \leq 3n$.

Next, we show that $\gamma_R^{wc}(H) \geq 3n$. Let g be a WCRDF on graph H. Clearly, $g(a_i) + g(b_i) + g(c_i) + g(d_i) \geq 2$, where $1 \leq i \leq n$. Since $(v_i, a_i), (v_i, b_i) \in E(H)$, from the definition of weaky connected Roman domination, it follows that $f(v_i)$ should be at least 1. Therefore $\gamma_R^{wc}(H) \geq 3n$. Hence $\gamma_R^{wc}(H) = 3n$. □

Lemma 6.1.7.1. *"For a GS graph H obtained from G, $\gamma(G) \leq k$ iff $\gamma(H) \leq k + n$."*

Proof. Suppose T be DS of G such that $|T| \leq k$. Then, clearly, $T' = T \cup \{d_i : 1 \leq i \leq n\}$ such that $|T'| \leq k + n$.

Conversely, suppose T' is a DS of H such that $|T'| \leq k + n$. Let T'' be a set which is formed by replacing all c_i's in T' by the corresponding d_i's, and all a_i's or b_i's in T' by the corresponding v_i's. Clearly, T'' is a DS of G such that $|T'| \leq k$. Hence the lemma. □

From the fact DDP is NP-complete [54] and Lemma 6.1.7.1, the following theorem is immediate.

Theorem 6.1.8. *"The DDP for GS graphs is NP-complete."*

6.2 Algorithmic Complexity of Roman {3}-Domination

In this section, we present complexity results for Roman {3}-domination.

6.2.1 Complexity in Subclasses of Bipartite Graphs

In this subsection, complexity results for R3DP in subclasses of bipartite is proved.

6.2.1.1 Star Convex Bipartite Graphs

In this section, NP-completeness of R3DP in star convex bipartite graphs is proved.

Theorem 6.2.1. *"R3DP is NP-complete for star convex bipartite graphs."*

Proof. Given a graph G and a function f, whether f is a R3DF of size at most k can be checked in polynomial time. Hence R3DP is a member of NP. Now we show that R3DP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of R3DP as in Theorem 3.1.1.

We show that, $X3SC$ has a solution if and only if G has a R3DF with weight at most $7q + 3$. Let $k = 7q + 3$. Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v = a \\ 2, & \text{if } v \in \{y_i : 1 \le i \le 3q\} \\ 1, & \text{if } v \in C' \\ 0, & \text{otherwise} \end{cases} \tag{6.2}$$

It can be easily verified that f is a R3DF of G and $f(V) = 7q + 3 = k$.

Conversely, suppose that G has a R3DF g with weight k. Let $M = \{a, a_1, a_2, a_3\}$. Clearly, $\sum_{u \in M} g(u) \geq 3$. The following claim holds.

Claim 6.2.1. *"If $g(V) = k$ then for each pair of vertices $\{x_i, y_i\}$, $g(x_i) = 0$ and $g(y_i) = 2$."*

Proof. (Proof by contradiction) Assume $g(V) = k$ and there exist some pairs $\{x_i, y_i\}$ such that $g(x_i)+g(y_i) > 2$. Let m (≥ 1) be the number of pairs of $\{x_i, y_i\}$ with $g(x_i)+g(y_i) \geq 3$. The number of pairs of $\{x_i, y_i\}$ with $g(x_i) = 0$ and $g(y_i) = 2$ is $3q - m$. Since g is a R3DF of G, each x_i with $g(x_i) = 0$, where $g(y_i) = 2$, should have a neighbor c_j with $g(c_j) = 1$. Then minimum number of c_j's required with $g(c_j) = 1$ is $\lceil \frac{3q-m}{3} \rceil$. Also, $g(a) + g(a_1) + g(a_2) + g(a_3) \geq 3$. Hence $g(V) \geq 3 + 6q + m + \lceil \frac{3q-m}{3} \rceil$, which is greater than k. Our assumption leads to a contradiction. Therefore for each pair $\{x_i, y_i\}$, $g(x_i) = 0$ and $g(y_i) = 2$. Hence the claim.

Since each c_i has exactly three neighbors in X, clearly, there exist at least q number of c_i's with weight exactly 1 such that $\left(\bigcup_{g(c_i)\geq 1} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. □

6.2.1.2 Comb Convex Bipartite Graphs

In this section, NP-completeness of R3DP in comb convex bipartite graphs is proved.

Theorem 6.2.2. *"R3DP is NP-complete for comb convex bipartite graphs."*

Proof. Clearly, R3DP for comb convex bipartite graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3SC$, where $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{c_1, c_2, ..., c_t\}$, to an instance $\langle G, k\rangle$ of R3DP as follows.

Create vertices x_i, x_i' and y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a', a_1, a_2, a_3 and b. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i, (x_i', b) for each x_i', (c_j, x_i) if $x_i \in c_j$ and (b, a'). Next add edges (c_j, a) and (c_j, a') for each c_j. Also add edges by joining each c_j to every x_i'. The graph constructed is shown in the Figure 6.3. Let $A = \{a, a'\} \cup \{x_i, x_i' : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. Assume, the set A induces a comb

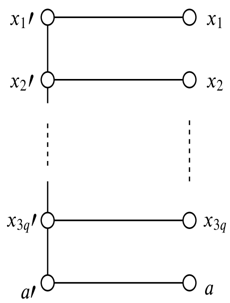

Figure 6.2: Comb graph

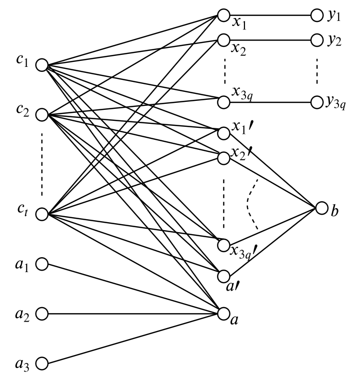

Figure 6.3: Construction of a comb convex bipartite graph from an instance of X3SC

with elements $\{x'_i : 1 \leq i \leq 3q\} \cup \{a'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\} \cup \{a\}$ as teeth, as shown in the Figure 6.2, and the neighbors of each element in B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3SC in polynomial time. Next, we show that, $X3SC$ has a solution if and only if G has a R3DF with weight at most $7q + 5$.

Suppose C' is a solution for $X3SC$ with $|C'| = q$. We define a function $f : V \rightarrow \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v = a \\ 2, & \text{if } v \in \{y_i : 1 \leq i \leq 3q\} \cup \{b\} \\ 1, & \text{if } v \in C' \\ 0, & \text{otherwise} \end{cases} \tag{6.3}$$

It can be easily verified that f is a R3DF of G and $f(V) = 7q + 5 = k$.

Conversely, suppose that G has a R3DF g with weight k. By contradiction, it can be easily shown that $g(b) \geq 2$ and $g(x'_i) = 0$, for $1 \leq i \leq 3q$. The rest of the proof is obtained with similar arguments as in the converse proof of the Theorem 6.2.1. □

From Theorems 6.2.1 and 6.2.2, the following corollary is immediate.

Corollary 6.2.1. *"R3DP is NP-complete for tree convex bipartite graphs."*

6.2.1.3 Chain Graphs

Here, MR3DP is proved to be linear time solvable for chain graphs. The following proposition has been proved in [18].

Proposition 6.2.1 ([18]). *"For any complete bipartite graph we have*

1. $\gamma_{\{R3\}}(K_{1,n}) = \gamma_{dR}(K_{1,n}) = 3$*,*

2. $\gamma_{\{R3\}}(K_{2,n}) = \gamma_{dR}(K_{2,n}) = 4$*,*

3. $\gamma_{\{R3\}}(K_{3,n}) = 5$ *and* $\gamma_{dR}(K_{3,n}) = 6$*, for* $n \geq 3$*,*

4. $\gamma_{\{R3\}}(K_{m,n}) = \gamma_{dR}(K_{m,n}) = 6$*, for* $m, n \geq 4$*."*

If G is a complete bipartite graph then $\gamma_{\{R3\}}(G)$ is obtained directly from Proposition 6.2.1. Otherwise, the following theorem holds.

Theorem 6.2.3. *"Let* $G\ (\neq K_{r,s})$ *be a connected chain graph. Then,*

$$\gamma_{\{R3\}}(G) = \begin{cases} 5, & \textit{if } |X| = 2 \textit{ or } |Y| = 2 \\ 6, & otherwise\text{"} \end{cases} \tag{6.4}$$

Proof. If $G \cong K_1$ then $\gamma_{\{R3\}}(G) = 2$. Otherwise, let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$ where $p, q \geq 2$. Now, define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

Case (1) : $|X| \geq 2$ and $|Y| = 2$ then $f(v) = \begin{cases} 3, & \text{if } v = y_1 \\ 2, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$

Case (2) : $|X| = 2$ and $|Y| > 2$ then $f(v) = \begin{cases} 3, & \text{if } v = x_2 \\ 2, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is a R3DF and $\gamma_{\{R3\}}(G) \leq 5$. From the definition of R3DF, it follows that $\gamma_{\{R3\}}(G) \geq 5$. Therefore $\gamma_{\{R3\}}(G) = 5$.

Case (3) : $|X| > 2$ and $|Y| > 2$ then $f(v) = \begin{cases} 3, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is a R3DF and $\gamma_{\{R3\}}(G) \leq 6$. By contradiction, it can be easily verified that $\gamma_{\{R3\}}(G) \geq 6$. Therefore $\gamma_{\{R3\}}(G) = 6$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $\gamma_{\{R3\}}(G) = \sum_{i=1}^{k} \gamma_{\{R3\}}(G_i)$. from Theorem 6.2.3, the result below follows.

Theorem 6.2.4. *"MR3DP can be solvable in linear time for chain graphs."*

6.2.2 Complexity in Chordal Graphs

Here, complexity results for R3DP in chordal graphs is proved.

Theorem 6.2.5. *"R3DP is NP-complete for chordal graphs."*

Proof. Clearly, R3DP is a member of NP. Now we show that R3DP is NP-hard for chordal graphs by transforming an instance $\langle X, C\rangle$ of X3SC, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of R3DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, c_i, b_i, p_i, q_i and r_i for each $c_i \in C$. Add edges (x_i, y_i) for each $x_i \in X$, (b_i, c_i), (b_i, p_i), (b_i, q_i), (b_i, r_i) for each b_i and (c_j, x_i) if $x_i \in c_j$. Also add edges (c_i, c_j), $\forall c_i, c_j \in C$, where $i \neq j$. The graph constructed is shown in the Figure 6.4. Since G admits a PEO $(y_1, y_2, \ldots, y_{3q}, x_1, x_2, \ldots, x_{3q}, p_1, p_2, \ldots, p_t, q_1, q_2, \ldots, q_t, r_1, r_2, \ldots, r_t, b_1, b_2, \ldots, b_t, c_1, c_2, \ldots, c_t)$, it is a chordal graph.

Next we show that, $X3SC$ has a solution if and only if G has a R3DF with weight at most $7q + 3t$. Let $k = 7q + 3t$. Suppose C' is a solution for $X3SC$ with $|C'| = q$. We

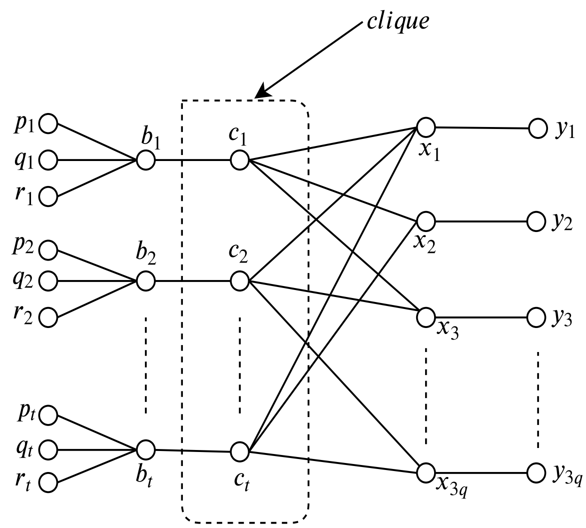

Figure 6.4: An illustration to the construction of chordal graph from an instance of X3SC

define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v \in \{b_i : 1 \leq i \leq t\} \\ 2, & \text{if } v \in \{y_i : 1 \leq i \leq 3q\} \\ 1, & \text{if } v \in C' \\ 0, & \text{otherwise} \end{cases} \tag{6.5}$$

It can be easily verified that f is a R3DF of G and $f(V) = 7q + 3t = k$.

Conversely, suppose that G has a R3DF g with weight k. Clearly, $\forall i, 1 \leq i \leq t, g(p_i) + g(q_i) + g(r_i) + g(b_i) \geq 3$. Hence $g(V) \geq 3t$. The following claim holds.

Claim 6.2.2. *"If $g(V) = k$ then for each pair of vertices $\{x_i, y_i\}$, $g(x_i) = 0$ and $g(y_i) = 2$."*

Proof. The proof is obtained with similar arguments as in the proof of Claim 6.2.1.

Since each c_i has exactly three neighbors in X, clearly, there exist at least q number of c_i's with weight at least 1 such that $\left(\bigcup_{g(c_i)\geq 1} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. $\square$

6.2.3 Complexity in Planar Graphs

Here, we show that R3DP is NP-complete for planar graphs by giving a polynomial time reduction from Planar Exact Cover by 3-Sets (Planar $X3SC$) [5], which is a NP-complete problem and is defined as follows.

"**Planar Exact Cover by 3 Sets (Planar $X3SC$)**

INSTANCE : A finite set $X = \{x_1, x_2, . . ., x_{3q}\}$ and a collection $C = \{c_1, c_2, . . ., c_t\}$ of 3-element subsets of X such that (i) every element of X occurs in at most three subsets and (ii) the induced graph is planar. (This induced graph $H(V, E)$ is defined as the graph such that $V = X \cup C$ and $E = \{(x_i, c_j)$ if $x_i \in c_j\})$.

QUESTION : Is there a subcollection C' of C such that every element of X appears in exactly one member of C' ?"

Theorem 6.2.6. *"R3DP is NP-complete for planar graphs."*

Proof. Clearly, R3DP is a member of *NP*. We transform an instance $\langle X, C\rangle$ of Planar $X3SC$, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of R3DP same as in Theorem 6.2.5.

Clearly, G is a planar graph and can be constructed from the given instance $\langle X, C\rangle$ of Planar $X3SC$ in polynomial time. Next we show that, Planar $X3SC$ has a solution if and only if G has a R3DF with weight at most $7q + 3t$.

Suppose C' is a solution for Planar $X3SC$ with $|C'| = q$. We construct a R3DF f, on G, same as in Equation 6.5. Clearly, $f(V) = 7q + 3t = k$.

The proof of the converse is similar to the proof given in Theorem 6.2.5. □

6.2.4 Complexity in Threshold Graphs

In this section, we determine the Roman $\{3\}$-domination number of threshold graph $G(V, E)$. If $|V| = 1$ then, clearly, $\gamma_{\{R3\}}(G) = 2$. Otherwise, the following theorem holds.

Theorem 6.2.7. *"Let G be a threshold graph. Then,*

$$\gamma_{\{R3\}}(G) = \begin{cases} 2k, & \text{if } |E(G)| = 0 \\ 2k+1, & otherwise, \end{cases} \tag{6.6}$$

where k is the number of connected components in G."

Proof. If a threshold graph G has k connected components but no edges, it implies G has k isolated vertices and the result follows. Otherwise, let G be a threshold graph with p clique vertices such that $N_G[c_1] \subseteq N_G[c_2] \subseteq N_G[c_3] \subseteq \ldots \subseteq N_G[c_p]$. Now, define a function $g : V \to \{0, 1, 2, 3\}$ on G as follows.

$$g(v) = \begin{cases} 2, & \text{if } deg(v) = 0 \\ 3, & \text{if } v = c_p \\ 0, & \text{otherwise} \end{cases} \tag{6.7}$$

Clearly, g is a R3DF and $\gamma_{\{R3\}}(G) \leq 2k + 1$.

Let $G_1, G_2, \ldots, G_k$ be the k components of G. Let G_1 be the component with at least one edge. From the definition of threshold graphs, it follows that each G_i for $2 \leq i \leq k$ is a single vertex graph. Clearly, $\gamma_{\{R3\}}(G_1) \geq 3$ and $\gamma_{\{R3\}}(G_i) = 2$ for $2 \leq i \leq k$. Hence $\gamma_{\{R3\}}(G) \geq 3 + 2(k-1) = 2k + 1$. □

From Theorem 6.2.7, the result below follows.

Theorem 6.2.8. *"MR3DP can be solvable in linear time for threshold graphs."*

6.2.5 Complexity in Bounded Tree-width Graphs

Here, we show that MR3DP for bounded tree-width graphs can be solvable in linear time.

Theorem 6.2.9. *"Given a graph G and a positive integer k, R3DP can be expressed in CMSOL."*

Proof. Let $g : V \to \{0, 1, 2, 3\}$ be a function on a graph $G(V, E)$, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. The CMSOL formula for the R3DP is expressed as follows.

$Rom\_3\_Dom(V) = (g(V) \leq k) \wedge \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r, s \in V_1 \wedge adj(p, q) \wedge adj(p, r) \wedge adj(p, s)) \vee ((\exists t \in V_1 \wedge \exists u \in V_2 \wedge adj(p, t) \wedge adj(p, u)) \vee (\exists q, r \in V_2 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists v \in V_3 \wedge adj(p, v))))) \vee (p \in V_1 \wedge (\exists w, x \in V_1 \wedge adj(p, w) \wedge adj(p, x)) \vee (\exists y \in (V_2 \cup V_3) \wedge adj(p, y))) \vee (p \in V_2) \vee (p \in V_3)),$

$ROM\_3\_Dom(V)$ ensures that for every vertex $p \in V$, either (i) $p \in V_2$ or (ii) $p \in V_3$, or (iii) if $p \in V_0$ then either there exist three vertices $q, r, s \in V_1$ such that p is adjacent to q, r and s, or there exists two vertives $t \in V_1$, $u \in V_2$ such that p is adjacent to both t and u, or there exist two vertices $q, r \in V_2$ such that p is adjacent to both q and r, or there exist a vertex $v \in V_3$ such that p is adjacent to v (iv) if $p \in V_1$ then either there exists two vertices $w, x \in V_1$ such that p is adjacent to both w and x or there exists a vertex $y \in V_2 \cup V_3$ such that p is adjacent to y. □

From Theorems 3.1.8 and 6.2.9, the result below follows.

Theorem 6.2.10. *"MR3DP can be solvable in linear time for bounded tree-width graphs."*

6.2.6 Complexity Contrast between Domination and Roman {3}-domination Problems

Here, we show the complexity difference between domination problem and R3DP by constructing a new class of graphs.

We construct a new class of graphs, called GI graph same as in Section 5.3.6, in which the MR3DP can be solved trivially, whereas the DDP is NP-complete.

Theorem 6.2.11. *"If G' is a GP graph obtained from a graph $G = (V, E)$ $(|V| = n)$, then $\gamma_{\{R3\}}(G') = 16n$."*

Proof. Let $G' = (V', E')$ is a GP graph constructed from G. Let $f : V' \to \{0, 1, 2, 3\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a_i, h_i, c_i, e_i, g_i, j_i, l_i, n_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{6.8}$$

Clearly, f is an R3DF and $\gamma_{\{R3\}}(G') \leq 16n$.

Next, we show that $\gamma_{\{R3\}}(G') \geq 16n$. Let g be a R3DF on graph G'. Then following claim holds.

Claim 6.2.3. *"If $g(V) = 16n$ then for each $v_i \in V$, $g(v_i) = 0$."*

Proof. (Proof by contradiction) Assume $g(V) = 16n$ and there exist m (≥ 1) v_i's such that $g(v_i) \neq 0$. Clearly, each $\langle\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, k_i, l_i, m_i, n_i : 1 \leq i \leq n\}\rangle$, requires a weight of at least 16. Hence $g(V) \geq 16n + m > 16n$, a contradiction. Therefore for each $v_i \in V$, $g(v_i) = 0$. □

Clearly, $g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) + g(f_i) + g(g_i) \geq 8$ and $g(h_i) + g(i_i) + g(j_i) + g(k_i) + g(l_i) + g(m_i) + g(n_i) \geq 8$, where $1 \leq i \leq n$. Hence $g(V) \geq 16n$. Therefore $g(V) = 16n$. □

Lemma 6.2.11.1. *"Let G' be a GP graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + 6n$."*

Proof. Suppose D be DS of G of size at most k, then it is clear that $D \cup \{b_i, d_i, f_i, i_i, k_i, m_i : 1 \leq i \leq n\}$ is a DS of G' of size at most $k + 6n$.

Conversely, suppose D' is a DS of G' such that $|D'| \leq k + 6n$. Then, from each pair of the vertices $\{b_i, c_i\}$, $\{d_i, e_i\}$, $\{f_i, g_i\}$, $\{i_i, j_i\}$, $\{k_i, l_i\}$, $\{m_i, n_i\}$ at least one vertex must be included in D'. Let D'' be the set formed by replacing all a_i's or h_i's in D' by the corresponding v_i's. Clearly, D'' is a DS of G such that $|D''| \leq k$. Hence the lemma. □

From the fact DDP is NP-complete [54] and above Lemma 6.2.11.1, the following theorem is immediate.

Theorem 6.2.12. *"The DOMINATION DECISION problem is NP-complete for GP graphs."*

6.2.7 Integer Linear Programming

In this section, we study the ILP formulations for MR3DP. Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$ and f be a R3DF on G. The MR3DP can now be modeled as an Integer Linear Program (ILP). The variables for this ILP are

$$a_v = \begin{cases} 1, & \text{if } f(v) = 0 \\ 0, & otherwise \end{cases} \qquad b_v = \begin{cases} 1, & \text{if } f(v) = 1 \\ 0, & otherwise \end{cases}$$

$$c_v = \begin{cases} 1, & \text{if } f(v) = 2 \\ 0, & otherwise \end{cases} \qquad d_v = \begin{cases} 1, & \text{if } f(v) = 3 \\ 0, & otherwise \end{cases}$$

The only constant in the ILP is n.

The ILP model of the MR3DP can now be formulated as

Determine : $min(\sum_{v\in V} b_v + 2\sum_{v\in V} c_v + 3\sum_{v\in V} d_v)$ (1)

subject to

$1 - (a_v + b_v) + \sum_{u\in N_G[v]} b_u + 2c_u + 3d_u \geq 3, \forall\ v \in V$ (2)

$a_v + b_v + c_v + d_v = 1, \forall\ v \in V$ (3)

$a_v, b_v, c_v, d_v \in \{0, 1\}, \forall\ v \in V$ (4)

In the above ILP formulation, the objective function (1) minimizes the weight of a R3DF. The constraint in (2), guarantees that the sum of labels of vertices in the closed neighborood of a vertex with label zero or one is at least three. The condition in (3), guarantees that exactly one label is assigned to a vertex. The condition in (4) ensures that the decision variables are binary in nature. The number of variables in the ILP formulated for a graph with n vertices are $4n$ and and the number of constraints are $2n$.

6.3 Algorithmic Complexity of Total Roman {2}-Domination

In this section, we present complexity results for Total Roman {2}-domination.

6.3.1 Complexity in Threshold Graphs

Total Roman {2}-domination number for connected threshold graphs can be determined in linear time, by using the same arguments as in Theorem 4.3.1.

6.3.2 Complexity in Chain Graphs

MTR2DP for chain graphs can be solved in linear time, by using the same arguments as in Theorem 4.3.3

6.3.3 Complexity in Bounded Tree-width Graphs

Here, we show that MTR2DP for bounded tree-width graphs can be solvable in linear time.

Theorem 6.3.1. *"TR2DP can be expressed in CMSOL."*

Proof. Let $G(V, E)$ be a graph and $g : V \to \{0, 1, 2\}$ be a function defined on G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0, 1, 2\}$. Then CMSOL for the TR2DP is specified as below.

$Tot\_Rom\_2\_Dom(V) = (g(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p((p \in V_0 \wedge ((\exists q, r \in V_1 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists t \in V_2 \wedge adj(p, t)))) \vee (p \in (V_1 \cup V_2)) \wedge ((p \in V_1 \wedge \exists q \in (V_1 \cup V_2) \wedge adj(p, q)) \vee (p \in V_2) \wedge \exists q \in (V_1 \cup V_2) \wedge adj(p, q)))$.

$Tot\_Rom\_2\_Dom(V)$ ensures that 1). $\forall p \in V$, either (i) $p \in V_1$ or (ii) $p \in V_2$, or (iii) if $p \in V_0$ then $\exists q, r \in V_1$ such that p is adjacent to q and r, and 2). every vertex $p \in V_1 \cup V_2$ is adjacent to some vertex q in $V_1 \cup V_2$. □

From Theorems 3.1.8 and 6.3.1, the result below follows.

Theorem 6.3.2. *"MTR2DP can be solvable in linear time for bounded tree-width graphs."*

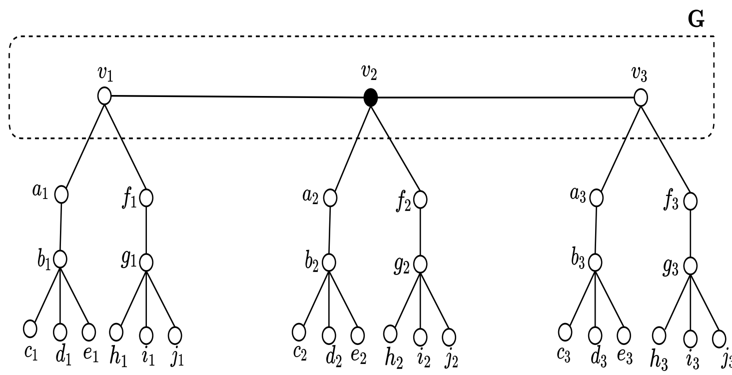

Figure 6.5: An illustration to the construction of GT graph from G

6.3.4 Computational Complexity Contrast between Total Roman {2}-domination and Domination Problems

In this section, we show the complexity difference between domination problem and TR2DP by constructing a new class of graphs.

We build a new graph class in which the DDP is NP-complete, whereas the MTR2DP can be solved trivially.

"**Definition**. (GT graph). Let $G = (V, E)$, where $|V| = n$ and $V = \{v_1, v_2, \ldots, v_n\}$ be a connected graph. A *GT graph* can be constructed from graph G in the following way :

1. Create two copies of P_2 graphs such as $a_i - b_i$ and $f_i - g_i$, for each i.
2. Consider six additional vertices $\{c_i, d_i, e_i, h_i, i_i, j_i\}$, for each i.
3. Add edges $\{(v_i, a_i), (v_i, f_i), (b_i, c_i), (b_i, d_i), (b_i, e_i), (g_i, h_i), (g_i, i_i), (g_i, j_i) : 1 \leq i \leq n\}$."

General GT graph construction is shown in Figure 6.5.

Theorem 6.3.3. "$\gamma_{tR2}(G') = 6n$."

Proof. Let $G' = (V', E')$ is a GT graph constructed from G. Let g be a function defined on G' as follows.

$$g(x) = \begin{cases} 1, & \text{if } x \in \{a_i, f_i : 1 \leq i \leq n\} \\ 2, & \text{if } x \in \{b_i, g_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{6.9}$$

Clearly, g is a TR2DF, $\gamma_{tR2}(G') \leq 6n$.

Next, we show that $\gamma_{tR2}(G') \geq 6n$. Let h be a TR2DF defined on G'. It can be easily verified that, the sum of the weights of the vertices in each set $\{a_i, b_i, c_i, d_i, e_i\}$, $\{f_i, g_i, h_i, i_i, j_i\}$, where $1 \leq i \leq n$ is greater than 2. Hence $h(V) \geq 6n$. Therefore $h(V) = 6n$. □

Lemma 6.3.3.1. *"G has a DS D such that $|D| \leq k$ iff G' has a DS D' such that $|D'| \leq k + 2n$."*

Proof. Suppose D be DS of G with $|D| \leq k$, then, clearly, $D' = D \cup \{b_i, g_i : 1 \leq i \leq n\}$ is a DS of G', where $|D'| \leq k + 2n$.

Let D' is a DS of G' with $|D'| \leq k + 2n$ Clearly, D' should contain at least one vertex from each set $\{b_i, c_i, d_i, e_i\}$ and $\{g_i, h_i, i_i, j_i\}$. Let D'' be the set formed by replacing all a_i's (f_i's) in D' by the corresponding v_i's. Clearly, D'' is a DS of G, where $|D| \leq k$. Hence the lemma. □

The following theorem follows from the fact DDP is NP-complete for general graphs [54] and above lemma.

Theorem 6.3.4. *"The DDP for GT graphs is NP-complete."*

6.4 Algorithmic Complexity of Total Roman {3}-Domination

In this section, we present complexity results for total Roman {3}-domination.

6.4.1 Complexity in Chain Graphs

Here, we solve MTR3DP for connected chain graphs in linear time. Let $G = (X, Y, E)$ be a chain graph, where the vertices of $X = \{x_1, x_2, \ldots, x_p\}$ form a chain such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$ and the vertices of $Y = \{y_1, y_2, \ldots, y_p\}$ form a chain such that $N_G(y_1) \supseteq N_G(y_2) \supseteq \ldots \supseteq N_G(y_q)$.

Theorem 6.4.1. *"Let $G(X, Y, E)(\neq K_{m,n})$ be a connected chain graph. Then $\gamma_{t\{R3\}}(G) = 6$."*

Proof. Let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$ where $p, q \geq 1$. Let g be a function on G defined as below.

$$g(v) = \begin{cases} 3, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases} \tag{6.10}$$

Clearly, g is a TR3DF and $\gamma_{t\{R3\}}(G) \leq 6$. Since $N_G(x_p) = Y$ and $N_G(y_1) = X$, from the definition of TR3DF, clearly, $\gamma_{t\{R3\}}(G) \geq 6$. Therefore $\gamma_{t\{R3\}}(G) = 6$. □

The theorem below follows from 6.4.1.

Theorem 6.4.2. *"MTR3DP for chain graphs is solvable in linear time."*

If chain graph G is disconnected i.e., G contains isolated vertices, then TR3DF can not be defined on G.

6.4.2 Complexity in Chordal Graphs

Here, complexity results for TR3DP in chordal graphs is proved.

Theorem 6.4.3. *"TR3DP is NP-complete for chordal graphs."*

Proof. Given a function $f : V \to \{0, 1, 2, 3\}$ on a chordal graph and an integer $s(s > 0)$, whether the function f is a TR3DF of graph G of weight at most s can be verified in

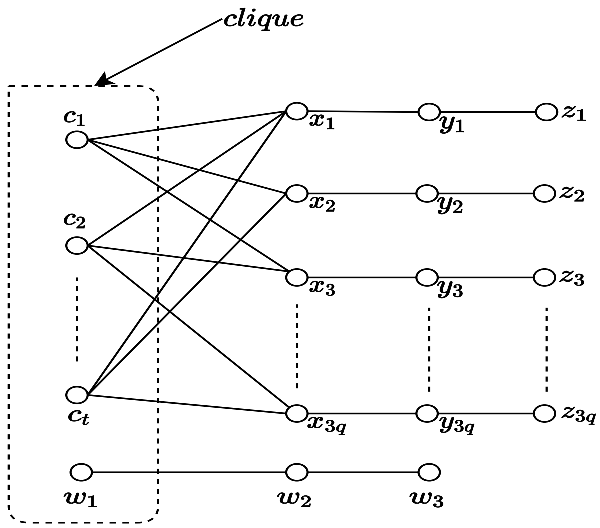

Figure 6.6: Construction of chordal graph from $X3SC$ instance

polynomial time. Hence TR3DP is a member of *NP*. To show that the problem is NP-hard, we transform an instance of $X3SC$ to TR3DP instance. Assume that we have an arbitrary instance of $X3SC$ as $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$. From the given instance of $X3SC$, we construct an instance of chordal graph $G(V, E)$, in polynimial time, as follows.

Create vertices x_i, y_i, z_i for each $x_i \in X$, c_i for each $c_i \in C$. Also create three new vertices w_1, w_2 and w_3. Add edges (c_j, x_i) if $x_i \in c_j$, $(x_i, y_i), (y_i, z_i)$ for all i, $1 \leq i \leq n$, (w_1, w_2) and (w_2, w_3). Also add edges (w_1, c_i) for each c_i and (c_i, c_j), $\forall c_i, c_j \in C$, where $i \neq j$. The graph constructed is shown in the Figure 6.6. Graph G is chordal since it admits a PEO $(z_1, z_2, \ldots, z_{3q}, y_1, y_2, \ldots, y_{3q}, x_1, x_2, \ldots, x_{3q}, w_3, w_2, w_1, c_1, c_2, \ldots, c_t)$. Next we show that, $X3SC$ has a solution iff G has a TR3DF with weight $10q + 4$ or less.

Assume C' is a solution for $X3SC$ and S be the set of vertices in G which corresponds to the elements of C'. A function h on G defined below is clearly a TR3DF with weight $10q + 4$. .

$$h(v) = \begin{cases} 3, & \text{if } v = w_2 \\ 2, & \text{if } v \in \{y_i : 1 \leq i \leq 3q\} \\ 1, & \text{if } v \in S \cup \{z_i : 1 \leq i \leq 3q\} \cup \{w_1\} \\ 0, & \text{otherwise} \end{cases} \tag{6.11}$$

Conversely, assume that G has a TR3DF g such that $g(V) \leq 10q + 4$. Clearly, each path $y_i - z_i$ requires a weight of at least 3 and the path $w_1 - w_2 - w_3$ requires a weight of at least 4. These make the size at least $9q + 4$.

Claim 6.4.1. *"If $g(V) \leq 10q + 4$ then $\forall x_i$, $g(x_i) = 0$."*

Proof. (Proof by contradiction) Assume $g(V) \leq 10q + 4$ and $\exists n$ (≥ 1) such that $|\{x_i : g(x_i) \neq 0\}| = n$. Then $|\{x_i : g(x_i) = 0\}| = 3q - n$. Clearly, number of c_j's required with $g(c_j) \geq 1$ is $\lceil \frac{3q-n}{3} \rceil$. Also $g(w_1)+g(w_2)+g(w_3) \geq 4$ and $\forall i, 1 \leq i \leq 3q, g(y_i)+g(z_i) \geq 3$. Therefore $g(V) \geq 4 + 9q + n + \lceil \frac{3q-n}{3} \rceil > 10q + 4$, a contradiction. Hence the claim. □

Therefore, the set $\{c_i : g(c_i) = 1\}$ is a solution for $X3SC$. □

6.4.3 Complexity in Threshold Graphs

In this section, we determine the total Roman $\{3\}$-domination number of threshold graph $G(V, E)$. The following results are obtained in [72].

Proposition 6.4.1. *([72]) "For any complete bipartite graph, we have*

1. $\gamma_{t\{R3\}}(K_{1,n}) = 4$, *where* $n \geq 2$.
2. $\gamma_{t\{R3\}}(K_{m,n}) = 5$ *for* $m \in \{2, 3\}$ *and* $n \geq 3$.
3. $\gamma_{t\{R3\}}(K_{m,n}) = 6$ *for* $m, n \geq 4$."

Observation 6.4.1. *([72]) "Let $m \geq 2$. Then*

$$\gamma_{t\{R3\}}(P_m) = \begin{cases} m+2, & \text{if } m \equiv 1 (\text{mod } 3), \\ m+1, & \text{otherwise"} \end{cases}$$

Observation 6.4.2. *([72]) "$\gamma_{t\{R3\}}(C_n) = n$."*

For integers $m, n \geq 1$ if $G = K_{m,n}$ then $\gamma_{t\{R3\}}(G)$ is obtained directly from Proposition 6.4.1 or Observation 6.4.1 or Observation 6.4.2. Otherwise, the following theorem holds.

Theorem 6.4.4. *"Let G be a connected threshold graph with clique vertex ordering such that $N_G[q_1] \subseteq N_G[q_2] \subseteq \ldots N_G[q_p]$. Then*

$$\gamma_{t\{R3\}}(G) = \begin{cases} 3, & \text{if } N_G[q_p] = N_G[q_{p-1}] \\ 4, & otherwise\text{''} \end{cases} \tag{6.12}$$

Proof. Let G be a connected threshold graph with i independent vertices and p clique vertices as described above. Now, define $h : V \to \{0, 1, 2, 3\}$ function as below.

Case 1 : If $N_G[q_p] = N_G[q_{p-1}]$ then

$$h(v) = \begin{cases} 2, & \text{if } v = q_p \\ 1, & \text{if } v = q_{p-1} \\ 0, & \text{otherwise} \end{cases} \tag{6.13}$$

Obviously, h is a TR3DF and $\gamma_{t\{R3\}}(G) \leq 3$. From TR3DF definition, $\gamma_{t\{R3\}}(G) \geq 3$. Hence $\gamma_{t\{R3\}}(G) = 3$.

Case 2 : Otherwise,

$$h(v) = \begin{cases} 3, & \text{if } v = q_p \\ 1, & \text{if } v = q_a, \text{ where } q_a \text{ is any one element in } V \setminus \{q_p\} \\ 0, & \text{otherwise} \end{cases} \tag{6.14}$$

Obviously, h is a TR3DF and $\gamma_{t\{R3\}}(G) \leq 4$.

If $N_G[q_p] \neq N_G[q_{p-1}]$ then G contains at least one pendant vertex and by the definition of TR3DF it follows that $\gamma_{t\{R3\}}(G) \geq 4$. Therefore $\gamma_{t\{R3\}}(G) = 4$. □

If a threshold graph G is disconnected then it contains an isolated vertex and TR3DF can not be defined on G. Now, result below is immediate from Theorem 3.1.6.

Theorem 6.4.5. *"MTR3DP for threshold graphs is linear time solvable."*

6.4.4 Complexity in Bounded Tree-width Graphs

Here, we show that MTR3DP for bounded tree-width graphs can be solvable in linear time.

Theorem 6.4.6. *"Let G be a graph and k be any positive integer. Then CMSOL representation exists for TR3DP."*

Proof. Let $h : V \rightarrow \{0, 1, 2, 3\}$ defined on G. Also, let $V_i = \{v|h(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. A CMSOL formula for the TR3DP is specified as below.

$Tot\_Rom\_3\_Dom(V) = (h(V) \leq k) \wedge \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r, s \in V_1 \wedge adj(p, q) \wedge adj(p, r) \wedge adj(p, s)) \vee ((\exists t \in V_1 \wedge \exists u \in V_2 \wedge adj(p, t) \wedge adj(p, u)) \vee (\exists q, r \in V_2 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists v \in V_3 \wedge adj(p, v))))) \vee (p \in V_1 \wedge (\exists w, x \in V_1 \wedge adj(p, w) \wedge adj(p, x)) \vee (\exists y \in (V_2 \cup V_3) \wedge adj(p, y))) \vee (p \in V_2 \wedge \exists q \in (V_1 \cup V_2 \cup V_3) \wedge adj(p, q)) \vee (p \in V_3) \wedge \exists q \in (V_1 \cup V_2 \cup V_3) \wedge adj(p, q))$.

$Tot\_Rom\_3\_Dom(V)$ ensures that 1). for every vertex $p \in V$, either (i) $p \in V_3$ or (ii) $p \in V_2$, or (iii) if $p \in V_0$ then either there exist three vertices $q, r, s \in V_1$ such that p is adjacent to q, r and s, or there exists two vertives $t \in V_1$, $u \in V_2$ such that p is adjacent to both t and u, or there exist two vertices $q, r \in V_2$ such that p is adjacent to both q and r, or there exist a vertex $v \in V_3$ such that p is adjacent to v, or (iv) if $p \in V_1$ then either there exists two vertices $w, x \in V_1$ such that p is adjacent to both w and x or there exists a vertex $y \in V_2 \cup V_3$ such that p is adjacent to y, which also ensures that always vertex with label one is adjacent to a vertex with label in ($V_1 \cup V_2 \cup V_3$), and 2). every vertex $p \in V_2 \cup V_3$ is adjacent to some vertex q in $V_1 \cup V_2 \cup V_3$. □

Now, the result below follows from Theorems 3.1.8 and 6.4.6.

Theorem 6.4.7. *"MTR3DP can be solvable in linear time for bounded tree-width graphs."*

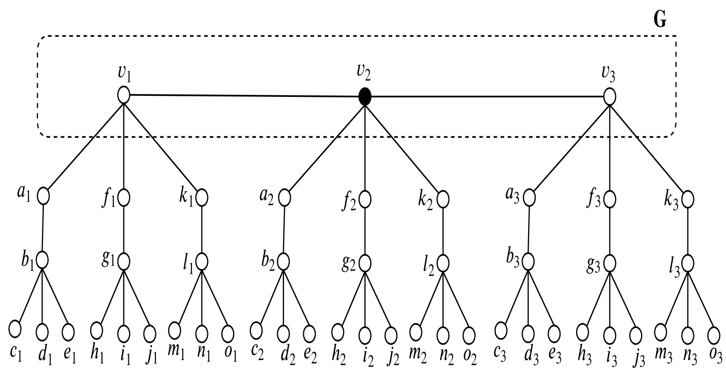

Figure 6.7: Construction of GT graph from G

6.4.5 Contrast between Domination and Total Roman {3}-domination Problems

Here, we show the complexity difference between domination problem and TR3DP by constructing a new class of graphs.

We build a new class of graphs in which the DDP is NP-complete, whereas the MTR3DP can be solved trivially.

"**Definition** . (GT graph). A graph is *GT graph* if it can be constructed from a connected graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, in the following way :

1. Create three copies of P_2 graphs such as $a_i - b_i$, $f_i - g_i$ and $k_i - l_i$, for each i.
2. Consider nine additional vertices $\{h_i, i_i, j_i, m_i, n_i, c_i, d_i, e_i, o_i\}$, for each i.
3. Add edges $\{(v_i, a_i), (v_i, f_i), (v_i, k_i), (b_i, d_i), (b_i, c_i), (b_i, e_i), (g_i, i_i), (g_i, h_i), (g_i, j_i), (l_i, n_i), (l_i, m_i), (l_i, o_i) : 1 \le i \le n\}$."

General GT graph construction is shown in Figure 6.7.

Theorem 6.4.8. *"If G' is a GT graph obtained from a graph $G = (V, E)$ $(|V| = n)$, then $\gamma_{t\{R3\}}(G') = 12n$."*

Proof. Let $G' = (V', E')$ be a GT graph constructed from G. Let $f : V' \to \{0, 1, 2, 3\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 3, & \text{if } v \in \{b_i, g_i, l_i : 1 \leq i \leq n\} \\ 1, & \text{if } v \in \{a_i, f_i, k_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{6.15}$$

Clearly, f is a TR3DF and $\gamma_{t\{R3\}}(G') \leq 12n$.

Next, we show that $\gamma_{t\{R3\}}(G') \geq 12n$. Let g be a TR3DF on graph G'. Clearly, from Proposition 6.4.1, $g(a_i)+g(b_i)+g(c_i)+g(d_i)+g(e_i) \geq 4$, $g(f_i)+g(g_i)+g(h_i)+g(i_i)+g(j_i) \geq 4$, $g(k_i)+g(l_i)+g(m_i)+g(n_i)+g(o_i) \geq 4$, where $1 \leq i \leq n$. Hence $g(V) \geq 12n$. Therefore $g(V) = 12n$. □

Lemma 6.4.8.1. *"Let G' be a GT graph built from a graph $G = (V, E)$. Then G has a DS T such that $|T| \leq k$ iff G' has a DS T' such that $|T'| \leq k + 3n$."*

Proof. Suppose T be DS of G such that $|T| \leq k$, then, clearly, $T' = T \cup \{b_i, g_i, l_i : 1 \leq i \leq n\}$ is a DS of G' such that $|T'| \leq k + 3n$.

Conversely, suppose T' is a DS of G' such that $|T'| \leq k+3n$. Then, from each subgraph $\langle l_i, m_i, n_i, o_i\rangle$, $\langle g_i, h_i, i_i, j_i\rangle$ and $\langle b_i, c_i, d_i, e_i\rangle$ at least one vertex must be included in T'. Let T'' be a set which is formed by replacing all a_i's (f_i's or k_i's) in T' by the corresponding v_i's. Clearly, T'' is a DS of G such that $|T''| \leq k$. Hence the lemma. □

From the fact DDP is NP-complete for general graphs [54] and above lemma, the following theorem is immediate.

Theorem 6.4.9. *"The DDP for GT graphs is NP-complete."*

6.4.6 Integer Linear Programming

In this section, we study the ILP formulations for MTR3DP. Let $G = (V, E)$ be an undirected graph, with $|V| = n$, $|E| = m$ and $f : V \to \{0, 1, 2, 3\}$ be a TR3DF on G. The MTR3DP can now be modeled as Integer Linear Program (ILP).

Here we present an ILP model for MTR3DP. This model uses four sets of binary variables. Specifically, for each vertex $v \in V$, we define

$$a_v = \begin{cases} 1, & f(v) = 0 \\ 0, & \text{otherwise} \end{cases} \qquad b_v = \begin{cases} 1, & f(v) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$c_v = \begin{cases} 1, & f(v) = 2 \\ 0, & \text{otherwise} \end{cases} \qquad d_v = \begin{cases} 1, & f(v) = 3 \\ 0, & \text{otherwise} \end{cases}$$

An ILP model of the MTR3DP can now be formulated as

Determine : $min(\sum_{v \in V}(b_v + 2c_v + 3d_v))$ (1a)

subject to

$$3(1 - a_v) + \sum_{u \in N(v)}(b_u + 2c_u + 3d_v) \geq 3, v \in V \tag{2a}$$

$$2(1 - b_v) + \sum_{u \in N(v)}(b_u + 2c_u + 3d_v) \geq 2, v \in V \tag{3a}$$

$$a_v + \sum_{u \in N(v)}(b_u + c_u + d_u) \geq 1, v \in V \tag{4a}$$

$$a_v + b_v + c_v + d_v = 1, v \in V \tag{5a}$$

$$a_v, b_v, c_v, d_v \in \{0, 1\}, v \in V \tag{6a}$$

The objective function (1a) minimizes the weight of a TR3DF. The condition in (2a), guarantees that for every vertex labeled zero, the sum of labels in its open neighborhood is three or more. The condition in (3a), guarantees that for every vertex labeled one, the sum of labels in its open neighborhood is at least two. The condition in (4a), ensures that every vertex with label greater than zero has at least one neighbor with non-zero label. The condition in (5a), guarantees that exactly one label is assigned to every vertex and the condition in (6a) ensures that the variables are binary in nature.

The number of variables and the constraints in the proposed ILP model is $4n$.

6.5 Summary

In this chapter, the WCRDP, R3DP, TR2DP and TR3DP complexity has been investigated in various graph classes and the results obtained are tabulated.

Graph Class	**WCRDP**	**R3DP**	**TR2DP**	**TR3DP**
Chordal graphs	-	NPC	-	NPC
Star convex bipartite graphs	-	NPC	-	-
Comb convex bipartite graphs	-	NPC	-	-
Chain graphs	P	P	P	P
Threshold graphs	P	P	P	P
Bounded tree-width graphs	P	P	P	P

Table 6.1: Complexity status of WCRDP, R3DP, TR2DP and TR3DP

Chapter 7

Approximation Algorithms and Hardness Results

In this chapter, we study the optimization version of these variants of Roman domination problems in approximation point of view.

7.1 Roman Domination

In approximation point of view, we deal the MRDP and give bounds on the approximation ratio in this section. Further, in graphs with $\Delta = 5$, we prove the APX-completeness of MRDP.

7.1.1 Lower Bound on the Approximation Ratio of MRDP in Star Convex and Comb Convex Bipartite Graphs

In Section 3.1.1, it has been shown that the RDP is NPC for the star convex and the comb convex bipartite graphs. In this section, we prove an approximation hardness result for the MRDP in star convex and comb convex bipartite graphs. To show the hardness result for the MRDP, we provide an approximation preserving reduction from the MIN SET COVER problem which is stated below.

"**Min Set Cover problem :** Let X be any non-empty set and C be a family of subsets of X. For the set system (X, C), a set $C' \subseteq C$ is called a cover of X, if every element of X belongs to at least one element of C'."

The MIN SET COVER problem is to find a minimum cardinality cover of X for a given set system (X, C). The following result is proved in [68].

Theorem 7.1.1 ([68]). *"The MIN SET COVER problem for the input instance (X, C) does not admit a $(1 - \epsilon) \ln |X|$-approximation algorithm for any $\epsilon > 0$ unless $P = NP$. Furthermore, this inapproximability result holds for the case when the size of the input collection C is no more than the size of the set X."*

Now we are ready to prove the following result:

Theorem 7.1.2. *"MRDP for a star convex bipartite graph G with n vertices does not admit a $(1 - \epsilon) \ln n$-approximation algorithm for any $\epsilon > 0$ unless $P = NP$."*

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $C = \{c_1, c_2, \ldots, c_q\}$ be an instance of the MIN SET COVER problem. From this, with similar arguments as in Theorem 3.1.1, we construct an instance $G = (V, E)$ of MRDP for star convex bipartite graphs. Next, we state the following claim.

Claim 7.1.1. *"MIN SET COVER instance (X, C) has a cover of cardinality m if and only if G has a RDF of size $2m + 2$."*

Proof. The proof is obtained with similar arguments as in Theorem 3.1.1. □

If f is a minimum RDF of G and C^* is a minimum set cover of X for the set system (X, C), then $f(V) = 2|C^*| + 2$. Suppose that the MRDP can be approximated within a ratio of α, where $\alpha = (1 - \epsilon) \ln n$ for some fixed $\epsilon > 0$, by using some approximation algorithm, say Algorithm A, that runs in polynomial time. Let k be a fixed positive integer. Then the algorithm SET-COVER-APPROX constructs solution for MIN SET COVER problem.

Algorithm 7.1 SET-COVER-APPROX(X, C)

Require: A set X and a collection C of subsets of X.
Ensure: A cover of X.

```
if there exists a cover C' of X of cardinality ≤ k then
    C_x = C';
else
    Build the graph G;
    Compute a RDF g on G by using algorithm A;
    Construct a cover C' of X from RDF g (as illustrated in the proof of
                                        the Claim 7.1.1);
    C_x = C';
end if
return C_x;
```

Clearly, SET-COVER-APPROX runs in polynomial time. If the cardinality of a minimum cover of X is at most k, then it can be computed in polynomial time. Next, we analyze the case, where the cardinality of a minimum cover of X is greater than k. Let C^* denotes a minimum cover of X and f be a minimum RDF of G. So, $|C^*| > k$. If C_x is a cover of X computed by the algorithm SET-COVER-APPROX, then, $|C_x| < g(V) \leq \alpha(f(V)) \leq \alpha(2 + 2|C^*|) \leq \alpha(2 + \frac{2}{|C^*|})|C^*|$. Therefore, SET-COVER-APPROX approximates a cover of X within a ratio of $\alpha(2 + \frac{2}{|C^*|})$. If $\frac{1}{|C^*|} < \epsilon/2$, then the approximation ratio becomes $\alpha(2 + \frac{2}{|C^*|}) < (1-\epsilon)(2 + 2(\epsilon/2)) \ln n = (1-\epsilon)(2+\epsilon) \ln n = (1-\epsilon') \ln n \approx (1-\epsilon') \ln p$ (since $\ln n \approx \ln p$ for sufficiently large values of p), where $\epsilon' = \epsilon^2 + \epsilon - 1$.

This proves that the algorithm APPROX-SET-COVER approximates set cover of X within ratio $(1-\epsilon') \ln p$ for some fixed $\epsilon' > 0$. By Theorem 7.1.1, if the MIN SET COVER problem can be approximated within a ratio of $(1-\epsilon') \ln p$, then $P = NP$. It follows that, if MRDP can be approximated within a ratio of $(1-\epsilon) \ln n$ for any $\epsilon > 0$, then $P = NP$. Hence, for a star convex bipartite graph $G = (V, E)$, the MRDP cannot be approximated within a ratio of $(1-\epsilon) \ln n$ for any $\epsilon > 0$ unless $P = NP$. □

Theorem 7.1.3. *"MRDP for a comb convex bipartite graph G with n vertices does not admit a $(1-\epsilon) \ln n$-approximation algorithm for any $\epsilon > 0$ unless $P = NP$."*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.2, in which replace the Theorem 3.1.1 by Theorem 3.1.2. □

7.1.2 Upper Bound on Approximation Ratio

Here, we design an approximation algorithm for MRDP based on the well known optimization problem called MINIMUM DOMINATION problem. The following theorem has been proved in [14].

> **Theorem 7.1.4** ([14])**.** *"The MINIMUM DOMINATION problem in a graph with maximum degree Δ can be approximated with an approximation ratio of* $1+\ln(\Delta+1)$*."*

Let APPROX-DOM-SET be an approximation algorithm that gives a dominating set D of a graph G such that $|D| \leq (1+\ln(\Delta+1))\gamma(G)$, where Δ is the maximum degree of a graph G.

Next, we propose an algorithm APPROX-RDF to compute an approximate solution of MRDP. In our algorithm, first we compute a dominating set D of the input graph G using the approximation algorithm APPROX-DOM-SET. Next, we construct a triple T_r in which every vertex in D will be assigned with weight 2 and the remaining vertices will be assigned with weight 0.

Now, let $T_r = (D', \emptyset, D)$ be the triple obtained by using the APPROX-RDF algorithm. It can be easily seen that every vertex $v \in V$ is assigned with weight either 0 or 2. Since D is a dominating set of G, every vertex $v \in D'$ having weight 0 is adjacent to a vertex $u \in D$ having weight 2. Thus, T_r gives a Roman dominating function of G. We note that

Algorithm 7.2 APPROX-RDF(G)

Input: A simple, undirected graph G.
Output: A Roman dominating triple (V_0, V_1, V_2) of vertices of G.

1: $D \leftarrow$ APPROX-DOM-SET(G)
2: $T_r \leftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

the algorithm APPROX-RDF computes a Roman dominating triple T_r of a given graph G in polynomial time. Hence, we have the following result.

Theorem 7.1.5. *"MRDP in a graph with maximum degree Δ can be approximated with an approximation ratio of* $2(1 + \ln(\Delta + 1))$*."*

Proof. Let D be the dominating set produced by the algorithm APPROX-DOM-SET, T_r be the Roman dominating triple produced by the algorithm APPROX-RDF and W_r be the weight of T_r.

It can be observed that $W_r = 2|D|$. It is known that $|D| \leq (1+\ln(\Delta+1))\gamma(G)$. Therefore, $W_r \leq 2(1+\ln(\Delta+1))\gamma(G)$. Since $\gamma(G) \leq \gamma_R(G)$ [23], it follows that $W_r \leq 2(1+\ln(\Delta+1))\gamma_R(G)$. Hence the result. □

From Theorem 7.1.5, the corollary below is immediate.

Corollary 7.1.1. *"MRDP is in the class of APX when the maximum degree Δ is fixed."*

7.1.3 APX-completeness for Bounded Degree Graphs

Here, we show that the MRDP is APX-complete for graphs with maximum degree 5 by giving an L-reduction from MINIMUM DOMINATING SET problem in graphs with maximum degree 3 (DOM-3) which has been proved as APX-complete [61]. Clearly, from Corollary 7.1.1, MRDP $\in$ APX.

Theorem 7.1.6. *"MRDP is APX-complete for graphs with maximum degree 5."*

Proof. It is known that MRDP is in APX. We construct an instance $G' = (V', E')$ of MRDP from a given instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, as follows.

Create n copies of P_3 with b_i as the central vertex and a_i, c_i as terminal vertices. Add the edges $\{(v_i, a_i), (v_i, c_i) : 1 \leq i \leq n\}$. Example construction of G' from G is shown in Figure 7.6. Note that G' is a graph with maximum degree 5. First we prove the following claim.

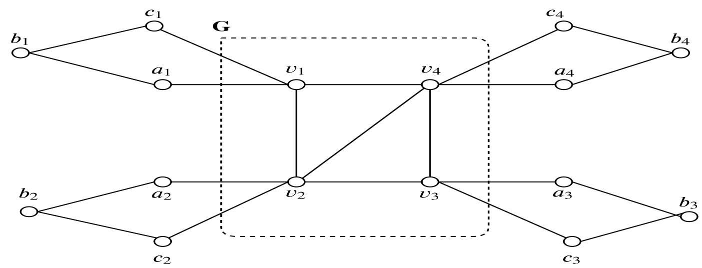

Figure 7.1: An illustration to the construction of G' from G

Claim 7.1.2. *"If D^* is a minimum dominating set of G then $\gamma_R(G') = 2n + |D^*|$, where $n = |V(G)|$."*

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be a graph and $G' = (V', E')$ is a graph constructed from G.

Let D^* be a minimum DS of G and $f : V \to \{0, 1, 2\}$ be a function on graph G', which is defined as below.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{v_i : v_i \in D^*\} \text{ or } v \in \{b_i : v_i \notin D^*\} \\ 1, & \text{if } v \in \{b_i : v_i \in D^*\} \\ 0, & \text{otherwise} \end{cases} \tag{7.1}$$

Clearly, f is a RDF and $\gamma_R(G') \leq 2n + |D^*|$.

Next, we show that $\gamma_R(G') \geq 2n + |D^*|$. Let g be a RDF on graph G'. Clearly if $g(v_i) = 0$ then $g(a_i) + g(b_i) + g(c_i) \geq 2$ and if $g(v_i) \geq 1$ then $g(v_i) + g(a_i) + g(b_i) + g(c_i) \geq 3$. Therefore $\gamma_R(G') \geq 2n + |D^*|$, where $D^* = \{v_i : g(v_i) \geq 1\}$ is a minimum dominating set of G. Hence $\gamma_R(G') = 2n + |D^*|$. □

Let D^* be a minimum dominating set of G and $f : V' \to \{0, 1, 2\}$ be a minimum RDF of G'. It is known that for any graph $G = (V, E)$ with maximum degree Δ, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. Thus, $|D^*| \geq \frac{n}{4}$. From the above claim it is evident that $f(V') = |D^*| + 2n \leq |D^*| + 8|D^*| = 9|D^*|$.

Now consider a RDF $g : V' \to \{0, 1, 2\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) \geq 1$ or $g(a_i) \geq 1$ or $g(c_i) \geq 1\}$ is a dominating set of G. Therefore, $|D| \leq g(V') - 2n$. Hence,

$|D| - |D^*| \leq g(V') - 2n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 9$ and $\beta = 1$. □

7.2 Roman {2}-Domination

In this section, we present upper bound on the approximation ratio of MR2DP based on the approximation result known for MINIMUM DOMINATION problem which is defined in Theorem 7.1.4.

7.2.1 Upper Bound on Approximation Ratio

Here, similar to Algorithm 7.2, we propose an approximation algorithm namely, APPROX-R2D, which produces a Roman $\{2\}$-dominating triple as follows.

Algorithm 7.3 APPROX-R2D(G)

Input: A simple, undirected graph G.

Output: A Roman 2-dominating triple T_r of G.

1: $D \leftarrow$ APPROX-DOM-SET(G)

2: $T_r \leftarrow (V \setminus D, \emptyset, D)$

3: return T_r.

We note that the algorithm APPROX-R2D computes a Roman $\{2\}$-dominating triple T_r of the given graph G in polynomial time. Hence, we have the following result.

Theorem 7.2.1. *"MR2DP in a graph with maximum degree Δ can be approximated with an approximation ratio of $2(1 + \ln(\Delta + 1))$."*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.5. □

7.3 Double Roman Dominaton

Here, we present an upper bound on the approximation ratio of MDRDP based on the approximation result known for MINIMUM DOMINATION problem which is defined in

Theorem 7.1.4.

7.3.1 Upper Bound on Approximation Ratio

Here, similar to Algorithm 7.2, we propose an approximation algorithm namely, APPROX-DRD, which produces a double Roman dominating quadruple as follows. We also note

Algorithm 7.4 APPROX-DRD(G)

Input: A simple, undirected graph G.
Output: A Double Roman Dominating Quadruple Q_r of G.

1: $D \leftarrow$ APPROX-DOM-SET(G)
2: $Q_r \leftarrow (V \setminus D, \emptyset, \emptyset, D)$
3: return Q_r.

that the algorithm APPROX-DRD computes a double Roman dominating quadruple Q_r of a given graph G in polynomial time. Hence, the following theorem holds.

Theorem 7.3.1. *"MDRDP in a graph with maximum degree Δ can be approximated with an approximation ratio of $3(1 + \ln(\Delta + 1))$."*

Proof. The proof is obtained with similar arguments as in Theorem 7.2.1. □

We have the following corollaries of Theorem 7.2.1 and Theorem 7.3.1, respectively.

Corollary 7.3.1. *"MR2DP for bounded degree graphs is in APX."*

Corollary 7.3.2. *"MDRDP for bounded degree graphs is in APX."*

7.4 Total Roman Dominaton and Total Double Roman Dominaton

Here, results related to obtaining approximate solutions to MTRDP and MTDRDP are presented.

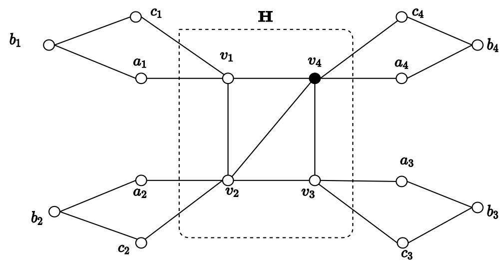

Figure 7.2: Construction of H' from H

7.4.1 Lower Bound on Approximation Ratio

In this section, we prove an approximation hardness result for the MTRDP and MTDRDP. To show the hardness result for the MTRDP and MTDRDP, we provide an approximation preserving reduction from the MDS problem. An existing result obtained on lower bound of approximation ratio of MDS is given below.

Theorem 7.4.1. *([39]) "For a graph $G = (V, E)$, unless $P = NP$, the MDS problem cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$."*

Theorem below provides a lower bound on approximation ratio of MTRDP.

Theorem 7.4.2. *"For a graph H, unless $P = NP$, the MTRDP cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$."*

Proof. We propose a reduction which preserves the approximation. Let $H(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an instance of the MDS problem. From H, an instance H' of MTRDP is constructed as follows.

Create n copies of P_3 with b_i as the central vertex and a_i, c_i as terminal vertices. Add the edges $\{(v_i, a_i), (v_i, c_i) : 1 \leq i \leq n\}$. An example construction of H' from H is shown in Figure 7.2. Next, we prove a claim.

Claim 7.4.1. "$\gamma_{tR}(H') = 3n + \gamma(H)$."

Proof. Let $H(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be a graph and $H' = (V', E')$ is a graph constructed from H.

Let T^* be a MDS of H i.e., $|T^*| = \gamma(H)$ and f be a function on H', defined as

$$f(v) = \begin{cases} 2, & \text{if } v \in \{v_i, a_i : v_i \in T^*\} \text{ or } v \in \{b_i : v_i \notin T^*\} \\ 1, & \text{if } v \in \{a_i : v_i \notin T^*\} \\ 0, & \text{otherwise} \end{cases} \tag{7.2}$$

Clearly, f is a TRDF and $\gamma_{tR}(H') \leq 3n + |T^*|$.

Next, we show that $\gamma_{tR}(H') \geq 3n + |T^*|$. Let g be a TRDF on graph H'. Clearly if $g(v_i) = 0$ then $g(a_i)+g(b_i)+g(c_i) \geq 3$ and if $g(v_i) \geq 1$ then $g(v_i)+g(a_i)+g(b_i)+g(c_i) \geq 4$. Therefore $\gamma_{tR}(H') \geq 3n + |T^*|$. Hence $\gamma_{tR}(H') = 3n + \gamma(H)$. □

Suppose that the MTRDP has an approximation algorithm A which runs in polynomial time with approximation ratio β, where $\beta = (1-\delta)\ln|V|$ for some fixed $\delta > 0$. Let l be a fixed positive integer. Next, we design an approximation algorithm, say DOM-SET-APPROX which runs in polynomial time to find a DS of a given graph H

Algorithm 7.5 DOM-SET-APPROX(G)

Input: A simple and undirected graph H.
Output: A DS T of H.

if there exists a DS T' of size at most l **then**
  $T \leftarrow T'$
else
  Build the graph H'
  Calculate a TRDF f on H' by using algorithm A
  Find a DS T of H from TRDF f (as
    illustrated in the proof of Claim 7.4.1)
end if
return T.

It can be noted that if T is a DS with $|T| \leq l$, then it is optimal. Otherwise, let T^* be a DS of H with minimum cardinality and g be a TRDF of H' with $g(V') = \gamma_{tR}(H')$. Clearly $g(V) \geq l$. If T is a DS of H obtained by the algorithm DOM-SET-APPROX, then $|T| \leq f(V) \leq \beta(g(V)) \leq \beta(3n+|T^*|) = \beta(1+\frac{3n}{|T^*|})|T^*|$. Therefore, DOM-SET-APPROX approximates a MDS within a ratio $\beta(1+\frac{3n}{|T^*|})$. If $\frac{1}{|T^*|} < \delta/2$, then the approximation ratio becomes $\beta(1 + \frac{3n}{|T^*|}) < (1-\delta)(1 + \frac{3n\delta}{2}) \ln n = (1-\delta') \ln n$, where $\delta' = \frac{3n\delta^2}{2} - \frac{3n\delta}{2} + \delta$.

By Theorem 7.4.1, if there exists an approximation algorithm for MDS problem with approximation ratio $(1-\delta)\ln|V|$ then $P = NP$. Similarly, if there exists an approximation algorithm for MTRDP with approximation ratio $(1-\delta)\ln|V|$ then $P = NP$. For large values of n, $\ln n \approx \ln(4n)$. Hence, in a graph $H'(V', E')$, where $|V'| = 4|V|$, the MTRDP cannot have an approximation algorithm with a ratio of $(1-\delta)\ln|V'|$ unless $P = NP$. □

Theorem 7.4.3. *"For a graph H, unless $P = NP$, the MTDRDP cannot have a solution with approximation ratio $(1-\delta)\ln|V|$ for any $\delta > 0$."*

Proof. The proof is obtained with similar arguments as in Theorem 7.8.1, in which replace the assigned value, for the vertices, 2 with 3. □

7.4.2 Upper Bound on Approximation Ratio

Here, an approximation algorithm for MT(D)RDP is designed based on the approximation result known for MTDS problem below.

Theorem 7.4.4 ([32])**.** *"The MTDS problem can be approximated with an approximation ratio of $\ln(\Delta - 0.5) + 1.5$."*

Here, similar to Algorithm 7.2, we propose an approximation algorithm namely, APP-TRDF, which produces a total Roman dominating triple (TRDT) as follows.

Algorithm 7.6 APP-TRDF(G)

Input: A simple, undirected graph G.
Output: A TRDT T_r of G.

1: $D \leftarrow$ APP-TD-SET(G)
2: $T_r \leftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

We note that the algorithm APP-TRDF computes a total Roman dominating triple T_r of the given graph G in polynomial time. Hence, we have the following result.

Theorem 7.4.5. *"MTRDP in a graph can be approximated with an approximation ratio of* $2(\ln(\Delta - 0.5) + 1.5)$*."*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.5. □

Similar to the Algorithm 7.6, we propose an approximation algorithm APP-TDRDF which produces a total double Roman dominating quadruple (TDRDQ). We also note that the

Algorithm 7.7 APP-TDRDF(G)

Input: A simple, undirected graph G.
Output: A TDRDQ Q_r of G.
1: $D \leftarrow$ APP-TD-SET(G)
2: $Q_r \leftarrow (V \setminus D, \emptyset, \emptyset, D)$
3: return Q_r.

algorithm APP-TDRDF computes a TDRDQ Q_r of a given graph G in polynomial time and the following theorem holds.

Theorem 7.4.6. *"MTDRDP in a graph can be approximated with an approximation ratio of* $3(\ln(\Delta - 0.5) + 1.5)$*."*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.5. □

The following corollary is from Theorems 7.4.5 and 7.4.6.

Corollary 7.4.1. *"$MT(D)RDP \in APX$ for graphs with* $\Delta = O(1)$*."*

7.4.3 APX-completeness for Bounded Degree Graphs

Here, we prove that MTRDP and MTDRDP are APX-complete for graphs with $\Delta = 5$ by providing an L-reduction from DOM-3 problem.

Theorem 7.4.7. *"MTRDP is APX-complete for graphs with* $\Delta = 5$*."*

Proof. From Corollary 7.4.1, it is clear that MTRDP is in APX. Given an instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V', E')$ of

MTRDP same as in Section 7.4.1. Note that G' is a graph with $\Delta = 5$. First we prove the following claim.

Claim 7.4.2. *"$\gamma_{tR}(G') = 3n + \gamma(G)$, where $n = |V|$."*

Proof. The proof is similar to the proof in claim 7.4.1. □

Let D^* be a minimum DS of G and $f : V' \to \{0, 1, 2\}$ be a minimum TRDF of G'. It is known that for any graph $G = (V, E)$, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. Thus, $|D^*| \geq \frac{n}{4}$. From the above claim it is evident that $f(V') = |D^*| + 3n \leq |D^*| + 12|D^*| = 13|D^*|$.

Now consider a TRDF $g : V' \to \{0, 1, 2\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) \geq 1$ or $g(a_i) \geq 1$ or $g(c_i) \geq 1\}$ is a DS of G. Therefore, $|D| \leq g(V') - 3n$. Hence, $|D| - |D^*| \leq g(V') - 3n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 13$ and $\beta = 1$. □

Theorem 7.4.8. *"MTDRDP is APX-complete for graphs with $\Delta = 5$."*

Proof. The proof is obtained with similar arguments as in Theorem 7.4.7, in which replace the assigned value 2 with 3. We get an L-reduction with $\alpha = 18$ and $\beta = 1$. □

7.5 Independent Roman Domination, Independent Roman {2}-Domination, Independent Double Roman Domination

Here, results related to obtaining approximate solutions to MIRDP, MIR2DP and MIDRDP are presented.

7.5.1 APX-hardness for Bounded Degree Graphs

In this section, we prove that MIRDP, MIR2DP and MIDRDP are APX-hard for graphs with maximum degree 4 by providing an L-reduction from MINIMUM INDEPENDENT

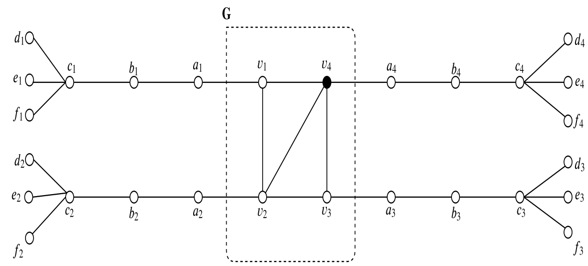

Figure 7.3: An illustration to the construction of G'

DOMINATING SET-3 (MIDS-3) problem which has been proved as APX-complete [47]. The MIDS-3 problem is to find a minimum independent dominating set of a graph with maximum degree 3.

Theorem 7.5.1. *"MIRDP is APX-hard for graphs with maximum degree 4."*

Proof. Given an instance $G = (V, E)$ of MIDS-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V', E')$ of MIRDP as follows.

Let $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$, $C = \{c_1, c_2, \ldots, c_n\}$, $D = \{d_1, d_2, \ldots, d_n\}$, $E = \{e_1, e_2, \ldots, e_n\}$ and $F = \{f_1, f_2, \ldots, f_n\}$. In the graph G', $V' = V \cup A \cup B \cup C \cup D \cup F$ and $E' = E \cup \{(v_i, a_i), (a_i, b_i), (b_i, c_i), (c_i, d_i), (c_i, e_i), (c_i, f_i) : 1 \leq i \leq n\}$. An example construction of G' is shown in Figure 7.3. First we prove the following claim.

Claim 7.5.1. *"If G' is the graph obtained from a graph $G = (V, E)$ ($|V| = n$) then $i_R(G') = 3n + i(G)$."*

Proof. Suppose D^* be a minimum independent dominating set of G i.e., $|D^*| = i(G)$ and $f : V' \to \{0, 1, 2\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 2, & \text{if } v \in C \cup D^* \\ 1, & \text{if } v \in \{a_i : v_i \notin D^*\} \\ 0, & \text{otherwise} \end{cases} \tag{7.3}$$

Clearly, f is an IRDF and $i_R(G') \leq 3n + |D^*|$.

Next, we show that $i_R(G') \geq 3n + |D^*|$. Let g be an IRDF on graph G'. Clearly, $g(b_i)+g(c_i)+g(d_i)+g(e_i)+g(f_i) \geq 2$, $g(a_i)+g(v_i) \geq 1$ and if $g(v_i) = 0$ then $g(a_i) \geq 1$. Therefore $i_R(G') \geq 3n + |D^*|$. Hence $i_R(G') = 3n + i(G)$. □

Let D^* be a minimum independent dominating set of G and $f : V' \rightarrow \{0, 1, 2\}$ be a minimum IRDF of G'. It is known that for any graph $G = (V, E)$ with maximum degree Δ, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. From [66], we know that $\gamma(G) \leq i(G)$. Thus, $|D^*| \geq \frac{n}{4}$. From the above claim it is evident that $f(V') = |D^*| + 3n \leq |D^*| + 12|D^*| = 13|D^*|$.

Now consider an IRDF $g : V' \rightarrow \{0, 1, 2\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) = 2$ or $g(a_i) = 2\}$ is an IDS of G. Therefore, $|D| \leq g(V') - 3n$. Hence, $|D| - |D^*| \leq g(V') - 3n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 13$ and $\beta = 1$. □

Theorem 7.5.2. *"MIR2DP is APX-hard for graphs with maximum degree 4."*

The proof is similar to the proof given in Theorem 7.5.1.

Theorem 7.5.3. *"MIDRDP is APX-hard for graphs with maximum degree 4."*

Proof. The proof is obtained with similar arguments as in Theorem 7.5.1, in which replace the assigned values, for the vertices, 1 with 2 and 2 with 3. We get an L-reduction with $\alpha = 21$ and $\beta = 1$. □

7.6 Weakly Connected Roman Dominaton

Here, results related to obtaining approximate solutions to MWCRDP are presented.

7.6.1 Lower Bound on Approximation Ratio

In this section, we prove an approximation hardness result for the MWCRDP. To show the hardness result for the MWCRDP, we provide an approximation preserving reduction from

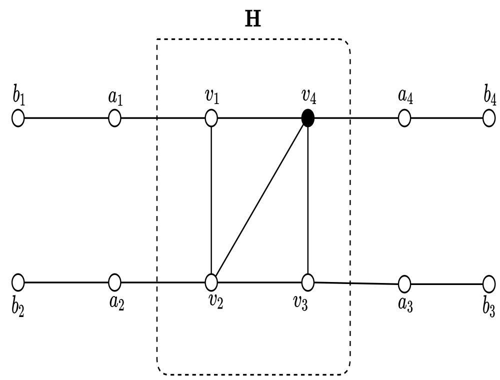

Figure 7.4: Construction of H' from H

the MDS problem given in Theorem 7.4.1. Theorem below provides a lower bound on approximation ratio of MWCRDP.

Theorem 7.6.1. *"For a graph H, unless $P = NP$, the MWCRDP cannot have a solution with approximation ratio $(1-\delta)\ln|V|$ for any $\delta > 0$."*

Proof. We propose a reduction which preserves the approximation. Let $H(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an instance of the MDS problem. From H, an instance H' of MWCRDP with the following vertex and edge sets is constructed.

$V(H') = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\} \cup V(H)$ and

$E(H') = \{(v_i, a_i), (a_i, b_i) : 1 \leq i \leq n\} \cup E(H)$.

Figure 7.4 shows a construction of H' from H. Next, we prove a claim.

Claim 7.6.1. "$\gamma_R^{wc}(H') = 2n + \gamma(H)$."

Proof. Let $H'(V', E')$ is a graph constructed from $H(V, E)$.

Let T^* be a MDS of H i.e., $|T^*| = \gamma(H)$ and h be a function on H', defined as

$$h(v) = \begin{cases} 2, & \text{if } v \in \{a_i : 1 \leq i \leq n\} \\ 1, & \text{if } v \in T^* \\ 0, & \text{otherwise} \end{cases} \tag{7.4}$$

Clearly, h is a WCRDF and $\gamma_R^{wc}(H') \leq 2n + |T^*|$.

Next, we show that $\gamma_R^{wc}(H') \geq 2n + |T^*|$. Let g be a WCRDF on graph H'. Clearly, $g(a_i) + g(b_i) \geq 2$. These make the size at least $2n$. Therefore $\gamma_R^{wc}(H') \geq 2n + |T^*|$. Hence $\gamma_R^{wc}(H') = 2n + \gamma(H)$. □

Suppose that the MWCRDP has an approximation algorithm A which runs in P with approximation ratio β, where $\beta = (1 - \delta)\ln|V|$ for some fixed $\delta > 0$. Next, we design an approximation algorithm, say DS-APX which runs in P to find a DS of a given graph H. Let l (≥ 0) be an integer.

Algorithm 7.8 DS-APX(H)

Input: A simple, undirected graph H.
Output: A DS of H.

```
if there exists a DS T' such that |T'| ≤ l then
    T ← T'
else
    Build the graph H'
    Calculate a WCRDF f on H' by using algorithm A
    Let T = {v_i : f(v_i) + f(a_i) ≥ 2}. (As illustrated in the Claim 7.6.1)
end if
return T.
```

It can be noted that if T is a DS with $|T| \leq l$, then it is optimal. Otherwise, let T^* be a DS of H with minimum cardinality and g be a WCRDF of H' with $w(g) = \gamma_R^{wc}(H')$. Clearly $g(V) \geq l$. If T is a DS of H obtained by the algorithm DS-APX, then $|T| \leq f(V) \leq \beta(g(V)) \leq \beta(2n + |T^*|) = \beta(1 + \frac{2n}{|T^*|})|T^*|$. Therefore, DS-APX approximates a MDS within a ratio $\beta(1 + \frac{2n}{|T^*|})$. If $\frac{1}{|T^*|} < \delta/2$, then the approximation ratio becomes $\beta(1 + \frac{2n}{|T^*|}) < (1 - \delta)(1 + \frac{2n\delta}{2})\ln n = (1 - \delta')\ln n$, where $\delta' = n\delta^2 + \delta - n\delta$.

By Theorem 7.4.1, if there exists an approximation algorithm for MDS problem with approximation ratio $(1-\delta)\ln|V|$ then $P = NP$. Similarly, if there exists an approximation algorithm for MWCRDP with approximation ratio $(1 - \delta)\ln|V|$ then $P = NP$. For large values of n, $\ln n \approx \ln(2n)$. Hence, in a graph $H'(V', E')$, where $|V'| = 2|V|$, the MWCRDP problem cannot have an approximation algorithm with a ratio of $(1 - \delta)\ln|V'|$ unless $P = NP$. □

7.6.2 Upper Bound on Approximation Ratio

Here, an approximation algorithm for MWCRDP is designed based on the approximation result known known for MCDS problem and the proposition below.

Theorem 7.6.2. *([19]) "The MCDS problem can be approximated with an approximation ratio of* $(1+\epsilon)(1+\ln(\Delta-1))$ *in a graph for any* $\epsilon > 0$*."*

Proposition 7.6.1. *"If* H *is a connected graph then* $\gamma_R^{wc}(H) \leq 2\gamma_c(H)$*."*

Proof. Let C be a CDS of H such that $|C| = \gamma_c(H)$. Then the function h defined as

$$h(v) = \begin{cases} 2, & \text{if } v \in C \\ 0, & \text{otherwise} \end{cases} \tag{7.5}$$

is a WCRDF of H. Thus $\gamma_R^{wc}(H) \leq w(h) = 2|C| = 2\gamma_c(H)$. Hence the proposition. □

Let ACDSET be an approximation algorithm that produces a CDS C of a graph H such that $|C| \leq (1+\epsilon)(1+\ln(\Delta-1))\gamma_c(H)$, where ϵ is any positive constant.

Next, we design AWCRDF algorithm to determine an approximate solution of MWCRDP. In this algorithm, we first determine a CDS C of H using the ACDSET algorithm. Next, we build a weakly connected Roman dominating triple (WCRDT) T_r such that weight 2 is assigned for all vertices in C and weight 0 is assigned for the remaining vertices.

Now, let $T_r = (V \setminus C, \emptyset, C)$ be the WCRDT obtained from the AWCRDF algorithm. Clearly, T_r gives a WCRDF of G and AWCRDF algorithm determines the WCRDT T_r of H in P. Hence, the result follows.

Algorithm 7.9 AWCRDF(H)

Input: A simple, undirected graph H.
Output: A WCRDT T_r of H.

```
C ← ACDSET(H)
T_r ← (V \ C, ∅, C)
return T_r.
```

Theorem 7.6.3. *"MWCRDP can be approximated with an approximation ratio of $2(1+\epsilon)(1+\ln(\Delta-1))$ in a graph for any $\epsilon > 0$."*

Proof. Let C be the CDS obtained from the ACDSET algorithm, T_r be the WCRDT produced by the AWCRDF algorithm and $W_r = |Q_r|$. Clearly, $W_r = 2|C|$. It is known that $|C| \leq (1+\epsilon)(1+\ln(\Delta-1))\gamma_c(H)$. Therefore, $W_r \leq 2(1+\epsilon)(1+\ln(\Delta-1))\gamma_c(H)$. Since $\gamma_c(H) \leq \gamma_R^{wc}(H)$, it follows that $W_r \leq 2(1+\epsilon)(1+\ln(\Delta-1))\gamma_R^{wc}(H)$. Hence the result. □

From Theorem 7.6.3, the corollary below follows.

Corollary 7.6.1. *"$MWCRDP \in APX$ for graphs with $\Delta = O(1)$."*

7.6.3 APX-completeness for Bounded Degree Graphs

Here, we prove that MWCRDP is APX-complete for graphs with $\Delta = 4$ by providing an L-reduction from DOM-3 problem.

Theorem 7.6.4. *"MWCRDP is APX-complete for graphs with $\Delta = 4$."*

Proof. From Corollary 7.6.1, it follows that MWCRDP $\in$ APX for graphs with $\Delta = 4$. From the given instance $H = (V, E)$ of DOM-3, we construct an instance $H' = (V', E')$ of MWCRDP same as in Theorem 7.6.1. Clearly, $\Delta(H') = 4$. We make use of the Claim 7.6.1 to complete the proof.

Let $f : V' \to \{0, 1, 2\}$ be a WCRDF of H', where $f(V') = \gamma_R^{wc}(H')$ and T^* be a MDS of H. It is known that for any graph $H = (V, E)$, $\gamma(G) \geq \frac{|V|}{\Delta+1}$. Thus, $|T^*| \geq \frac{n}{4}$. From the Claim 7.6.1, clearly, $f(V') = 2n + |T^*| \leq 8|T^*| + |T^*| = 9|T^*|$.

Now consider a WCRDF $h : V' \to \{0, 1, 2\}$ of H'. Clearly, $T = \{v_i : h(v_i) + h(a_i) \geq 1\}$ is a DS of H. Therefore, $|T| \leq h(V') - 2n$. Hence, $|T| - |T^*| \leq h(V') - 2n - |T^*| \leq h(V') - f(V')$. This infers there exists an L-reduction with $\alpha = 9$ and $\beta = 1$. □

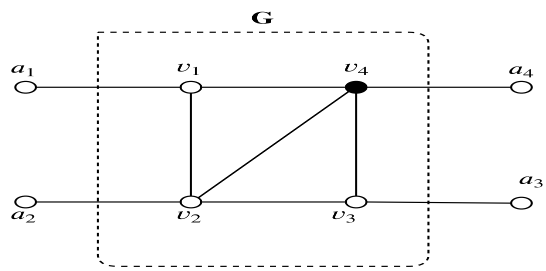

Figure 7.5: An illustration to the construction of G' from G

7.7 Roman {3}-Dominaton

Here, results related to obtaining approximate solutions to MR3DP are presented.

7.7.1 Lower Bound on Approximation Ratio

In this section, we prove an approximation hardness result for the MR3DP. To show the hardness result for the MR3DP, we provide an approximation preserving reduction from the MDS problem given in Theorem 7.4.1. The following theorem provides a lower bound for approximation ratio of MR3DP.

Theorem 7.7.1. *"For a graph $G = (V, E)$, the MR3DP cannot be approximated within a factor of $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $P = NP$."*

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an instance of the MINIMUM DOMINATING SET problem. From this, we construct an instance $G' = (V', E')$ of MR3DP as follows.

Create a vertex set $\{a_1, a_2, \ldots, a_n\}$. Add the edges $\{(v_i, a_i) : 1 \leq i \leq n\}$. Example construction of G' from G is shown in Figure 7.5. First we need to prove the following claim.

Claim 7.7.1. *"If G' is the graph obtained from a graph $G = (V, E)$ ($|V| = n$) then $\gamma_{\{R3\}}(G') = 2n + \gamma(G)$."*

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be a graph and $G' = (V', E')$ is a graph constructed from G.

Let D^* be a MDS of G i.e., $|D^*| = \gamma(G)$ and $f : V \rightarrow \{0, 1, 2, 3\}$ be a function on graph G', defined as

$$f(v) = \begin{cases} 3, & \text{if } v \in D^* \\ 2, & \text{if } v \in \{a_i : v_i \notin D^*\} \\ 0, & \text{otherwise} \end{cases} \tag{7.6}$$

Clearly, f is a R3DF and $\gamma_{\{R3\}}(G') \leq 2n + |D^*|$.

Next, we show that $\gamma_{\{R3\}}(G') \geq 2n + |D^*|$. Let g be a R3DF on graph G'. Clearly, $g(v_i) + g(a_i) \geq 2$, if $g(v_i) = 0$ then $g(a_i) \geq 2$, if $g(a_i) = 0$ then $g(v_i) = 3$ and if $|E(G')| = 1$ then $g(a_i) + g(v_i) \geq 3$. Therefore $\gamma_{\{R3\}}(G') \geq 2n + |D^*|$. Hence $\gamma_{\{R3\}}(G') = 2n + \gamma(G)$. □

Suppose that the MR3DP has an approximation algorithm A which runs in polynomial time with approximation ratio α, where $\alpha = (1 - \epsilon) \ln |V|$ for some fixed $\epsilon > 0$. Let k be a fixed positive integer. Next, we design an approximation algorithm, say DOM-SET-APPROX which runs in polynomial time to find a DS of a given graph G.

Algorithm 7.10 DOM-SET-APPROX(G)

Input: A simple and undirected graph G.
Output: A DS D of G.

```
if there exists a DS D' of size at most k then
    D ← D'
else
    Build the graph G'
    Find a R3DF g on G' by using algorithm A
    Let D = {v_i : g(v_i) + g(a_i) ≥ 3}. (from Claim 7.7.1)
end if
return D.
```

Clearly, DOM-SET-APPROX runs in polynomial time. It can be noted that if D is a MDS of size at most k, then it is optimal. Otherwise, let D^* be a MDS of G and f be a R3DF of G' with $f(V') = \gamma_{\{R3\}}(G')$. Clearly $f(V') \geq k$. If D is a DS of G produced by the algorithm DOM-SET-APPROX, then $|D| \leq g(V') \leq \alpha(f(V')) \leq \alpha(2n + |D^*|)$ = $\alpha(1 + \frac{2n}{|D^*|})|D^*|$. Therefore, DOM-SET-APPROX approximates a DS within a ratio

$\alpha(1+\frac{2n}{|D^*|})$. If $\frac{1}{|D^*|} < \epsilon/2$, then the approximation ratio becomes $\alpha(1+\frac{2n}{|D^*|}) < (1-\epsilon)(1+n\epsilon)\ln n = (1-\epsilon')\ln n$, where $\epsilon' = n\epsilon^2+\epsilon-n\epsilon$. Hence DOM-SET-APPROX approximates minimum dominating set within $(1-\epsilon')\ln|V|$. So by Theorem 7.4.1 and the fact that $\ln(2|V|) \approx \ln|V'|$ for $|V| \to \infty$, unless $P = NP$, MR3DP cannot be approximated within a ratio of $(1-\epsilon)\ln|V|$ for any $\epsilon > 0$. □

7.7.2 Upper Bound on Approximation Ratio

Here, similar to Algorithm 7.2, we propose an approximation algorithm namely, APP-R3D, which produces a Roman $\{3\}$-dominating quadruple as follows.

Algorithm 7.11 APP-R3D(G)

Input: A simple, undirected graph G.

Output: A Roman $\{3\}$-dominating quadruple Q_r of G.

1: $D \leftarrow$ APP-DOM-SET(G)

2: $Q_r \leftarrow (V \setminus D, \emptyset, \emptyset, D)$

3: return Q_r.

We note that the algorithm APP-R3D computes a Roman $\{3\}$-dominating quadruple Q_r of the given graph G in polynomial time. Hence, we have the following result.

Theorem 7.7.2. *"MR3DP in a graph with maximum degree Δ can be approximated with an approximation ratio of* $3(1+\ln(\Delta+1))$*."*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.5. □

From Theorem 7.7.2, the corollary below follows.

Corollary 7.7.1. *"$MR3DP \in APX$ for graphs with $\Delta = O(1)$."*

7.7.3 APX-completeness for Bounded Degree Graphs

Here, we prove that MR3DP is APX-complete for graphs with $\Delta = 4$ using the L-reduction [16]. By providing an L-reduction from MDS problem with $\Delta = 3$ i.e., DOM-3 which is known to be APX-complete [61], we show that MR3DP $\in$ APX-hard for graphs with $\Delta = 4$.

Theorem 7.7.3. *"MR3DP is APX-complete for graphs with maximum degree 4."*

Proof. By using Corollary 7.7.1, we can say that MR3DP is in APX for graphs with maximum degree 4. Given an instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V', E')$ of MR3DP same as in Section 7.7.1. Note that G' is a graph with maximum degree 4. We make use of the Claim 7.7.1 to complete the proof.

Let D^* be a MDS of G and $f : V' \to \{0, 1, 2, 3\}$ be a minimum R3DF of G'. It is known that for any graph $G = (V, E)$, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. Thus, $|D^*| \geq \frac{n}{4}$. From Claim 7.7.1, it is evident that $f(V') = 2n + |D^*| \leq 8|D^*| + |D^*| = 9|D^*|$.

Now consider a R3DF $g : V' \to \{0, 1, 2, 3\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) + g(a_i) \geq 3\}$ is a DS of G. Therefore, $|D| \leq g(V') - 2n$. Hence, $|D| - |D^*| \leq g(V') - 2n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 9$ and $\beta = 1$. □

7.8 Total Roman {2}-Dominaton

Here, results related to obtaining approximate solutions to MTR2DP are presented.

7.8.1 Lower Bound on Approximation Ratio

In this section, we prove an approximation hardness result for the MTR2DP. To show the hardness result for the MTR2DP, we provide an approximation preserving reduction from the MDS problem given in Theorem 7.4.1. Theorem below provides a lower bound on approximation ratio of MTR2DP.

Theorem 7.8.1. *"For a graph H, unless $P = NP$, the MTR2DP cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$."*

Proof. We propose a reduction which preserves the approximation. Let $H(V, E)$, where

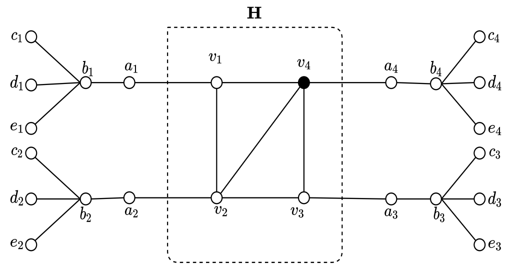

Figure 7.6: Construction of H' from H

$V = \{v_1, v_2, \ldots, v_n\}$ be a MDS problem instance. From H, an instance H' of MTR2DP is constructed as below.

Create n copies of star graphs with b_i as the central vertex and a_i, c_i, d_i and e_i as terminal vertices. Add the edges $\{(v_1, a_1), (v_2, a_2), \ldots, (v_n, a_n)\}$. Figure 7.6, shows an example construction of H' from H. Next, we prove a claim.

Claim 7.8.1. "$\gamma_{tR2}(H') = 3n + \gamma(H)$."

Proof. Let $H(V, E)$ be a graph, where $V = \{v_1, v_2, \ldots, v_n\}$ and $H' = (V', E')$ be a graph constructed from H.

Let T^* be a MDS of H i.e., $|T^*| = \gamma(H)$ and h be a function on H', defined as

$$h(v) = \begin{cases} 1, & \text{if } v \in \{a_i : 1 \leq i \leq n\} \cup T^* \\ 2, & \text{if } v \in \{b_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{7.7}$$

Clearly, h is a TR2DF and $\gamma_{tR2}(H') \leq 3n + |T^*|$.

Next, we show that $\gamma_{tR2}(H') \geq 3n + |T^*|$. Let g be a TR2DF on graph H'. Clearly, irrespective of v_i's, $g(a_i)+g(b_i)+g(c_i)+g(d_i)+g(e_i) \geq 3$. Therefore $\gamma_{tR2}(H') \geq 3n+|T^*|$. Hence $\gamma_{tR2}(H') = 3n + \gamma(H)$. □

Suppose that the MTR2DP has an approximation algorithm L which runs in polynomial time with approximation ratio β, where $\beta = (1 - \delta) \ln |V|$ for some fixed $\delta > 0$. Let l be a fixed positive integer. Next, we design an approximation algorithm, say DSA which runs in polynomial time to find a DS of a given graph H.

Algorithm 7.12 DSA(G)

Input: A simple and undirected graph H.
Output: A DS T of H.

if there exists a DS T' of size at most l **then**
  $T \leftarrow T'$
else
  Build the graph H'
  Calculate a TR2DF f on H' by using algorithm L
  Find a DS T of H from TR2DF f (from Claim 7.8.1) **end if**
return T.

It can be noted that if T is a DS with $|T| \leq l$, then it is optimal. Otherwise, let T^* be a DS of H with minimum cardinality and g be a TR2DF of H' with $g(V') = \gamma_{tR2}(H')$. Clearly $g(V) \geq l$. If T is a DS of H obtained by the algorithm DSA, then $|T| \leq f(V) \leq \beta(g(V)) \leq \beta(3n + |T^*|) = \beta(1 + \frac{3n}{|T^*|})|T^*|$. Therefore, DSA approximates a MDS within a ratio $\beta(1 + \frac{3n}{|T^*|})$. If $\frac{1}{|T^*|} < \delta/2$, then the approximation ratio becomes $\beta(1 + \frac{3n}{|T^*|}) < (1-\delta)(1 + \frac{3n\delta}{2}) \ln n = (1-\delta') \ln n$, where $\delta' = \frac{3n\delta^2}{2} - \frac{3n\delta}{2} + \delta$.

By Theorem 7.4.1, if there exists an approximation algorithm for MDS problem with approximation ratio $(1-\delta) \ln |V|$ then $P = NP$. Similarly, if there exists an approximation algorithm for MTR2DP with approximation ratio $(1-\delta) \ln |V|$ then $P = NP$. For large values of n, $\ln n \approx \ln(5n)$. Hence, in a graph $H'(V', E')$, where $|V'| = 5|V|$, unless $P = NP$, the MTR2DP cannot have an approximation algorithm with a ratio of $(1 - \delta) \ln |V'|$. □

7.8.2 Upper Bound on Approximation Ratio

Here, an approximation algorithm for MTR2DP is designed based on the approximation result known for MTDS problem given in Theorem 7.4.4. Let APP-TD-SET be an approximation algorithm that produces a TDS D of a graph G such that $|D| \leq (\ln(\Delta - 0.5) + 1.5)\gamma_t(G)$. Similar to Algorithm 7.2, we propose an approximation algorithm namely, APP-TR2DF, which produces a total Roman $\{2\}$-dominating triple (T_r) as follows.

We note that the algorithm APP-TR2DF computes a total Roman $\{2\}$-dominating triple T_r of the given graph G in polynomial time. Hence, we have the following result.

Algorithm 7.13 APP-TR2DF(G)

Input: A simple, undirected graph G.
Output: A TR2DT T_r of G.
1: $D \leftarrow$ APP-TD-SET(G)
2: $T_r \leftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

Theorem 7.8.2. *"MTR2DP in a graph can be approximated with an approximation ratio of* $2(\ln(\Delta - 0.5) + 1.5)$*."*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.5. □

From Theorem 7.8.2, the corollary below follows.

Corollary 7.8.1. *"$MTR2DP \in APX$ for graphs with $\Delta = O(1)$."*

7.8.3 APX-completeness for Bounded Degree Graphs

Here, we prove that MTR2DP is APX-complete for graphs with $\Delta = 4$ using the L-reduction [16]. By providing an L-reduction from MDS problem with $\Delta = 3$ i.e., DOM-3 which is known to be APX-complete [61], we show that MTR3DP $\in$ APX-hard for graphs with $\Delta = 4$.

Theorem 7.8.3. *"MTR2DP is APX-complete for graphs with* $\Delta = 4$*."*

Proof. From Corollary 7.4.1, clearly, MTR2DP$\in$APX. From the given instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct a MTR2DP instance $G' = (V', E')$ same as in Section 7.8.1. Clearly, $\Delta(G') = 4$.

Claim 7.8.2. *"$\gamma_{tR2}(G') = 3n + \gamma(G)$, where $n = |V|$."*

Proof. The proof is same as in Claim 7.8.1. □

Assume g be a TR2DF on G', where $g(V') = \gamma_{tR2}(G')$ and D^* be a MDS of G. For any graph G, it is known that $\gamma(G) \geq \frac{n}{\Delta+1}$. Clearly, $|D^*| \geq \frac{n}{4}$. From the claim 7.8.2,

$g(V') = |D^*| + 3n \leq |D^*| + 12|D^*| = 13|D^*|$.

Let $h : V' \to \{0, 1, 2\}$ be a TR2DF of G'. Then, clearly, $D = \{v_i : h(v_i) \geq 1$ or $h(a_i) \geq 1\}$ is a DS of G. Hence, $|D| \leq h(V') - 3n$. Therefore, $|D| - |D^*| \leq h(V') - 3n - |D^*| \leq h(V') - g(V')$. This infers that there exists an L-reduction with $\beta = 1$ and $\alpha = 13$. □

7.9 Total Roman {3}-Dominaton

Here, results related to obtaining approximate solutions to MTR3DP are presented.

7.9.1 Lower Bound on Approximation Ratio

In this section, we prove an approximation hardness result for the MTR3DP. To show the hardness result for the MTR3DP, we provide an approximation preserving reduction from the MDS problem given in Theorem 7.4.1. Theorem below provides a lower bound on approximation ratio of MTR3DP.

Theorem 7.9.1. *"MTR3DP for a graph H cannot have an approximation algorithm with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$ unless $P = NP$."*

Proof. We propose a reduction which preserves the approximation. Let $H(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an instance of the MDS problem. From H, an instance H' of MTR3DP with the following vertex and edge sets is constructed.

$V(H') = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n,\} \cup V(H)$ and

$E(H') = \{(v_i, a_i), (a_i, b_i) : 1 \leq i \leq n\} \cup E(H)$.

Next, we prove a claim.

Claim 7.9.1. "$\gamma_{t\{R3\}}(H') = 3n + \gamma(H)$."

Proof. Let $H'(V', E')$ is a graph constructed from $H(V, E)$ as described above.

Let T^* be a MDS of H i.e., $|T^*| = \gamma(H)$ and f be a function on H', defined as

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a_i : 1 \leq i \leq n\} \\ 1, & \text{if } v \in \{b_i : 1 \leq i \leq n\} \text{ or } v \in \{v_i : v_i \in T^*\} \\ 0, & \text{otherwise} \end{cases} \tag{7.8}$$

Clearly, f is a TR3DF and $\gamma_{t\{R3\}}(H') \leq 3n + |T^*|$.

Next, we show that $\gamma_{t\{R3\}}(H') \geq 3n + |T^*|$. Let g be a TR3DF on graph H'. Clearly if $g(v_i) = 0$ then $g(a_i) + g(b_i) \geq 3$ and if $g(v_i) \geq 1$ then $g(v_i) + g(a_i) + g(b_i) \geq 4$. Therefore $\gamma_{t\{R3\}}(H') \geq 3n + |T^*|$. Hence $\gamma_{t\{R3\}}(H') = 3n + \gamma(H)$. □

Suppose that the MTR3DP has an approximation algorithm A which runs in polynomial time with approximation ratio β, where $\beta = (1 - \delta) \ln |V|$ for some fixed $\delta > 0$. Let l (> 1) be an integer. Next, we design an approximation algorithm, say DS-APP which runs in polynomial time to determine a DS of a H.

Algorithm 7.14 DS-APP(H)

Input: A simple, undirected graph H.
Output: A DS T of H.

```
if there exists a DS T' such that |T'| ≤ l then
  T ← T'
else
  Build the graph T'
  Calculate a TR3DF g on T' by using algorithm A
  Find a DS T of H from TR3DF g (as illustrated in the proof of Claim 7.9.1)
end if
return T.
```

It can be noted that if T is a DS with $|T| \leq l$, then it is optimal. Otherwise, let T^* be a DS of H with minimum cardinality and f be a TR3DF of H' with $f(V(H')) = \gamma_{t\{R3\}}(H')$. Clearly $f(V) \geq l$. If T is a DS of H obtained by the algorithm DS-APP, then $|T| \leq g(V) \leq \beta(f(V)) \leq \beta(3n + |T^*|) = \beta(1 + \frac{3n}{|T^*|})|T^*|$. Therefore, DS-APP approximates a MDS within a ratio $\beta(1 + \frac{3n}{|T^*|})$. If $\frac{1}{|T^*|} < \delta/2$, then the approximation ratio becomes $\beta(1 + \frac{3n}{|T^*|}) < (1 - \delta)(1 + \frac{3n\delta}{2}) \ln n = (1 - \delta') \ln n$, where $\delta' = \frac{3n\delta^2}{2} - \frac{3n\delta}{2} + \delta$. Hence DOM-SET-APP approximates MDS within $(1 - \delta') \ln |V|$. So by Theorem 7.4.1

and the fact that $\ln(3|V|) \approx \ln|V|$ for $|V| \to \infty$, unless $P = NP$, MTR3DP cannot be approximated within a ratio of $(1-\delta)\ln|V|$ for any $\delta > 0$. □

7.9.2 Upper Bound on Approximation Ratio

Here, an approximation algorithm for MTR3DP is designed based on the approximation result known for MTDS problem given in Theorem 7.4.4. Similar to Algorithm 7.2, we propose an approximation algorithm namely, APP-TR3DF, which produces a total Roman $\{3\}$-dominating quadruple as follows.

Algorithm 7.15 APP-TR3DF(G)

Input: A simple, undirected graph G.
Output: A TR3DQ Q_r of G.

1: $D \leftarrow$ APP-TDOM-SET(G)
2: $Q_r \leftarrow (V \setminus D, \emptyset, \emptyset, D)$
3: return Q_r.

We note that the algorithm APP-TR3DF computes a total Roman $\{3\}$-dominating quadruple Q_r of the given graph G in polynomial time. Hence, we have the following result.

Theorem 7.9.2. *"MTR3DP in a graph can be approximated with an approximation ratio of* $3(\ln(\Delta - 0.5) + 1.5)$.*"*

Proof. The proof is obtained with similar arguments as in Theorem 7.1.5. □

From Theorem 7.9.2, the corollary below follows.

Corollary 7.9.1. *"$MTR3DP \in APX$ for graphs with $\Delta = O(1)$."*

7.9.3 APX-completeness for Bounded Degree Graphs

Here, we prove that MTR3DP is APX-complete for graphs with $\Delta = 4$ using the L-reduction [16]. By providing an L-reduction from MDS problem with $\Delta = 3$ i.e., DOM-3 which is known to be APX-complete [61], we show that MTR3DP $\in$ APX-hard for graphs with $\Delta = 4$.

Theorem 7.9.3. *"MTR3DP is APX-complete for graphs with $\Delta = 4$."*

Proof. From Corollary 7.9.1, it is clear that MTR3DP $\in$ APX. From the given instance $G = (V, E)$ of DOM-3, we construct an instance $G' = (V', E')$ of MTR3DP same as in Section 7.9.1. Clearly, G' is a graph with $\Delta = 4$. The following claim holds.

Claim 7.9.2. *"$\gamma_{t\{R3\}}(G') = 3n + \gamma(G)$, where $n = |V|$."*

Proof. The proof is same as in Claim 7.9.1. ☐

Let $f : V' \to \{0, 1, 2, 3\}$ be a TR3DF of G', where $f(V') = \gamma_{t\{R3\}}(G')$ and T^* be a MDS of G. It is known that for any graph $G = (V, E)$, $\gamma(G) \geq \frac{|V|}{\Delta+1}$. Thus, $|T^*| \geq \frac{n}{4}$. From the above claim, clearly, $f(V') = |T^*| + 3n \leq |T^*| + 12|T^*| = 13|T^*|$.

Now consider a TR3DF $h : V' \to \{0, 1, 2, 3\}$ of G'. Clearly, $h(b_i) + h(c_i) \geq 3$ and $T = \{v_i : h(v_i) + h(a_i) + h(c_i) \geq 4\}$ is a DS of G. Therefore, $|T| \leq h(V') - 3n$. Hence, $|T| - |T^*| \leq h(V') - 3n - |T^*| \leq h(V') - f(V')$. This infers there exists an L-reduction with $\alpha = 13$ and $\beta = 1$. ☐

7.10 Summary

In this chapter, $\Delta+1$-approximation algorithms for the MRDP, MR2DP, MDRDP, MTRDP, MTR2DP, MTDRDP, MR3DP, MTR3DP and MWCRDP have been proposed. Lower bounds on approximation ratio for the MRDP, MTRDP, MTDRDP, MR3DP, MTR2DP, MTR3DP and MWCRDP have been obtained. APX-hardness of MIRDP, MIR2DP and MIDRDP have been proved for graphs with $\Delta = 4$. Finally, it has been proved that MRDP, MTRDP and MTDRDP are APX-complete for graphs with $\Delta = 5$, and MR3DP, MTR2DP, MTR3DP and MWCRDP are APX-complete for graphs with $\Delta = 4$.

Chapter 8

Conclusion and Future Research

Here, we present the summary of contributions made in this thesis and mention the open problems triggered out of the study.

8.1 Conclusions

We have studied the algorithmic aspects of the Roman domination and its variants, namely Roman $\{2\}$-domination, double Roman domination, perfect Roman domination, perfect double Roman domination, independent Roman domination, independent Roman $\{2\}$-domination, independent double Roman domination, total Roman domination, total double Roman domination, weakly connected Roman domination, Roman $\{3\}$-domination, total Roman $\{2\}$-domination and total Roman $\{3\}$-domination.

In chapter 3, we have proved that the RDP, R2DP and DRDP are NPC for star convex bipartite graphs and comb convex bipartite graphs, and R2DP is NPC for bisplit graphs. We have shown that theses problems are linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs. In chapter 4, we have proved that the PRDP is NPC for star convex and comb convex bipartite graphs and PDRDP is NPC for chordal and bipartite graphs. We have proved that MPRDP, MPDRDP, MTRDP and MTDRDP are linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs. We have exhibited the case where the complexity of DOMINATION problem and PDRDP differ, that is, when the input graph is GI graph.

In chapter 5, algorithmic aspects of independent Roman domination, independent Roman $\{2\}$-domination and independent double Roman domination are studied. We have proved that IRDP, IR2DP and IDRDP are NPC for, dually chordal graphs, star convex bipartite graphs and comb convex bipartite graphs, and IR2DP and IDRDP are NPC for chordal graphs and are linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs, a subclass of split graphs. We have studied the complexity difference of IRDP (IR2DP, IDRDP) with DOMINATION problem.

In chapter 6, algorithmic aspects of WCRDP, R3DP, TR2DP and TR3DP are studied. We have proved that these problems are linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs. We have proved that R3DP and TR3DP are NP-complete for chordal graphs and R3DP is NP-complete for planar graphs. We have studied the complexity difference of WCRDP (R3DP, TR2DP, TR3DP) with DOMINATION problem.

It has been proved that decision versions of variant Roman domination problems are NP-complete in the previous chapters. In chapter 7, we have presented some approximation results of these domination parameters. We have proposed $2(1+\ln(\Delta+1))$-approximation algorithm for the MRDP and MR2DP, and $3(1+\ln(\Delta+1))$-approximation algorithm for the MDRDP, $2(\ln(\Delta-0.5)+1.5)$-approximation algorithm for MTRDP and MTR2DP, $3(\ln(\Delta-0.5)+1.5)$-approximation algorithm for the MTDRDP and MTR3DP, $3(1+\ln(\Delta-1))$-approximation algorithm for the MR3DP and $2(1+\epsilon)(1+\ln(\Delta-1))$-approximation algorithm for the MWCRDP, where Δ is the maximum degree of G. We also have proved that these problems cannot be approximated within $(1-\epsilon)\ln(|V|)$ for any $\epsilon > 0$ unless $P = NP$. We have shown that MRDP, MTRDP and MTDRDP are APX-complete for graphs with $\Delta = 5$, and MR3DP, MTR2DP, MTR3DP and MWCRDP are APX-complete for graphs with $\Delta = 4$. Finally, MIRDP, MIR2DP and MIDRDP have been proved as APX-hard for graphs with $\Delta = 4$.

8.2 Some Open Problems

Some of the open problems triggered from the study of Roman domination and its thirteen variants in this thesis are mentioned below.

- In chapter 3, RDP, R2DP and DRDP are proved as NPC for star convex bipartite graphs and comb convex bipartite graphs and polynomial time solvable in subclasses of chordal graphs namely threshold graphs. Investigating the algorithmic complexity of these problems for other subclasses of bipartite graphs like bipartite permutation graphs, chordal bipartite, bipartite chain graphs and subclasses of chordal graphs like block graphs, strongly chordal graphs and directed path graphs remains open.

- In chapter 4, it is proved that PRDP is NPC for star convex and comb convex bipartite graphs. Investigating the complexity status of it in bipartite subclasses is an open problem.

- PDRDP is proved as NPC for chordal and bipartite graphs. Investigating the algorithmic complexity of PDRDP in subclasses of bipartite graphs and chordal graphs is interesting.

- In chapter 5, it is proved that IRDP, IR2DP and IDRDP are NPC for dually chordal graphs, star convex bipartite graphs and comb convex bipartite graphs, and IR2DP and IDRDP are NPC for chordal graphs. Investigating the complexity of these problems in other graph classes is interesting.

- In chapter 6, complexity aspects of WCRDP, R3DP, TR2DP and TR3DP are studied in subclasses of bipartite and chordal graphs. Investigating the complexity of these problems in other graph classes is interesting.

- In chapter 7, lower and upper bounds of approximation ratio of MTRDP, MTDRDP, MR3DP, MTR2DP, MTR3DP and MWCRDP are obtained. Similarly, it is interesting to investigate lower and upper bounds for MIRDP, MIR2DP and MIDRDP, for which the decision version, IRDP, IR2DP and IDRDP, have been proved as NP-complete.

- It is proved that MRDP, MTRDP, MTDRDP are APX-complete for graphs with $\Delta = 5$. The complexity status of these problems are still open for graphs with $\Delta = 4$.
- It is proved that MR3DP, MTR2DP, MTR3DP and MWCRDP are APX-complete for graphs with $\Delta = 4$. The complexity status of these problems are still open for graphs with maximum degree other than 4.

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List of Publications

Journal Papers

1. **P. Chakradhar** and P. Venkata Subba Reddy, ”Algorithmic aspects of Roman domination in graphs”, *Journal of Applied Mathematics and Computing*, 2020 (1-14), Springer. (SCIE). https://doi.org/10.1007/s12190-020-01345-4

2. **P. Chakradhar** and P. Venkata Subba Reddy, ”Complexity of Roman $\{2\}$-domination and the double Roman domination in graphs”, *AKCE International Journal of Graphs and Combinatorics*, 2020 (1-6), Taylor & Francis, (SCIE). https://doi.org/10.1016/j.akcej.2020.01.005

3. **P. Chakradhar** and P. Venkata Subba Reddy, ”Complexity aspects of variants of independent Roman domination in graphs”, *Bulletin of the Iranian Mathematical Society*, 2020, Springer. (SCIE). https://doi.org/10.1007/s41980-020-00468-5

4. **P. Chakradhar** and P. Venkata Subba Reddy, ”Algorithmic aspects of total Roman $\{3\}$-domination in graphs”, *Discrete Mathematics, Algorithms and Applications*. (World Scientific, Scopus). https://doi.org/10.1142/S1793830921500634

5. **P. Chakradhar** and P. Venkata Subba Reddy, ”Complexity isssues of perfect Roman domination in graphs”, *Kyungpook Mathematical Journal*. (Scopus). (*Accepted*).

6. **P. Chakradhar** and P. Venkata Subba Reddy, ”Algorithmic aspects of Roman $\{3\}$-domination in graphs”, *RAIRO – Operations Research*. (SCIE). (*Under Review*).

7. **P. Chakradhar** and P. Venkata Subba Reddy, ”Algorithmic complexity of weakly connected Roman domination in graphs”, *Discrete Mathematics, Algorithms and Applications*. (World Scientific, Scopus). (*Under Review*).

8. **P. Chakradhar** and P. Venkata Subba Reddy, ”Algorithmic aspects of total Roman $\{2\}$-domination in graphs”, *Communications in Combinatorics and Optimization*. (Scopus). (*Under Review*).

Conference Papers

1. **P. Chakradhar** and P. Venkata Subba Reddy, ”Algorithmic aspects of total Roman domination and total double Roman domination in graphs”, 7^{th} *Annual International Conference on Algorithms and Discrete Applied Mathematics* (CALDAM 2021), February 11-13, 2021, IIT Ropar, India. (*Accepted*).

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