Quantitative Synthesis to Tracking Error Problem Based on Nominal Sensitivity Formulation

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***Abstract*—This brief presents a simple and robust design technique based on quantitative feedback theory (QFT) to solve the tracking error specification (TES) problem. The TES problem is converted into an equivalent disturbance attenuation (EDA) problem wherein the plant uncertainties are translated into equivalent external disturbance sets. The design problem then becomes a simple *nominal* disturbance rejection problem. The main advantage of the proposed technique is that they are simple, computationally efficient, and require *less* control effort (overdesign) due to the *nominal* sensitivity function formulation. The computational simplicity of the proposed methods makes them potentially more suitable for handling design problems involving a *large* number of uncertain plant parameters. Experiments are conducted on a laboratory electro-mechanical plant emulator system to demonstrate the efficacy of the proposed design method.**

***Index Terms*—Quantitative feedback theory (QFT), robust control, tracking error specification, uncertain system.**

I. Introduction

QUANTITATIVE feedback theory (QFT) is a frequency domain based robust loopshaping technique which is applicable to both single and multivariable control system design. The primary reason for employing the feedback is plant uncertainty and/or unknown (bounded) disturbances [1]. This method uses the two degrees of freedom (2-DOF) structure as shown in Fig. 1, in which $\mathcal{P} = \{P\}$ is the set of plant transfer functions, $G(s)$ is the controller to be designed to reduce system sensitivity wherein the laplace variable is denoted as 's', and the prefilter $F(s)$ is designed to shift the overall responses to the desired limits. QFT has been implemented to various range of practical problems, such as servo hydraulic system [2], Boost converter [3], suspension control [4], and electro-hydrostatic actuators [5]. The graphical design procedure involved in QFT makes the designer to understand the tradeoffs clearly between the tuning and the closed loop response, which is very important aspect from the industrial application point of view.

The reference tracking problem in QFT can be solved using two different approaches [6]. In the first approach, the desired tracking specifications are given in the form of the magnitude of closed-loop transfer function that must lie between the desired lower and upper tracking transfer functions. The second approach is based on the tracking error specification (TES), which is the difference between a given desired reference model $M(s)$ and the closed loop transfer function. That is, the magnitude response of the closed-loop tracking system $T_R(s)$ should not deviate from the reference model $M(s)$ by more than the tolerance B_e.

For designing the multivariable control system, the most preferred method is to design the single input single output (SISO) controller with decouplers, despite the advanced control methods, because of the simple structure and ease in implementation [7]. A brief overview of the existing methods for the SISO TES problem is discussed subsequently. Eitelberg [8] assumed the reference model to be unity (i.e., $M(s) = 1$) and found the controller bounds based on the sensitivity constraint. The prefilter is designed based on the assumption of a zero nominal tracking error and it may result in a prefilter of unnecessarily high order and sometimes leads to an impractical solution.

Boje [9] extended the brief in [8] to a non-unity reference model (i.e., $M(s) \neq 1$). The approach is based on the distance between the tracking error function of two plants which leads to the restriction on controller bounds. After designing the controller, the prefilter is designed (in the Nichols chart) using the specification inequality (1).

Elso *et al.* [10] proposed a design method to overcome the above mentioned drawbacks. The controller bounds are generated using a quadratic inequality (see [10, Eq. (41)]). The controller bounds are non-conservative and the nominal plant choice is arbitrary. The prefilter design is carried as reported in the Boje method [9]. However, the Elso/Gil/Mario method is computationally intensive and often prohibitive for a large number of uncertain plant parameters due to the 'pairing' requirement between the plants in the quadratic inequality. To simplify the prefilter design, the Boje method and Elso/Gil/Mario method tolerates the overdesign for certain cases as reported in [10]. Recently, QFT with inversion feedforward for the TES problem is formulated in [11], but it is restricted for the nominal

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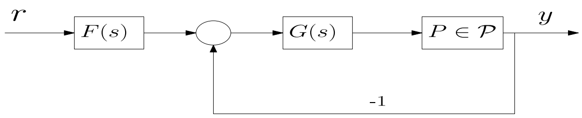

Fig. 1. The two degrees of freedom feedback structure for the linear plant P from a set $\mathcal{P}$.

plant case and not applicable to the non-minimum phase (NMP) system.

The limitations mentioned above motivated authors to propose computationally simple, less conservative design method for the TES problem and applicable to both minimum phase (MP)/NMP systems. To achieve quick servo response, the controllers are designed with large controller gains and this in turn produces more control effort. This makes the plant input to operate closely to the controller constraint boundary. Under this condition, even a small external perturbation may lead to the plant input saturation and can cause either the deteriorate closed-loop performance or instability. To overcome such problems when dealing with practical systems, it is vital to design control algorithms that are computational simple and produce minimal control effort while achieving the quick set-point tracking. The computational simple methods are especially useful for large number of uncertain parameters. The main contribution of this brief can be described as follows:

- The TES problem is transformed into a nominal disturbance rejection problem, based on the equivalent disturbance attenuation (EDA) approach [12] which has not been explored for the TES in the literature, except the brief in [11] but limited to minimum phase system.
- Qualitatively, a *single* design inequality needs to be solved at each frequency, irrespective of the original number of plant parameters. Moreover, the inequality is on the nominal output disturbance rejection spec, without the need for plant template generation, and hence is much easier to compute and solve.
- The prefilter design is also easily done in the Nichols chart (similar to [9], [10]) to satisfy the zero nominal tracking error.
- Experimental study on a laboratory electro-mechanical emulator system with 12 uncertain parameters is demonstrated. The proposed TES method is found to be computationally much simpler than the Method in [10], and requires *less* control effort as compared to the Boje method [9] while achieving the similar servo response.

This brief is structured as follows. Section II presents the problem statement. The proposed design procedure for TES problem is given in Section III and it is demonstrated on an industrial emulator system in Section IV. Finally, Section V concludes this brief.

II. Problem Statement

Consider the 2-DOF feedback control structure in Fig. 1 with an uncertain plant set $\mathcal{P}$. The aim is to find a controller $G(s)$ and a prefilter $F(s)$ so that the overall system satisfies the following specifications:

- Tracking error specification: The closed loop tracking transfer function denoted as T_R (from r to y) and the desired specification is written as

$$|M(j\omega) - T_R(j\omega)| = \left|M(j\omega) - \frac{P(j\omega)G(j\omega)F(j\omega)}{1 + P(j\omega)G(j\omega)}\right|$$
$$|M(j\omega) - T_R(j\omega)| \leq B_e(\omega) \quad (1)$$

where B_e is a given model matching tolerance. The TES design frequency set is denoted as $\Omega_{TES} = [0, \omega_{TES}]$. Assume that the nominal tracking error is zero, as in [8]; i.e.,

$$M(j\omega) = T_{R0}(j\omega), \quad \forall \omega \in \Omega_{TES} \quad (2)$$

The subscript $'0'$ denotes the nominal case.
- the system is closed loop stable for every plant $P \in \mathcal{P}$.

III. QFT Tracking Error Specification (TES) Problem

A. Feedback Control Design

Consider the 2-DOF feedback structure shown in Fig. 1 and Let P_0 be the nominal plant transfer function. The tracking transfer function T_R can be expressed as

$$T_R = \frac{PGF}{(1 + PG)}. \quad (3)$$

After rearranging the terms and multiply both the numerator and denominator of the above equation by P_0 to get

$$T_R = \frac{P_0 GF}{P_0\left(\frac{1}{P} + G\right)}.$$

After manipulation, the above expression can be written as

$$\left(\frac{P_0}{P} + L_0\right) T_R = L_0 F.$$

where $L_0 := P_0 G$ is the nominal loop transmission function. Further define the following

$$V := 1 - \frac{P_0}{P}, \quad (4)$$

and the above equation becomes

$$(1 + L_0)T_R = L_0 F + V T_R. \quad (5)$$

In terms of the nominal closed-loop transfer function $T_{R0} = \dfrac{L_0 F}{1 + L_0}$, equation (5) becomes as

$$(1 + L_0)(T_R - T_{R0}) = V T_R.$$

Let $\Delta T_R = T_R - T_{R0}$. Then, we have

$$\Delta T_R = \frac{V T_R}{(1 + L_0)}. \quad (6)$$

The plant uncertainty has been transformed into a disturbance term VT_R, and the uncertain plant is implicit in the equivalent disturbance term [12]. From equation (6), in terms of magnitude:

$$\left|\frac{1}{1 + L_0}\right| = \left|\frac{T_R - T_{R0}}{V T_R}\right|. \quad (7)$$

Using (2) for T_{R0} gives[1]:

$$\left|\frac{1}{1+L_0}\right| = \left|\frac{T_R - M}{VT_R}\right|.$$

Using specification (1), the above one becomes

$$\left|\frac{1}{1+L_0}\right| \le \frac{B_e}{|VT_R|}.$$

Since this must hold for all $P \in \mathcal{P}$, the following inequality must be satisfied as

$$\left|\frac{1}{1+L_0}\right| \le \frac{B_e}{|V|_{max}|T_R|_{max}}.$$

Using (1), and (4), the above inequality can be written as

$$\left|\frac{1}{1+L_0}\right| \le \frac{B_e}{\left|1-\frac{P_0}{P}\right|_{max}(|M|+B_e)}. \quad (8)$$

Note that this inequality represents the nominal output disturbance rejection. So, there is only one quadratic inequality to be solved at each $\omega \in \Omega_{TES}$, unlike in Boje's method where an inequality for each $P \in \mathcal{P}$ is solved at each $\omega \in \Omega_{TES}$. The design equation (8) is to be satisfied over the TES frequency range Ω_{TES}. The robust stability bounds are generated over Ω using the standard QFT quadratic inequality as in [13].

Remark 1: The proposed inequality (8) looks similar to the one given in one of the author's paper [11] which is solely based on the inversion feedforward-feedback ideas. The equivalence of solving the TES problem using both the frameworks (feedforward-feedback concept) are mentioned in the paper [14]. But, the use of inversion feedforward in [11] to obtain the inequality *restricts* its usage only for MP fixed system. The proposed inequality in this brief does not have the mentioned limitation and applicable to both MP and NMP uncertain systems.

B. Prefilter Design

A prefilter $F(s)$ needs to be designed to satisfy (2). This is done by adjusting the frequency response of the filter $F(s)$ such that $T_{R0}(j\omega) = M(j\omega)$ in the Nichols chart, at each $\omega \in \Omega_{TES}$.

IV. Industrial Plant (Flexible) Emulator System: Experimental Study

The proposed technique in Section III is illustrated with an experimental implementation on the much studied industrial plant (flexible) emulator system [15]. All the computations are performed in MATLAB R2015b with the QFT Toolbox [13] on a server machine (i7 CPU, 3.40 GHz with 8 cores and 16 GB RAM). An emulator system considered for the brief (see Fig. 2) represents an important class of industrial systems such as automated assembly machines, spindle drives, conveyors and machine tools [16]. The drive disk of the system is controlled by a brushless DC motor (BLDC). The motion of another disk connected to the load is controlled by a drive disk via a speed reduction arrangement and a flexible belt as shown in Fig. 2. The disk positions are measured using a high resolution encoders. The schematic diagram of the experimental setup is shown in Fig. 2, wherein the control algorithm is implemented using the MATLAB/Simulink interfaced with the ECP software with a digital signal processor (DSP-M56001)/data acquisition board [16]. The real-time model is build and executed using the Real-Time Windows Target (RTWT) which is converted to C++ code, further, downloaded into DSP board via RTWT.

[1]If the unknown term $|T_R - T_{R0}|$ in (7) is replaced by the overbounding step (usual in QFT), then it becomes $2(|M| + B_e)$, i.e., overbounded twice. It will result in huge over design and to avoid this, its reasonable to assume the zero nominal tracking error even though its not always possible to meet this assumption exactly at every design frequency.

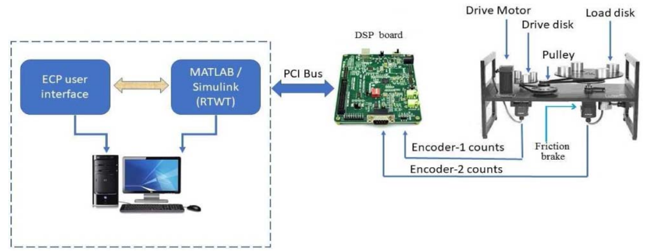

Fig. 2. Experimental setup schematic diagram.

The uncertain system (flexible) transfer function relates the drive disk input torque T_D to the position of the load disk (output) θ_2 as:

$$\frac{\theta_2(s)}{T_D(s)} = \frac{K_{hw}}{g_r}\frac{c_{12}s + k}{(\mathcal{B}s^4 + \mathcal{C}s^3 + \mathcal{D}s^2 + \mathcal{E}s)},$$

where, $\mathcal{B} = J_{dp}J_l$, $\mathcal{C} = c_2J_{dp} + c_1J_l + \left(J_{dp} + \frac{J_l}{g_r^2}\right)c_{12}$, $\mathcal{D} = k\left(J_{dp} + \frac{J_l}{g_r^2}\right) + c_1c_2 + \left(c_1 + \frac{c_2}{g_r^2}\right)c_{12}$, $\mathcal{E} = k\left(c_1 + \frac{c_2}{g_r^2}\right)$. Here, the hardware gain is $k_{hw} = 24.87k_ak_t$, drive and load disk weights inertias are $J_{wdo} = \frac{1}{2}m_{wd}r_{wdo}^2$ and $J_{wlo} = \frac{1}{2}m_{wl}r_{wlo}^2$, the drive inertia is $J_d = 0.004 + m_{wd}r_{wd}^2 + J_{wdo}$, gear ratio is $g_r = 0.25$, load inertia is $J_l = 0.0065 + m_{wl}r_{wl}^2 + J_{wlo}$, and the combined inertias due to pulleys and drive is $J_{dp} = J_d + 0.000156$. The nominal values of $c_1, c_2, c_{12}, r_{wd}, r_{wl}, m_{wd}, m_{wl}, k_a, k_t, k, r_{wdo}, r_{wlo} = [0.004, 0.05, 0.017, 0.05, 0.1, 0.8, 2, 2, 0.1, 8.45, 0.02, 0.02]$ and 10% uncertainty is considered for each parameter.

The following design specifications are considered:

- Tracking error specification: Rise time = 4.25 sec, no offset and no overshoot. The specifications are represented by the desired response model:

$$M(s) = \frac{\left(\frac{s}{12.5} + 1\right)}{\left(\frac{s}{0.55} + 1\right)}, \quad (9)$$

with the error tolerance[2] as

$$B_e(\omega) = \left|\frac{0.095s(s+3)}{3.75}\right|_{s=j\omega}, \quad \forall \omega \le \omega_{TES} = 1.2. \quad (10)$$

- Robust stability margin: Phase margin $\ge 45^o$, Gain margin ≥ 5 dB.

[2]Due to page limitation, the detailed discussion about the selection of $B_e(\omega)$ and $M(s)$ are discussed in the paper [11].

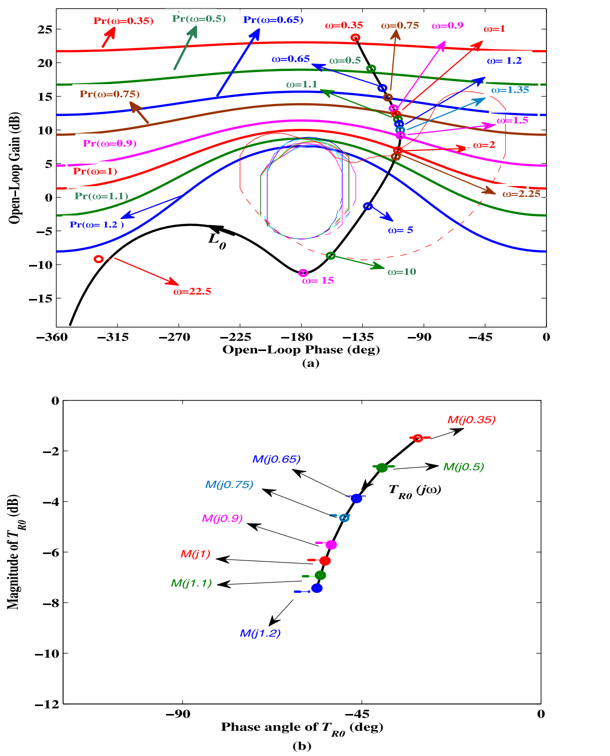

Fig. 3. Design plots using the proposed method (a) the nominal loop shaping (L_0) and (b) the prefilter shaping.

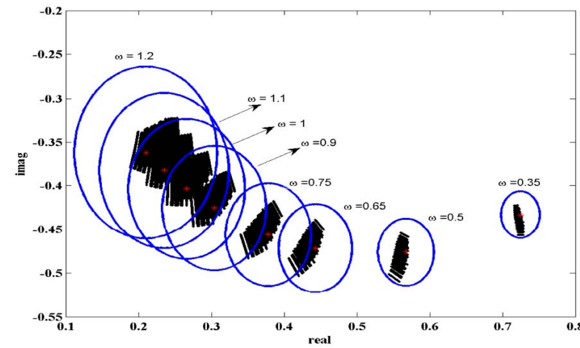

Fig. 4. Frequency domain validation of proposed design with the tracking error specifications at each TES frequency. Here, red dot indicates the reference model $M(j\omega)$ (equation (9)).

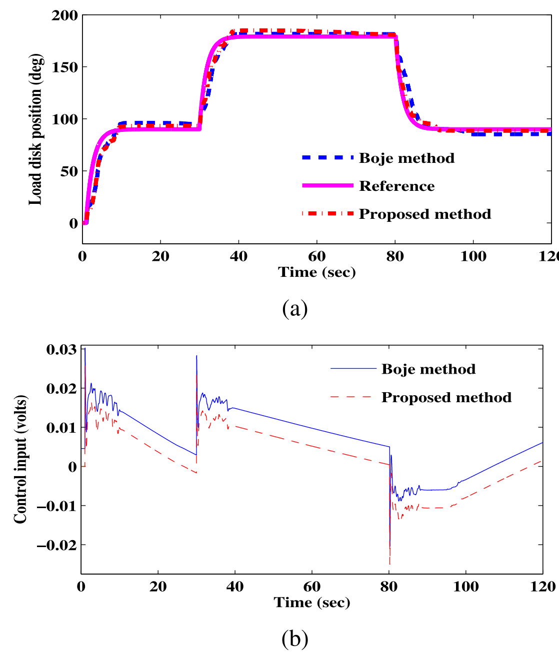

Fig. 5. Tracking response (a) Full weight: weights on the load disk (2x200gm, 2x500gm), and (b) the controller magnitude comparison for full weight case.

The Elso/Gil/Mario method [10] could not generate the bounds even after 24 hours due to the requirement of 'pairing' between all the plants in the plant set, and was therefore aborted.[3] The proposed method (equation 8) computes the TES bounds in 160 seconds which is two and half times faster than the Boje method (375 sec) for the 531,441 plant elements (3 grid points for each parameter), and atleast a couple of orders faster than the Elso/Gil/Mario method.

Figure 3(a) shows the tracking bounds obtained using the proposed method along with the loop shaping. The prefilter shaping is shown in Fig. 3(b) and the designed controller and the prefilter are as follows

$$G(s) = \frac{0.007\left(\frac{s}{0.6}+1\right)\left(\frac{s}{0.7}+1\right)}{s\left(\frac{s}{7}+1\right)\left(\frac{s}{50}+1\right)};\;\; F(s) = \frac{\left(\frac{s}{2.84}+1\right)}{\left(\frac{s}{0.48}+1\right)}$$

For comparison purpose, the design using the Boje method [9] is carried out and the designed controller G_b and the prefilter F_b are also given (plots are omitted for simplicity) below

$$G_b(s) = \frac{0.011\left(\frac{s}{0.6}+1\right)\left(\frac{s}{0.7}+1\right)}{s\left(\frac{s}{7}+1\right)\left(\frac{s}{50}+1\right)};\;\; F_b(s) = \frac{\left(\frac{s}{2.1}+1\right)}{\left(\frac{s}{0.45}+1\right)}$$

Figure 4 illustrates that the proposed design satisfies the tracking error specification over Ω_{TES}. The performance of the proposed control system is now experimentally tested and compared with that of the Boje method, for different experimental cases. The friction clamp placed below in the load disk assembly (see Fig. 2) is tightened in this configuration.

Full Weight on Load Disk: The proposed design is tested for its capability to handle the uncertainty. Known parametric uncertainties are introduced by adding weights (2x200 gm and 2x500 gm) on the load disk. This weights changes the load disk inertia and subsequently changes the dynamics of the plant. As shown in Fig. 5(a), the closed-loop response (dashed-dotted line) tracks the setpoint despite the uncertainty. The closed-loop response using the Boje method also tracks the setpoint as shown in the same figure (dashed line). Figure 5(b) compares the control effort (volts) and it shows that the proposed design demands *less* control effort than the Boje method.

Half Weight on Load Disk/Drive Disk: Next, the proposed control system is further tested with weights on both the load and the drive disks. Here, the weights of (2x500 gm) on the

[3]The total number of pairs to be considered in the design quadratic inequality in the Elso/Gil/Mario method is $3^{12} \times 3^{12} = 282,429,536,481$.

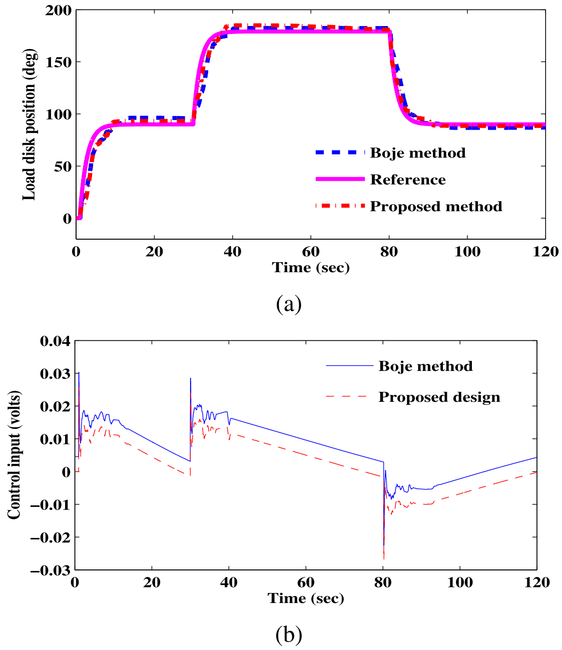

Fig. 6. Tracking response for Half weight case: weights on the load (2x500gm) and drive disk (2x200gm), and, (b) the controller magnitude comparison plot for half weight case.

TABLE I
COMPARISON OF CONTROL EFFORT NORMS BETWEEN THE BOJE [9] AND THE PROPOSED DESIGNS

Norms	Weight configuration	Friction	
		Boje method	Proposed TES method
1-norm	Load disk (Full)	263	199
	Load + Drive disk(Half)	258	200
2-norm	Load disk(Full)	1.74	1.36
	Load + Drive disk(Half)	1.77	1.4
∞ -norm	Load disk(Full)	0.0303	0.0257
	Load + Drive disk(Half)	0.03	0.027

load disk at 10 cm from the disk center and (2x200 gm) on the drive disk are added. As shown in Fig. 6 (a), the closed-loop response (dashed-dotted line) follows the setpoint despite this large parametric uncertainty. The Boje design also satisfies the tracking specification, as shown in the same figure (denoted as a dashed line). However, in this configuration also, the less control effort is required to meet the specification over its counterpart, as shown in Fig. 6(b). Table I compares the control effort (in terms of norms) with the proposed method and the Boje method. For both the weight configurations, it is experimentally concluded that the required control effort in the proposed method is less than that with the Boje method.

V. CONCLUSION

This brief presented a novel QFT design methodology for the TES problem. The key advantage of the proposed method over the existing method is computational simplicity and less overdesign. The proposed design requires less control effort as compared to its counterpart to achieve the desired specification. The computational advantages have been illustrated using a practical implementation on a laboratory electro-mechanical system. It was also shown that the computational advantage of the proposed method becomes crucial for the more number of uncertain plant parameter. The possible extension could be on reducing the overdesign problem by an iterative design. Further work is underway to extend the proposed methods for the multivariable case as well as experimental implementation on DC-DC NMP boost converter system. And also combining the automatic loopshaping for proposed TES method with the actuator saturation problem is worth for pursuing.

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