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Nonlinear rotating convection in a sparsely packed porous medium

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ABSTRACT

We investigate linear and weakly nonlinear properties of rotating convection in a sparsely packed Porous medium. We obtain the values of Takens–Bogdanov bifurcation points and co-dimension two bifurcation points by plotting graphs of neutral curves corresponding to stationary and oscillatory convection for different values of physical parameters relevant to rotating convection in a sparsely packed porous medium near a supercritical pitchfork bifurcation. We derive a nonlinear two-dimensional Landau–Ginzburg equation with real coefficients by using Newell–Whitehead method [16]. We investigate the effect of parameter values on the stability mode and show the occurrence of secondary instabilities viz., Eckhaus and Zigzag Instabilities. We study Nusselt number contribution at the onset of stationary convection. We derive two nonlinear one-dimensional coupled Landau–Ginzburg type equations with complex coefficients near the onset of oscillatory convection at a supercritical Hopf bifurcation and discuss the stability regions of standing and travelling waves.

1. Introduction

Nonlinear rotating convection in a porous medium uniformly heated from below is of considerable interest in geophysical fluid dynamics, as this phenomena may occur within the Earth's outer core. Earth's outer core consists of molten Iron and lighter alloying element, sulphur in its molten form. This lighter alloying element present in the liquid phase is released as the new iron freezes due to supercooling onto the solid Inner core. Hence we get a mushy layer near the inner core boundary, where the problem becomes convective instability in a porous medium [19]. The mushy layer is a region of coexisting liquid and solid phases, forming as a consequence of constitutional supercooling, when a binary alloy solidifies directionally [25]. The effect of rotating field on the convective instability is of interest in geophysics, particular in the study of Earth's interior where the molten liquid Iron is electrically conducting, which can become convectively unstable as a result of differential diffusion.

Rotating convection in an electrically conducting fluid in a nonporous medium has been studied extensively [6,3–5,26,22–24,9,8]. But its counterpart in a porous medium has not been given much attention inspite of its geophysical applications. The multiplicity of control parameters makes this system an interesting one for the study of hydrodynamic stability, bifurcation and turbulence [11]. Palm et al. [18] investigated Rayleigh–Benard convection problem in a porous medium. Brand and Steinberg [1] investigated convecting instabilities in binary liquid in a porous medium. However, Palm et al. [18], Brand and Steinberg [2] and Steinberg and Brand [21] have made use of Darcy's law ($-\nu\nabla^2\overline{V}$ is replaced by $K\overline{V}$ where K is the permeability of a porous medium. For nonporous medium K is infinity). They have also not considered usual convective nonlinearity. It is well known that Darcy's law breaks down in situations where in other effects like viscous shear and inertia

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come into play. In fact Darcy's law is applicable to densely packed porous medium. An alternative to Darcy's equation is Brinkman equation and is of the form

$$\nabla'\rho' - \rho'\overline{g} = -\frac{\mu}{K}\overline{V}' + \mu_e\nabla'^2\overline{V}',$$

where μ is the fluid viscosity and μ_e is the effective fluid viscosity, Brinkman model is valid for a sparsely packed porous medium wherein there is more window fluid to flow so that the distortion of velocity give rise to the usual shear force. Lapwood [14] was the first to suggest the inclusion of convective term $(\overline{V}'.\nabla')\overline{V}'$ in the momentum equation and study the Rayleigh–Benard convection in a sparsely packed porous medium. Recently, Tagare and Benerji Babu [22] have investigated the problem of nonlinear convection in a sparsely packed porous medium due to thermal and compositional buoyancy.

In this paper, we investigate the problem of rotating convection in a sparsely packed porous medium. Rudraiah and Srimani [20] have studied linear stability analysis in the case of thermal convection in a rotating fluid saturated porous medium using Brinkman model but they have taken effective viscosity μ_e same as fluid viscosity μ, However, experiments show that the ratio of effective viscosity μ_e takes the value ranging from 0.5 to 10.9 [7]. In Section 2, we write basic dimensionless equations in Boussinesq approximation for rotating convection a in a sparsely packed medium by using for a momentum equation Darcy–Lapwood-Brinkman model with effective viscosity different from fluid viscosity. In Section 3, we study linear stability analysis. In Section 4, by using multiple-scale analysis of Newell and Whitehead [16], we derive two dimensional nonlinear Landau–Ginzburg equation in complex amplitude $A(X,Y,T)$ with real coefficients near the supercritical pitchfork bifurcation, where X,Y and T are slow space and time variables. In Section 4.1, we show the occurrence of secondary instabilities such as Eckhaus and Zigzag Instabilities and also we study the Nusselt number contribution at the onset of stationary convection from Landau–Ginzburg equation. In Section 5, we derive two nonlinear one-dimensional time-dependent coupled Landau–Ginzburg type equations with complex coefficients near the onset of oscillatory convection at supercritical Hopf bifurcation, here $A_{1R}(X,Y,T)$ and $A_{1L}(X,Y,T)$ stands for amplitudes of right-hand and left-hand travelling waves. In Section 5.1, following Matthews and Rucklidge [15], we neglect the slow space dependence and obtain two ordinary differential equations in $A_{1R}(T)$ and $A_{1L}(T)$ with complex coefficients and discuss the stability regions of travelling and standing waves. In Section 5.2, we also discuss Benjamin–Feir instability and occurrence of tri-critical points. In Section 6, we write conclusions of the paper.

2. Basic equations

Consider a horizontal, infinitely extended layer of fluid in a porous medium of depth d which is kept rotating at a constant angular velocity Ω about z-axis, this layer is heated from below. The upper and lower bounding surfaces of the layer are assumed to be stress-free. Physical properties of the fluid are assumed constant, except density in the buoyancy term, so that the Boussinesq approximation is valid. The porous medium is considered homogeneous and isotropic. The onset of convection is such a layer is governed by the following equations [17]

$$\nabla' \cdot \overline{V'} = 0, \tag{2.1}$$

$$\rho'_0\left[\frac{1}{\phi}\frac{\partial\overline{V'}}{\partial t'} + \frac{1}{\phi^2}(\overline{V'}\cdot\nabla')\overline{V'} + \overline{\Omega}\times(\overline{\Omega}\times\overline{r'}) + \frac{2}{\phi}\left(\overline{\Omega}\times\overline{V'}\right)\right] = -\nabla'P' + \rho'\overline{g} - \frac{\mu}{K}\overline{V'} - \mu_e\nabla'^2\overline{V'}, \tag{2.2}$$

$$M\frac{\partial T'}{\partial t'} + (\overline{V}'.\nabla')T' = \kappa_T\nabla'^2 T' \tag{2.3}$$

and

$$\rho' = \rho'_0[1 - \alpha(T' - T'_b)]. \tag{2.4}$$

Here $\overline{\Omega} = \Omega\hat{e}_z$ is angular velocity about z-axis, $\overline{r'}$ is a position vector of a fluid particle, $\alpha = -\rho_0'^{-1}(\partial\rho'/\partial T')$ is thermal expansion coefficient, P' is pressure, $\overline{V'}$ is mean flow velocity, $\overline{g}$ is an acceleration due to gravity, K is permeability of porous medium, μ is fluid velocity and μ_e is coefficient of effective fluid viscosity. Eq. (2.2) is known as Darcy–Lapwood–Brinkman equation and is valid for $0.8 < \phi < 1$. Givler and Altobelli [7] shown that the range of $\Lambda = (\mu_e/\mu)$ varies from 0.5 to 10.9. M is dimensionless heat capacity and is defined as the ratio of the effective heat capacity of the porous medium to the heat capacity $(\rho' C_p)_f$ of the fluid. In a nonporous medium, $\Lambda = M = \phi = 1$ and $K \to \infty$ and Eq. (2.2) reduces to Navier–Stokes equation. In this paper, for sparsely packed porous medium, we consider $M = 0.9$ and $\phi = 0.9$. The conduction state is characterized by

$$\overline{V}'_s = 0, T'_s = T'_b - (\Delta T'/d)z' \tag{2.5}$$

and we take the temperature perturbation as $\theta' = T' - T'_s$. We use the scaling

$$x = x'/d, \quad y = y'/d, \quad z = z'/d, \quad t = \frac{t'}{Md^2/\kappa_T}, \quad u = \frac{u'}{\kappa_T/Md}, \quad v = \frac{v'}{\kappa_T/Md}, \quad w = \frac{w'}{\kappa_T/Md}, \theta = \frac{\theta'}{\Delta T'},$$

$$P = \frac{P'}{\rho'_0 M^{-2}\kappa_T^2 d^{-2}}. \tag{2.6}$$

Here Md^2/κ_T is thermal diffusion time in a porous medium. Using Eqs. (2.4) and (2.6), we can write basic dimensionless equations of rotating fluid in a porous medium as

$$\nabla \cdot \overline{V} = 0, \tag{2.7}$$

$$\frac{1}{M^2\phi Pr}\left[\frac{\partial \overline{V}}{\partial t} + \frac{1}{\phi}(\overline{V} \cdot \nabla)\overline{V}\right] = -\nabla\left(\frac{P}{MPr} - \frac{TaPr}{8M\phi^2}|\hat{e}_z \times \overline{r}|^2\right) - \frac{1}{MD_a}\overline{V} + \frac{Ta^{\frac{1}{2}}}{\phi}(\overline{V} \times \hat{e}_z) + \frac{\Lambda}{M}\nabla^2\overline{V} + R\theta\hat{e}_z, \tag{2.8}$$

$$\frac{\partial \theta}{\partial t} + \frac{1}{M}(\overline{V} \cdot \nabla)\theta = \frac{w}{M} + \nabla^2\theta. \tag{2.9}$$

The dimensionless numbers required for the description of the motion are Rayleigh number: $R = g\alpha\Delta Td^3/\kappa\nu$, Prandtl number $Pr = \nu/\kappa_T$, Darcy number $D_a = K/d^2$ and Taylor number $Ta = 4\Omega^2 d^4/\nu^2$. The Curl of Eq. (2.8) gives

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t} + \frac{1}{MD_a} - \frac{\Lambda}{M}\nabla^2\right)\omega - R\nabla \times (\theta\hat{e}_z) - \frac{Ta^{\frac{1}{2}}}{\phi}\nabla \times (\overline{V} \times \hat{e}_z) = -\frac{1}{M^2\phi^2 Pr}[\nabla \times (\overline{V} \cdot \nabla)\overline{V}], \tag{2.10}$$

where vorticity $\omega = \nabla \times \overline{V}$ and $\nabla \times [(\overline{V}.\nabla)\overline{V}] = [(\overline{V} \cdot \nabla)\omega - (\omega \cdot \nabla)\overline{V}]$. The Curl of Eq. (2.10) in turn gives, after use of Eq. (2.10).

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t} + \frac{1}{MD_a} - \frac{\Lambda}{M}\nabla^2\right)\nabla^2\overline{V} - R\left[\nabla^2(\theta\hat{e}_z) - \nabla\frac{\partial \theta}{\partial z}\right] + \frac{Ta^{\frac{1}{2}}}{\phi}\frac{\partial \omega}{\partial z} = \frac{1}{M^2\phi^2 Pr}[\nabla \times \nabla \times (\overline{V}.\nabla)\overline{V}]. \tag{2.11}$$

The z-component of Eqs. (2.10) and (2.11) are

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t} + \frac{1}{MD_a} - \frac{\Lambda}{M}\nabla^2\right)\omega_z - \frac{Ta^{\frac{1}{2}}}{\phi}\frac{\partial w}{\partial z} = -\frac{1}{M^2\phi^2 Pr}\hat{e}_z \cdot [\nabla \times (\overline{V} \cdot \nabla)\overline{V}], \tag{2.12}$$

$$\left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t} + \frac{1}{MD_a} - \frac{\Lambda}{M}\nabla^2\right)\nabla^2 w - R\nabla_h^2\theta + \frac{Ta^{\frac{1}{2}}}{\phi}\frac{\partial \omega_z}{\partial z} = \frac{1}{M^2\phi^2 Pr}\hat{e}_z \cdot [\nabla \times \nabla \times (\overline{V} \cdot \nabla)\overline{V}], \tag{2.13}$$

where ω_z and w are the z-components of vorticity and velocity respectively and $\nabla^2 = (\partial^2/\partial x^2 + \partial^2/\partial z^2)$ is a horizontal Laplacian operator. Eliminating θ and ω_z from the linear part of Eqs. (2.9), (2.12) and (2.13), we get

$$\mathcal{L}w = \mathcal{N}, \tag{2.14}$$

where

$$\mathcal{L} = \mathcal{D}\mathcal{D}_{Pr}^2\nabla^2 + \mathcal{D}\frac{Ta}{\phi^2}\frac{\partial^2}{\partial z^2} - \mathcal{D}_{Pr}\frac{R}{M}\nabla_h^2, \tag{2.15}$$

$$\mathcal{N} = -\mathcal{D}_{Pr}\frac{R}{M}\nabla_h^2(\overline{V}.\nabla)\theta + \mathcal{D}\frac{Ta^{\frac{1}{2}}}{M^2\phi^3 Pr}\frac{\partial}{\partial z}\hat{e}_z \cdot [\nabla \times (\overline{V} \cdot \nabla)\overline{V}] + \mathcal{D}\mathcal{D}_{Pr}\frac{1}{M^2\phi^2 Pr}\hat{e}_z \cdot [\nabla \times \nabla \times (\overline{V} \cdot \nabla)\overline{V}], \tag{2.16}$$

here $\mathcal{D} = \left(\frac{\partial}{\partial t} - \nabla^2\right)$ and $\mathcal{D}_{Pr} = \left(\frac{1}{M^2\phi Pr}\frac{\partial}{\partial t} + \frac{1}{MD_a} - \frac{\Lambda}{M}\nabla^2\right)$ and $\nabla_h^2 = \frac{\partial^2}{\partial x^2}$.

2.1. Boundary conditions

We assume that fluid is confined between $z = 0$ and $z = 1$ corresponds to a mantle boundary. For perfectly conducting boundary with temperature, we have

$$\theta = 0 \quad \text{on} \quad z = 0, \quad z = 1 \quad \text{for all} \quad x, y.$$

Also the normal component of the velocity would vanish on $z = 0,\ z = 1$,

$$\text{i.e.,} \quad w = 0 \quad \text{on} \quad z = 0,\ z = 1 \quad \text{for all} \quad x, y.$$

However, there are two more conditions to be imposed on velocity depending on the nature of the surface. In this paper we consider free–free boundary conditions, i.e., on surfaces the tangential stresses vanish, which is equivalent to

$$P_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0, \quad P_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0,$$

where $\mu = \gamma\rho_0$ is dynamic viscosity. Since w vanishes for x, y on $z = 0,\ z = 1$, it follows that $\partial u/\partial z = \partial v/\partial z = 0$ on a free surface $z = 0,\ z = 1$. Hence from equation of continuity we have $\partial^2 w/\partial z^2 = 0$ on $z = 0,\ z = 1$ for all x, y. In this paper we have considered only the idealized stress-free conditions on the surface and vanishing of temperature fluctuations. Thus $w = D^2 w = D^4 w = 0$ at $z = 0, 1$. w and its even derivatives vanish at $z = 0$ and $z = 1$.

3. Linear stability analysis

We perform a linear stability analysis of the problem by substituting

$$w = W(z)e^{iqx+pt}, \tag{3.1}$$

into linearized version of Eq. (2.14) viz., $\mathcal{L}w = 0$, and obtain an equation

$$\left[(D^2-q^2)(p-D^2+q^2)\left[\frac{p}{M^2\phi Pr}+\frac{1}{MD_a}-\frac{\Lambda}{M}(D^2-q^2)\right]^2+\frac{Ta}{\phi^2}D^2(p-D^2+q^2)\right.$$
$$\left.+\frac{Rq^2}{M}\left[\frac{p}{M^2\phi Pr}+\frac{1}{MD_a}-\frac{\Lambda}{M}(D^2-q^2)\right]\right]W(z)=0, \tag{3.2}$$

where $D = (d/dz)$. We consider stress-free boundary conditions, then $W = D^2W = 0$ on $z = 0,\ z = 1$ for all x, y.

3.1. Determination of marginal stability when Rayleigh number R is a dependent variable

Substituting $W(z) = \sin \pi z$ and $p = i\omega$ into (3.2), we get

$$R = \frac{M}{q^2}\left[A_1 + i\omega(A_2\omega^2 + A_3)\right], \tag{3.3}$$

where

$$A_1 = \sigma_2\left[\sigma_1\left\{\sigma_1^2\delta_{sc}^4 - \frac{\omega^2\delta_{sc}^4}{M^4\phi^2Pr^2} + \frac{Ta}{\phi^2}\pi^2\delta_{sc}^2 - \frac{2\delta_{sc}^2\omega^2}{M^2\phi Pr}\sigma_1\right\} + \frac{\omega^2}{M^2\phi Pr}\left\{\sigma_1^2\delta_{sc}^2 - \frac{\omega^2\delta_{sc}^2}{M^4\phi^2Pr^2} + \frac{Ta}{\phi^2}\pi^2 + \frac{2\delta_{sc}^4}{M^2\phi Pr}\sigma_1\right\}\right], \tag{3.4}$$

$$A_2 = \sigma_2\left[\frac{\delta_{sc}^4}{M^6\phi^3Pr^3} + \frac{\sigma_1\delta_{sc}^2}{M^4\phi^2Pr}\right], \tag{3.5}$$

$$A_3 = \sigma_2\left[\sigma_1\left\{\sigma_1^2\delta_{sc}^2 + \frac{Ta}{\phi^2}\pi^2 + \frac{2\delta_{sc}^4}{M^2\phi Pr}\sigma_1\right\} - \frac{1}{M^2\phi Pr}\left\{\sigma_1^2\delta_{sc}^4 + \frac{Ta}{\phi^2}\pi^2\delta_{sc}^2\right\}\right], \tag{3.6}$$

here $\delta_{sc}^2 = (\pi^2 + q_{sc}^2), \sigma_1 = \left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_{sc}^2\right)$ and $\sigma_2 = \left(\sigma_1^2 + \frac{\omega^2}{M^4\phi^2Pr^2}\right)^{-1}$, from Eq. (3.5), $A_2 > 0$.

3.1.1. Stationary convection ($\omega = 0$)

Substituting $\omega = 0$ in Eq. (3.3), we get

$$R_s = \frac{M}{q_s^2}\left[\frac{\delta_s^4\left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_s^2\right)^2 + \frac{Ta}{\phi^2}\pi^2\delta_s^2}{\left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_s^2\right)}\right], \tag{3.7}$$

where $\delta_s^2 = (\pi^2 + q_s^2)$. Here R_s is the value of the Rayleigh number for stationary convection. The minimum value of R_s is obtained for $q_s = q_{sc}$, where

$$2\left(\frac{q_{sc}}{\pi}\right)^6 + 3\left(\frac{q_{sc}}{\pi}\right)^4 = 1 + \frac{Ta}{\pi^4}\left(\frac{M}{\Lambda\phi}\right)^2. \tag{3.8}$$

Threshold for the onset of stationary convection is given by Eq. (3.7) with $q_s = q_{sc}$,

$$R_{sc} = \frac{M}{q_{sc}^2}\left[\frac{\delta_{sc}^4\sigma_1^2 + \frac{Ta}{\phi^2}\pi^2\delta_{sc}^2}{\sigma_1}\right], \tag{3.9}$$

where $\delta_{sc}^2 = (\pi^2 + q_{sc}^2)$. For $Ta/\pi^4 \gg 1$ (for large Taylor number), The required root of Eq. (3.8) becomes

$$\left(\frac{q_{sc}}{\pi}\right) \simeq \left(\frac{TaM^2}{2\pi^4\Lambda^2\phi^2}\right)^{\frac{1}{6}}.$$

The corresponding asymptotic values of q_{sc} and R_{sc} are

$$q_{sc} \simeq \left(\frac{\pi^2TaM^2}{2\Lambda^2\phi^2}\right)^{\frac{1}{6}}, \tag{3.10a}$$

$$R_{sc} \simeq 3\Lambda\pi^4\left(\frac{M^2Ta}{2\Lambda^2\phi^2\pi^4}\right)^{\frac{2}{3}}. \tag{3.10b}$$

In the free–free boundary conditions, for large Taylor number, we have

$$R_{sc} \propto Ta^{\frac{2}{3}} \quad \text{and} \quad q_{sc} \propto Ta^{\frac{1}{6}}. \tag{3.11}$$

This is also true for rigid-rigid and rigid-free boundary conditions.

3.1.2. Oscillatory convection ($\omega^2 > 0$)

For the oscillatory convection ($\omega \neq 0$) and from Eq. (3.3), R will be complex. But the physical meaning of R requires it to be real. The condition that R is real implies that imaginary part of Eq. (3.3) is zero, i.e.,

$$A_2\omega^2 + A_3 = 0, \tag{3.12}$$

where A_2 and A_3 are given by Eqs. (3.5) and (3.6). For oscillatory convection $\omega^2 = (-A_3/A_2) > 0$ i.e.,

$$\omega^2 = \frac{M^2Pr^2}{\delta_o^2(1 + MPr\Lambda\phi)}\left[Ta\pi^2M^2(1 - MPr\Lambda\phi) - \delta_o^6\Lambda^2\phi^2(1 + MPr\Lambda\phi)\right], \tag{3.13}$$

where $\delta_o^2 = \pi^2 + q_o^2$. Substituting ω^2 from Eq. (3.13) into the real part of Eq. (3.3), we get

$$R_o = \frac{2\Lambda(1 + MPr\Lambda\phi)}{Mq_o^2}\left[\delta_o^6 + \frac{M^4\pi^2Pr^2Ta}{(1 + MPr\Lambda\phi)^2}\right]. \tag{3.14}$$

A necessary condition for $\omega^2 > 0$ is $Pr < 1$. However, this is not sufficient condition and one must have in addition

$$Ta > \frac{(1 + MPr\Lambda\phi)\Lambda^2\phi^2\delta_o^6}{M^2\pi^2(1 - MPr\Lambda\phi)}.$$

$$Ta = Ta_c = \frac{(1 + MPr\Lambda\phi)\Lambda^2\phi^2\delta_o^6}{M^2\pi^2(1 - MPr\Lambda\phi)}, \quad q = q_c \tag{3.15}$$

is a solution of $A_3(Ta_c) = 0$ and corresponds to a Takens–Bogdanov bifurcation point. At Takens–Bogdanov bifurcation point $q_o = q_s = q_c$ and $A_3(q_c) = 0$. We note that from Eq. (3.13), if $\omega^2 > 0$ then $R_o(q_o)$ will be less than $R_s(q_o)$ and not $R_s(q_s)$ given by Eq. (3.7), which corresponds to stationary convection. However, at Takens–Bogdanov bifurcation point

$$R_o(q_o) = R_s(q_s) = R_c(q_c), \quad q_o = q_s = q_c$$

and $\omega^2 = 0$ is a double zero at $Ta = Ta_c(q_c)$. The Takens–Bogdanov bifurcation point occurs where neutral curves for Hopf and pitchfork bifurcation meet and only a single wave number is present viz., $q_o = q_s = q_c$. If $q_c > q_{sc}$ then for all $q < q_c$ the first instability to set in is an oscillatory instability.

The asymptotic value q_c is obtained from Eq. (3.15) and is given by

$$q_c \to \left[\frac{\pi^2M^2Ta(1 - MPr\Lambda\phi)}{(1 + MPr\Lambda\phi)\Lambda^2\phi^2}\right]^{\frac{1}{6}}.$$

Thus for large Taylor number ($Ta \to \infty$), we have

$$q_c \to \left[\frac{\pi^2M^2Ta(1 - MPr\Lambda\phi)}{(1 + MPr\Lambda\phi)\Lambda^2\phi^2}\right]^{\frac{1}{6}} \quad \text{and} \quad q_{sc} \to \left(\frac{\pi^2TaM^2}{2\Lambda^2\phi^2}\right)^{\frac{1}{6}} \tag{3.16}$$

from Eq. (3.16), $q_c \propto Ta^{\frac{1}{6}}$ and $q_{sc} \propto Ta^{\frac{1}{6}}$ for $Ta \to \infty$. The critical wave number corresponding to the onset of oscillatory convection for given parameters Pr and Ta is obtained for $q = q_{oc}$ from the following equation

$$2\left(\frac{q_{oc}}{\pi}\right)^6 + 3\left(\frac{q_{oc}}{\pi}\right)^4 = 1 + \frac{TaPr^2M^4}{\pi^4(1 + MPr\Lambda\phi)^2}. \tag{3.17}$$

For large Taylor number, the required root of Eq. (3.17) becomes

$$\left(\frac{q_{oc}}{\pi}\right) \simeq \left(\frac{TaPr^2M^4}{2\pi^4(1 + MPr\Lambda\phi)^2}\right)^{\frac{1}{6}}.$$

The corresponding asymptotic behavior of q_{oc} and R_{oc} for large Taylor number are [6]

$$q_{oc} \simeq \left(\frac{TaPr^2M^4\pi^2}{2(1 + MPr\Lambda\phi)^2}\right)^{\frac{1}{6}}. \tag{3.18a}$$

$$R_{oc} \simeq \frac{2\pi^4 \Lambda}{M}(1 + MPr\Lambda\phi)\left[2\left(\frac{TaPr^2M^4}{2\pi^4(1 + MPr\Lambda\phi)^2}\right)\right]^{\frac{2}{3}}. \tag{3.18b}$$

From Eqs. (3.10b) and (3.18b), $R_{oc} \to R_{sc}$ as $Ta \to \infty$ implies that for large Taylor number

$$\frac{2M^{\frac{8}{3}}Pr^{\frac{4}{3}}}{(1 + MPr\Lambda\phi)^{\frac{1}{3}}} = 1. \tag{3.19}$$

Root of Eq. (3.19) is (Chandhrasekhar [6]) $Pr = Pr_c = 0.783813$. Thus $R_{oc}(q_{oc}) \to R_{sc}(q_{sc})$ at $Pr = Pr_c$. From the monotonic dependence of q_{oc} and q_{sc} on Ta, we may conclude that for $Pr > Pr_c, R_{oc} > R_{sc}$ for all Ta. Hence for $1 > Pr > Pr_c$, instability will always manifest itself, first as stationary convection. For $Pr < Pr_c$, there exist a $Ta(Pr)$ such that for $Ta \leqslant Ta(Pr)$ the onset instability will be stationary convection at pitchfork bifurcation while for $Ta > Ta(Pr)$ it will be oscillatory convection at Hopf bifurcation. $Ta(Pr)$ is a function of Prandtl number Pr and for $Ta = Ta(Pr)$,

$$R_{oc}(q_{oc}) = R_{sc}(q_{sc}) \quad \text{but} \quad q_{oc} \neq q_{sc}. \tag{3.20}$$

This condition (3.20) gives codimension two bifurcation point where critical Rayleigh numbers of stationary convection and oscillatory convection coincide at distinct critical wave numbers. Thus Takens–Bogdanov bifurcation point and codimension two bifurcation point are different. There is no simple formula to give $Ta(Pr)$ as a function of Pr. In the next subsection we obtain Ta as a function of Pr at the codimension two bifurcation point by assuming R as an independent variable. Such kind of interesting relation is not available in [6].

In Fig. 1(a–d), solid line represents stationary convection (pitchfork bifurcation) and dotted line denotes oscillatory convection (Hopf bifurcation) which are plotted in (q, R)-plane. On solid line $\omega^2 = 0$ and dotted line $\omega^2 > 0$. the value of ω^2 decreases on dotted line when q increases and ω^2 takes zero value at the intersection of solid and dotted line.

In Fig. 1(a–d) we have shown the effect of Taylor number Ta, over the onset of both stationary and oscillatory convection. From these figures we can say that when Ta increases, then the onset of both stationary and oscillatory convection will increase. This implies that rotation rate inhibits the onset of convection. This result is true for other parameter Pr also. In Fig. 1(a–d), we can see three types of bifurcations like pitchfork bifurcation, Hopf bifurcation, Takens–Bogdanov bifurcation

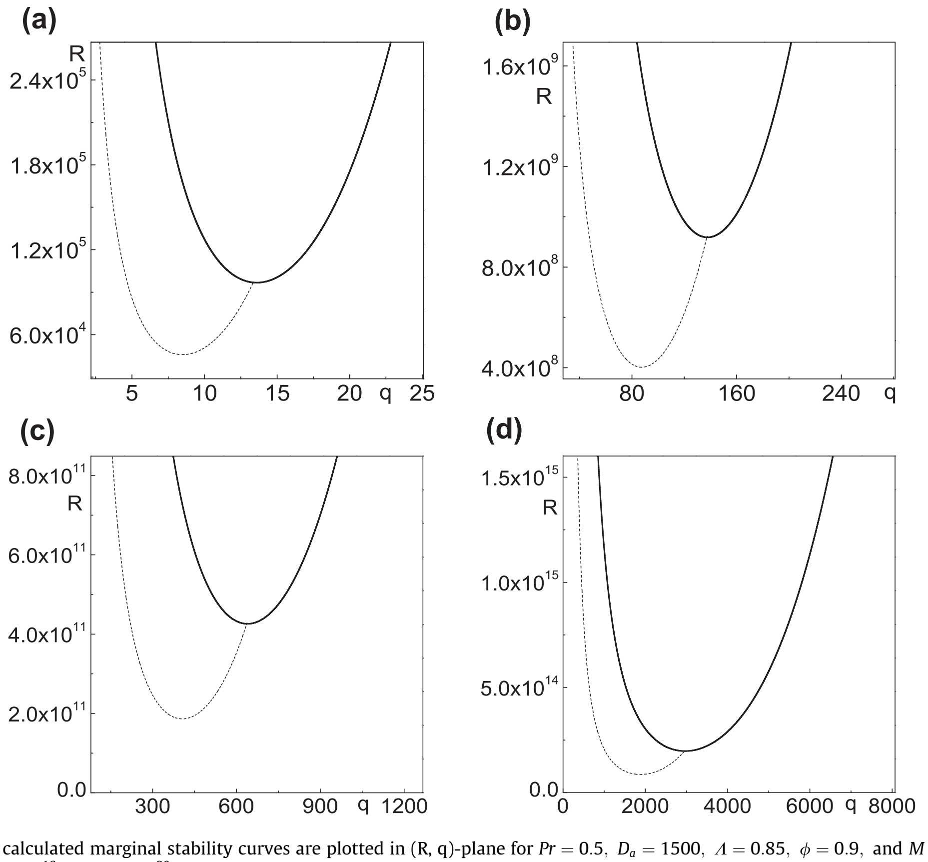

Fig. 1. Numerically calculated marginal stability curves are plotted in (R, q)-plane for $Pr = 0.5$, $D_a = 1500$, $\Lambda = 0.85$, $\phi = 0.9$, and $M = 0.9$, (a) $Ta = 10^6$, (b) $Ta = 10^{12}$, (c) $Ta = 10^{16}$, (d) $Ta = 10^{20}$, then the onset of stationary convection and the onset of oscillatory convection increases (stationary convection stands for solid lines and oscillatory convection stands dotted lines).

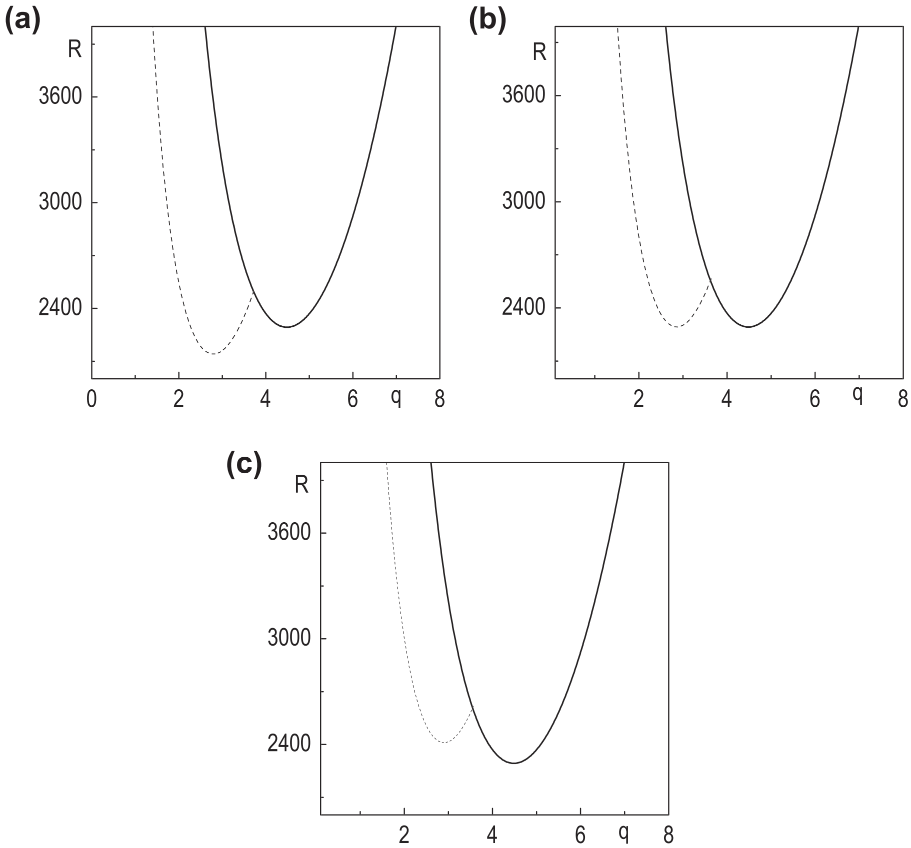

Fig. 2. Neutral curves for the stationary bifurcation (solid lines) and for the Hopf bifurcation (dashed lines) near the codimension two point for $Ta = 2000,\ D_a = 1500,\ \Lambda = 0.85,\ \phi = 0.9$ and $M = 0.9$, (a) $Pr = 0.5$, (b) $Pr = 0.557$, (c) $Pr = 0.6$. x-axis wave number, y-Rayleigh numbers R_s, R_o.

point, (the intersection point of solid and dotted line). In Fig. 2, when Pr_2 increases then the onset of oscillatory convection decreases. In Fig. 2b, we can observe the appearance of both primary bifurcations (pitchfork bifurcation, Hopf bifurcation) and secondary bifurcations (Takens–Bosgdanov bifurcation point, co-dimension two bifurcation point).

Eq. (3.9) shows the stabilizing or inhibiting effect of rotation at the onset of stationary convection. The increase of R_{sc} and R_{oc} with the Taylor number Ta implies that disturbances in the fluid will not move upward or downward easily due to the presence of Coriolis force. We have neglected the effect of centrifugal force. For inviscid fluid the Taylor number Ta is infinite and consequently the critical Rayleigh number R_{sc} for the onset of stationary convection in a rotating fluid. Inviscid fluid with rotation is stable for all vertical temperature gradients. This is a consequence of the Taylor–Proudman theorem. The patterns of convection in the presence of rotation depend on both horizontal co-ordinates and Taylor number. An infinite number of patterns are theoretically possible at same critical Rayleigh number. The patterns can be rolls, square cells, rectangular cells of all side ratios and hexagonal cells. Chandrasekhar [6] calculated the velocity fields for these various patterns. Küppers and Lortz [13] showed that with rotation and under slightly supercritical conditions all three-dimensional convective flows are unstable.

3.2. Determination of marginal stability when Rayleigh number R is an independent variable

Putting $W = \sin \pi z$, into Eq. (3.2) we get a third degree polynomial equation in p of the following form:

$$p^3 + Bp^2 + Cp + D = 0, \tag{3.21}$$

where

$$B = \delta^2 + 2MPr\phi\left(\Lambda\delta^2 + \frac{1}{D_a}\right), \tag{3.22a}$$

$$C = \delta^4 \Lambda MPr\phi(2 + \Lambda MPr\phi) + \frac{MPr}{\delta^2}\left(Ta\pi^2 M^3 Pr - Rq^2\phi\right) + \frac{MPr\phi}{Da}\left(2\Lambda\delta^2 + 2\Lambda MPr\phi\delta^2 + \frac{MPr\phi}{D_a}\right), \tag{3.22b}$$

$$D = M^2 Pr^2\left(Ta\pi^2 M^2 + \delta^6 \Lambda^2\phi^2 - Rq^2\Lambda\phi^2\right) + \frac{M^2 Pr^2 \phi^2}{D_a}\left(2\delta^4\Lambda + \frac{\delta^2}{D_a} - \frac{Rq^2}{\delta^2}\right). \tag{3.22c}$$

Table 1
Classifications of stability modes.

	$D < 0$	$D > 0$	$D = 0$
$BC - D < 0$	Unstable	Unstable	Unstable
$BC - D > 0$	Unstable	Stable	$p = 0,\ Re(p) < 0,\ Re(p) < 0$
$BC - D = 0$	Unstable	$p = -d_1,\ p = i\omega,\ p = -i\omega$	$p = 0,\ p = 0,\ p < 0$

From Eq. (3.22a), B is always positive. The system will be stable when three roots of cubic Eq. (3.21) have $Re(p) < 0$. If $Re(p) > 0$ for at least one root of the cubic equation then the system will be unstable. With each root of the cubic equation there is an associated combination of a flow field and temperature distribution. The instability can set in as stationary convection if one root of the Eq. (3.21) is zero or oscillatory convection if two roots are purely imaginary. The classification of stability modes of the system are given Table 1 [10] from the roots of Eq. (3.21).

In Table 1, ‘unstable’ means there exists at least one root of Eq. (3.21) with $Re(p) > 0$, ‘stable’ means all roots of Eq. (3.21) with $Re(p) < 0$. We get pitchfork bifurcation when $D = 0$ and $BC - D > 0$. When $D > 0$ and $BC - D = 0$, we get Hopf bifurcation.

3.2.1. Stationary convection ($\omega = 0$)

The stability of the system is determined by the sign of D and $BC - D$. Eq. (3.22c) shows that $D < 0$ when, for given q and Ta, R is large enough. $D > 0$ for small enough R and $D = 0$ when $p = 0$. Equation $D = 0$ gives

$$(r+\pi^2)^3 + \frac{Ta\pi^2 M^2}{\phi^3 \Lambda^2} - \frac{Rr}{\Lambda} = 0, \quad \text{where} \quad r = q^2. \tag{3.23}$$

For a fixed Ta, Eq. (3.23) determines a curve

$$R = \frac{M}{r\Lambda}\left[\frac{\Lambda^2}{M^2}(r+\pi^2)^3 + \frac{\pi^2 Ta}{\phi^2}\right] \tag{3.24}$$

in Rr-plane (critical Rayleigh number for the onset of stationary convection) there are two positive values of r say r_1, r_2, between which $D < 0$ and $R < R_{sc}, D > 0$ for all r (see Fig. 3). The system is stable for $D > 0$ and unstable $D < 0$.

On differentiating Eq. (3.24) with respect r we get Eq. (3.8), from which we find critical wave number for a given Taylor number for the onset of stationary convection. Geometrically, there is another way is to find the critical Rayleigh number and critical wave number for a given Taylor number under the condition that the straight line Rr is tangent to the curve $\left[\Lambda^2 M^{-2}(r+\pi^2)^3 + Ta\pi^2\phi^{-2}\right]$ as shown in Fig. 4a. The straight line Rr becomes tangent to the curve only if $R = R_{sc}$. At the tangent say $r = r'$, then the critical wave number $q = q_{sc}$ obtained as $q_{sc} = r'^{\frac{1}{2}}$. Above discussions are done under the assumption that R as a dependent variable and calculated critical Rayleigh number and critical wave number at a fixed Taylor number for the onset of stationary convection. Similarly, by assuming Rayleigh number as an independent variable we can compute critical Taylor number and critical wave number for the onset of stationary convection. The analytical expressions for critical Taylor number and critical wave number can be computed as follow.

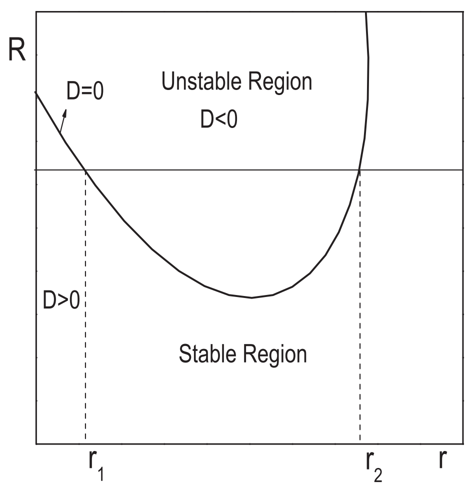

Fig. 3. A typical diagram showing the stability regions of the system for stationary convection.

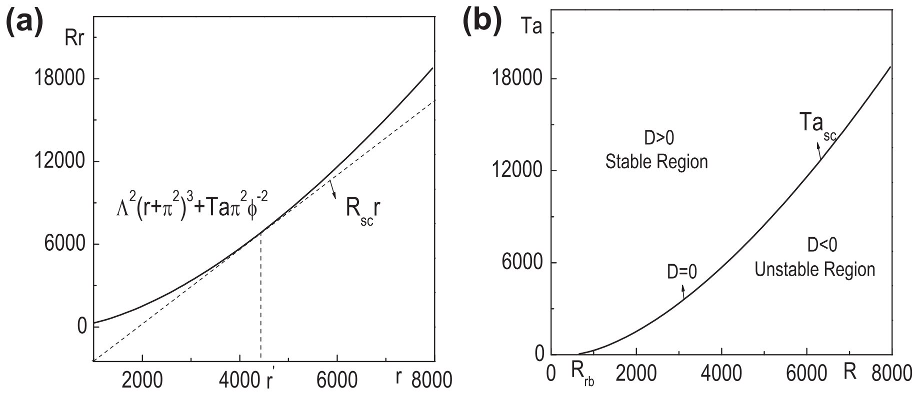

Fig. 4. At the intersection point of the curve and straight line in figure (a) we get the critical wave number $q_{sc} = r'^{\frac{1}{2}}$ corresponding to the critical Rayleigh number at a given Taylor number. In figure (b) the system (stationary convection) is stable in $D > 0$ region, unstable $D < 0$ region and $D = 0$ on the curve Ta_{sc}.

The derivative of Eq. (3.23) with respect to r gives

$$R = 3\Lambda(r + \pi^2)^2. \tag{3.25}$$

On substituting Eq. (3.25) into Eq. (3.23), we get

$$2\frac{r^3}{\pi^6} + 3\frac{r^2}{\pi^4} = 1 + \frac{Ta}{\pi^4}\left(\frac{M}{\Lambda\phi}\right)^2. \tag{3.26}$$

Eq. (3.26) is nothing but Eq. (3.8) at $r = q^2$ and $q = q_{sc}$. We can write Eq. (3.25) in terms of r as

$$r = \left(\frac{R}{3\Lambda}\right)^{\frac{1}{2}} - \pi^2. \tag{3.27}$$

From Eq. (3.27) we consider only positive values of r, since $r = q^2(> 0)$. Substituting Eq. (3.27) into Eq. (3.23), we get critical Taylor number $Ta = Ta_{sc}(R)$, where

$$Ta = Ta_{sc} = R\left[\frac{\Lambda^{\frac{1}{2}}\phi^2}{M^2}\left(\frac{R}{R_{rb}}\right)^{\frac{1}{2}} - \frac{\Lambda\phi^2}{M^2}\right] \quad \text{where} \quad R_{rb} = \frac{27\pi^4}{4}. \tag{3.28}$$

Here R_{rb} is the critical Rayleigh number for the onset of stationary convection of Rayleigh–Benard convection without rotation. Fig. 4b is plotted in (R, Ta)-plane for the curve (3.28). In this figure $Ta_{sc} = 0$ on R-axis. From R-axis the curve (3.28) starting from $R = R_{rb}$. In (R, Ta)-plane we check the sign of D in a range of R with $q = q_{sc}(R)$ at a fixed $Ta = Ta_{sc}$. For the values $\{R, Ta\}$ which are left to the curve (3.28). $D > 0$ and $D < 0$ for the values $\{R, Ta\}$ which are right to the curve (3.28) and $D = 0$ on the curve (3.28).

3.2.2. Oscillatory convection ($\omega^2 > 0$)

For oscillatory convection, substituting $p = i\omega$ into Eq. (3.21) and equating real and imaginary parts to zero we get $\omega^2 = D/B$ and $\omega^2 = C$. ω^2 is positive only if $D > 0$ or $C > 0$. From these two equations we get $BC - D = 0$. From Table 1, the condition $D > 0$ is not enough to discuss the stability of the system. So we have to also check the sign of D for the stability of the system. Thus $BC - D = 0$ gives

$$(r + \pi^2)^3 + \frac{Ta\pi^2 Pr^2 M^4}{(1 + MPr\Lambda\phi)^2} - \frac{MRr}{2\Lambda(1 + MPr\Lambda\phi)} = 0, \tag{3.29}$$

or

$$R_o = \frac{2\Lambda(1 + MPr\Lambda\phi)}{Mq_o^2}\left[\delta_o^2 + \frac{Ta\pi^2 Pr^2 M^4}{(1 + MPr\Lambda\phi)^2}\right],$$

where R_o is the Rayleigh number for oscillatory convection. The frequency for the oscillations is given by $\omega^2 = C$. Using Eq. (3.14) into C we get ω^2 as

$$\omega^2 = \frac{M^2 Pr^2}{\delta_o^2(1 + MPr\Lambda\phi)}\left[Ta\pi^2 M^2(1 - MPr\Lambda\phi) - \delta_o^6\Lambda^2\phi^2(1 + MPr\Lambda\phi)\right].$$

We follow similar procedure to compute analytical expressions of critical Taylor number Ta_{oc} and critical wave number q_{oc} for the oscillatory convection as we have obtained Ta_{sc} and q_{sc} for the stationary convection. Here we compute the analytical expressions Ta_{oc} and q_{oc} directly by comparing the Eqs. (3.23) and (3.29). Substituting

$$Ta = \frac{TaPr^2M^2\Lambda^2\phi^2}{(1+MPr\Lambda\phi)^2} \quad \text{and} \quad R = \frac{MR}{(1+MPr\Lambda\phi)},$$

into Eq. (3.28), we get

$$Ta = Ta_{oc}(R, Pr, \Lambda, \phi, M) = R\left[\left(\frac{(1+MPr\Lambda\phi)}{2^3Pr^4M^5\Lambda^3}\right)^{\frac{1}{2}}\left(\frac{R}{R_{rb}}\right)^{\frac{1}{2}} - \frac{(1+MPr\Lambda\phi)}{2Pr^2M^3\Lambda}\right], \tag{3.30}$$

and

$$q = q_{oc}(R, Pr, \Lambda, \phi, M) = \left[\left(\frac{MR}{6\Lambda(1+MPr\Lambda\phi)}\right)^{\frac{1}{2}} - \pi^2\right]^{\frac{1}{2}}. \tag{3.31}$$

As we have checked the sign of D in stationary convection, similarly we have to check the signs of D and $BC-D$ for the stability region of oscillatory convection. Here we can use $Ta = Ta_{sc}$ or $Ta = Ta_{oc}$ to identify the signs of $D, BC-D$. In the (R, Ta)-plane, on the R-axis $Ta = 0$ and the curve corresponds to (3.30) always starts from $R = R' = 2(1+Pr)R_{rb}$. In the plane $BC-D>0$ for the values R, Ta which are right to the curve (3.30) (see Fig. 5). From Table 1, when $BC-D>0$ and $D>0$ we get one damped mode and two oscillatory modes. Thus the system is stable in $BC-D>0$ and $D>0$ region. The coefficient

$$\left(\frac{(1+MPr\Lambda\phi)}{2^3Pr^4M^5\Lambda^3}\right)^{\frac{1}{2}},$$

of $R^{\frac{3}{2}}$ in Eq. (3.30) is equal to unity at $Pr = Pr_c = 0.783813$ and it is less than unity for $Pr > Pr_c$. When $Pr < Pr_c$. Eq. (3.28) intersect with (3.30) at

$$R = R_{ct} = (1+\Upsilon)^2R_{rb}, \quad Ta_{ct} = \frac{\Lambda^{\frac{1}{2}}\phi^2}{M^2}(1+\Upsilon-\Lambda^{\frac{1}{2}})(1+\Upsilon)^2R_{rb},$$

$$\Upsilon = \frac{2^{\frac{1}{2}}(1+MPr\Lambda\phi) - (M(1+MPr\Lambda\phi))^{\frac{1}{2}}}{(M(1+MPr\Lambda\phi))^{\frac{1}{2}} - 2^{\frac{3}{2}}Pr^2\Lambda^2\phi^2M^2}. \tag{3.32}$$

The suffix ct in Eq. (3.32) stands for parameter at codimension two bifurcation point. The Rayleigh number $R = R_{ct}$ is obtained by equating Eqs. (3.28) and (3.30). By substituting $R = R_{ct}$ either into Eq. (3.28) or into Eq. (3.30), we get $Ta = Ta_{ct}$. At $Ta_{ct}, Ta_{sc} = Ta_{oc}$ and $q_{sc} \neq q_{oc}$. At $Pr = Pr_c, Ta_{oc}$ approaches to Ta_{sc} asymptotically as $R \to \infty$ i.e., the intersection between Eqs. (3.28) and (3.30) appears at infinity. In Table 2, we have given the values of $Ta_{ct}(Pr)$ and $R_{ct}(Pr)$ for some values of Pr computed from Eq. (3.32). Fig. 6(a–d), show that with decreasing $Pr < Pr_c, Ta_{ct}$ and R_{ct} decreases. Thus at $Pr = 0$, we get codimension two bifurcation point at $R_{ct} = 2M^{-1}R_{rb}$ and $Ta_{ct} = 2M^{\frac{-3}{2}}(2^{\frac{1}{2}} - M^{\frac{1}{2}})R_{rb}$. When $Ta < Ta_{ct}$ we get stationary convection as a first instability while for $Ta > Ta_{ct}$ the first instability will be oscillatory convection. By eliminating Ta from equations $C = 0$ and $D = 0$, we get

$$q^6 + 3q^4\pi^2 + \left[3\pi^4 + \frac{R(MPr\Lambda\phi - 1)}{2\Lambda}\right]q^2 + \pi^6 = 0. \tag{3.33}$$

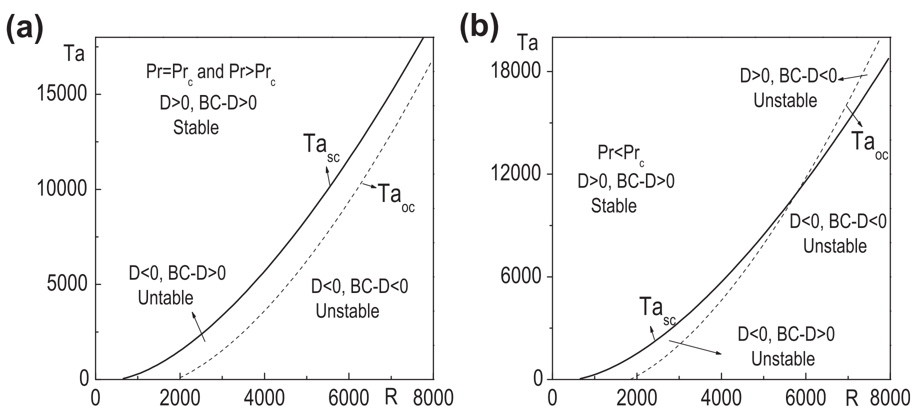

Fig. 5. The typical diagram show the stability regions of the system on the solid lines $D = 0$ and on the dotted lines $BC-D = 0$. In each figure for the values $\{R, Ta\}$ which are left to the solid line $D>0$ and $D<0$ for the values $\{R, Ta\}$ which are left to the dotted line and $BC-D<0$ for the values $\{R, Ta\}$ which are right to the dotted line.

Table 2
The values of Ta_{ct}, Ra_{ct} at different Prandtl number Pr.

Pr	Ta_{ct}	R_{ct}	Pr	Ta_{ct}	R_{ct}
0	766.174	1461.14	0.55	3624.21	3124.72
0.1	911.189	1576.97	0.6	4779.04	3630.42
0.2	1119.61	1732.1	0.63	5782.01	4033.51
0.4	1944.76	2261.63	0.65	6647.93	4361.03
0.5	2860.95	2756.75	0.6766	8080.86	4870.31

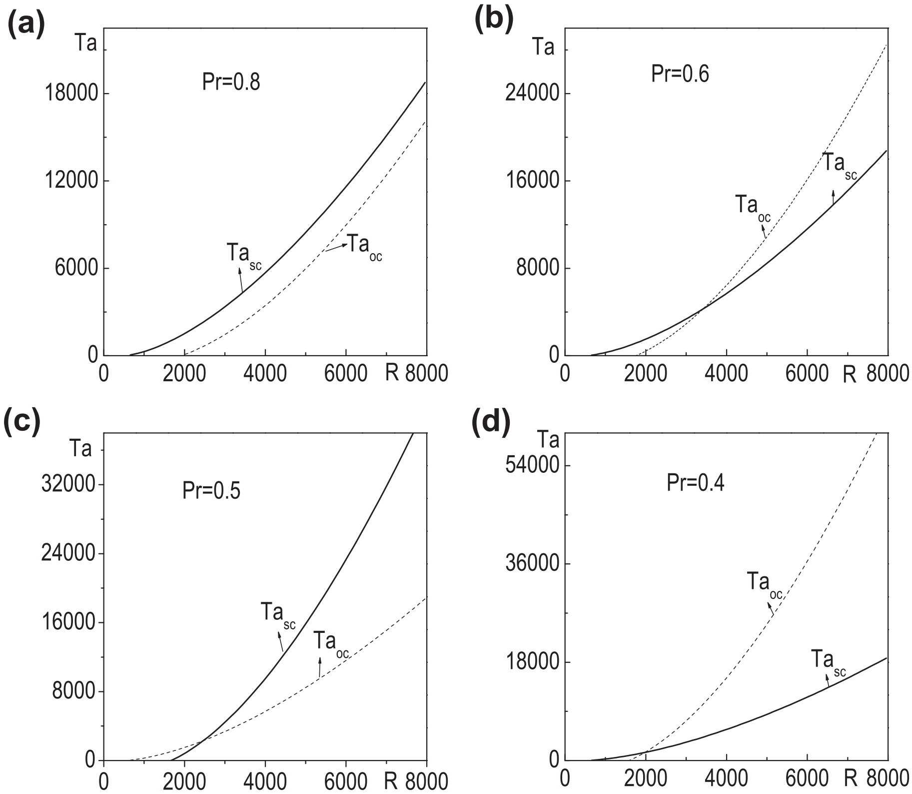

Fig. 6. In above figures solid lines are plotted for the curve Ta_{sc} (stationary convection) and dotted lines are plotted for the curve Ta_{oc} (oscillatory convection) at different values of Pr. When $Pr \to 0$ then the intersection point appear at $R_{ct} = \frac{2}{M} R_{rb}$ and $Ta_{ct} = 2M^{\frac{-3}{2}}(2^{\frac{1}{2}} - M^{\frac{1}{2}})R_{rb}$, whereas for $Pr \to Pr_c, R_{ct} \to \infty$ and $Ta_{ct} \to \infty$.

From Eq. (3.33), for $Pr < 1$, we get two positive roots which are correspond to two Takens–Bogdanov bifurcation points and for $Pr > 1$, we do not get positive roots. This implies that we do not get oscillatory convection for $Pr > 1$.

4. Derivation of nonlinear two-dimensional Landau–Ginzburg equation near the onset of stationary convection

In this Section the evolution of a general pattern is developed by means of a multiple scale analysis used by Newell and Whitehead [16]. A small amplitude convection cell is imposed on the basic flow. If this amplitude is of the size $O(\epsilon)$ then the interaction of the cell with itself forces a second harmonic and mean state correction of size $O(\epsilon^2)$ and then in turn drives an $O(\epsilon^3)$ correction to the fundamental component of the imposed roll. A solvability criteria for this correction yields the two-dimensional nonlinear Landau–Ginzburg equation of the complex valued amplitude $A(X, Y, T)$ of the imposed disturbance with real coefficients. To simplify the problem we assume the formulation of cylindrical rolls with axis parallel to y-axis, so that y-dependence disappears from Eq. (2.14). The z-dependence is contained entirely in the *sine* and *cosine* functions, which ensures that stress-free boundary conditions are satisfied. We use the expansion parameter ϵ as

$$\epsilon^2 = \frac{R - R_{sc}}{R_{sc}} \tag{4.1}$$

for the values of R close to threshold value R_{sc} i.e., $\epsilon \ll 1$, the structure of the slow length scales will be insensitive to ϵ, but a slow modulation in space and time is possible by making use of the band of the unstable solutions and linear growth rate is likely to saturate due to nonlinear effects. This behavior can be analyzed by writing solutions of Eqs. (2.7)–(2.9) in power series ϵ as

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + \cdots, \tag{4.2}$$

where $f = f(u, v, w, \theta)$ with the first approximation is given by the eigenvector of the linearized problem:

$$\begin{aligned} u_0 &= \frac{i\pi}{q_{sc}}\left[A(X,Y,T)e^{iq_{sc}x}\cos\pi z - c\cdot c\cdot\right],\\ v_0 &= \frac{-i\pi Ta^{\frac{1}{2}}}{\phi\sigma_1 q_{sc}}\left[A(X,Y,T)e^{iq_{sc}x}\cos\pi z - c\cdot c\cdot\right],\\ w_0 &= A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c\cdot c\cdot,\\ \theta_0 &= \frac{1}{M\delta_{sc}^2}\left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c\cdot c\cdot\right], \end{aligned} \tag{4.3}$$

where $\delta_{sc}^2 = (\pi^2 + q_{sc}^2)$. Here $c \cdot c\cdot$ stands for complex conjugate, $e^{iq_{sc}\sin\pi z}$ is the critical mode for the linear problem at $R = R_{sc}$ and $q = q_{sc}$. The complex amplitude $A(X, Y, T)$ depends on the slow variables. The independent variables x, y, z, t are scaled by introducing multiple scales

$$X = \epsilon x, \quad Y = \epsilon^{\frac{1}{2}} y, \quad Z = z \quad \text{and} \quad T = \epsilon^2 t \tag{4.4}$$

and these formally separate the fast and slow dependent variables in f. It should be noted that difference in scaling in the two directions reflects the inherent symmetry breaking of instability which was chosen here with wave vector in x-direction. The differential operators can be expressed as

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} \to \epsilon^{\frac{1}{2}}\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z} \to \frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial t} \to \epsilon^2\frac{\partial}{\partial T}, \tag{4.5}$$

with the assumption (4.5), the operators (2.15) and (2.16) are transformed into a set of linear inhomogeneous equations. The solvability conditions for the latter yields the amplitude equation using Eq. (4.3) the linear operator (2.15) can be written as

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 \ldots, \tag{4.6}$$

where

$$\mathcal{L}_0 = -\sigma_3^2\nabla^4 - Ta\nabla^2\frac{\partial^2}{\partial z^2} - \frac{R_{sc}}{M}\nabla_h^2\sigma_3, \tag{4.7}$$

$$\mathcal{L}_1 = -\left(2\frac{\partial^2}{\partial x\partial X} + \frac{\partial^2}{\partial Y^2}\right)\left[2\sigma_3^2\nabla^2 - 2\frac{\Lambda}{M}\sigma_3\nabla^4 + \frac{Ta}{\phi^2}\frac{\partial^2}{\partial z^2} - \frac{R_{sc}\Lambda}{M^2}\nabla_h^2 + \frac{R_{sc}}{M}\sigma_3\right], \tag{4.8}$$

$$\begin{aligned} \mathcal{L}_2 = &\frac{\partial}{\partial T}\left[-\frac{2}{M^2\phi P_r}\sigma_3\nabla^4 + \sigma_3^2\nabla^2 + \frac{Ta}{\phi^2}\frac{\partial^2}{\partial z^2} - \frac{R_{sc}}{M^3\phi Pr}\nabla_h^2\right]\\ &- \frac{\partial^2}{\partial X^2}\left[2\sigma_3^2\nabla^2 - 2\frac{\Lambda}{M}\sigma_3\nabla^4 + \frac{Ta}{\phi^2}\frac{\partial^2}{\partial z^2} - \frac{R_{sc}\Lambda}{M^2}\nabla_h^2 + \frac{R_{sc}}{M}\sigma_3\right]\\ &+ \left(2\frac{\partial^2}{\partial x\partial X} + \frac{\partial^2}{\partial Y^2}\right)^2\left[4\frac{\Lambda}{M}\sigma_3\nabla^2 - \sigma_3^2 - \frac{\Lambda^2}{M^2}\nabla^4 + \frac{R_{sc}\Lambda}{M^2}\right] - \frac{R_{sc}}{M}\nabla_h^2\sigma_3 \end{aligned} \tag{4.9}$$

and $\sigma_3 = \left(\frac{1}{MD_a} - \frac{\Lambda}{M}\nabla^2\right)$. Similarly nonlinear term $\mathcal{N}$ is given by

$$\mathcal{N} = \epsilon^2\mathcal{N}_0 + \epsilon^3\mathcal{N}_1 + \cdots \tag{4.10}$$

substituting Eqs. (4.6), (4.10) and (4.2) into Eq. (2.14), we get by equating the coefficients of $\epsilon, \epsilon^2, \epsilon^3$,

$$\mathcal{L}_0 w_0 = 0, \tag{4.11}$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \tag{4.12}$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \tag{4.13}$$

Eq. (4.7) gives the critical Rayleigh number for the onset of stationary convection

$$R_s = \frac{M}{q_s^2}\left[\frac{\delta_s^4\left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_s^2\right)^2 + \frac{Ta}{\phi^2}\pi^2\delta_s^2}{\left(\frac{1}{MD_a} + \frac{\Lambda}{M}\delta_s^2\right)}\right].$$

In Eq. (4.12), $\mathcal{N}_0 = 0,\ \mathcal{L}_1 w_0 = 0$ and hence $w_1 = 0$. From equation of continuity we find that $u_1 = 0$. The relevant equations for θ_1 and v_1 are

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta_1 = \frac{w_1}{M} - \frac{1}{M}\left[u_0\frac{\partial \theta_0}{\partial x} + w_0\frac{\partial \theta_0}{\partial z}\right], \tag{4.14}$$

$$\mathcal{D}_{Pr}\frac{\partial v_1}{\partial x} = \frac{Ta^{\frac{1}{2}}}{\phi}\frac{\partial w_1}{\partial z} - \frac{1}{M^2\phi Pr}\frac{\partial}{\partial x}\left[u_0\frac{\partial v_0}{\partial x} + w_0\frac{\partial v_0}{\partial z}\right]. \tag{4.15}$$

Substituting zeroth order approximations from Eq. (4.3) into Eqs. (4.14) and using $w_1 = 0$, we get

$$\theta_1 = \frac{-1}{2M^2\pi\delta_{sc}^2}|A|^2\sin 2\pi z,$$
$$v_1 = \frac{-i\pi^2 Ta^{\frac{1}{2}}}{M^2\phi^2 Pr q_{sc}\sigma_1\sigma_4}\left[A^2 e^{2iq_{sc}x} - c\cdot c\cdot\right] \tag{4.16}$$

and $\sigma_4 = \left(\frac{1}{MD_a} + 4\frac{\Lambda}{M}q_{sc}^2\right)$. Substituting zeroth order and first order solutions in (4.9) and equating coefficients of $\sin\pi z$ in $\mathcal{N}_1 - \mathcal{L}_2 w_0$ to zero, we get

$$\lambda_0\frac{\partial A}{\partial T} - \lambda_1\left(\frac{\partial}{\partial X} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial Y^2}\right)^2 A - \lambda_2 A + \lambda_3|A|^2 A = 0, \tag{4.17}$$

where

$$\lambda_0 = \frac{2\sigma_1}{M^2\phi Pr}\delta_{sc}^4 + \sigma_1^2\delta_{sc}^2 + \frac{Ta}{\phi^2}\pi^2 - \frac{R_{sc}}{M^3\phi Pr}q_{sc}^2,$$
$$\lambda_1 = 4q_{sc}^2\left[\frac{\Lambda^2}{M^2}\delta_{sc}^4 + 4\frac{\Lambda}{M}\sigma_1\delta_{sc}^2 + \sigma_1^2 - \frac{R_{sc}\Lambda}{M^2}\right],$$
$$\lambda_2 = \frac{R_{sc}\sigma_1}{M}q_{sc}^2,$$
$$\lambda_3 = \frac{R_{sc}q_{sc}^2\sigma_1}{2M^3\delta_{sc}^2} - \frac{2\pi^4 Ta\delta_{sc}^2}{M^4\phi^5 Pr^2\sigma_1\sigma_4}. \tag{4.18}$$

Eq. (4.17) is two-dimensional, nonlinear, time dependent Landau–Ginzburg equation describing the effect of rotating field in a sparsely packed porous medium near the onset of stationary convection at supercritical pitchfork bifurcation. Here λ_0 is always positive for $Pr < \frac{1}{\phi}$ and for any Ta but if $Pr > \frac{1}{\phi}$ then λ_0 is positive only if $Ta < Ta_c$. Thus for supercritical pitchfork bifurcation λ_0 is always positive. For $Pr > \frac{1}{\phi}$, λ_0 decreases as Ta increases and becomes zero at $Ta = Ta_c$. λ_1 and λ_2 are always positive. λ_3 is positive only if

$$Ta < \frac{MR_{sc}q_{sc}^2\sigma_1^2\sigma_4\phi^5 Pr^2}{4\pi^4\delta_{sc}^4}, \tag{4.19}$$

the pitchfork bifurcation is supercritical if $\lambda_3 > 0$ and subcritical if $\lambda_3 < 0$. At $\lambda_3 = 0$, we get tricritical bifurcation point (see Fig. 7). Dropping the time-dependent term from Eq. (4.17), we get

$$\frac{d^2A}{dX^2} + \frac{\lambda_2}{\lambda_1}\left(1 - \frac{\lambda_3}{\lambda_2}|A|^2\right)A = 0, \tag{4.20}$$

since $\lambda_1 > 0$, the solution of Eq. (4.20) is given by

$$A(X) = A_0\tanh(X/\Lambda_1), \tag{4.21}$$

where

$$A_0 = (\lambda_2/\lambda_3)^{\frac{1}{2}} \quad \text{and} \quad \Lambda_1 = (2\lambda_1/\lambda_2)^{\frac{1}{2}}. \tag{4.22}$$

4.1. Long wave-length instabilities (secondary instabilities)

The two-dimensional Landau–Ginzburg equation (4.17), can be written in fast variables x, y, t and $A(X, Y, T) = A(x, y, t)/\epsilon$, as

$$\lambda_0\frac{\partial A}{\partial t} - \lambda_1\left(\frac{\partial}{\partial x} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial y^2}\right)^2 A - \epsilon^2\lambda_2 A + \lambda_3|A|^2 A = 0, \tag{4.23}$$

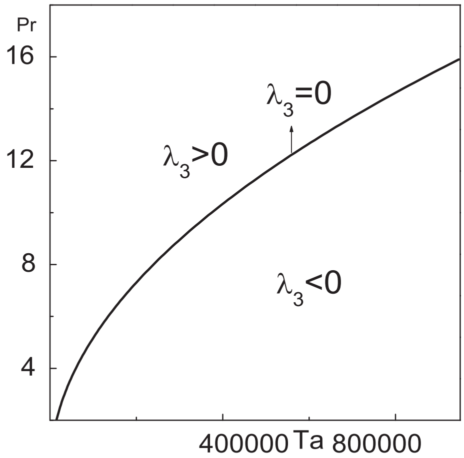

Fig. 7. Above figure is plotted for $D_a = 1500,\ \Lambda = 0.85,\ \phi = 0.9,\ M = 0.9$ and $Pr = 0.5$. λ_3 is the nonlinear coefficient of Landau–Ginzburg equation at the onset of stationary convection. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$, subcritical if $\lambda_3 < 0$ and $\lambda_3 = 0$ on the curve.

In order to study the properties of a structure with a given phase winding number δk, we substitute

$$A(x,y,t) = A_1(x,y,t)e^{i\delta kx}, \tag{4.24}$$

into the Eq. (4.23) and we obtain

$$\lambda_0\frac{\partial A_1}{\partial t} = \left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right)A_1 + 2i\lambda_1\delta k\left(\frac{\partial}{\partial x} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial y^2}\right)A_1 + \lambda_1\left(\frac{\partial}{\partial x} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial y^2}\right)^2 A_1 - \lambda_3|A_1|^2A_1 = 0. \tag{4.25}$$

The steady state uniform solution of Eq. (4.25) is

$$A_1 = A_{1o} = \left[\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right)\lambda_3^{-1}\right]^{\frac{1}{2}}. \tag{4.26}$$

Let $\tilde{u}(x,y,t) + i\tilde{v}(x,y,t)$ be an infinitesimal perturbation from a uniform steady state solution A_{1o} given by Eq. (4.26). Now substituting

$$A_1 = A_{1o} = \left[\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right)\lambda_3^{-1}\right]^{\frac{1}{2}} + \tilde{u} + i\tilde{v},$$

into Eq. (4.25) and equating real and imaginary parts, we obtain

$$\lambda_0\frac{\partial\tilde{u}}{\partial t} = \left[-2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)\right)^2 + \lambda_1\left(\frac{\partial^2}{\partial x^2} + \frac{\delta k}{q_{sc}}\frac{\partial^2}{\partial y^2} - \frac{1}{4q_{sc}^2}\frac{\partial^4}{\partial y^4}\right)\right]\tilde{u} - \left(2\lambda_1\delta k - \frac{\lambda_1}{q_{sc}}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\tilde{v}}{\partial x}, \tag{4.27a}$$

$$\lambda_0\frac{\partial\tilde{v}}{\partial t} = \left(2\lambda_1\delta k - \frac{\lambda_1}{q_{sc}}\frac{\partial^2}{\partial y^2}\right)\frac{\partial\tilde{u}}{\partial x} + \lambda_1\left(\frac{\partial^2}{\partial x^2} + \frac{\delta k}{q_{sc}}\frac{\partial^2}{\partial y^2} - \frac{1}{4q_{sc}^2}\frac{\partial^4}{\partial y^4}\right)\tilde{v}. \tag{4.27b}$$

We analyze Eqs. (4.27a) and (4.27b) by using normal modes of the form

$$\tilde{u} = Ue^{St}\cos(q_x x)\cos(q_y y) \quad \text{and} \quad \tilde{v} = Ve^{St}\sin(q_x x)\cos(q_y y). \tag{4.28}$$

Putting Eq. (4.28) in Eqs. (4.27a) and (4.27b) we get,

$$\left[\lambda_0 S + 2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \chi_1\right]U + \lambda_1 q_x\chi_2 V = 0, \tag{4.29a}$$

$$\lambda_1 q_x\chi_2 U + (\lambda_0 S + \chi_1)V = 0. \tag{4.29b}$$

Here $\chi_1 = \lambda_1[q_x^2 + (q_y^2\delta k)/q_{sc} + q_y^4/4q_{sc}^2],\ \chi_2 = (2\delta k + q_y^2/q_{sc})$. On solving Eq. (4.29a) and Eq. (4.29b) we get,

$$\lambda_0^2S^2 + 2S\left[2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\chi_1\right] + \left[2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \chi_1\right]\psi_1 - q_x^2\lambda_1\chi_2 = 0,$$

whose roots $(S\pm)$ are real. Here $(S\pm)$ is defined as

$$(S\pm) = -\frac{1}{\lambda_0^2}\left\{\left(2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\chi_1\right) \pm \left(2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right)^2 + \lambda_1^2q_x^2\chi_2^2\right)^{\frac{1}{2}}\right\}. \tag{4.30}$$

Solution $S(-)$ is clearly negative, thus the corresponding mode is stable and if $S(+)$ is positive then rolls can be unstable. Symmetry considerations help us to restrict the study of $S(+)$ to a domain $q_x \geqslant 0, q_y \geqslant 0$.

4.1.1. Longitudinal perturbations and Eckhaus instability

Inserting $q_y = 0$ into Eq. (4.30), we get

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\lambda_1 q_x^2\right] + \lambda_1 q_x^2\left[2\left(\epsilon^2\lambda_2 - 3\lambda_1(\delta k)^2\right) + q_x^2\right] = 0,$$

since the roots are real and their sum always negative, the pattern is stable as long as both roots are negative, i.e., their product is positive. The cell pattern becomes unstable when the product is negative, i.e., when

$$q_x^2 \leqslant 2\left(3\lambda_1\delta k^2 - \epsilon^2\lambda_2\right)$$

for this requires $|\delta k| \geqslant \sqrt{(\epsilon^2\lambda_2/3\lambda_1)}$, this condition defines the domain of Eckhaus instability. The above condition implies that the most unstable wave vector tends to zero, when $|\delta k| \to \sqrt{(\epsilon^2\lambda_2/3\lambda_1)}$.

4.1.2. Transverse perturbations and Zigzag Instability

Let us consider $q_x = 0$ into Eq. (4.30), we get

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\chi_1^y\right] + \left[2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \chi_1^y\right]\chi_1^y = 0,$$

where $\chi_1^y = \lambda_1\left(q_y^2\delta k/q_{sc} + q_y^4/4q_{sc}^2\right)$. The two eigenmodes are uncoupled and we have $S(-)$,

$$S(-) = -2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) - \frac{\lambda_1}{q_{sc}}\delta k q_y^2 - \frac{\lambda_1}{4q_{sc}^2}q_y^2 < 0$$

for one of them. The other is amplified when

$$S(+) = -\lambda_1 q_y^2\left(\delta k + \frac{q_y^2}{4q_{sc}}\right) > 0.$$

This implies that $\delta k < 0$, the above condition defines the domain of the Zigzag Instability. When $\delta k \to 0$ from below the wave vector q_y of the instability also tends to zero, while the growth rate varies as q_y^2. We have studied the effect of rotating on long wave length instabilities. We have observed that Eckhaus instability and Zigzag Instability regions increases when Ta increases (see Fig. 8).

4.1.3. Heat transport by convection

The maximum of steady amplitude A is denoted by $|A_{max}|$ which is given as

$$|A_{max}| = \left(\frac{\epsilon^2\lambda_2}{\lambda_3}\right)^{\frac{1}{2}}, \quad (4.31)$$

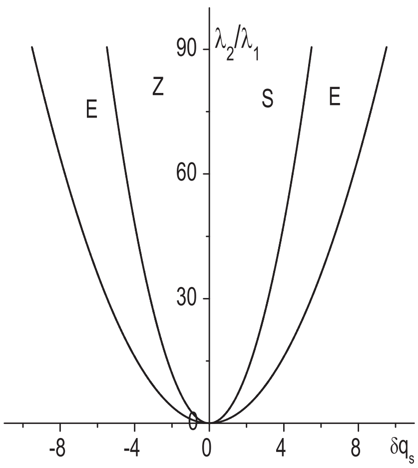

Fig. 8. Numerically computed secondary instability regions of Eckhaus instability (E), Zigzag Instability (Z) and stable regions (S) are plotted in $(\lambda_2/\lambda_1, \delta q_s)$-plane for $Ta = 2000,\ D_a = 1500,\ \Lambda = 0.85,\ \phi = 0.9,\ M = 0.9,$ and $Pr = 0.5$. As $|\delta q_s|$ increases then the secondary instability regions increases.

Eq. (4.31) is obtained from Eq. (4.21) with $\tanh(X/\Lambda_1) = 1$. We use $|A_{max}|$ to calculate Nusselt number Nu. To discuss the heat transfer near the neutral region, we express it through the Nusselt number is defined as $Nu = (Hd/\kappa\Delta T)$, which is the ratio of the heat transported across any layer to the heat which would be transported by conduction alone. Here H is the rate of heat transfer per unit area and is defined as

$$H = -\left\langle \frac{\partial T_{total}}{\partial z'} \right\rangle_{z'=0}, \tag{4.32}$$

In Eq. (4.32), angular brackets correspond to a horizontal average. The Nusselt number can be calculated in terms of amplitude A and is given as

$$Nu = 1 + \frac{\epsilon^2}{\delta_{sc}^2}|A_{max}|^2. \tag{4.33}$$

From Eq. (4.33), we get conduction for $R \leqslant R_{sc}$ and convection for $R > R_{sc}$. Since the amplitude equation is valid for $\lambda_3 > 0$, which is possible for $R > R_{sc}$ (supercritical pitchfork bifurcation), Thus we get $Nu > 1$ for $R > R_{sc}$. We get convection for $Nu > 1$ and conduction for $Nu \leqslant 1$. In stationary convection Nu increases implies that heat conducted by steady mode increases. In the problem of double diffusive convection in porous medium with rotating field, Nu depends on $Pr, \Lambda, M, \phi, D_a$ and Ta. We have computed Nu for different values of Ta, for some fixed values of other parameters and observed that Nu increases as Ta decreases (see Fig. 9). This implies that rotation inhibits the heat transport. The parameters Pr, Λ, M, ϕ and D_a show the same result as Ta shows on Nu.

5. Oscillatory convection at the supercritical Hopf bifurcation

The existence of a threshold (critical value of Rayleigh number for the onset of oscillatory convection $R = R_{oc}$) and a cellular structure (critical wave number $q = q_{oc}$) are main characteristics of the oscillatory convection. In this Section, we treat region near the onset of oscillatory convection. Here the axis of cylindrical rolls is taken as y-axis, so that y-dependence disappears from equation $\mathcal{L}w = \mathcal{N}$. The z-dependence contained entirely in *sine* and *cosine* functions which ensure that the free-free boundary conditions are satisfied. The purpose of this section is to derive coupled one dimensional nonlinear time dependent Landau–Ginzburg type equations near the onset of oscillatory convection at supercritical Hopf bifurcation. We introduce ϵ as

$$\epsilon^2 = \frac{R - R_{oc}}{R_{oc}} \ll 1. \tag{5.1}$$

We assume that

$$w_0 = \left[A_{1L}e^{i(q_{oc}x+\omega_{oc}t)} + A_{1R}e^{i(q_{oc}x-\omega_{oc}t)} + c \cdot c.\right] \sin \pi z$$

is a solution to linearized equation $\mathcal{L}w = 0$, which satisfies free–free boundary conditions. Here A_{1L} denotes the amplitude of left travelling wave of the roll and A_{1R} denotes the amplitude of right travelling wave of the roll, which depends on slow space and time variables [12]

$$X = \epsilon x, \quad \tau = \epsilon t, \quad T = \epsilon^2 t \tag{5.2}$$

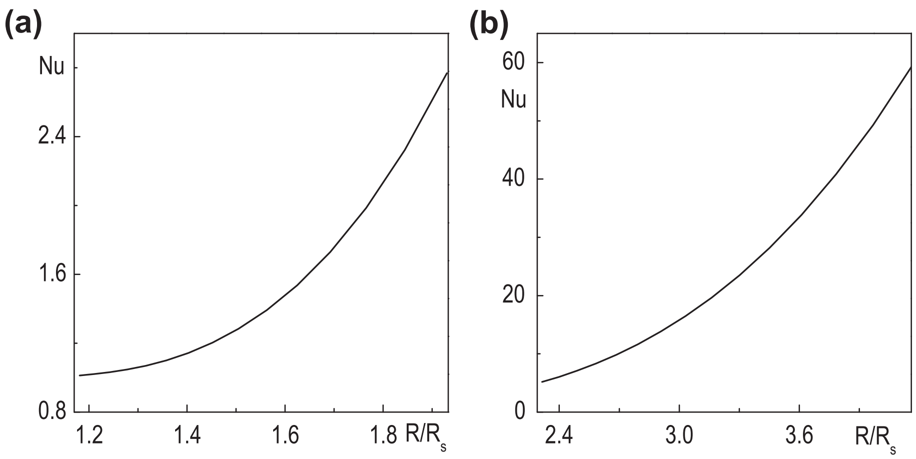

Fig. 9. Graph (a) is plotted for $Ta = 10^6$ and graph (b) is plotted for $Ta = 10^7$ for the fixed values of $D_a = 1500,\ \Lambda = 0.85,\ \phi = 0.9,\ Pr = 0.5$ and $M = 0.9$. in $(Nu, R/R_{sc})$-plane. In graphs (a) and (b), as R/R_{sc} increases then Nu increases.

and assume that $A_{1L} = A_{1L}(X, \tau, T)$ and $A_{1R} = A_{1R}(X, \tau, T)$. The differential operators can be expressed as

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau} + \epsilon^2 \frac{\partial}{\partial T}. \tag{5.3}$$

The solution of basic equations can be sought as power series in ϵ,

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + \cdots, \tag{5.4}$$

where $f = f(u, v, w, \theta)$ with the first approximation is given by eigenvector of the linearized problem:

$$\begin{aligned} u_0 &= \frac{i\pi}{q_{oc}} \left[A_{1L} e^{i(q_{oc}x + \omega_{oc}t)} + A_{1R} e^{i(q_{oc}x - \omega_{oc}t)} - c \cdot c. \right] \cos \pi z, \\ v_0 &= \frac{-i\pi Ta^{\frac{1}{2}}}{\phi q_{oc}} \left[\frac{A_{1L}}{e_2} e^{i(q_{oc}x + \omega_{oc}t)} + \frac{A_{1R}}{e_2^*} e^{i(q_{oc}x - \omega_{oc}t)} - c \cdot c \cdot, \right] \cos \pi z, \\ \theta_0 &= \frac{1}{M} \left[\frac{1}{e_1} A_{1L} e^{i(q_{oc}x + \omega_{oc}t)} + \frac{1}{e_1^*} A_{1R} e^{i(q_{oc}x - \omega_{oc}t)} + c \cdot c. \right] \sin \pi z, \end{aligned} \tag{5.5}$$

where $\delta_{oc}^2 = (\pi^2 + q_{oc}^2),\ e_1 = (\delta_{oc}^2 + i\omega_{oc}), e_2 = \left[\left(\frac{1}{MDa} + \frac{\Lambda}{M} \delta_{oc}^2 \right) + \frac{i\omega_{oc}}{M^2 \phi Pr} \right]$ here e_1^* and e_2^* are complex conjugate of e_1 and e_2, respectively.

We expand the linear operator $\mathcal{L}$ and nonlinear term $\mathcal{N}$ as the following power series

$$\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1 + \epsilon^2 \mathcal{L}_2 \cdots, \tag{5.6}$$

$$\mathcal{N} = \epsilon^2 \mathcal{N}_0 + \epsilon^3 \mathcal{N}_1 + \cdots \tag{5.7}$$

substituting Eqs. (5.3) and (5.4) into $\mathcal{L}w = \mathcal{N}$, for each order of ϵ, we get

$$\mathcal{L}_0 w_0 = 0, \tag{5.8}$$
$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \tag{5.9}$$
$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \tag{5.10}$$

Here

$$\begin{aligned} \mathcal{L}_0 &= \mathcal{D}\mathcal{D}_{Pr} \nabla^2 + \mathcal{D} \frac{Ta}{\phi^2} \frac{\partial^2}{\partial z^2} - \mathcal{D}_{Pr} \frac{R_{oc}}{M} \nabla_h^2, \\ \mathcal{L}_1 &= \frac{\partial \mathcal{F}_1}{\partial \tau} + 2 \frac{\partial^2 \mathcal{F}_2}{\partial x \partial X}, \\ \mathcal{L}_2 &= \frac{\partial \mathcal{F}_1}{\partial T} + 4 \frac{\partial^4}{\partial x^2 X^2} \left[\mathcal{D} \frac{\Lambda^2}{M^2} \nabla^2 - 2\mathcal{D}_{Pr} \frac{\Lambda}{M} \left(\frac{\partial}{\partial t} - 2\nabla^2 \right) - \mathcal{D}_{Pr}^2 + \frac{R_{oc}\Lambda}{M^2} \right] \\ &\quad + 2 \frac{\partial}{\partial \tau} \frac{\partial^2}{\partial x X} \left[-\mathcal{D} \frac{2\Lambda}{M^3 \phi Pr} \nabla^2 - \mathcal{D}_{Pr} \frac{2}{M^2 \phi Pr} \nabla^2 - 2\mathcal{D}_{Pr} \frac{\Lambda}{M} \nabla^2 + \mathcal{D}_{Pr}^2 + \mathcal{D}\mathcal{D}_{Pr} \frac{2}{M^2 \phi Pr} - \frac{R_{oc}}{M^3 \phi Pr} \right] \\ &\quad + \frac{\partial}{\partial \tau^2} \left[\mathcal{D} \frac{1}{M^4 \phi^2 Pr^2} \nabla^2 + \mathcal{D}_{Pr} \frac{2}{M^2 \phi Pr} \right] + \frac{\partial \mathcal{F}_2}{\partial X^2} - \mathcal{D}_{Pr} \frac{R_{oc}}{M} \nabla_h^2, \end{aligned} \tag{5.11}$$

where

$$\mathcal{F}_1 = \mathcal{D}\mathcal{D}_{Pr} \frac{2}{M^2 \phi Pr} \nabla^2 + \mathcal{D}_{Pr}^2 \nabla^2 + \frac{Ta}{\phi^2} \frac{\partial^2}{\partial z^2} - \frac{R_{oc}}{M^3 \phi Pr} \nabla_h^2$$

and

$$\mathcal{F}_2 = -2\mathcal{D}\mathcal{D}_{Pr} \frac{\Lambda}{M} \nabla^2 + \mathcal{D}_{Pr}^2 \left(\frac{\partial}{\partial t} - 2\nabla^2 \right) - \frac{Ta}{\phi^2} \frac{\partial^2}{\partial z^2} + \frac{R_{oc}\Lambda}{M^2} \nabla_h^2 - \mathcal{D}_{Pr} \frac{R_{oc}}{M}.$$

Eq. (5.8) is linear problem. We get critical Rayleigh number for the onset of oscillatory convection by using the zeroth order solution w_0 in Eq. (5.8). At $O(\epsilon^2), \mathcal{N}_0 = 0$ and $\mathcal{L}_1 w_0 = 0$ gives

$$\frac{\partial A_{1L}}{\partial \tau} - v_g \frac{\partial A_{1L}}{\partial X} = 0 \quad \text{and} \quad \frac{\partial A_{1R}}{\partial \tau} + v_g \frac{\partial A_{1R}}{\partial X} = 0, \tag{5.12}$$

where $v_g = (\partial \omega / \partial q)_{q = q_{oc}}$ is the group velocity and is real. Hence from Eq. (5.9), we get $w_1 = 0$. From equation of continuity we find that $u_1 = 0$. Substituting the zeroth order and first order approximation into Eqs. (4.14) and (4.15) we get

$$\theta_1 = \frac{-\pi}{M^2}\left[\left(|A_{1L}|^2 + |A_{1R}|^2\right)t_1 + \frac{2}{e_1 t_2}J_1 + \frac{2}{e_1^* t_2^*}J_1^*\right]\sin 2\pi z,$$

$$v_1 = \frac{-i\pi^2 Ta^{\frac{1}{2}}}{M^2\phi^3 Pr q_{oc}}\left[\frac{A_{1L}^2}{e_2 e_3}e^{2i(q_{oc}x+\omega_{oc}t)} + \frac{A_{1R}^2}{e_2^* e_3^*}e^{2i(q_{oc}x-\omega_{oc}t)} + \left(\frac{1}{e_2}+\frac{1}{e_2^*}\right)\left(\frac{1}{MD_a}+4\frac{\Lambda}{M}q_{oc}^2\right)^{-1}A_{1L}A_{1R}e^{2iq_{oc}x} - c\cdot c\cdot\right], \quad (5.13)$$

where $J_1 = A_{1L}A_{1R}^* e^{2i\omega_{oc}t},\ t_1 = \frac{1}{4\pi^2}\left(\frac{1}{e_1}+\frac{1}{e_1^*}\right), t_2 = (4\pi^2 + 2i\omega_{oc})$, and $e_3 = \left[\left(\frac{1}{MD_a} + 4\frac{\Lambda}{M}q_{oc}^2\right) + \frac{2i\omega_{oc}}{M^2\phi Pr}\right]$, here e_3 and J_1 are complex conjugates of e_3^* and J_1^*, respectively.

The Eq. (5.10) is solvable when $\mathcal{L}_0 w_0 = 0$, one requires that its right hand side be orthogonal to w_0, which is ensured that if the coefficients of $\sin\pi z$ in $\mathcal{N}_1 - \mathcal{L}_2 w_0$ are equal to zero. This implies that

$$\Lambda_0\frac{\partial A_{1L}}{\partial T} + \Lambda_1\left(\frac{\partial}{\partial\tau} - v_g\frac{\partial}{\partial X}\right)A_{2L} - \Lambda_2\frac{\partial^2 A_{1L}}{\partial X^2} - \Lambda_3 A_{1L} + \Lambda_4|A_{1L}|^2 A_{1L} + \Lambda_5|A_{1R}|^2 A_{1L} = 0, \quad (5.14)$$

$$\Lambda_0\frac{\partial A_{1R}}{\partial T} + \Lambda_1\left(\frac{\partial}{\partial\tau} - v_g\frac{\partial}{\partial X}\right)A_{2R} - \Lambda_2\frac{\partial^2 A_{1R}}{\partial X^2} - \Lambda_3 A_{1R} + \Lambda_4|A_{1R}|^2 A_{1R} + \Lambda_5|A_{1L}|^2 A_{1R} = 0, \quad (5.15)$$

where

$$\Lambda_0 = \frac{2\delta_{oc}^2}{M^2\phi Pr}e_1 e_2 + e_2^2\delta_{oc}^2 + \frac{Ta}{\phi^2}\pi^2 - \frac{R_{oc}q_{oc}^2}{M^2\phi Pr},$$

$$\Lambda_1 = \frac{\delta_{oc}^2}{M^4\phi^2 Pr^2}e_1 + \frac{2\delta_{oc}^2}{M^2\phi Pr}e_2,$$

$$\Lambda_2 = 4q^2\left[\frac{\Lambda^2}{M^2}\delta_{oc}^2 e_1 + 2\frac{\Lambda}{M}\delta_{oc}^2 e_2 - 2\frac{\Lambda}{M}e_1 e_2 + \frac{R_{oc}\Lambda}{M^2} - e_2^2\right],$$

$$\Lambda_3 = \frac{R_{oc}q_{oc}^2}{M}e_2,$$

$$\Lambda_4 = \frac{R_{oc}q_{oc}^2\pi^2}{M^3}e_2 t_1 - \frac{2\pi^4 Ta}{M^4\phi^6 Pr^2}\cdot\frac{e_1}{e_2 e_3},$$

$$\Lambda_5 = \frac{R_{oc}q_{oc}^2\pi^2}{M^3}\left(\frac{2}{e_1 t_2} + t_1\right)e_2 - \frac{2\pi^4 e_2 Ta}{M^4\phi^6 Pr^2}\left(\frac{1}{e_2}+\frac{1}{e_2^*}\right)\left(\frac{1}{MD_a} + 4\frac{\Lambda}{M}q_{oc}^2\right)^{-1}. \quad (5.16)$$

It should be noted that A_{1L} and A_{1R} are of order ϵ and A_{2L} and A_{2R} are of order ϵ^2. If $\omega_{oc} = 0$ in $\Lambda_0, \Lambda_2, \Lambda_3$ and Λ_4 then these expressions match with the coefficients $\lambda_0, \lambda_1, \lambda_2$ and λ_3 of Landau-Ginzburg equation at the onset of stationary convection.

From Eq. (5.12), we get $A_{1L}(\xi', T)$ and $A_{1R}(\eta', T)$, where $\xi' = v_g\tau + X, \eta' = v_g\tau - X$. Eqs. (5.14) and (5.15) can be written as

$$2v_g\Lambda_1\frac{\partial A_{2L}}{\partial\eta'} = -\Lambda_0\frac{\partial A_{1L}}{\partial T} + \Lambda_2\frac{\partial A_{1L}}{\partial X^2} + \lambda_3 A_{1L} - \left(\Lambda_4|A_{1L}|^2 + \Lambda_5|A_{1R}|^2\right)A_{1L}, \quad (5.17)$$

$$2v_g\Lambda_1\frac{\partial A_{2R}}{\partial\eta'} = -\Lambda_0\frac{\partial A_{1R}}{\partial T} + \Lambda_2\frac{\partial A_{1R}}{\partial X^2} + \lambda_3 A_{1R} - \left(\Lambda_4|A_{1R}|^2 + \Lambda_5|A_{1L}|^2\right)A_{1R}. \quad (5.18)$$

Let $\xi'\epsilon[0, l_1], \eta'\epsilon[0, l_2]$ where l_1 and l_2 are periods of A_{1L} and A_{1R}, respectively. Expansion (5.4) remains asymptotic for times $t = O(\epsilon^{-2})$ only if an appropriate solvability condition holds. This condition obtained integrating Eq. (5.17) over η' and Eq. (5.18) over ξ', we get

$$\Lambda_0\frac{\partial A_{1L}}{\partial T} = \Lambda_2\frac{\partial A_{1L}}{\partial X^2} + \lambda_3 A_{1L} - \left(\Lambda_4|A_{1L}|^2 + \Lambda_5|A_{1R}|^2\right)A_{1L}, \quad (5.19)$$

$$\Lambda_0\frac{\partial A_{1R}}{\partial T} = \Lambda_2\frac{\partial A_{1R}}{\partial X^2} + \lambda_3 A_{1R} - \left(\Lambda_4|A_{1R}|^2 + \Lambda_5|A_{1L}|^2\right)A_{1R}. \quad (5.20)$$

5.1. Travelling wave and standing wave convection

To study the stability regions of travelling waves and standing waves we proceed as follows:

On dropping slow variable X from Eqs. (5.19) and (5.20), we get a pair of first ODE's

$$\frac{dA_{1L}}{dT} = \frac{\Lambda_3}{\Lambda_0}A_{1L} - \frac{\Lambda_4}{\Lambda_0}A_{1L}|A_{1L}|^2 - \frac{\Lambda_5}{\Lambda_0}A_{1L}|A_{1R}|^2, \quad (5.21)$$

$$\frac{dA_{1R}}{dT} = \frac{\Lambda_3}{\Lambda_0}A_{1R} - \frac{\Lambda_4}{\Lambda_0}A_{1R}|A_{1R}|^2 - \frac{\Lambda_5}{\Lambda_0}A_{1R}|A_{1L}|^2 \quad (5.22)$$

put

$$\beta' = \frac{\Lambda_3}{\Lambda_0}, \quad \gamma' = -\frac{\Lambda_4}{\Lambda_0} \quad \text{and} \quad \delta' = -\frac{\Lambda_5}{\Lambda_0}.$$

Then Eqs. (5.21) and (5.22) take the following form

$$\frac{dA_{1L}}{dT} = \beta' A_{1L} + \gamma' A_{1L}|A_{1L}|^2 + \delta' A_{1L}|A_{1R}|^2, \tag{5.23}$$

$$\frac{dA_{1R}}{dT} = \beta' A_{1R} + \gamma' A_{1R}|A_{1R}|^2 + \delta' A_{1R}|A_{1L}|^2. \tag{5.24}$$

Consider $A_{1L} = a_L e^{i\phi_L}$ and $A_{1R} = a_L e^{i\phi_R}$ (we can write a complex number in the amplitude and phase form), where $a_L = |A_{1L}|,\ \phi_L = arg(A_{1L}) = \tan^{-1}(Im(A_{1L})/Re(A_{1L}))$ and $a_R = |A_{1R}|,\ \phi_R = arg(A_{1R}) = \tan^{-1}(Im(A_{1R})/Re(A_{1R}))$, here a_L, a_R, ϕ_L and ϕ_R are functions of time T, since A_{1L} and A_{1R} are functions of T. Thus a_L and a_R are positive functions.

Substituting the definitions of A_{1L}, A_{1R} and $\beta' = \beta_1 + i\beta_2,\ \gamma' = \gamma_1 + i\gamma_2,\ \delta' = \delta_1 + i\delta_2$ into Eqs. (5.23) and (5.24) we get,

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L|a_L|^2 + \delta_1 a_L|a_R|^2, \tag{5.25}$$
$$\frac{d\phi_L}{dT} = \beta_2 + \gamma_2|a_L|^2 + \delta_2|a_R|^2, \tag{5.26}$$
$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R|a_R|^2 + \delta_1 a_R|a_L|^2, \tag{5.27}$$
$$\frac{d\phi_R}{dT} = \beta_2 + \gamma_2|a_R|^2 + \delta_2|a_L|^2. \tag{5.28}$$

Eqs. (5.25) and (5.27) not contain phase term, so we take these two equations for the future discussions. We have Eqs. (5.25) and (5.27) as

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L^3 + \delta_1 a_R^2,$$
$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R^3 + \delta_1 a_L^2$$

since a_L and a_R are positive functions. Put

$$\frac{da_L}{dT} = F_1(a_L, a_R), \quad \frac{da_R}{dT} = F_2(a_L, a_R). \tag{5.29}$$

Now we discuss the stability of equilibrium points of above Eq. (5.29). We get four equilibrium points like $(a_L, a_R) = (0, 0)$ [conduction state], $(a_L, a_R) = (a_L, 0)$ [a_L = amplitude of left travelling waves, here we get $F_2 = 0$, and we get one condition from $F_1 = 0$ i.e., $a_L^2 = -\beta_1/\gamma_1 (= |A_{1L}|^2)$], $(a_L, a_R) = (0, a_R)$ [a_R = amplitude of right travelling waves, here we get $F_1 = 0$, and we get one condition from $F_2 = 0$ i.e., $a_R^2 = -\beta_1/\gamma_1 (= |A_{1R}|^2)$], and for $a_L \neq 0$ and $a_R \neq 0$ we get $(a_L, a_R) = (-\beta_1/(\gamma_1 + \delta_1), -\beta_1/(\gamma_1 + \delta_1))$ [this gives condition for standing waves. At standing waves we have $A_L = A_R$, so $a_L = a_R$]. For the pair of Eqs. (5.21) and (5.22), we do not get $a_L \neq a_R \neq 0$ [modulated waves]. now the Jacobian of F_1 and F_2 is given by

$$\begin{pmatrix} \partial F_1/\partial a_L & \partial F_1/\partial a_R \\ \partial F_2/\partial a_L & \partial F_2/\partial a_R \end{pmatrix},$$

If real parts of all eigenvalues of the Jacobian are negative at an equilibrium point, then that point is a stable equilibrium [Lyapounov's theorem or principle of linearized stability]. Some valuable conditions for travelling waves and standing waves are: Travelling waves are stable if $\beta_1 > 0,\ \gamma_1 < 0$ and $\delta_1 < \gamma_1 < 0$. Standing waves are stable if $\beta_1 > 0, \gamma_1 < 0$ and (i) if $\delta_1 > 0$, then $-\gamma_1 > \delta_1 > 0$, (ii) if $\delta_1 < 0$, then $-\gamma_1 > -\delta_1 > 0$.

The stability regions of travelling waves and standing waves are summarized in Fig. 10. Here E is total amplitude and defined as $E = a_L^2 + a_R^2$. We do not distinguish between left travelling waves and right travelling waves. For rest state (steady state) $E = 0$, for travelling waves $E = -\beta_1/\gamma_1$, for standing waves $E = -2\beta_1/(\gamma_1 + \varsigma_1)$. Travelling waves are supercritical if $\gamma_1 < 0$ and standing waves are supercritical if $\gamma_1 + \varsigma_1 < 0$. Fig. 10a is drawn for stable travelling wave conditions and Fig. 10b is drawn for stable standing wave conditions in (β_1, E)-plane. The symbols $(-,-)$ and $(+,+)$ in Fig. 10a and b indicate that both roots of Jacobian are negative and at least one root is positive between two roots.

In Fig. 10a and b, travelling wave solution and standing wave solution bifurcate simultaneously from the steady state solution ($\beta_1 \geqslant 0$ at this bifurcation point). In these Fig. 10a and b, steady state solution is stable for $\beta_1 < 0$ and unstable $\beta_1 > 0$. These figures show that for $\beta_1 > 0$ both travelling waves and standing waves are supercritical. When travelling waves and standing waves bifurcate supercritically then at most one solution among travelling waves and standing waves will be stable. Thus, for $\beta_1 > 0$ (Fig. 10a) travelling waves are stable and (Fig. 10b) standing waves are stable. In more detail we reproduce results of the stability analysis of equilibrium solutions in Fig. 10c, which is plotted in (γ_1, ς_1)-plane. From this figure we can observe that travelling waves are subcritical for $\gamma_1 > 0$ and standing waves are subcritical for $\gamma_1 + \varsigma_1 > 0$. In

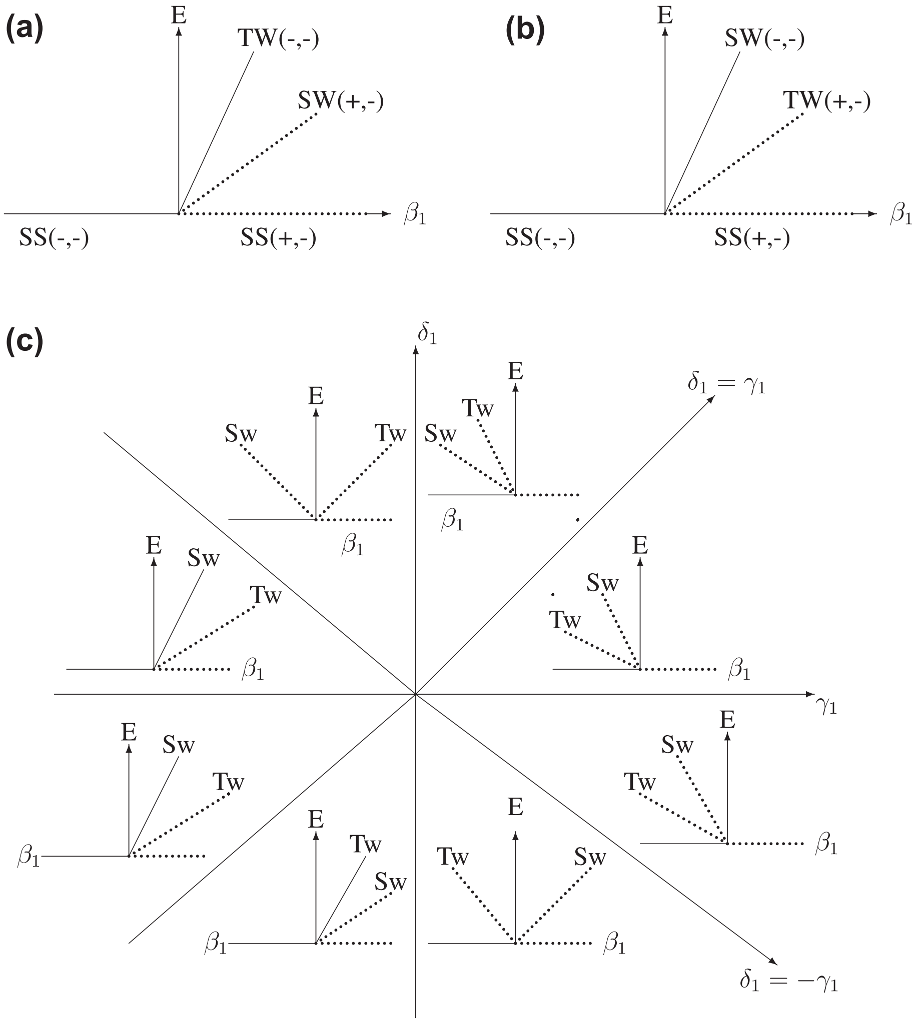

Fig. 10. (a), (b) and (c) are typical diagrams showing the stability of equilibrium solutions SS (steady state), SW (standing waves) and TW (travelling waves). On solid lines equilibrium solutions are stable and on dotted lines they are unstable.

Fig. 11, we have shown the stability regions for both travelling and standing waves for $Pr \ll 1, Pr = 0.5$ there is an intersection between standing waves and travelling waves.

5.2. Long wave-length instabilities for the onset of travelling wave convection (Benjamin–Feir instability)

For right travelling wave $A_R(X,T) = A(X,T)$ and $A_L(X,T) = 0$, for left travelling wave $A_R(X,T) = 0$ and $A_L(X,T) = A(X,T)$. Thus for travelling waves we get a single amplitude equation from Eqs. (5.19) and (5.20), given as

$$\Lambda_0 \frac{\partial A}{\partial T} - \Lambda_2 \frac{\partial^2 A}{\partial X^2} - \Lambda_3 A + \Lambda_4 |A|^2 A = 0. \tag{5.30}$$

For standing waves $A_{1L}(X,T) = A_{1R}(X,T) = A(X,T)$ and we get a single amplitude equation from Eqs. (5.19) and (5.20), given as

$$\Lambda_0 \frac{\partial A}{\partial T} - \Lambda_2 \frac{\partial^2}{\partial X^2} A - \Lambda_3 A + (\Lambda_4 + \Lambda_5)|A|^2 A = 0, \tag{5.31}$$

The above Eq. (5.31) possesses a family of planar wave solutions and solutions containing phase singular points. We study the Benjamin–Feir instability of travelling waves (which is similar to Eckhaus instability for onset of stationary convection) from complex Landau–Ginzburg equation (5.30). Eq. (5.30) can be written as

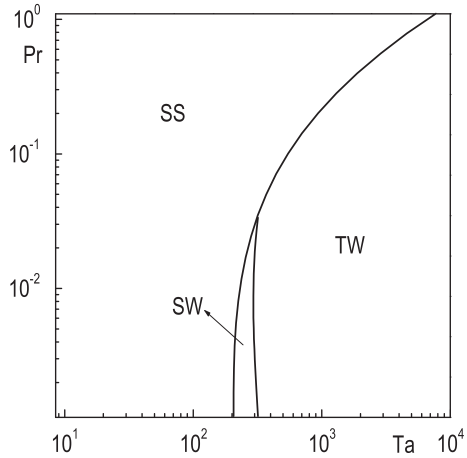

Fig. 11. Fig. 11 is plotted for $Pr = 0.5$. Stability regions of steady state (SS), travelling waves (TW) and standing waves (SW) are plotted (Q, Pr)-plane.

$$\frac{\partial A}{\partial T} = \xi \frac{\partial^2 A}{\partial X^2} + \beta A + \gamma |A|^2 A, \tag{5.32}$$

where $\xi = \xi_1 + i\xi_2,\ \beta = \beta_1 + i\beta_2,\ \gamma = \gamma_1 + i\gamma_2$. The phase winding solutions are obtained by substituting $A = \tilde{A_o} e^{i(\delta q_o X - \delta\omega T)}$, into Eq. (5.32) and equating real and imaginary parts we get

$$|\tilde{A_o}|^2 = \frac{\xi_1 \delta q_o^2 - \beta_1}{\gamma_1},$$

$$\delta\omega = \xi_2 \delta q_o^2 - \beta_2 + \frac{\gamma_2(\beta_1 - \xi_1 \delta q_o^2)}{\gamma_1}.$$

Here $\tilde{A_o}$ is constant and $\delta q_o = qX - q_{oc}$. We consider a modulated solution in the form: $A(X,T) = \widetilde{A}(X,T) e^{i(\delta q_o X - \delta\omega T)}$. Substituting the modulated into Eq. (5.32) which gives

$$\frac{\partial \widetilde{A}}{\partial T} = (\gamma_1 + i\gamma_2)\left[\left(\frac{\beta_1 - \delta q_o^2 \xi_1}{\gamma_1}\right) + |\widetilde{A}|^2\right]\widetilde{A} + (\gamma_1 + i\gamma_2)\left(\frac{\partial^2}{\partial X^2} + 2i\delta q_o \frac{\partial}{\partial X}\right)\widetilde{A}. \tag{5.33}$$

It is possible to conduct a general investigation of the linear stability of $A(X,T)$, but this is very difficult task, and therefore our primary concern here is to treat the stability of the uniformly oscillating solution $\tilde{A_o}$. Inserting $\widetilde{A} = \tilde{A_o} + \tilde{u} + i\tilde{v}$ into Eq. (5.33) and equating real and imaginary parts we get

$$\frac{\partial \tilde{u}}{\partial T} = -2(\beta_1 - \delta q_o^2 \xi_1)\tilde{u} + \xi_1\left(\frac{\partial^2 \tilde{u}}{\partial X^2} - 2\delta q_o \frac{\partial \tilde{v}}{\partial X}\right) - \xi_2\left(2\delta q_o \frac{\partial \tilde{u}}{\partial X} + \frac{\partial^2 \tilde{u}}{\partial X^2}\right), \tag{5.34}$$

$$\frac{\partial \tilde{v}}{\partial T} = \frac{-2\gamma_2(\beta_1 - \delta q_o^2 \xi_1)}{\gamma_1}\tilde{u} + \xi_1\left(2\delta q_o \frac{\partial \tilde{u}}{\partial X} + \frac{\partial^2 \tilde{u}}{\partial X^2}\right) + \xi_2\left(\frac{\partial^2 \tilde{u}}{\partial X^2} - 2\delta q_o \frac{\partial \tilde{v}}{\partial X}\right). \tag{5.35}$$

Consider $(\tilde{u}, \tilde{v}) = (U, V)e^{ST} \cos q_X X$ and S in the growth rate of disturbances. Using solutions of $\tilde{u}$, $\tilde{v}$ and $\delta q_o = 0$ into Eq. (5.34) and (5.35) we get

$$(S + 2\beta_1 + \xi_1 q_X^2)U - q_X^2 \xi_2 V = 0, \tag{5.36}$$

$$(S + q_X^2 \xi_1)V + \left(\frac{2\beta_1\gamma_2}{\gamma_1} + q_X^2 \xi_2\right)U = 0. \tag{5.37}$$

Solving Eqs. (5.36) and (5.37), we get

$$S^2 + 2S(\beta_1 + \xi_1 q_X^2) + q_X^2 \xi_1 (2\beta_1 + \xi_1 q_X^2) + q_X^2 \xi_2 \left(\frac{2\beta_1\gamma_2}{\gamma_1} + q_X^2 \xi_2\right). \tag{5.38}$$

There will be an instability only when a root of Eq. (5.38) is positive i.e.,

$$2\beta_1\left(\xi_1 + \frac{\gamma_2 \xi_2}{\gamma_1}\right) + q_X^2(\xi_1^2 + \xi_{12}^2) < 0, \tag{5.39}$$

$\beta_1 > 0$ when travelling waves or standing waves are stable. The instability of waves against long wavelength longitudinal modes is often called the Benjamin–Feir instability. Thus we get Benjamin–Feir instability for travelling waves when

$\xi_1 + \gamma_2\xi_2/\gamma_1 < 0$. Similarly by considering Eq. (5.32) instead of Eq. (5.31) and proceeding in the same way as above we get Benjamin-Feir instability for standing waves when $\xi_1 + (\gamma_2 + \delta_2)\xi_2/(\gamma_1 + \delta_1) < 0$.

6. Conclusions

In this paper we have considered both linear and weakly nonlinear analysis of rotating convection in a sparsely packed porous medium in Earth's outer core by using free-free (stress-free) boundary conditions. Even though free-free boundary conditions can not be achieved in laboratory, one can use it in geophysical fluid dynamic applications to Earth's outer core since they allow simple trigonometric eigenfunctions. Our goal is to identify the region of parameter values, for which roll emerge at the onset of convection.

Following Chandrasekhar [6], we have described the stationary convection and oscillatory convection as curves $R_s(q)$ and $R_o(q, Pr)$ vs wave numbers. The critical wave numbers for stationary convection and oscillatory convection are $q_{sc} = q_{oc} = \pi/\sqrt{2}$. For the problem of rotating convection in a sparsely packed porous medium, we get Takens–Bogdanov bifurcation point (TBBP) and codimension-two bifurcation point. In the case of linear theory both marginal and overstable motions are discussed. In the Figs. 1 and 2 is shown that the effect of Taylor number and porous parameters is to make the system more stable. By drawing stability boundaries in the Rayleigh number plan it is shown that the effect of rotating field and porous parameter is to decrease the region of stabilities. In the non-linear Eq. (4.17), $\lambda_0 = 0$ gives the TBBP at $q_s = q_{sc}$ and when $\lambda_0 = 0$ Eq. (4.17) is not valid. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$, subcritical if $\lambda_3 < 0$, and we get tricritical point if $\lambda_3 = 0$. We have obtained from Eq. (4.17), long wave length instabilities viz., Eckhaus and Zigzag Instabilities. From Eq. (4.17) which is valid only for $\lambda_3 > 0$, we have calculated Nusselt number Nu and studied heat transport by rotating convection. We have also derived two one-dimensional nonlinear coupled Landau–Ginzburg type equations viz., (5.14) and (5.15) at the onset of oscillatory convection at supercritical Hopf bifurcation. Weakly nonlinear theory must be used to resolve which of the standing and travelling waves will occur at the onset of convection. The coefficients in Eqs. (5.21) and (5.22) are complicated functions of the parameters Ta, Pr, Λ, ϕ, M and D_a, so it is not possible to give a simple criterion for the stability of the standing and travelling waves. We have computed stability regions of SW and TW at both Hopf bifurcation. The conditions for SW and TW are $A_L = A_R$ and $A_L = 0$ or $A_R = 0$, respectively. TW exist if $|A_L|^2 = -\beta_1/\gamma_1 > 0$ and they are supercritical if $\gamma_1 < 0$. SW exist if $|A_L|^2 = |A_R|^2 = -\beta_1/\gamma_1 + \delta_1 > 0$ and SW are supercritical if $\gamma_1 + \delta_1 < 0$. When both SW and TW are supercritical then at most one equilibrium solution is stable. At Takens–Bogdanov bifurcation point we get both TW and SW. By deriving one-dimensional Landau-Ginzburg equations with complex coefficients viz. Eqs. (5.30) and (5.31), we have shown the existence of Benjamin–Feir type of instability for both TW and SW. Near the Takens–Bogdanov bifurcation point the conducting state becomes unstable against both stationary and oscillatory mode, i.e., the real parts of two eigenvalues pass through zero simultaneously. This violates the assumption made for deriving amplitude equations (4.17), (5.14) and (5.15). Instead a new equation, which is second order in time, has to be used near the TBBP.

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