An Exponentially Fitted Non Symmetric Finite Difference Method for Singular Perturbation Problems

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Abstract: In this paper, we have presented an exponentially fitted non symmetric numerical method for singularly perturbed differential equations with layer behaviour. We have introduced a fitting factor in a non symmetric finite difference scheme which takes care of the rapid changes occur that in the boundary layer. This fitting factor is obtained from the theory of singular perturbations. The discrete invariant imbedding algorithm is used to solve the tridiagonal system of the fitted method. This method controls the rapid changes that occur in the boundary layer region and it gives good results in both cases i.e., $h \leq \varepsilon$ and $\varepsilon << h$. The existence and uniqueness of the discrete problem along with stability estimates are discussed. Also we have discussed the convergence of the method. Maximum absolute errors in numerical results are presented to illustrate the proposed method for $\varepsilon << h$.

Key–Words: Singularly perturbed two point boundary value problem, Boundary layer, Taylor series, Fitting factor, Maximum absolute error

1 Introduction

During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. Typically, these problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography, and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, Wentzel, Kramers and Brillouin (WKB) problems, the modelling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, etc.

The numerical treatment of singular perturbation problems has always been far from trivial, because of the boundary layer behaviour of the solutions. However, the area of singular perturbations is a field of increasing interest to applied mathematicians. Much progress has been made recently in developing finite difference and finite element methods for solving singular perturbation problems. Several authors Eckhaus [4], Natesan and Ramanujam [9], Valanarasu and Ramanujam [13] have investigated solving singular perturbation problems by numerically constructing asymptotic solutions. The general motivation is to provide simpler efficient computational techniques to solve singular perturbation problems. A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these, we mention Bawa [1], Bellman [2], Bender [3], Hemker et.al. [5], Kadalbajoo, Reddy [6], Kadalbajoo and Patidar [7], Kevorkian and Cole [8], Nayfeh [10], O'Malley [11], Ramos et.al. [12], Van Dyke [14] and Vigo-Aguiar, Natesan [15].

The fitted technique is one such tool to reach these goals in an optimum way. There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there, whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Present work deals with the second approach.

In this paper, we have presented an exponentially fitted non symmetric numerical method for singularly perturbed differential equations with layer behaviour. We have introduced a fitting factor in a non symmetric finite difference scheme which takes care of the rapid changes occur that in the boundary layer. This fitting factor is obtained from the theory of singular perturbations. The discrete invariant imbedding algorithm is used to solve the tridiagonal system of the fitted method. The existence and uniqueness of the discrete problem along with stability estimates are discussed. Also we have discussed the convergence of the method. Maximum absolute errors in numerical results are presented to illustrate the proposed method for $\varepsilon << h$.

2 Numerical Scheme

2.1 Left-end boundary layer problems

Consider a linearly singularly perturbed two point boundary value problem of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x)\ ,\ x \in [0,1] \tag{1}$$

with the boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta \tag{2}$$

whereε is a small positive parameter ($0 < \varepsilon << 1$) and α, β are known constants.

We assume that *a(x), b(x)* and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that *b(x)* ≤ 0, *a(x)* $\geq M >$ 0 throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$ which in general, displays a boundary layer of width O(ε) at $x = 0$ for small values of ε.

From the theory of singular perturbations it is known that the solution of (1) - (2) is of the form (O'Malley [11])

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{-\int\limits_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} \tag{3}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), y(1) = \beta \tag{4}$$

By taking Taylor's series expansion for $a(x)$ and $b(x)$ about the point '0' and restricting to their first terms, (3) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x} + O(\varepsilon) \tag{5}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0=$x_1, x_2, ...x_N$=1 be the mesh points. Then we have $x_i = ih$: i =0, 1, 2,. . . ,N. From (5), we get

$$y(x_i) = y_0(x_i) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x_i} + O(\varepsilon)$$

$$i.e., y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)ih} + O(\varepsilon)$$

Therefore,

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)i\rho} \tag{6}$$

where $\rho = \frac{h}{\varepsilon}$.

From finite differences, we have

$$y_{i-1} - 2y_i + y_{i+1} = h^2\left(y_i'' + \frac{h^2}{12}y_i^{(4)}\right) + O(h^6) \tag{7}$$

$$y_{i-1}'' - 2y_i'' + y_{i+1}'' = h^2 y_i^{(4)} + \frac{h^4}{12}y_i^{(6)} + ...$$

Substituting $h^2 y_i^{(4)}$ from the above equation in (7), we get

$$y_{i-1} - 2y_i + y_{i+1} = h^2\left(y_i'' + \frac{h^2}{12}y_i^{(4)}\right) + O(h^6)$$

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h^2}{12}\left(y_{i-1}'' + 10y_i'' + y_{i+1}''\right) \tag{8}$$

Now from the equation (1), we have

$$\varepsilon y_{i+1}'' = -a_{i+1}{y_{i+1}'}^{*} - b_{i+1}y_{i+1} + f_{i+1} \tag{9}$$

$$\varepsilon y_i'' = -a_i y_i' - b_i y_i + f_i \tag{10}$$

$$\varepsilon y_{i-1}'' = -a_{i-1}{y_{i-1}'}^{*} - b_{i-1}y_{i-1} + f_{i-1} \tag{11}$$

where we approximate ${y_{i+1}'}^{*}$ and ${y_{i-1}'}^{*}$ using non symmetric finite differences

$${y_{i+1}'}^{*} = \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} \tag{12}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \tag{13}$$

$${y_{i-1}'}^{*} = \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} \tag{14}$$

Substituting (12), (13) and (14) in (9), (10) and (11) respectively, and simplifying, we get

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}\left(-3y_{i-1} + 4y_i - y_{i+1}\right)$$
$$+\frac{10a_i}{24h}\left(y_{i+1} - y_{i-1}\right) + \frac{a_{i+1}}{24h}\left(y_{i-1} - 4y_i + 3y_{i+1}\right)$$
$$+\frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i}{12}y_i + \frac{b_{i+1}}{12}y_{i+1} = \frac{(f_{i-1} + 10f_i + f_{i+1})}{12}$$

Now introducing the fitting factor $\sigma(\rho)$in the above scheme, we have

$$\sigma(\rho)\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)+\frac{a_{i-1}}{24h}\left(-3y_{i-1}+4y_i-y_{i+1}\right)$$
$$+\frac{10a_i}{24h}\left(y_{i+1}-y_{i-1}\right)+\frac{a_{i+1}}{24h}\left(y_{i-1}-4y_i+3y_{i+1}\right)$$
$$+\frac{b_{i-1}}{12}y_{i-1}+\frac{10b_i}{12}y_i+\frac{b_{i+1}}{12}y_{i+1}=\frac{(f_{i-1}+10f_i+f_{i+1})}{12} \quad (15)$$

The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of difference scheme converges uniformly to the solution of (1) - (2).

Multiplying (15) by h and taking the limit as $h \to 0$, we get

$$\frac{\sigma(\rho)}{\rho}\underset{h\to 0}{Lt}\left(y_{i-1}-2y_i+y_{i+1}\right)+$$
$$\frac{a(0)}{24}\underset{h\to 0}{Lt}\left(-3y_{i-1}+4y_i-y_{i+1}\right)+$$
$$\frac{10a(0)}{24}\underset{h\to 0}{Lt}\left(y_{i+1}-y_{i-1}\right) \quad (16)$$
$$+\frac{a(0)}{24}\underset{h\to 0}{Lt}\left(y_{i-1}-4y_i+3y_{i+1}\right)=0$$

Let $B=\frac{a^2(0)-\varepsilon b(0)}{a(0)}$.

By using (6), we get

$$\underset{h\to 0}{Lt}\left(y_{i-1}-2y_i+y_{i+1}\right)=$$
$$\left(\alpha-y_0(0)\right)e^{-Bi\rho}\left(e^{B\rho}+e^{-B\rho}-2\right)$$

$$\underset{h\to 0}{Lt}\left(-3y_{i-1}+4y_i-y_{i+1}\right)=$$
$$\left(\alpha-y_0(0)\right)e^{-Bi\rho}\left(-3e^{B\rho}-e^{-B\rho}+4\right)$$

$$\underset{h\to 0}{Lt}\left(y_{i-1}-4y_i+3y_{i+1}\right)=$$
$$\left(\alpha-y_0(0)\right)e^{-Bi\rho}\left(e^{B\rho}+3^{-B\rho}-4\right)$$

$$\underset{h\to 0}{Lt}\left(y_{i+1}-y_{i-1}\right)=$$
$$\left(\alpha-y_0(0)\right)\ e^{-Bi\rho}\left(e^{-B\rho}-e^{B\rho}\right)$$

By using the above equations in equation (16), we get

$$\frac{\sigma(\rho)}{\rho}\left(e^{\frac{B\rho}{2}}-e^{-\frac{B\rho}{2}}\right)^2=\frac{a(0)}{2}\left(e^{B\rho}-e^{-B\rho}\right)$$

Therefore,

$$\sigma(\rho)=\rho\frac{a(0)}{2}\coth\left(\frac{\left(a^2(0)-\varepsilon b(0)\right)\rho}{2a(0)}\right) \quad (17)$$

which is a constant fitting factor. The tridiagonal system of the equation (15) is given by

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i,\quad i=1,2,\ldots,N-1 \quad (18)$$

where

$$E_j=\frac{\varepsilon\sigma}{h^2}-\frac{3a_{i-1}}{24h}+\frac{b_{i-1}}{12}-\frac{10a_i}{24h}+\frac{a_{i+1}}{24h}$$

$$F_j=\frac{2\varepsilon\sigma}{h^2}-\frac{4a_{i-1}}{24h}-\frac{10b_i}{12}+\frac{4a_{i+1}}{24h}$$

$$G_j=\frac{\varepsilon\sigma}{h^2}-\frac{a_{i-1}}{24h}+\frac{b_{i+1}}{12}+\frac{10a_i}{24h}+\frac{3a_{i+1}}{24h}$$

$$H_j=\frac{1}{12}\left(f_{i-1}+10f_i+f_{i+1}\right)$$

where $\sigma(\rho)$ is given by (17). We solve this tridiagonal system by the discrete invariant imbedding algorithm [6].

2.2 Right-end boundary layer problems

Finally, we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. We consider the singular perturbation problem

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)=f(x),\quad x\in[0,1] \quad (19)$$

with boundary conditions

$$y(0)=\alpha,\quad y(1)=\beta \quad (20)$$

where ε is a small positive parameter (0< ε <<1) and α, β are known constants.

We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of x=1.

From the theory of singular perturbations the solution of (19)-(20) is of the form

$$y(x)=y_0(x)+\frac{a(1)}{a(x)}(\beta-y_0(1))e^{\int_x^1\left(\frac{a(x)}{\varepsilon}-\frac{b(x)}{a(x)}\right)dx} \quad (21)$$

where $y_0(x)$ is the solution of the reduced problem

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), y_0(0) = \alpha \quad (22)$$

By taking the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '1' and restricting to their first terms, (21) becomes,

$$y(x) = y_0(x)+(\beta-y_0(1))\ e^{\left(\frac{a(1)}{\varepsilon}-\frac{b(1)}{a(1)}\right)(1-x)}\ +O(\varepsilon) \quad (23)$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0 = x_0, x_1,..., x_N =1 be the mesh points. Then we have x_i = *ih*, i =0,1,...,N.

From (23), we get

$$y(ih) = y_0(ih)+(\beta-y_0(1))\ e^{\left(\frac{a(1)}{\varepsilon}-\frac{b(1)}{a(1)}\right)(1-ih)}\ +O(\varepsilon).$$

Therefore,

$$\lim_{h\to 0} y(ih) = y_0(0)+(\beta-y_0(1))\ e^{\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\left(\frac{1}{\varepsilon}-i\rho\right)} \quad (24)$$

where $\rho = \frac{h}{\varepsilon}$.

Let $\hat{B} = \frac{a^2(1)-\varepsilon b(1)}{a(1)}$.

By using (24), we get

$$\underset{h\to 0}{Lt}\,(y_{i-1} - 2y_i + y_{i+1}) =$$

$$(\beta - y_0(1))\, e^{\hat{B}(\frac{1}{\varepsilon}-i\rho)} \left(e^{\hat{B}\rho} + e^{-\hat{B}\rho} - 2\right)$$

$$\underset{h\to 0}{Lt}\,(-3y_{i-1} + 4y_i - y_{i+1}) =$$

$$(\beta - y_0(1))\, e^{\hat{B}(\frac{1}{\varepsilon}-i\rho)} \left(-3e^{\hat{B}\rho} - e^{-\hat{B}\rho} + 4\right)$$

$$\underset{h\to 0}{Lt}\,(y_{i-1} - 4y_i + 3y_{i+1}) =$$

$$(\alpha - y_0(0))\, e^{\hat{B}(\frac{1}{\varepsilon}-i\rho)} \left(e^{\hat{B}\rho} + 3^{-\hat{B}\rho} - 4\right)$$

$$\underset{h\to 0}{Lt}\,(y_{i+1} - y_{i-1}) =$$

$$(\alpha - y_0(0))\ e^{\hat{B}(\frac{1}{\varepsilon}-i\rho)} \left(e^{-\hat{B}\rho} - e^{\hat{B}\rho}\right)$$

By using the above equations in equation (16), we get

$$\frac{\sigma(\rho)}{\rho}\left(e^{\frac{\hat{B}\rho}{2}} - e^{-\frac{\hat{B}\rho}{2}}\right)^2 = \frac{a(0)}{2}\left(e^{\hat{B}\rho} - e^{-\hat{B}\rho}\right)$$

Therefore, we get the fitting factor as

$$\sigma = \rho\frac{a(0)}{2}\coth\left(\frac{\left(a^2(1) - \varepsilon b(1)\right)\rho}{2a(1)}\right) \quad (25)$$

which is a constant fitting factor. Here, we have the same tridiagonal system of the (18), where σ is given by (25). We solve this tridiagonal system by the discrete invariant imbedding algorithm.

3 Stability and Convergence Analysis

Theorem 1 *Under the assumptions* $\varepsilon > 0$, $a(x) \geq M > 0$ *and b(x) < 0,*$\forall x \in [0, 1]$*, the solution to the system of the difference equations (18), together with the given boundary conditions exists, is unique and satisfies*

$$\|y\|_{h,\infty} \leq 2M^{-1}\|H\|_{h,\infty} + (\ |\alpha| + |\beta|\ )$$

where $\|\cdot\|_{h,\infty}$*is the discrete* $l_\infty - norm$*, given by*

$$\|x\|_{h,\infty} = \max_{0 \leq i \leq N}\{\ |x_i|\}.$$

Proof: Let $L_h(.)$ denote the difference operator on left hand side of (10) and w_i be any mesh function satisfying $L_h(w_i) = f_i$. By rearranging the difference scheme (18) and using non-negativity of the coefficients $E_i,\ F_i$ and G_i, we obtain

$$F_i\ |w_i|\ \leq\ |H_i| + E_i\ |w_{i-1}| + G_i\ |w_{i+1}|$$

$$\sigma\,\varepsilon\frac{(|w_{i+1}| - 2\,|w_i| + |w_{i-1}|)}{h^2}$$

$$+\frac{a_i}{12}\left(\frac{-\,|w_{i+1}| + 4\,|w_i| - 3\,|w_{i-1}|}{2h}\right)$$

$$+\frac{b_{i-1}}{12}\,|w_{i-1}| + \frac{10b_i}{12}\,|w_i| + \frac{b_{i+1}}{12}\,|w_{i+1}|$$

$$+\frac{10a_i}{12}\frac{(|w_{i+1}| - |w_{i-1}|)}{2h}$$

$$+\frac{a_i}{12}\left(\frac{3\,|w_{i+1}| - 4\,|w_i| + |w_{i-1}|}{2h}\right) + |H_i|\ \geq 0$$

Now using the assumption $\varepsilon > 0$ and $a_i \geq M$, the definition of l_∞-norm and manipulating the above inequality, we get

$$\sigma\,\varepsilon\frac{(|w_{i+1}|-2|w_i|+|w_{i-1}|)}{h^2}+\frac{M}{12}\left(\frac{-|w_{i+1}|+4|w_i|-3|w_{i-1}|}{2h}\right)$$
$$+\frac{b_{i-1}}{12}\,|w_{i-1}|+\frac{10b_i}{12}\,|w_i|+\frac{b_{i+1}}{12}\,|w_{i+1}|$$
$$+\frac{10M}{12}\frac{(|w_{i+1}|-|w_{i-1}|)}{2h}+\frac{M}{12}\left(\frac{3|w_{i+1}|-4|w_i|+|w_{i-1}|}{2h}\right)$$
$$+|H_i|\ \geq 0 \tag{26}$$

To prove the uniqueness and existence, let $\{u_i\}$, $\{v_i\}$ be two sets of solution of the difference equation (26) satisfying boundary conditions.

Then $w_i = u_i - v_i$ satisfies $L_h(w_i) = H_i$ where $H_i = 0$ and $w_0 = w_N = 0$.

Summing (26) over i = 1, 2, …, N-1, we get

$$-\sigma\varepsilon\frac{|w_1|}{h^2}-\sigma\varepsilon\frac{|w_{N-1}|}{h^2}+\|a\|_{h,\infty}\frac{|w_1|}{24h}$$
$$+\|a\|_{h,\infty}\frac{3|w_{N-1}|}{24h}+\frac{1}{12}\sum_{i=1}^{N-1}b_{i-1}\,|w_{i-1}|$$
$$+\frac{10}{12}\sum_{i=1}^{N-1}b_i\,|w_i|+\frac{1}{12}\sum_{i=1}^{N-1}b_{i+1}\,|w_{i+1}|-\frac{10M}{24h}\ |w_1|$$
$$-\frac{10M}{24h}\ |w_{N-2}|-\frac{3M}{24h}\ |w_1|-\frac{M}{24h}\ |w_{N-1}|$$
$$+\sum_{i=1}^{N-1}|H_i|\geq 0 \tag{27}$$

Since, $\varepsilon > 0, \|a\|_{h,\infty} \geq 0, b_i < 0$ and $|w_i| \geq 0, \forall i, i = 1, 2, ..., N-1$, therefore for inequality (27) to hold, we must have

$w_i = 0\ \ \forall i,\ i = 1, 2, ...N-1.$

This implies the uniqueness of the solution of the tridiagonal system of difference equations (18). For linear equations, the existence is implied by uniqueness.

Now to establish the estimate, let $w_i = y_i - l_i$, where y_i satisfies difference equations (18), the boundary conditions and $l_i = (1-ih)\,\alpha + (ih)\,\beta$, then $w_0 = w_N = 0$, and $w_i,\ i = 1, 2, ...N-1$.

Now let $|w_n| = \|w\|_{h,\infty} \geq |w_i|,\ i = 0, 1, ..., N.$

Then summing (26) from $i = n$ to $N-1$ and using the assumption on $a(x)$, which gives

$$-\sigma\varepsilon\frac{(|w_n|-|w_{n-1}|)}{h^2}-\sigma\varepsilon\frac{|w_{N-1}|}{h^2}+$$
$$\frac{M}{12}\left(\frac{-|w_{N-1}|+4|w_n|-3|w_{n-1}|}{2h}\right)+\frac{1}{12}\sum_{i=n}^{N-1}b_{i-1}\,|w_{i-1}|+$$
$$\frac{10}{12}\sum_{i=n}^{N-1}b_i\,|w_i|+\frac{1}{12}\sum_{i=n}^{N-1}b_{i+1}\,|w_{i+1}|+$$
$$\frac{10M}{12}\left(\frac{|w_{N-1}|-|w_n|-|w_{n-1}|}{2h}\right)$$
$$+\frac{M}{12}\left(\frac{-|w_{N-1}|-3|w_n|-|w_{n-1}|}{2h}\right)+\sum_{i=n}^{N-1}|H_i|\geq 0 \tag{28}$$

Inequality (28), together with the condition on $b(x)$ implies that

$$\frac{M}{2}\,|w_n|\leq h\sum_{i=n}^{N-1}|H_i|\leq h\sum_{i=0}^{N}|h_i|\leq\|H\|_{h,\infty},$$

i.e., we have

$$|w_n|\leq 2M^{-1}\,\|H\|_{h,\infty} \tag{29}$$

Also, we have $y_i = w_i + l_i$

$$\begin{aligned}\|y\|_{h,\infty} &= \max_{0\leq i\leq N}\{\ |y_i|\ \}\\ &\leq\ \|w\|_{h,\infty}+\|l\|_{h,\infty}\\ &\leq\ |w_n|+\|l\|_{h,\infty}.\end{aligned} \tag{30}$$

Now to complete the estimate, we have to find out the bound on l_i

$$\begin{aligned}\|l\|_{h,\infty} &= \max_{0\leq i\leq N}\{\ |l_i|\ \}\\ &\leq \max_{0\leq i\leq N}\{\ |(1-ih)|\ \ |\alpha|+|ih|\ \ |\beta|\}\\ &\leq \max_{0\leq i\leq N}\{\ (1-ih)\ \ |\alpha|+(ih)\ \ |\beta|\},\end{aligned}$$

i.e., we have

$$\|\ l\ \|_{h,\infty}\leq|\alpha|+|\beta| \tag{31}$$

From Eqs. (30) – (31), we get the estimate

$$\|y\|_{h,\infty}\leq 2M^{-1}\,\|H\|_{h,\infty}+(|\alpha|+|\beta|).$$

This theorem implies that the solution to the system of the difference equations (18) are uniformly bounded, independent of mesh size h and the perturbation parameterε. Thus the scheme is stable for all step sizes.

Corollary 2 *Under the conditions for theorem 1, the error $e_i = y(x_i) - y_i$ between the solution $y(x)$ of the continues problem and the solution y_i of the discretized problem, with boundary conditions, satisfies the estimate*

$$\|e\|_{h,\infty} \le 2M^{-1} \|\tau\|_{h,\infty}, \text{ where}$$

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{\sigma h^2 \varepsilon^2}{12} |y^{(4)}(x)| \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{10ah^2}{72} |y^{(3)}(x)| \right\}$$

Proof: Truncation error τ_i in the difference scheme is given by

$$\tau_i = \sigma\varepsilon \left\{ \left(\frac{y_{i+1} - y_i + y_{i-1}}{h^2} \right) - y_i'' \right\} + \frac{a_{i-1}}{12} \left(\frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} - y_{i-1}' \right) + \frac{10a_i}{12} \left(\frac{y_{i+1} - y_{i-1}}{2h} - y_i' \right) + \frac{a_{i+1}}{12} \left(\frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} - y_{i+1}' \right)$$

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \tfrac{\sigma h^2 \varepsilon^2}{12} \left| y^{(4)}(x) \right| \right\} - \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \tfrac{ah}{12} \left| y^{(2)}(x) \right| \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \tfrac{10ah^2}{72} \left| y^{(3)}(x) \right| \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \tfrac{ah}{12} \left| y^{(2)}(x) \right| \right\}$$

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}} \{ \frac{\sigma h^2 \varepsilon^2}{12} |y^{(4)}(x)| \} + \max_{x_{i-1}\le x\le x_{i+1}} \{ \frac{10ah^2}{72} |y^{(3)}(x)| \} \tag{32}$$

One can easily show that the error $e_i$satisfies

$$L_h(e(x_i)) = L_h(y(x_i)) - L_h(y_i) = \tau_i,$$

$i = 1, 2, ..., N-1$ and $e_0 = e_N = 0$.

Then Theorem 1 implies that

$$\|e\|_{h,\infty} \le 2M^{-1} \|\tau\|_{h,\infty} \tag{33}$$

The estimate (33) establishes the convergence of the difference scheme for the fixed values of the parameter ε.

Theorem 3 *Under the assumptions $\varepsilon > 0$, $a(x) \le M < 0$ and $b(x)$ <0, $\forall x \in [0,\ 1]$, the solution to the system of the difference equations (18), together with the given boundary conditions exists, is unique and satisfies*

$$\|y\|_{h,\infty} \le 2M^{-1} \|H\|_{h,\infty} + (|\alpha| + |\beta|).$$

The proof of estimate can be done on similar lines as we did in theorem 1.

4 Numerical Examples

To demonstrate the applicability of the method we have applied it to three linear singular perturbation problems with left-end boundary layer, two linear singular perturbation problems with right-end boundary layer and two nonlinear singular perturbation problems. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The numerical solutions are compared with the exact solutions and maximum absolute errors with and without fitting factor are presented to support the given method.

Example 4 *Consider the following homogeneous singular perturbation problem from Bender and Orszag [3]*

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad x \in [0, 1]$$

with y(0) = 1 and y(1) = 1.

Clearly this problem has a boundary layer at x = 0. i.e. at the left end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$$

where

$$m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$$

and

$$m_2 = (-1 - \sqrt{1+4\varepsilon})/(2\varepsilon)$$

The maximum absolute errors are presented in table 1 for different values of ε with and without fitting factor.

Example 5 *Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity*

$$\varepsilon y''(x) + y'(x) = 1 + 2x; \quad x \in [0,1]$$

with y(0) = 0 and y(1) = 1.

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + \frac{(2\varepsilon-1)\left(1-e^{-x/\varepsilon}\right)}{(1-e^{-1/\varepsilon})}$$

The maximum absolute errors are presented in table 2 for different values of ε with and without fitting factor.

Example 6 *Finally we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [8]*

$$\varepsilon y''(x) + (1-\frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0; \quad x \in [0,1]$$

with y(0) = 0 and y(1) = 1.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [10] as our 'exact' solution

$$y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-(x-x^2/4)/\varepsilon}$$

The maximum absolute errors are presented in table 3 for different values of ε with and without fitting factor.

Example 7 *Consider the following singular perturbation problem*

$$\varepsilon y''(x) - y'(x) = 0; \quad x \in [0,1]$$

with y(0) = 1 and y(1) = 0.

Clearly, this problem has a boundary layer at x=1. i.e., at the right end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{\left(e^{(x-1)/\varepsilon} - 1\right)}{(e^{-1/\varepsilon} - 1)}$$

The maximum absolute errors are presented in table 4 for different values of ε with and without fitting factor.

Example 8 *Now we consider the following singular perturbation problem*

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0; \quad x \in [0,1]$$

with y(0) = 1+exp(-(1+ε)/ε); and y(1) =1+1/e.

The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$$

The maximum absolute errors are presented in table 5 for different values of ε with and without fitting factor.

Example 9 *Now consider the following non singular perturbation problem from Kevorkian and Cole [[8], page 56; equation 2.5.1]*

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0; x \in [0,1]$$

with y(0)= -1 and y(1)=3.9995

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x+2.9995)y'(x) = x + 2.9995$$

We have chosen to use the Kivorkian and Cole's uniformly valid approximation [[8], pages 57 and 58; equations (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right)$$

where c_1 = 2.9995 and $c_2 = (1/c_1)\log_e[(c_1-1)/(c_1+1)]$

For this example also we have a boundary layer of width O(ε) at x = 0.

The maximum absolute errors are presented in table 6 for different values of ε with and without fitting factor.

Example 10 *Finally we consider the following non singular perturbation problem from O' Malley [[11], page 9; equation (1.10) case 2]:*

$$\varepsilon y''(x) - y(x)y'(x) = 0; x \in [-1,1]$$

with y(-1) = 0 and y(1) = -1.

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0$$

We have chosen to use O' Malley's approximate solution [[11], pages 9 and 10; equations 1.13 and 1.14] for comparison,

$$y(x) = -\frac{\left(1-e^{-(x+1)/\varepsilon}\right)}{(1+e^{-(x+1)/\varepsilon})}$$

For this example, we have a boundary layer of width O(ε) at x = -1.

The maximum absolute errors are presented in table 7 for different values of ε with and without fitting factor.

5 Discussions and Conclusions

We have described an exponentially fitted non symmetric second order numerical method for singularly perturbed problems. We have introduced a fitting factor in a non symmetric second order finite difference scheme which takes care of the rapid changes occur that in the boundary layer. This fitting factor is obtained from the theory of singular perturbations.

The discrete invariant imbedding algorithm is used to solve the tridiagonal system of the fitted method. The existence and uniqueness of the discrete problem along with stability estimates are discussed. We have presented maximum absolute errors for the standard examples chosen from the literature and also presented maximum absolute errors for the some of the examples with and without fitting factor to show the efficiency of the method when $\varepsilon << h$.

The computational rate of convergence is also obtained by using the double mesh principle defined below.

Let

$$Z_h = \max_j \left| y_j^h - y_j^{h/2} \right|, j = 0, 1, \ldots, N-1,$$

where y_j^h is the computed solution on the mesh $\{\, x_j\}_0^N$ at the nodal point x_j where

$$x_j = x_{j-1} + h,\ j = 1, 2, ..., N$$

and $y_j^{h/2}$ is the computed solution at the nodal point $x_j$on the mesh$\{\, x_j\}_0^{2N}$ where

$$x_j = x_{j-1} + h/2 for j = 1(1)2N$$

.

In the same way we can define $Z_{h/2}$ by replacing h by h/2 and N by $2N$

i.e.,

$$Z_{h/2} = \max_j \left| y_j^{h/2} - y_j^{h/4} \right|, j = 0, 1, ..., 2N-1.$$

The computed order of convergence is defined as

$$\text{Order} = \frac{\log Z_h - \log Z_{h/2}}{\log(2)}$$

We have taken $h = 2^{-3}$ for finding the computed order of convergence and results are shown in Table 8.

Table 1: The maximum absolute errors in solution of example 4

h	$\varepsilon = 10^{-3}$ with f.f	without f.f	$\varepsilon = 10^{-5}$ with f.f	without f.f
1/8	2.02e-002	7.82e-001	2.05e-002	8.82e-001
1/16	1.06e-002	6.58e-001	1.09e-002	8.99e-001
1/32	5.26e-003	5.48e-001	5.63e-003	8.99e-001
1/64	2.48e-003	4.83e-001	2.84e-003	8.65e-001
1/128	1.06e-003	3.72e-001	1.42e-003	7.62e-001

f.f.=fitting factor

Table 2: The maximum absolute errors in solution of example 5

h	$\varepsilon = 10^{-3}$ with f.f	without f.f	$\varepsilon = 10^{-5}$ with f.f	without f.f
1/8	1.07e-001	15.41	1.09e-001	1.56(+3)
1/16	5.67e-002	4.043	5.85e-002	390.49
1/32	2.83e-002	1.8208	3.02e-002	97.60
1/64	1.34e-002	1.5446	1.53e-002	24.434
1/128	6.17e-003	1.1839	7.73e-003	6.2991

Table 3: The maximum absolute errors in solution of example 6

h	$\varepsilon = 10^{-3}$ with f.f	without f.f	$\varepsilon = 10^{-5}$ with f.f	without f.f
1/8	4.48e-002	1.0511	4.48e-002	1.8761
1/16	2.44e-002	6.01e-001	2.44e-002	1.8884
1/32	1.28e-002	4.39e-001	1.28e-002	1.7658
1/64	6.62e-003	3.84e-001	6.62e-003	1.3177
1/128	3.77e-003	2.94e-001	3.36e-003	0.7490

Table 4: The maximum absolute errors in solution of example 7

h	$\varepsilon = 10^{-3}$ with f.f	$\varepsilon = 10^{-3}$ without f.f	$\varepsilon = 10^{-5}$ with f.f	$\varepsilon = 10^{-5}$ without f.f
1/8	0	7.71e+000	0	7.81e+002
1/16	0	2.02e+000	0	1.95e+002
1/32	1.11e-016	9.11e-001	0	4.88e+001
1/64	1.11e-016	7.73e-001	0	1.22e+0
1/128	2.78e-014	5.92e-001	0	3.14e+000

Table 5: The maximum absolute errors in solution of example 8

h	$\varepsilon = 10^{-3}$ with f.f	$\varepsilon = 10^{-3}$ without f.f	$\varepsilon = 10^{-5}$ with f.f	$\varepsilon = 10^{-5}$ without f.f
1/8	2.02e-002	1.23e+000	2.06e-002	1.39e+000
1/16	1.06e-002	1.04e+000	1.10e-002	1.42e+000
1/32	5.27e-003	8.67e-001	5.63e-003	1.42e+000
1/64	2.48e-003	7.61e-001	2.84e-003	1.36e+000
1/128	1.06e-003	5.89e-001	1.42e-003	1.20e+000

Table 6: The maximum absolute errors in solution of example 9

h	$\varepsilon = 10^{-3}$ with f.f	$\varepsilon = 10^{-3}$ without f.f	$\varepsilon = 10^{-5}$ with f.f	$\varepsilon = 10^{-5}$ without f.f
1/8	5.30e-002	29.69	5.30e-002	41.39
1/16	2.71e-002	16.41	2.71e-002	40.89
1/32	1.37e-002	6.46	1.37e-002	39.26
1/64	6.90e-003	3.71	6.90e-003	33.55
1/128	3.50e-003	3.36	3.50e-003	21.23

Table 7: The maximum absolute errors in solution of example 10

h	$\varepsilon = 10^{-3}$ with f.f	$\varepsilon = 10^{-3}$ without f.f	$\varepsilon = 10^{-5}$ with f.f	$\varepsilon = 10^{-5}$ without f.f
1/8	0	3.91	0	390.56
1/16	0	1.22	0	97.162
1/32	2.67e-014	8.80e-001	0	24.41
1/64	1.63e-007	7.73e-001	0	6.14
1/128	4.04e-004	5.93e-001	0	1.73

Table 8. Numerical order of convergence for examples 4- 10.

	h	$h/2$	Z_h	Order of conv.
Example 1	2^{-3}	2^{-4}	6.49E-02	
	2^{-4}	2^{-5}	1.06E-02	2.6123
Example 2	2^{-3}	2^{-4}	1.62e-02	
	2^{-4}	2^{-5}	5.42e-003	1.5803
Example 3	2^{-3}	2^{-4}	1.05e-001	
	2^{-4}	2^{-5}	1.85e-002	2.5137
Example 4	2^{-3}	2^{-4}	1.11e-001	
	2^{-4}	2^{-5}	1.84e-002	2.5873
Example 5	2^{-3}	2^{-4}	1.04e-001	
	2^{-4}	2^{-5}	1.70e-002	2.6115
Example 6	2^{-3}	2^{-4}	4.55e-001	
	2^{-4}	2^{-5}	8.00e-002	2.5077
Example 7	2^{-3}	2^{-4}	1.11e-001	
	2^{-4}	2^{-5}	1.85e-002	2.5873

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