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NUMERICAL SOLUTION OF SIXTH ORDER BOUNDARY VALUE PROBLEMS BY PETROV-GALERKIN METHOD WITH QUARTIC B-SPLINES AS BASIS FUNCTIONS AND QUINTIC B-SPLINES AS WEIGHT FUNCTIONS

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ABSTRACT

In this paper, a finite element method involving Petrov-Galerkin method with quartic B-splines as basic functions and quintic B-splines as weight functions has been developed to solve a general sixth order boundary value problem with a particular case of boundary conditions. The basic functions are redefined into a new set of basic functions which vanish on the boundary where the Dirichlet and Neumann or mixed types of boundary conditions are prescribed. The weight functions are also redefined into a new set of weight functions which in number match with the number of redefined basis functions. The proposed method was applied to solve several examples of sixth order linear and nonlinear boundary value problems. The obtained numerical results were found to be in good agreement with the exact solutions available in the literature.

Keywords: Petrov-galerkin method, quartic B-spline, quintic B-spline, sixth order boundary value problem, absolute error.

1. INTRODUCTION

In this paper, we consider a general sixth order linear boundary value problem

$$p_0(t)v^{(6)}(t)+p_1(t)v^{(5)}(t)+p_2(t)v^{(4)}(t)+p_3(t)v'''(t)+p_4(t)v''(t) \\ +p_5(t)v'(t)+p_6(t)v(t)=q(t),\quad a<x<b \tag{1}$$

subject to boundary conditions

$$v(a)=A_0,\ v(b)=C_0,\ v'(a)=A_1,\ v'(b)=C_1,\ v''(a)=A_2,\ v''(b)=C_2 \tag{2a}$$

or

$$v(a)=A_0,\ v(b)=C_0,\ v'(a)+\sigma_1 v(a)=A_1,\ v'(b)+\sigma_2 v(b)=C_1, \\ v''(a)=A_2, v''(b)=C_2 \tag{2b}$$

where A_0, C_0, A_1, C_1, A_2, σ_1 and σ_2 are real constants and $p_0(t)$, $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$, $p_5(t)$ and $q(t)$ are continuous functions defined in [a, b].

The sixth order boundary value problems occur in astrophysics [1]. Chandrasekhar [2] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is of sixth order. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [3]. Finding the analytical solutions of such type of boundary value problems in general is not possible. Over the years, many researchers have worked on boundary value problems by using different methods for numerical solutions. Wazwaz [4] developed the solution of special type of sixth order boundary value problems by using the modified Adomain decomposition method. Huan [5] presented variational approach technique to solve a special case of sixth order boundary value problems. Noor *et al.* [6] presented the variational iteration principle to solve a special case of sixth order boundary value problems after transforming the given differential equation into a system of integral equations. Ghazala and Siddiqi [7], Ramadan *et al.* [8] presented the solution of a special case of sixth order boundary value problems by using non-polynomial spline functions and septic non-polynomial spline functions respectively. Siddiqi *et al.* [9], Siddiqi and Ghazala [10] developed quintic spline funtions and septic spline functions techniques to solve a special case of linear sixth order boundary value problems respectively. Lamnii *et al.* [11], Kasi Viswanadham and Showri raju [12] developed septic spline collocation and quintic B-spline collocation method to solve sixth order boundary value problems respectively. Loghmani and Ahmadinia [13] used sixth degree B-spline functions to construct an approximation solution for sixth order boundary value problems. Waleed [14] presented Adomain decomposition method with Green's function to solve a special case of sixth order boundary value problems. Liang and Jefferey [15] presented Homotopy analysis method to solve a parameterized sixth order boundary value problem for large parameter values. Kasi Viswanadham and Murali krishna [16] developed septic B-spline collocation method to solve a special case of sixth order boundary value problems. Kasi Viswanadham and Sreenivasulu [17] developed quintic B-spline Galerkin method to solve a general sixth order boundary value problem. Kasi Viswanadham and Reddy [18] solved a general sixth order boundary value problem with quartic B-splines as basis functions and sextic B-splines as weight functions. So far, sixth order boundary value problems have not been solved by using Petrov-Galerkin method with quartic B-splines as basis functions and quintic B-splines as weight functions. This motivated us to solve a sixth order boundary value problem by Petrov-Galerkin method with quartic B-splines as basis functions and quintic B-splines as weight functions.

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In this paper, we try to present a simple finite element method which involves Petrov-Galerkin approach with quartic B-splines as basis functions and quintic B-splines as weight functions to solve a general sixth order boundary value problem of the type (1)-(2). This paper is organized as follows. Section 2 deals with the justification for using Petrov-Galerkin Method. In Section 3, a description of Petrov-Galerkin method with quartic B-splines as basis functions and quintic B-splines as weight functions is explained. In particular we first introduce the concept of quartic B-splines, quintic B-splines and followed by the proposed method with the specified boundary conditions. In Section 4, the procedure to solve the nodal parameters has been presented. In section 5, the proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [19]. Finally, in the last section, the conclusions are presented.

2. JUSTIFICATION FOR USING PETROV-GALERKIN METHOD

In Finite Element Method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz method, Galerkin method, Least Squares method, Petrov-Galerkin method and Collocation method etc. In Petrov-Galerkin method, the residual of approximation is made orthogonal to the weight functions. When we use Petrov-Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [20, 21] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to the boundary conditions [22]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed and also the number of weight functions should match with the number of basis functions. Hence in this paper we employed the use of Petrov-Galerkin method with quartic B-splines as basis functions and quintic B-splines as weight functions to approximate the solution of sixth order boundary value problem.

3. DESCRIPTION OF THE METHOD

3.1 Definition of quartic B-splines and quintic B-splines

The quartic B-splines and quintic B-splines are described in [23-25]. Space variable domain [a, b] is divided into spaced knots (which need not be spaced evenly) given by the partition $a=t_0<t_1<\ldots<t_{n-1}<t_n=b$. Eight additional knots t_{-4}, t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}, t_{n+3} and t_{n+4} are introduced which satisfy the relation

$t_{-4}<t_{-3}<t_{-2}<t_{-1}<t_0$ and $t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4}$.

Now the quartic B-splines $S_i(t)$'s are defined by

$$S_i(t)=\begin{cases}\sum_{r=i-2}^{i+3}\dfrac{(t_r-t)_+^4}{\pi'(t_r)}, & t\in[t_{i-2},\,t_{i+3}]\\ 0, & otherwise\end{cases}$$

where $(t_r-t)_+^4=\begin{cases}(t_r-t)^4, & if\ t_r\geq t\\ 0, & if\ t_r\leq t\end{cases}$

and $\pi(t)=\prod_{r=i-2}^{i+3}(t-t_r)$

where $\{S_{-2}(t), S_{-1}(t), S_0(t), S_1(t),\ldots, S_{n-1}(t), S_n(t), S_{n+1}(t)\}$ forms a basis for the space $S_4(\pi)$ of quartic polynomial splines. Schoenberg [25] has shown that quartic B-splines are the unique nonzero splines of smallest compact support with the knots at $t_{-4}<t_{-3}<t_{-2}<t_{-1}<t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4}$.

In the same way, the quintic B-splines $R_i(t)$'s are defined by

$$R_i(t)=\begin{cases}\sum_{r=i-3}^{i+3}\dfrac{(t_r-t)_+^5}{\pi'(t_r)}, & x\in[t_{i-3},\,t_{i+3}]\\ 0, & otherwise\end{cases}$$

where $(t_r-t)_+^5=\begin{cases}(t_r-t)^5, & if\ t_r\geq t\\ 0, & if\ t_r\leq t\end{cases}$

and $\pi(t)=\prod_{r=i-3}^{i+3}(t-t_r)$

where $\{R_{-2}(t), R_{-1}(t), R_0(t), R_1(t),\ldots,R_{n-1}(t), R_n(t), R_{n+1}(t), R_{n+2}(t)\}$ forms a basis for the space $S_5(\pi)$ of quintic polynomial splines by introducing two more additional knots t_{-5}, t_{n+5} to the already existing knots t_{-4} to t_{n+4}. Schoenberg [25] has shown that quintic B-splines are the unique nonzero splines of smallest compact support with the knots at $t_{-5}<t_{-4}<t_{-3}<t_2<t_1<t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<t_{n+2}<t_{n+3}<t_{n+4.}<t_{n+5}$.

We define the approximation for $v(t)$ as

$$v(t)=\sum_{j=-2}^{n+1}\alpha_j S_j(t) \tag{3}$$

where α_j's are the nodal parameters to be determined and $S_j(t)$'s are the quartic B-spline basis functions. In Petrov-Galerkin method, the basis functions should vanish on the boundary where the essential types of boundary conditions are prescribed. In the set of quartic B-splines $\{S_{-2}(t), S_{-1}(t), S_0(t), S_1(t),\ldots, S_{n-1}(t), S_n(t), S_{n+1}(t)\}$, the basis functions $S_{-2}(t), S_{-1}(t), S_0(t),S_1(t)$ do not vanish on the left boundary and $S_{n-2}(t), S_{n-1}(t), S_n(t)$ and $S_{n+1}(t)$ do not vanish on the right boundary. So, it is necessary to redefine the basis functions into a new set of basis functions which vanish on the boundary where the essential type boundary conditions are specified. When the chosen approximation satisfies the prescribed boundary conditions or most of the boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, the Neumann or mixed boundary conditions are prescribed.

3.2 Redefinition of basis functions with boundary conditions (2a)

Applying the essential boundary conditions of (2), we get the approximate solution *v(t)* at the boundary points as

$$A_0 = v(a) = v(t_0) = \alpha_{-2}S_{-2}(t_0) + \alpha_{-1}S_{-1}(t_0) + \alpha_0 S_0(t_0) + \alpha_1 S_1(t_0) \quad (4)$$

$$C_0 = v(b) = v(t_n) = \alpha_{n-2}S_{n-2}(t_n) + \alpha_{n-1}S_{n-1}(t_n) + \alpha_n S_n(t_n) + \alpha_{n+1}S_{n+1}(t_n) \quad (5)$$

Eliminating α_{-2} and α_{n+1} from the equations (3), (4) and (5) we get

$$v(t) = w_1(t) + \sum_{j=-1}^{n} \alpha_j P_j(t) \quad (6)$$

where

$$w_1(t) = \frac{A_0}{S_{-2}(t_0)} S_{-2}(t) + \frac{C_0}{S_{n+1}(t_n)} S_{n+1}(t) \quad (7)$$

$$P_j(t) = \begin{cases} S_j(t) - \dfrac{S_j(t_0)}{S_{-2}(t_0)} S_{-2}(t), & j = -1, 0, 1 \\ S_j(t), & j = 2, 3, \ldots, n-3 \\ S_j(t) - \dfrac{S_j(t_n)}{S_{n+1}(t_n)} S_{n+1}(t), & j = n-2, n-1, n \end{cases} \quad (8)$$

Using the Neumann boundary conditions of (2a) to the approximate solution *v(t)* in (6), we get

$$A_1 = v'(a) = v'(t_0) = w_1'(t_0) + \alpha_{-1}P_{-1}'(t_0) + \alpha_0 P_0'(t_0) + \alpha_1 P_1'(t_0) \quad (9)$$

$$C_1 = v'(b) = v'(t_n) = w_1'(t_n) + \alpha_{n-2}P_{n-2}'(t_n) + \alpha_{n-1}P_{n-1}'(t_n) + \alpha_n P_n'(t_n) \quad (10)$$

Eliminating α_{-1} and α_n from the equations (6), (9) and (10), we get the approximation for *v (t)* as

$$v(t) = w(t) + \sum_{j=0}^{n-1} \alpha_j \tilde{B}_j(t) \quad (11)$$

where

$$w(t) = w_1(t) + \frac{A_1 - w_1'(t_0)}{P_{-1}'(t_0)} P_{-1}(t) + \frac{C_1 - w_1'(t_n)}{P_n'(t_n)} P_n(t) \quad (12)$$

$$\tilde{B}_j(t) = \begin{cases} P_j(t) - \dfrac{P_j'(t_0)}{P_{-1}'(t_0)} P_{-1}(t), & j = 0, 1 \\ P_j(t), & j = 2, 3, \ldots, n-3 \\ P_j(t) - \dfrac{P_j'(t_n)}{P_n'(t_n)} P_n(t), & j = n-2, n-1 \end{cases} \quad (13)$$

3.3 Redefinition of basis functions with boundary conditions (2b)

Using the mixed boundary conditions of (2b) to the approximate solution *v(t)* in (6), we get,

$$A_1 = v'(a) + \sigma_1 v(a) = v'(t_0) + \sigma_1 v(t_0) = w_1'(t_0) + \alpha_{-1}P_{-1}'(t_0) + \alpha_0 P_0'(t_0) + \alpha_1 P_1'(t_0) + \sigma_1 w_1(t_0) \quad (14)$$

$$C_1 = v'(b) + \sigma_2 v(b) = v'(t_n) + \sigma_2 v(t_n) = w_1'(t_n) + \alpha_{n-2}P_{n-2}'(t_n) + \alpha_{n-1}P_{n-1}'(t_n) + \alpha_n P_n'(t_n) + \sigma_2 w_1(t_n) \quad (15)$$

Eliminating α_{-1} and α_n from the equations (14), (15) and (6), we get approximation for *v(t)* as

$$v(t) = w(t) + \sum_{j=0}^{n-1} \alpha_j \tilde{B}_j(t) \quad (16)$$

where

$$w(t) = w_1(t) + \frac{A_1 - w_1'(t_0) - \sigma_1 w_1(t_0)}{P_1'(t_0)} P_{-1}(t) + \frac{C_1 - w_1'(t_n) - \sigma_2 w_1(t_n)}{P_n'(t_n)} P_n(t) \quad (17)$$

and $\tilde{B}_j(t)$'s are as defined in (13).

$\{\tilde{B}_j(t), j = 0, 1, \ldots, n-1\}$ is the new set of basis functions for the approximation *v(t)*. Here *w(t)* takes care of given set of essential and Neumann or mixed type of boundary conditions and $\tilde{B}_j(t)$'s are vanishing at the boundary and their first derivative or mixed derivative vanish on boundary. In the proposed method, the new set of basis functions and weight functions should be equal in number. Here the number of basis functions in the approximation for *v(t)* in (6) is n and the number of weight functions is n+5. So, it is necessary to redefine the weight functions into a new set of weight functions which are equal in number of the basis functions.

Let us write the approximation for *u(t)* as

$$u(t) = \sum_{j=-2}^{n+2} \beta_j R_j(t) \quad (18)$$

where $R_j(t)$'s are the quintic B-splines.

Let us assume that approximation *u(t)*, given by (18), satisfies the conditions

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$$u(a)=0, u(b)=0, u'(a)=0, u'(b)=0, u''(a)=0 \tag{19}$$

Using (18) and (19), we get the approximate solution for *u(t)* at the boundary points as

$$u(a)=u(t_0)=\sum_{j=-2}^{2}\beta_j R_j(t_0)=0 \tag{20}$$

$$u(b)=u(t_n)=\sum_{j=n-2}^{n+2}\beta_j R_j(t_n)=0 \tag{21}$$

$$u'(a)=u'(t_0)=\sum_{j=-2}^{2}\beta_j R'_j(t_0)=0 \tag{22}$$

$$u'(b)=u'(t_n)=\sum_{j=n-2}^{n+2}\beta_j R'_j(t_n)=0 \tag{23}$$

$$u''(a)=u'(t_0)=\sum_{j=-2}^{2}\beta_j R''_j(t_0)=0 \tag{24}$$

Eliminating β_{-2}, β_{-1}, β_0, β_{n+1} and β_{n+2} from the equations (18) and (20) to (24), we get the approximation for *u(t)* as

$$u(t)=\sum_{j=1}^{n}\beta_j T_j(t) \tag{25}$$

where

$$T_j(t)=\begin{cases} V_j(t)-\dfrac{V_j''(t_0)}{V_0''(t_0)}V_0(t), & j=1,2\\ V_j(t), & j=3,4,\dots,n\end{cases} \tag{26}$$

$$V_j(t)=\begin{cases} Q_j(t)-\dfrac{Q_j'(t_0)}{Q_{-1}'(t_0)}Q_{-1}(t), & j=0,1,2\\ Q_j(t), & j=3,4,\dots,n-3\\ Q_j(t)-\dfrac{Q_j'(t_n)}{Q_{n+1}'(t_n)}Q_{n+1}(t), & j=n-2,n-1,n\end{cases} \tag{27}$$

$$Q_j(t)=\begin{cases} R_j(t)-\dfrac{R_j(t_0)}{R_{-2}(t_0)}R_{-2}(t), & j=-1,0,1,2\\ R_j(t), & j=3,4,\dots,n-3\\ R_j(t)-\dfrac{R_j(t_n)}{R_{n+2}(t_n)}R_{n+2}(t), & j=n-2,n-1,n,n+1\end{cases} \tag{28}$$

Now the new set of weight functions for the approximation *u(t)* is $\{T_j(t), j=1,2,\dots,n\}$.

Here $T_j(t_0)=T_j(t_n)=T_j'(t_0)=T_j'(t_n)=T_j''(t_0)=0$ for all j.

Applying the proposed method to (1) with the new set of basis functions $\{\tilde{B}_j(t), j=0,1,\dots,n-1\}$ defined in (13) and with the new set of weight functions $\{T_j(t), j=1,2,\dots,n\}$ defined in (26), we get

$$\int_{t_0}^{t_n}\{p_0(t)v^{(6)}(t)+p_1(t)v^{(5)}(t)+p_2(t)v^{(4)}(t)+p_3(t)v'''(t)+ p_4(t)v''(t)+p_5(t)v'(t)+p_6(t)v(t)\}T_i(t)\,dt=\int_{t_0}^{t_n}q(t)T_i(t)dt \tag{29}$$

for i = 1,2,…, n.

Integrating by parts the first three terms on the left hand side of (29) and after applying the boundary conditions mentioned in (2a), we get

$$\begin{aligned}\int_{t_0}^{t_n}p_0(t)T_i(t)v^{(6)}(t)dt &= \frac{d^2}{dt^2}\left[p_0(t)T_i(t)\right]_{t_n}v'''(t_n)\\ &-\frac{d^3}{dt^3}\left[p_0(t)T_i(t)\right]_{t_n}C_2+\frac{d^3}{dt^3}\left[p_0(t)T_i(t)\right]_{t_0}A_2\\ &+\int_{t_0}^{t_n}\frac{d^4}{dt^4}\left[p_0(t)T_i(t)\right]v''(t)dt\end{aligned} \tag{30}$$

$$\begin{aligned}\int_{t_0}^{t_n}p_1(t)T_i(t)v^{(5)}(t)dt &= \frac{d^2}{dt^2}\left[p_1(t)T_i(t)\right]_{t_n}C_2\\ &-\int_{t_0}^{t_n}\frac{d^3}{dt^3}\left[p_1(t)T_i(t)\right]v''(t)dt\end{aligned} \tag{31}$$

$$\int_{t_0}^{t_n}p_2(t)T_i(t)v^{(4)}(t)dt=\int_{t_0}^{t_n}\frac{d^2}{dt^2}\left[p_2(t)T_i(t)\right]v''(t)dt \tag{32}$$

Using (30), (31), (32) and (11) in (29) and after rearrangement, we get a system of equations in the matrix form as

$$K\alpha=f \tag{33}$$

where

$K=[k_{ij}]$;

$$\begin{aligned}k_{ij}&=\int_{t_0}^{t_n}\{[\frac{d^4}{dt^4}[p_0(t)T_i(t)]-\frac{d^3}{dt^3}[p_1(t)T_i(t)]+\frac{d^2}{dt^2}[p_2(t)T_i(t)]\\ &+p_4(t)T_i(t)]\tilde{B}_j''(t)+p_3(t)T_i(t)\tilde{B}_j'''(t)+p_5(t)T_i(t)\tilde{B}_j'(t)\\ &+p_6(t)T_i(t)\tilde{B}_j(t)\}dt+\frac{d^2}{dt^2}[p_0(t)T_i(t)]_{t_n}\tilde{B}_j'''(t_n)\end{aligned} \tag{34}$$

for i=1,2,…,n; j=0,1,2,…,n-1.

$\boldsymbol{f}=[f_i]$;

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$$f_i = \int_{t_0}^{t_n} \{q(t)T_i(t) + [-\frac{d^4}{dt^4}[p_0(t)T_i(t)] + \frac{d^3}{dt^3}[p_1(t)T_i(t)]$$
$$-\frac{d^2}{dt^2}[p_2(t)T_i(t)] - p_4(t)T_i(t)]w''(t) - p_3(t)T_i(t)w'''(t) \tag{35}$$
$$-p_5(t)T_i(t)w'(t) - p_6(t)T_i(t)w(t)\}dt - \frac{d^2}{dt^2}[p_0(t)T_i(t)]_{t_n} w'''(t_n)$$
$$+\frac{d^3}{dt^3}[p_0(t)T_i(t)]_{t_n} C_2 - \frac{d^2}{dt^2}[p_1(t)T_i(t)]_{t_n} C_2 - \frac{d^3}{dt^3}[p_0(t)T_i(t)]_{t_0} A_2$$

for i= 1,2,..., n and $\alpha = [\alpha_0\, \alpha_1 \ldots \alpha_{n-1}]^T$.

4. PROCEDURE TO FIND SOLUTION FOR NODAL PARAMETERS

A general element in the matrix $\boldsymbol{K}$ is given by $\sum_{m=0}^{n-1} I_m$, where $I_m = \int_{t_m}^{t_{m+1}} u_i(t) r_j(t) M(t) dt$, $r_j(t)$ are the quartic B-spline basis functions or their derivatives and $u_i(t)$ are the quintic B-spline weight functions or their derivatives. Here $I_m = 0$ if $(t_{j-2}, t_{j+3}) \cap (t_{i-3}, t_{i+3}) \cap (t_m, t_{m+1}) = \varnothing$. For the evaluation of each I_m, we have used 5-point Gauss-Legendre quadrature formula. Due to this, the stiffness matrix K is a ten diagonal band matrix. Solving the system $K\alpha = f$ *by* using the band matrix solution package, we get the nodal parameter vector $\boldsymbol{\alpha}$. We have used the FORTRAN-90 code to solve the boundary value problems (1) - (2) by the proposed method.

5. NUMERICAL EXAMPLES

To test the accuracy and efficiency of the developed method, we solved three linear and two nonlinear sixth order boundary value problems. The obtained numerical results for each problem are presented in tabular forms.and compared with the exact solutions available in the literature.

Example 1:

Consider the linear boundary value problem

$$v^{(6)} + e^{-t}v = -720 + (t - t^2)^3 e^{-t}, \quad 0 < t < 1 \tag{36}$$

subject to

$v(0) = v(1) = 0,\ v'(0) = 0,\ v'(1) = 0,\ v''(0) = 0,\ v''(1) = 0.$

The exact solution for the above problem is

$v = t^3(1-t)^3.$

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table-1. The maximum absolute error obtained by the proposed method is 1.173466 x10^{-07}.

Table-1. Numerical results for example 1.

t	Absolute error by the proposed method
0.1	6.053597E-09
0.2	3.539026E-08
0.3	9.313226E-08
0.4	1.117587E-07
0.5	1.173466E-07
0.6	1.043081E-07
0.7	7.823110E-08
0.8	4.330650E-08
0.9	1.461012E-08

Example 2:

Consider the linear boundary value problem

$$v^{(6)} + v^{(5)} + \sin t\ v^{(4)} + tv = (2 + \sin t + t)e^t, \quad 0 < t < 1 \tag{37}$$

subject to

$v(0) = 1, v(1) = e,\ v'(0) = 1,\ v'(1) = e,\ v''(0) = 1,\ v''(1) = e.$

The exact solution for the above problem is

$v = e^t$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table-2. The maximum absolute error obtained by the proposed method is 2.396107 x10^{-05}.

Table-2. Numerical results for Example 2.

t	Absolute error by the proposed method
0.1	1.668930E-06
0.2	8.344650E-06
0.3	1.919270E-05
0.4	2.396107E-05
0.5	2.300739E-05
0.6	1.692772E-05
0.7	1.025200E-05
0.8	2.861023E-06
0.9	7.152557E-07

Example 3:

Consider the linear boundary value problem

$$v^{(6)} + v''' + v'' - v = (-15t^2 + 78t - 114)e^{-t}, \quad 0 < t < 1 \tag{38}$$

subject to

$v(0) = 0, v(1) = \frac{1}{e},\ v'(0) = 0,\ v'(1) = \frac{2}{e},\ v''(0) = 0,\ v''(1) = \frac{1}{e}.$

The exact solution for the above problem is $v = t^3 e^{-t}$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table-3. The maximum absolute error obtained by the proposed method is 1.206994 x10^{-06}.

Table-3. Numerical results for Example 3.

t	Absolute error by the proposed method
0.1	3.725290E-09
0.2	5.122274E-09
0.3	6.891787E-08
0.4	4.284084E-07
0.5	8.717179E-07
0.6	1.206994E-06
0.7	1.028180E-06
0.8	7.450581E-07
0.9	3.874302E-07

Example 4:

Consider the nonlinear boundary value problem

$$v^{(6)} + e^{-t} v^2 = e^{-t} + e^{-3t}, \quad 0 < t < 1 \tag{39}$$

subject to

$$v(0) = 1,\, v(1) = \frac{1}{e},\, v'(0) = -1,\, v'(1) = \frac{-1}{e},\, v''(0) = 1,\, v''(1) = \frac{1}{e}.$$

The exact solution for the above problem is

$$v = e^{-t}.$$

The nonlinear boundary value problem (39) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [19] as

$$v_{(n+1)}^{(6)} + 2e^{-t} v_{(n)} v_{(n+1)} = e^{-t} v_{(n)}^2 + e^{-t} + e^{-3t}, \quad n = 0,1,2,... \tag{40}$$

subject to

$$v_{(n+1)}(0) = 0,\, v_{(n+1)}(1) = \frac{1}{e},\, v'_{(n+1)}(0) = -1,\, v'_{(n+1)}(1) = -\frac{1}{e},$$
$$v''_{(n+1)}(0) = 1,\, v''_{(n+1)}(1) = \frac{1}{e}.$$

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for $v(t)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (40). The obtained numerical results for this problem are presented in Table-4. The maximum absolute error obtained by the proposed method is 3.099442 x10^{-06}.

Table-4. Numerical results for Example 4.

t	Absolute error by the proposed method
0.1	3.576279E-07
0.2	1.251698E-06
0.3	3.099442E-06
0.4	3.039837E-06
0.5	2.503395E-06
0.6	1.430511E-06
0.7	9.238720E-07
0.8	2.086163E-07
0.9	9.490116E-07

Example 5:

Consider the nonlinear boundary value problem

$$v^{(6)} + v'v^{(5)} - \pi^3 \sin(\pi t) v''' + vv'' + \pi^2 v^2 = -\pi^6 \cos(\pi t), \quad 0 < t < 1 \tag{41}$$

subject to

$$v(0) = 1,\, v(1) = -1,\, v'(0) = 0,\, v'(1) = 0,\, v''(0) = -\pi^2,\, v''(1) = \pi^2.$$

The exact solution for the above problem is

$$v = \cos(\pi t).$$

The nonlinear boundary value problem (41) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [19] as

$$\begin{aligned} & v_{(n+1)}^{(6)} + v'_{(n)} v_{(n+1)}^{(5)} - \pi^3 \sin(\pi t) v'''_{(n+1)} + v_{(n)} v''_{(n+1)} + v_{(n)}^{(5)} v'_{(n+1)} \\ & + (2\pi^2 v_{(n)} + v''_{(n)}) v_{(n+1)} = v_{(n)} v''_{(n)} + \pi^2 v_{(n)}^2 + v'_{(n)} v_{(n)}^{(5)} - \pi^6 \cos(\pi t) \end{aligned} \tag{42}$$

for n = 0,1,2,…

subject to

$$v_{(n+1)}(0) = 1,\, v_{(n+1)}(1) = -1,\, v'_{(n+1)}(0) = 0,\, v'_{(n+1)}(1) = 0,$$
$$v''_{(n+1)}(0) = -\pi^2,\, v''_{(n+1)}(1) = \pi^2.$$

Here $v_{(n+1)}$ is the $(n+1)^{th}$ approximation for $v(t)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (42). The obtained numerical results for this problem are presented in Table-5. The maximum absolute error obtained by the proposed method is 7.2121621x10^{-06}.

Table-5. Numerical results for Example 5.

t	Absolute error by the proposed method
0.1	5.960464E-07
0.2	1.788139E-06
0.3	3.814697E-06
0.4	5.692244E-06
0.5	7.129289E-06
0.6	7.212162E-06
0.7	4.947186E-06
0.8	2.563000E-06
0.9	8.940697E-07

6. CONCLUSIONS

In this paper, we have solved a general sixth order two point boundary value problem with two different cases of boundary conditions by the proposed method with quartic B-splines as basis functions and quintic B-splines as weight functions. The quartic B-splines and quintic B-splines are redefined into new sets of functions which contain the equal number of functions. To test the accuracy and efficiency of the developed method, it has been tested on three linear and two nonlinear sixth order boundary value problems. It is found that the obtained results are giving a little error. The strength of the developed method lies in the easiness of its application, accuracy and efficiency.

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