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Moore-Penrose inverses of Gram matrices leaving a cone invariant in an indefinite inner product space

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Abstract: In this paper we characterize Moore-Penrose inverses of Gram matrices leaving a cone invariant in an indefinite inner product space using the indefinite matrix multiplication. This characterization includes the acuteness (or obtuseness) of certain closed convex cones.

Keywords: Gram matrix; Moore-Penrose inverse; acute cone; Indefinite inner product space

MSC: 46C20; 15A09

1 Introduction

The notion of indefinite inner product has been rather well studied for the past four decades or so. Along with studies on spaces which possess an indefinite inner product, a host of other questions have been considered in the literature. Let us consider these briefly. These include for instance, the spectral theorem for normal matrices, the indefinite least squares problem and solution of matrix equations. Certain aspects of generalized inverses were studied in [9]. A specific example of an indefinite inner product space is the so-called Minkowski space which is of interest to physicists. Principal pivot transformations and range symmetric matrices over such spaces have been studied in [8]. A good source for results on indefinite inner product spaces are the excellent books [3] and [6].

While one studies matrices in an indefinite inner product space, the usual matrix multiplication is employed. This gives rise to a mismatch when one computes the inner product of vectors. To rectify this deficiency, the authors of [11] defined a new matrix product and called it the indefinite matrix product. Their stance was vindicated in the sense that quite a few results for matrices in the setting of a real Euclidean space were obtained in the setting of indefinite inner product spaces with a feature that one could obtain the results in the Euclidean space as particular cases. This aspect was exemplified in [11] in connection with the proof of the existence of Moore-Penrose inverses, in [12] in the proof of the Farkas lemma and nonnegativity of the Moore-Penrose inverse of Gram operators in [13]. This new matrix product proved fruitful in other considerations as well. Let us cite a few results in this regard. The author in [14] studied EP matrices with respect to the new multiplication and obtained characterizations. A host of questions on nonnegative generalized inverses were considered in the work [15]. He also considered the reverse order law and obtained necessary and sufficient conditions for this law to hold. Relationships with certain matrix partial orders were also obtained [16]. Finally, the author of [10] again considered EP matrices and extended many results of [14].

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Let us now turn to the main object of study in the present work. A real square matrix A is called *monotone* if it satisfies the implication: $Ax \geq 0 \Longrightarrow x \geq 0$, where the order is the usual component wise order of vectors. Monotone matrices were studied by Collatz in connection with applying finite difference methods for elliptic differential equations [5]. He showed that A is monotone if and only if A is invertible and all the entries of the inverse are nonnegative. Since then monotonicity has been extended to characterize nonnegativity of generalized inverses. We refer the reader to the excellent source [2] for much more details on this aspect. In particular, nonnegativity of the inverse of Gram operators has been studied in connection with certain optimization problems [4], where a characterization is proved. This characterization has been extended to operators between Hilbert spaces [7] and [17]. In the latter, a completely new approach was proposed. The sole aim of the present work is to extend this characterization of nonnegativity of the Moore-Penrose inverse of a Gram operator in an indefinite inner product space with the indefinite product of matrices, adopting the approach taken as in [17]. Here, nonnegativity should be interpreted in terms of taking one cone into another. This result is proved in Theorem 3.8. In the next section, we collect certain preliminary results and fix the notation that will be used in the rest of the article.

2 Notations, Definitions and Preliminaries

We begin this section with the definition of an indefinite inner product space.

Definition 2.1. *Let N be a real symmetric matrix of order $n \times n$ such that $N = N^{-1}$. Such a matrix N is called a weight. An indefinite inner product in $\mathbb{R}^n$ is defined by $[x, y] = \langle x, Ny \rangle$ for all $x, y \in \mathbb{R}^n$, where $\langle ., . \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^n$. A space with an indefinite inner product is called an indefinite inner product space.*

In the rest of the paper, $\mathbb{R}^m$ and $\mathbb{R}^n$ represent indefinite inner product spaces with the corresponding weights M and N, respectively.

Next, we define the notion of an indefinite matrix product. We refer the reader to [11] for the detailed study of properties of this product.

Definition 2.2. *Let A and B be $m \times n$ and $n \times l$ real matrices, respectively. Let N be an arbitrary but fixed weight matrix of order $n \times n$. An indefinite matrix product of A and B (relative to N) is defined by $A \circ B = ANB$.*

Note that for $N = I$ the above product becomes the usual matrix product.

Definition 2.3. *Let $A \in \mathbb{R}^{m \times n}$, where $\mathbb{R}^{m \times n}$ denotes the set of all real matrices of order $m \times n$. The adjoint $A^{[\star]}$ of A (relative to weights N, M) is defined by $A^{[\star]} = NA^{\star}M$, where $A^{\star}$ stands for the transpose of A.*

For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times l}$, it easily follows that $(A^{[\star]})^{[\star]} = A$ and $(A \circ B)^{[\star]} = B^{[\star]} \circ A^{[\star]}$.

Next, we introduce the definition of a Gram matrix that plays a key role in this article.

Definition 2.4. *Let $A \in \mathbb{R}^{m \times n}$. Then $A^{[\star]} \circ A$ is called the Gram matrix of A with respect to the indefinite matrix product in an indefinite inner product space.*

Definition 2.5. *Let $A \in \mathbb{R}^{m \times n}$. The range space of A with respect to the indefinite matrix product, $\mathcal{R}(A)$ is defined by $\mathcal{R}(A) = \{y \in \mathbb{R}^m : y = A \circ x, x \in \mathbb{R}^n\}$ and the null space of A with respect to the indefinite matrix product, $\mathcal{N}(A)$ is defined by $\mathcal{N}(A) = \{x \in \mathbb{R}^n : A \circ x = 0\}$.*

Let $R(A)$ and $N(A)$ denote the range and null spaces of A with respect to the usual matrix product, respectively. Then it follows that $\mathcal{R}(A) = R(A)$ and $\mathcal{N}(A^{[\star]}) = N(A^{\star})$.

We now move on to the definition of the Moore-Penrose inverse in an indefinite inner product space with respect to the indefinite matrix product.

Definition 2.6. *([11]) Let $A \in \mathbb{R}^{m\times n}$. Then the matrix $X \in \mathbb{R}^{n\times m}$ is called the Moore-Penrose inverse of A if it satisfies the following equations:*
$A \circ X \circ A = A$, $X \circ A \circ X = X$, $(A \circ X)^{[*]} = A \circ X$, $(X \circ A)^{[*]} = X \circ A$.

Such an X will be denoted by $A^{[\dagger]}$. It is shown in [11] that the Moore-Penrose inverse of any matrix exists over an indefinite inner product space with respect to the indefinite matrix product, whereas a similar result is false with the usual matrix product. It easily follows from the definition that for $A \in \mathbb{R}^{m\times n}$, $(A^{[\dagger]})^{[\dagger]} = A$ and $A^{[\dagger]} = NA^{\dagger}M$. If $N = M = I$ then $A^{[\dagger]} = A^{\dagger}$. We refer the reader to [1] (and the references cited there in) for a detailed study of $A^{\dagger}$.

In the next lemma, we collect some more properties of $A^{[\dagger]}$ that will be used in proving main results of this paper. These properties can be proved easily, by the direct verification of definitions.

Lemma 2.7. *Let $A \in \mathbb{R}^{m\times n}$. Then*
(i) $(A^{[*]})^{[\dagger]} = (A^{[\dagger]})^{[*]}$.
(ii) $(A^{[*]} \circ A)^{[\dagger]} = A^{[\dagger]} \circ (A^{[\dagger]})^{[*]}$.
(iii) $(A \circ I)^{[\dagger]} = I \circ A^{[\dagger]}$ and $(I \circ A)^{[\dagger]} = A^{[\dagger]} \circ I$.
(iv) $\mathcal{R}(A \circ A^{[\dagger]}) = \mathcal{R}(A)$ and $\mathcal{R}(A^{[\dagger]} \circ A) = \mathcal{R}(A^{[*]})$.
(v) $(A^{[*]} \circ A)^{[\dagger]} \circ (A^{[*]} \circ A) = A^{[\dagger]} \circ A$.

Proof. (i), (ii) and (iii) follow directly from definitions of the Moore-Penrose inverse and the adjoint of a matrix.
(iv) $\mathcal{R}(A \circ A^{[\dagger]}) = \mathcal{R}(ANNA^{\dagger}M) = \mathcal{R}(AA^{\dagger}M) = R(AA^{\dagger}) = R(A) = \mathcal{R}(A)$. Similarly, $\mathcal{R}(A^{[\dagger]} \circ A) = \mathcal{R}(A^{[*]})$.
(v) From part (ii), $(A^{[*]} \circ A)^{[\dagger]} \circ (A^{[*]} \circ A) = A^{[\dagger]} \circ (A^{[\dagger]})^{[*]} \circ A^{[*]} \circ A = A^{[\dagger]} \circ (A \circ A^{[\dagger]})^{[*]} \circ A = A^{[\dagger]} \circ A \circ A^{[\dagger]} \circ A = A^{[\dagger]} \circ A$. □

We now briefly discuss the notions of a cone and its dual.

Definition 2.8. *Let K be a subset of $\mathbb{R}^n$. Then K is called a cone if (i) $x, y \in K \Rightarrow x + y \in K$ and (ii) $x \in K$, and $\alpha \in \mathbb{R}$, $\alpha \geq 0 \Rightarrow \alpha x \in K$. The dual of a cone K in an indefinite inner product space is defined by $K^{[*]} = \{x \in \mathbb{R}^n : [x, t] \geq 0,$ for all $t \in K\}$. K is self dual if $K^{[*]} = K$.*

Let K be a cone, closed in $\mathbb{R}^n$ with usual topology and let K^{*} denote the dual of the cone K, in the Euclidean setting. Then

$$K^{*} = \{x \in \mathbb{R}^n : \langle x, t\rangle \geq 0, \text{ for all } t \in K\}$$

and $K^{**} = K$. Note that $K^{[*]} = NK^{*}$ and $K^{[*][*]} = (K^{[*]})^{[*]} = N^2K = K$. In particular, if $K = \mathbb{R}^n_+$ then $K^{[*]} = I \circ \mathbb{R}^n_+ = N\mathbb{R}^n_+$ and $K^{[*][*]} = \mathbb{R}^n_+$.

In the setting of an indefinite inner product space, a cone C is said to be acute if $[x, y] \geq 0$ for all $x, y \in C$ and C is said to be obtuse if $C^{[*]} \cap \{cl\ span\ C\}$ is acute. In particular, let $C = A \circ I \circ K$ then we say that $C = \{A \circ I \circ x : x \in K\}$ is obtuse if $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. According to Novikoff, the acuteness of a cone C in $\mathbb{R}^n$ is defined by the inclusion $C \subseteq C^{*}$. We can easily verify this condition in indefinite inner product spaces as $C \subseteq C^{[*]}$.

Definition 2.9. *Let K_1 and K_2 be cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A \in \mathbb{R}^{m\times n}$. Then A leaves a cone invariant (relative to K_1, K_2) with respect to the indefinite matrix product, if $A \circ K_1 \subseteq K_2$.*

Finally, we conclude this section with the following lemma which will be used frequently in this paper.

Lemma 2.10. *([13], Lemma 2.2) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then, the linear equation $A \circ x = b$ has a solution iff $b \in \mathcal{R}(A)$. In this case, the general solution is given by $x = A^{[\dagger]} \circ b + z$ where $z \in \mathcal{N}(\mathcal{A})$.*

3 Main Results

In the setting of an indefinite inner product space, for a given $A \in \mathbb{R}^{m\times n}$, Ramanathan and Sivakumar [13] derived a set of necessary and sufficient conditions for a cone to be invariant under $(A^{[*]} \circ A)^{[\dagger]}$. These conditions include pairwise acuteness (or pairwise obtuseness) of certain cones. In this article, we avoid the notion of pairwise acuteness of cones and characterize the Moore-Penrose inverses of Gram matrices leaving a cone invariant in the approach of Sivakumar [17]. These results generalize the existing results of Sivakumar [17] in the finite dimensional setting from Euclidean spaces to indefinite inner product spaces.

In this section, we prove a series of results that lead up to the main theorem (Theorem 3.8). Some of these are interesting in their own right. First, we fix some notations. Throughout this section, we consider an $m \times n$ real matrix A satisfying the condition $A \circ I = I \circ A$ (i.e., $AN = MA$, where M, N are weight matrices) and K be a cone, closed in $\mathbb{R}^n$ with respect to the indefinite matrix (or vector) product. Also, we make a note that for any $A \in \mathbb{R}^{m\times n}$, if $A \circ I = I \circ A$ then $A^{[*]} = NA^*M = (MAN)^* = (ANN)^* = A^*$.

Lemma 3.1. *Let $A \in \mathbb{R}^{m\times n}$ be such that $A\circ I = I\circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product. Then*
(i) $[A \circ x, y] = [x, A^{[*]} \circ y]$ for all $x \in \mathbb{R}^n$ and for all $y \in \mathbb{R}^m$.
(ii) $u \in (A \circ I \circ K)^{[*]} \Rightarrow (A \circ I)^{[*]} \circ u \in K^{[*]}$.
(iii) $A^{[\dagger]} \circ A \circ K \subseteq K \Leftrightarrow A^{[\dagger]} \circ A \circ K^{[*]} \subseteq K^{[*]}$.

Proof.
(i) $[A\circ x, y] = \langle A\circ x, My\rangle = \langle ANx, My\rangle = \langle x, NA^*My\rangle = [x, A^*My] = [x, A^{[*]}My] = [x, A^{[*]} \circ y]$, since $A^{[*]} = A^*$.
(ii) Let $u \in (A\circ I\circ K)^{[*]}$ and $r \in K$. Then $0 \le [u, A\circ I\circ r] = [(A\circ I)^{[*]} \circ u, r]$, by part (i). Thus $(A\circ I)^{[*]} \circ u \in K^{[*]}$.
(iii) Let $A^{[\dagger]}\circ A\circ K \subseteq K$, $y = A^{[\dagger]}\circ A\circ x$ with $x \in K^{[*]}$, $u \in K$ and $u^1 = A^{[\dagger]}\circ A\circ u \in K$. Then $[y, u] = [A^{[\dagger]}\circ A\circ x, u] = [x, (A^{[\dagger]}\circ A)^{[*]}\circ u] = [x, A^{[\dagger]}\circ A\circ u] = [x, u^1] \ge 0$. This shows that $y \in K^{[*]}$. Hence $A^{[\dagger]}\circ A\circ K^{[*]} \subseteq K^{[*]}$. Similarly, one can easily prove the converse part. □

The condition (iii) in Lemma 3.1 is equivalent to "K is invariant under $A^{[\dagger]} \circ A$ if and only if $K^{[*]}$ is invariant under $A^{[\dagger]} \circ A$".

In the next result, we determine the set $((A^{[\dagger]})^{[*]} \circ I \circ K^{[*]})^{[*]}$ under the condition $A^{[\dagger]} \circ A \circ K \subseteq K$.

Theorem 3.2. *Let $A \in \mathbb{R}^{m\times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product satisfying the condition $A^{[\dagger]}\circ A\circ K \subseteq K$. Then $((A^{[\dagger]})^{[*]}\circ I\circ K^{[*]})^{[*]} = A\circ I\circ K + \mathcal{N}((A\circ I)^{[*]})$.*

Proof. First, we prove that

$$(A \circ I \circ K)^{[*]} = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} + \mathcal{N}((A \circ I)^{[*]}). \tag{3.0.1}$$

For this, let $y \in (A \circ I \circ K)^{[*]}$. Then by part (ii) of Lemma 3.1, $z = (A \circ I)^{[*]} \circ y \in K^{[*]}$. So, by Lemma 2.10, $y = ((A\circ I)^{[*]})^{[\dagger]}\circ z + w$ for some $w \in \mathcal{N}((A\circ I)^{[*]})$. Then $y \in ((A\circ I)^{[*]})^{[\dagger]}\circ K^{[*]} + \mathcal{N}((A\circ I)^{[*]}) = (A^{[\dagger]})^{[*]}\circ I\circ K^{[*]} + \mathcal{N}((A\circ I)^{[*]})$, by part (i) and (iii) of Lemma 2.7. This proves $(A \circ I \circ K)^{[*]} \subseteq (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} + \mathcal{N}((A \circ I)^{[*]})$.

Next, let $u = u^1 + u^2$, where $u^1 = (A^{[\dagger]})^{[*]} \circ I \circ l$ with $l \in K^{[*]}$ and $u^2 \in \mathcal{N}((A \circ I)^{[*]})$. Let $v = A \circ I \circ t$, $t \in K$ and set $t' = A^{[\dagger]} \circ A \circ t \in K$. Then $[u, v] = [u^1 + u^2, v] = [u^1, v] + [u^2, v] = [u^1, A \circ I \circ t] = [(A^{[\dagger]})^{[*]} \circ I \circ l, A \circ I \circ t] = [l, (A \circ I)^{[\dagger]} \circ A \circ I \circ t] = [l, A^{[\dagger]} \circ A \circ t] = [l, t'] \ge 0$, since $[u^2, v] = [u^2, A \circ I \circ t] = 0$ and by part (iii) of Lemma 2.7. Thus $u \in (A \circ I \circ K)^{[*]}$. This proves (1).

Now, we replace A by $((A^{[\dagger]})^{[*]}$ and K by $K^{[*]}$ in the equation (3.0.1), and use part (iii) of Lemma 3.1 to get the desired result. □

Remarks 3.3. *The following example shows that Theorem 3.2 may not hold in the absence of the condition $A^{[\dagger]} \circ A \circ K \subseteq K$.*

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. *Then* $A^\dagger = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$ *and* $A^{[\dagger]} = NA^\dagger M =$

$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$. *Let* $K = \mathbb{R}^3_+$ *then* $K^{[*]} = N\mathbb{R}^3_+$. *Suppose* $x = (1,2,3)^t$. *Then* $A^{[\dagger]} \circ A \circ x = (1, \frac{-1}{2}, \frac{1}{2}) \notin K$. *Thus* $A^{[\dagger]} \circ A \circ K \nsubseteq K$. *Also* $A \circ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. *Therefore,* $\mathcal{N}((A \circ I)^{[*]})$ *contains only the zero vector. Let* $y = (2,5,8)^t \in K$ *and set* $y^1 = A \circ I \circ y = Ay = (2,3)^t \in A \circ I \circ K$. *Let* $v = N(1,2,0)^t = (1,-2,0)^t \in K^{[*]}$ *and* $z = (A^{[\dagger]})^{[*]} \circ I \circ v = (1,1)^t \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$. *Then* $[y^1, z] = \langle y^1, Mz\rangle = \langle (2,3)^t, (1,-1)^t\rangle < 0$, *so that* $y^1 \notin ((A^{[\dagger]})^{[*]} \circ I \circ K^{[*]})^{[*]}$.

The next result is used in proving the acuteness of certain cones.

Lemma 3.4. *Let* $A \in \mathbb{R}^{m\times n}$ *be such that* $A \circ I = I \circ A$ *and let* K *be a closed cone in* $\mathbb{R}^n$ *with respect to the indefinite matrix product satisfying the condition* $A^{[\dagger]} \circ A \circ K \subseteq K$. *Then* $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$.

Proof. Let $y = A \circ I \circ x \in (A \circ I \circ K)^{[*]}$. Then by part (*ii*) of Lemma 3.1, $(A \circ I)^{[*]} \circ y \in K^{[*]}$. Also, $y = (A\circ I)\circ(A\circ I)^{[\dagger]}\circ y = ((A\circ I)\circ(A\circ I)^{[\dagger]})^{[*]}\circ y = ((A\circ I)^{[\dagger]})^{[*]}\circ(A\circ I)^{[*]}\circ y = (A^{[\dagger]})^{[*]}\circ I\circ(A\circ I)^{[*]}\circ y \in (A^{[\dagger]})^{[*]}\circ I\circ K^{[*]}$, proving that $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$.

Conversely, suppose that $x \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$. Then $x = ((A \circ I)^{[\dagger]})^{[*]} \circ u$ for some $u \in K^{[*]}$. This implies $x \in \mathcal{R}(A \circ I)$. Let $w \in K$, $v = A \circ I \circ w \in A \circ I \circ K$ and $w^1 = A^{[\dagger]} \circ A \circ w \in K$. Then we have $[x, v] = [(A^{[\dagger]})^{[*]} \circ I \circ u, A \circ I \circ w] = [u, A^{[\dagger]} \circ A \circ w] = [u, w^1] \geq 0$. Thus $x \in (A \circ I \circ K)^{[*]}$. □

Next, we obtain an equivalent condition for the acuteness of the cone $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$.

Lemma 3.5. *Let* $A \in \mathbb{R}^{m\times n}$ *be such that* $A \circ I = I \circ A$ *and let* K *be a closed cone in* $\mathbb{R}^n$ *with respect to the indefinite matrix product satisfying the condition* $A^{[\dagger]} \circ A \circ K \subseteq K$. *Then* $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ *is acute if and only if* $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq A \circ I \circ K$.

Proof. Suppose that $L = (A\circ I\circ K)^{[*]}\cap\mathcal{R}(A\circ I)$ is acute. Then $L \subseteq L^{[*]}$. By Lemma 3.4 and Theorem 3.2, it follows that $L^{[*]} = ((A\circ I\circ K)^{[*]}\cap\mathcal{R}(A\circ I))^{[*]} = ((A^{[\dagger]})^{[*]}\circ I\circ K^{[*]})^{[*]} = A\circ I\circ K + \mathcal{N}((A\circ I)^{[*]})$. So, $(A\circ I\circ K)^{[*]}\cap\mathcal{R}(A\circ I) \subseteq A\circ I\circ K+\mathcal{N}((A\circ I)^{[*]})$. However, we have to show that $(A\circ I\circ K)^{[*]}\cap\mathcal{R}(A\circ I) \subseteq A\circ I\circ K$. Let $x \in (A\circ I\circ K)^{[*]}\cap\mathcal{R}(A\circ I)$. Then $x = A \circ I \circ u + z$, with $u \in K$, $z \in \mathcal{N}((A \circ I)^{[*]})$. Since x and $A \circ I \circ u \in \mathcal{R}(A \circ I)$, it follows that $z \in \mathcal{R}(A \circ I) \cap \mathcal{N}((A \circ I)^{[*]}) = \{0\}$. Thus $x \in A \circ I \circ K$.

Conversely, let $x, y \in (A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I) \subseteq A \circ I \circ K$. Then $x = A \circ I \circ u$, $u \in K$. We also have $(A \circ I)^{[*]} \circ y \in K^{[*]}$. Now, $[x, y] = [A \circ I \circ u, y] = [u, (A \circ I)^{[*]} \circ y] \geq 0$. Thus $(A \circ I \circ K)^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. □

We next obtain a necessary and sufficient condition for a cone to be invariant under $(A^{[*]} \circ A)^{[\dagger]}$.

Lemma 3.6. *Let* $A \in \mathbb{R}^{m\times n}$ *be such that* $A \circ I = I \circ A$ *and let* K *be a closed cone in* $\mathbb{R}^n$ *with respect to the indefinite matrix product satisfying the condition* $A^{[\dagger]} \circ A \circ K \subseteq K$. *Then the following are equivalent:*
(i) $(A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} \subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$.
(ii) $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.
(iii) $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K$.

Proof. $(i) \Rightarrow (ii)$:
For $x \in K^{[*]}$, let $y = (A^{[*]} \circ A)^{[\dagger]} \circ x = ((A \circ I)^{[*]} \circ (A \circ I))^{[\dagger]} \circ x = (A \circ I)^{[\dagger]} \circ ((A \circ I)^{[\dagger]})^{[*]} \circ x$. Then

$$\begin{aligned} A \circ I \circ y &= (A \circ I) \circ (A \circ I)^{[\dagger]} \circ ((A \circ I)^{[\dagger]})^{[*]} \circ x \\ &= ((A \circ I)^{[\dagger]})^{[*]} \circ x \\ &= (A^{[\dagger]})^{[*]} \circ I \circ x \in (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} \\ &\subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]}). \end{aligned}$$

Therefore $A \circ I \circ y = A \circ I \circ v + w$, $v \in K$, $w \in \mathcal{N}((A \circ I)^{[*]})$. So , $A \circ I \circ (y - v) \in \mathcal{R}(A \circ I) \cap \mathcal{N}((A \circ I)^{[*]}) = \{0\}$. Then $A \circ I \circ (y - v) = 0$. This implies, $y - v = u \in \mathcal{N}(A \circ I)$. Then $y = u + v$, $v \in K$, $u \in \mathcal{N}(A \circ I)$. This shows that $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.

$(ii) \Rightarrow (i)$:

Let $y = (A^{[\dagger]})^{[*]} \circ I \circ x$, $x \in K^{[*]}$. Then $y = ((A \circ I)^{[\dagger]})^{[*]} \circ x$ and $(A \circ I)^{[\dagger]} \circ y = (A \circ I)^{[\dagger]} \circ ((A \circ I)^{[\dagger]})^{[*]} \circ x = ((A \circ I)^{[*]} \circ (A \circ I))^{[\dagger]} \circ x = (A^{[*]} \circ A)^{[\dagger]} \circ x = u + v$, $u \in K$, $v \in \mathcal{N}(A \circ I)$. This implies that $y = ((A \circ I)^{[\dagger]})^{[\dagger]} \circ (u + v) + w$, $w \in \mathcal{N}((A \circ I)^{[\dagger]})$. Thus $y = A \circ I \circ u + w \in A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$.

$(ii) \Rightarrow (iii)$:

Let $x \in K^{[*]}$ and $y = (A^{[*]} \circ A)^{[\dagger]} \circ x$. Then $(A^{[*]} \circ A)^{[\dagger]} \circ x = u + v$ where $u \in K$, $v \in \mathcal{N}(A \circ I)$. This implies $x = (A^{[*]} \circ A) \circ (u + v) + w$, $w \in \mathcal{N}(A \circ I)$, so that $y = (A^{[*]} \circ A)^{[\dagger]} \circ (A^{[*]} \circ A) \circ u = A^{[\dagger]} \circ A \circ u \in K$, by part (v) of Lemma 2.7.

$(iii) \Rightarrow (ii)$:

This part is obvious. □

We also have a stronger one-way implication, given below.

Lemma 3.7. *Let $A \in \mathbb{R}^{m\times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product. If $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$ then $K^{[*]} \cap \mathcal{R}(A \circ I)^{[*]} \subseteq A^{[*]} \circ A \circ K + \mathcal{N}((A \circ I)$.*

Proof. Let $y = (A \circ I)^{[*]} \circ x \in K^{[*]}$. Then $(A^{[*]} \circ A)^{[\dagger]} \circ y = u + z$, $u \in K$, $z \in \mathcal{N}(A \circ I)$. From this $y = (A^{[*]} \circ A) \circ (u + z) + w$, $w \in \mathcal{N}(A^{[*]} \circ A)^{[\dagger]}$. Since $A^{[*]} \circ A = (A \circ I)^{[*]} \circ (A \circ I)$ and $z \in \mathcal{N}(A \circ I)$, we get $y = A^{[*]} \circ A \circ u + w \in A^{[*]} \circ A \circ K + \mathcal{N}(A \circ I)$. □

We are now in a position to prove the main result of this article.

Theorem 3.8. *(Main Result) Let $A \in \mathbb{R}^{m\times n}$ be such that $A \circ I = I \circ A$ and let K be a closed cone in $\mathbb{R}^n$ with respect to the indefinite matrix product satisfying the condition $A^{[\dagger]} \circ A \circ K \subseteq K$. Let $C = A \circ I \circ K$ and $D = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$. Then the following conditions are equivalent:*

(i) D is acute.

(ii) $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.

(iii) C is obtuse.

Proof. $(i) \Rightarrow (ii)$:

Suppose D is acute then by definition, $D \subseteq D^{[*]}$. By Theorem 3.2, $D^{[*]} = A \circ I \circ K + \mathcal{N}(A \circ I)^{[*]}$. Thus $D \subseteq A \circ I \circ K + \mathcal{N}(A \circ I)^{[*]}$. Now, by Lemma 3.6, we obtain $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$.

$(ii) \Rightarrow (i)$:

Suppose $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$. By Lemma 3.6, $D \subseteq A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]})$. Since $A \circ I \circ K + \mathcal{N}((A \circ I)^{[*]}) = D^{[*]}$ by Theorem 3.2, we get $D \subseteq D^{[*]}$. Hence D is acute.

$(ii) \Rightarrow (iii)$ Suppose $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$. Note that $C = A \circ I \circ K$ is obtuse if $C^{[*]} \cap \mathcal{R}(A \circ I)$ is acute. By Lemma 3.5, it is enough to show that $C^{[*]} \cap \mathcal{R}(A \circ I) \subseteq C$.

Let $y \in C^{[*]} \cap \mathcal{R}(A \circ I)$. Then $y = A \circ I \circ x$ and by part (ii) of Lemma 3.1, $(A \circ I)^{[*]} \circ y \in K^{[*]}$. So, $(A \circ I)^{[*]} \circ y \in K^{[*]} \cap \mathcal{R}(A \circ I)^{[*]}$. By Lemma 3.7, $(A \circ I)^{[*]} \circ y = A^{[*]} \circ A \circ u + z$ with $u \in K$, $z \in \mathcal{N}(A \circ I)$. Since $A^{[*]} \circ A = (A \circ I)^{[*]} \circ (A \circ I)$, it follows that $(A \circ I)^{[*]} \circ y$, $A^{[*]} \circ A \circ u \in \mathcal{R}(A \circ I)^{[*]}$. Thus $z \in \mathcal{R}(A \circ I)^{[*]} \cap \mathcal{N}(A \circ I) = \{0\}$. This implies $z = 0$. Then $(A \circ I)^{[*]} \circ y = A^{[*]} \circ A \circ u$. From this,

$$\begin{aligned} y &= ((A \circ I)^{[\dagger]})^{[*]} \circ ((A \circ I)^{[*]} \circ A \circ I \circ u) + w \\ &= ((A \circ I) \circ (A \circ I)^{[\dagger]})^{[*]} \circ (A \circ I) \circ u + w \\ &= (A \circ I) \circ (A \circ I)^{[\dagger]} \circ (A \circ I) \circ u + w \\ &= (A \circ I) \circ u + w, \end{aligned}$$

where $w \in \mathcal{N}((A \circ I)^{[*]})$.

Since $y \in \mathcal{R}(A \circ I)$, it follows that $w \in \mathcal{R}(A \circ I) \cap \mathcal{N}(A \circ I)^{[*]}) = \{0\}$. Thus $y \in A \circ I \circ K = C$.

$(iii) \Rightarrow (ii)$:
Let $C = A \circ I \circ K$ be obtuse. Then by definition, $C^{[*]} \cap \mathcal{R}(A \circ I) \subseteq C$. By Lemma 3.4, $(A^{[\dagger]})^{[*]} \circ I \circ K^{[*]} \subseteq C$. Now by Lemma 3.6, $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$. □

Corollary 3.9. *In addition to the conditions of Theorem 3.8, suppose that K is self dual (i.e., $K^{[*]} = K$). Then the conditions (i) and (iii) are equivalent to $(A^{[*]} \circ A)^{[\dagger]} \circ K \subseteq K + \mathcal{N}(A \circ I)$.*

The above corollary and Lemma 3.6 shows that $(A^{[*]} \circ A)^{[\dagger]}$ is cone invariant that justifies the title of the article.

Remarks 3.10.

(i) The inclusion $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)$ does not appear to imply $(A^{[*]} \circ A)^{[\dagger]}$ leaves a cone invariant. However, due to Lemma 3.6 this inclusion is equivalent to $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \subseteq K$ which clearly shows our requirement.

(ii) The following example illustrates Theorem 3.8. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $K = \mathbb{R}^3_+$. *Then* $A^\dagger = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$, $A^{[\dagger]} = NA^\dagger M = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $K^{[*]} = N\mathbb{R}^3_+$. *Note that for* $x^1 = (x, y, z)^t \in K$, $A^{[\dagger]} \circ A \circ x^1 = A^{[\dagger]}Ax^1 = \frac{1}{2}(x+z, 0, x+z)^t \in K$. *Thus* $A^{[\dagger]} \circ A \circ K \subseteq K$. *And* $(A^{[*]} \circ A)^\dagger = \frac{1}{16}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$. *Therefore* $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} = N(A^{[*]} \circ A)^\dagger NK^{[*]} \subseteq K$. *Also one can easily verify that* $C = A \circ I \circ K$ *is obtuse and* $D = (A^{[\dagger]})^{[*]} \circ I \circ K^{[*]}$ *is acute.*

(iii) Here, we show by an example that in the absence of the condition $A \circ I = I \circ A$, Theorem 3.8 may not hold. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = N$. *Then clearly* $A \circ I \neq I \circ A$. *Let* $K = \{(x, 0) : x \geq 0\}$ *then* $K^* = \{(x, y) : x \geq 0, y \in \mathbb{R}\}$ *and* $K^{[*]} = \{(y, x) : x \geq 0,\ y \in \mathbb{R}\}$. *Also,* $A^\dagger = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $A^{[\dagger]} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. *Clearly* $A^{[\dagger]} \circ A \circ K \subseteq K$ *and* $D = \left\{(\frac{x}{2}, \frac{x}{2}) : x \geq 0\right\}$ *is acute but* $(A^{[*]} \circ A)^{[\dagger]} \circ K^{[*]} \nsubseteq K$ *where* $(A^{[*]} \circ A)^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

(iv) In [13], authors derived a set of necessary and sufficient conditions for a cone to be invariant under $(A^{[*]} \circ A)^{[\dagger]}$ in terms of pairwise acuteness of cones D and $I \circ D$. However, it is easy to verify that pairwise acuteness of D and $I \circ D$ in an indefinite inner product space with respect to the indefinite matrix product is same as the acuteness of the cone D in usual inner product space with respect to the usual matrix product. Thus the results in this article are different from the results in [13].

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