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Mathematical model of flow in a tube with an overlapping constriction and permeability

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Abstract

Perturbation solution is given for the laminar flow of viscous incompressible fluid through a non-uniform tube with an overlapping constriction and permeable wall. The effect of fluid reabsorption through permeable wall is considered by taking flux as a function of axial distance. The effects of various parameters on the velocity profiles at different cross sections of the tube, the axial distribution of wall shear stress, mean pressure drop and streamlines are discussed and presented graphically.

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Keywords: renal tubules; overlapping constriction; perturbation method; permeable wall.

1. Introduction

Mathematical models of flow in renal tubule has been studied by various authors. An earlier work on mathematical model of flow in renal tubule was done by Kelman [2] and Macey [1,3]. Marshal and Trowbridge [4] used the physical condition existing at the permeable wall instead of prescribing the flux/radial velocity at the wall. Pallat *et al.* [5] studied the flow in an infinite permeable cylinder assuming the loss of fluid from permeable tube to be a function of pressure gradients across the wall. The above studies modeled renal tubule as cylindrical tube of uniform cross section, while in general, such tubes may not have uniform cross section throughout their length. Radhakrishnamacharya *et al.*[6] discussed to understand the hydrodynamical aspects of an incompressible viscous fluid in a circular tube of varying cross section with reabsorption at the wall. In another study, Chathurani and Ranganatha [7] considered fluid flow through a diverging/converging tube with variable wall permeability. Muthu and Tesfahun [8] developed a mathematical model for the flow in a non-uniform channel with non-zero Reynolds number. They have given an approximate analytical solution using a perturbation method to understand the basic concepts of the flow.

In this present work, the objective of the study is to understand the impact of an overlapping constriction on the steady flow of a Newtonian fluid through a tube with permeable wall.

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2. Geometry

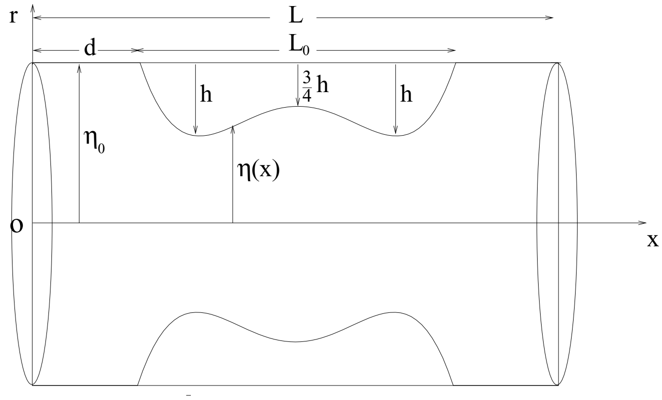

Fig. 1. Geometry of the tube with an overlapping constriction.

The boundary of the tube wall is assumed to be axisymmetric about $x-$axis and vary with x as illustrated in Figure 1. It is taken as [9],

$$\eta(x) = \begin{cases} \eta_0 - \frac{3}{2}h\left[11(\frac{x-d}{L_0}) - 47(\frac{x-d}{L_0})^2 + 72(\frac{x-d}{L_0})^3 - 36(\frac{x-d}{L_0})^4\right], & d \leq x \leq d + L_0 \\ \eta_0, & \text{otherwise.} \end{cases} \tag{1}$$

where η_0 is the radius of the tube inlet (at $x = 0$), h is the maximum height of the constriction, d is the location of constriction, L_0 is the length of the spread of constriction and L is the length of the tube.

3. Mathematical Formulation

Consider an incompressible, viscous Newtonian fluid flow through an overlapping constricted tube as given by equation (1). The motion of the fluid is assumed to be laminar, steady and symmetric. The tube is assumed to be long enough to neglect both the entrance and end effects. For this situation the equations of continuity and momentum are given by

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{3}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right) \tag{4}$$

where u and v are the velocity components along the x and r axes respectively, p is the pressure, ρ density of the fluid and $\nu = \frac{\mu}{\rho}$ is kinematic viscosity.

These are the subjected to the following boundary conditions,

(i) The tangential velocity at the wall is zero. That is,

$$u + \frac{d\eta}{dx}v = 0 \quad \text{at} \quad r = \eta(x) \tag{5}$$

(ii) The regularity condition requires,

$$v = 0 \quad \text{and} \quad \frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{6}$$

(iii) The reabsorption has been accounted by considering the bulk flow as a decreasing function of x. That is, the flux across a cross section is given as

$$Q(x) = \int_0^{\eta(x)} 2\pi r u(x,r)\, dr = Q_0 F(\alpha x) \tag{7}$$

where $F(\alpha x) = 1$ when $\alpha = 0$ and decreases with x. Further, $\alpha \geq 0$ is the reabsorption coefficient, a constant and Q_0 is the flux across the cross section at $x = 0$.

The relation between the stream function $\psi(x,r)$ and velocity components is given as

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} \quad , \quad v = -\frac{1}{r}\frac{\partial\psi}{\partial x}. \tag{8}$$

Substituting (8) in (2) - (4) and eliminating p from (3) and (4) and using the following non-dimensional quantities,

$$x' = \frac{x}{L}, \quad r' = \frac{r}{\eta_0}, \quad \eta' = \frac{\eta}{\eta_0}, \quad \psi' = \frac{2\pi\psi}{Q_0}, \quad \alpha' = \alpha L, \quad p' = \frac{2\pi\eta_0^3}{\mu Q_0}p.$$

we have the non-dimensional form of equations of motion(after dropping primes):

$$\nabla^2\psi = \frac{\delta Re}{2}\left[\frac{1}{r}\frac{\partial\psi}{\partial x}\nabla\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\nabla\frac{\partial\psi}{\partial x} - \frac{2}{r^2}\frac{\partial\psi}{\partial x}\nabla\psi + \frac{1}{r^3}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial x}\right] \tag{9}$$

where $\nabla = \delta^2\frac{\partial^2}{\partial x^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$

Further the boundary conditions and equation of the boundary of the overlapping constricted tube in non-dimensional form are defined as

$$\frac{\partial\psi}{\partial r} = -\frac{3}{2}H\delta\left[11 - 94\left(\frac{x-a}{\epsilon}\right) + 216\left(\frac{x-a}{\epsilon}\right)^2 - 144\left(\frac{x-a}{\epsilon}\right)^3\right]\frac{\partial\psi}{\partial x} \quad \text{at} \quad r = \eta(x) \tag{10}$$

$$\psi = 0 \quad \text{and} \quad -\frac{\partial\psi}{\partial r} + r\frac{\partial^2\psi}{\partial r^2} = 0 \quad \text{at} \quad r \to 0 \tag{11}$$

$$\psi = F(\alpha x) \quad \text{at} \quad r = \eta(x) \tag{12}$$

$$\eta(x) = \begin{cases} 1 - \frac{3}{2}H\delta\left[11(\frac{x-a}{\epsilon}) - 47(\frac{x-a}{\epsilon})^2 + 72(\frac{x-a}{\epsilon})^3 - 36(\frac{x-a}{\epsilon})^4\right], & a \leq x \leq a+\epsilon \\ 1, & \text{otherwise.} \end{cases} \tag{13}$$

where, $Re = \frac{Q_0}{\pi\eta_0\nu}$ is the Reynolds number, $\delta = \frac{\eta_0}{L}$ is the geometric parameter, $H = \frac{h}{L_0}$ is the the constriction parameter, $\epsilon = \frac{L_0}{L}$ is the parameter describing the length of spread of the overlapping constriction and $a = \frac{d}{L}$ is a parameter indicating the location of the spread of overlapping constriction. In this problem, we consider exponentially decreasing bulk flow, that is, in equation (7), following Kelman [2], F is taken as,

$$F(\alpha x) = e^{-\alpha x} \tag{14}$$

4. Method of Solution

To solve (9) in the present analysis, we assume the geometrical parameter $\delta \ll 1$ and we shall seek a solution for stream function $\psi(x,r)$ in the form of a power series in terms of δ, as

$$\psi(x,r) = \psi_0(x,r) + \delta\psi_1(x,r) + ... \tag{15}$$

Substituting (15) in equations (9) - (12) and collecting coefficients of various like powers of δ, we get the following sets of equations for $\psi_0(x,r)$, $\psi_1(x,r)$, ...

δ^0 case:

$$\left(\delta^2\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)^2\psi_0 = 0 \tag{16}$$

The boundary conditions are

$$\frac{\partial \psi_0}{\partial r} = 0 \quad \text{at} \quad r = \eta(x) \tag{17}$$

$$\psi_0 = 0 \quad \text{and} \quad \frac{\partial^2 \psi_0}{\partial r^2} = 0 \quad \text{at} \quad r = 0 \tag{18}$$

$$\psi_0 = e^{-\alpha x} \quad \text{at} \quad r = \eta(x) \tag{19}$$

δ^1 case:

$$\nabla_1^2 \psi_1 = \frac{Re}{2}\left[\frac{1}{r}\frac{\partial \psi_0}{\partial x}\nabla_1 \frac{\partial \psi_0}{\partial x} - \frac{1}{r}\frac{\partial \psi_0}{\partial r}\nabla_1 \frac{\partial \psi_0}{\partial x} - 2\frac{1}{r^2}\frac{\partial \psi_0}{\partial x}\nabla_1 \psi_0 + \frac{1}{r^3}\frac{\partial \psi_0}{\partial r}\frac{\partial \psi_0}{\partial x}\right] \tag{20}$$

where $\nabla_1 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$
The boundary conditions are

$$\frac{\partial \psi_1}{\partial r} = -\frac{3}{2}H\left[11 - 94\left(\frac{x-a}{\epsilon}\right) + 216\left(\frac{x-a}{\epsilon}\right)^2 - 144\left(\frac{x-a}{\epsilon}\right)^3\right]\frac{\partial \psi_0}{\partial x} \quad \text{at} \quad r = \eta(x) \tag{21}$$

$$\psi_1 = 0 \quad \text{and} \quad \frac{\partial \psi_1}{\partial x} = 0 \quad \text{at} \quad r = 0 \tag{22}$$

$$\psi_1 = 0 \quad \text{at} \quad r = \eta(x) \tag{23}$$

However, since we are looking for an approximate analytical solution for the problem, we consider up to order of δ^1 equations. The solution of equation (16) together with boundary conditions (17) - (19) is

$$\psi_0(x, r) = A_1(x)r^2 + A_2(x)r^4 \tag{24}$$

where $A_1(x) = \frac{-2}{\eta^2}e^{-\alpha x}$ and $A_2(x) = -\frac{1}{\eta^4}e^{-\alpha x}$.
The solution of equation (20) together with boundary conditions (21) - (23) is

$$\psi_1(x, r) = A_3(x)r^2 + A_4(x)r^4 - Re\left[\left(A_1(x)\frac{dA_2(x)}{dx}\frac{r^6}{24} + A_2(x)\frac{dA_1(x)}{dx}\right)\frac{r^8}{72}\right] \tag{25}$$

where

$$A_3(x) = -Re\left[A_1\frac{dA_2}{dx}\frac{\eta^4}{24} + A_2\frac{dA_2}{dx}\frac{\eta^6}{36}\right] + \frac{3}{4}H\left[11 - 94\left(\frac{x-a}{\epsilon}\right) + 216\left(\frac{x-a}{\epsilon}\right)^2 - 144\left(\frac{x-a}{\epsilon}\right)^3\right]\left[\frac{dA_1}{dx}\eta + \frac{dA_2}{dx}\eta^3\right]$$

$$A_4(x) = Re\left[A_1\frac{dA_2}{dx}\frac{\eta^2}{12} + A_2\frac{dA_2}{dx}\frac{\eta^4}{24}\right] - \frac{3}{4}H\left[11 - 94\left(\frac{x-a}{\epsilon}\right) + 216\left(\frac{x-a}{\epsilon}\right)^2 - 144\left(\frac{x-a}{\epsilon}\right)^3\right]\left[\frac{dA_1}{dx}\frac{1}{\eta} + \frac{dA_2}{dx}\eta\right]$$

Hence, substituting ψ_0 and ψ_1 in equation (15), we get

$$\psi(x, r) = A_1 r^2 + A_2 r^4 + \delta\left(A_3 r^2 + A_4 r^4 - Re\left(A_1\frac{\partial A_2}{\partial x}\frac{r^6}{24} + A_2\frac{\partial A_2}{\partial x}\frac{r^8}{72}\right)\right) \tag{26}$$

The velocity components along x and r directions are obtained by substituting equation (26) in (8). The non dimensional pressure $p(x, r)$ can be obtained by using equations (26), (8) and (3). It is given as

$$p(x, r) = \delta\frac{\partial u}{\partial x} + \frac{1}{\delta}\int \frac{\partial^2 u}{\partial r^2}dx + \frac{1}{\delta}\int \frac{1}{r}\frac{\partial u}{\partial r}dx - \frac{Re}{2}\left[\int u\frac{\partial u}{\partial x}dx + \int v\frac{\partial u}{\partial r}dx\right] \tag{27}$$

The mean pressure is given as

$$\bar{p}(x) = \frac{1}{\pi\eta^2(x)} \int_0^{\eta(x)} 2\pi r p(x,r) dr \tag{28}$$

Further, the mean pressure drop between $x = 0$ and $x = x_0$

$$\triangle\bar{p}(x_0) = \bar{p}(0) - \bar{p}(x_0), \quad 0 \leq x_0 \leq 1 \tag{29}$$

The wall shear stress $\tau_w(x)$ is defined as,

$$\tau_w(x) = \frac{(\sigma_{rr} - \sigma_{xx})\frac{d\eta}{dx} + \sigma_{xr}(1 - (\frac{d\eta}{dx})^2)}{1 + (\frac{d\eta}{dx})^2} \quad at \quad r = \eta(x) \tag{30}$$

where $\sigma_{xx} = 2\mu\frac{\partial u}{\partial x}$, $\sigma_{rr} = 2\mu\frac{\partial v}{\partial r}$ and $\sigma_{xr} = \mu(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x})$

Using the non-dimensional quantity $\tau'_w = \frac{2\pi\eta_0^3}{\mu Q_0}\tau_w$, the wall shear stress becomes,

$$\tau_w(x) = \frac{2\delta^2(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial x})\frac{d\eta}{dx} + (\frac{\partial u}{\partial r} + \delta^2\frac{\partial v}{\partial x})(1 - \delta^2(\frac{d\eta}{dx})^2)}{1 + \delta^2(\frac{d\eta}{dx})^2} \tag{31}$$

In the equation (27), the integrals are calculated by numerical integration because it is difficult to evaluate analytically to get closed form expression for $p(x, r)$.

5. Results and Discussions

The aim of this present study is to observe the behaviour of an incompressible fluid flow through an overlapping constricted tube with reabsorbing wall. The parameter H characterizes the constriction of the wall, a represent the entrance length of the tube and α represents reabsorption coefficient of wall. We discuss the effects of H and α on the transverse velocity $v(x, y)$ and mean pressure drop ($\Delta\bar{p}$), wall shear stress ($|\tau_w|$) and streamlines. In all our calculations, the following parameters are fixed as $\delta = 0.1$, $Re = 1.0$ and $a = 0.1$.

5.1. Velocity v :

The velocity profile of the flow is obtained by taking different values of constriction parameter H at different cross-sections $x = 0.25$, $x = 0.50$ and $x = 0.75$ of the tube. The reabsorption coefficient is taken as $\alpha = 1.0$. As the value of H increases from 0.01 to 0.03, the narrowness of constriction increases. From the figures 2(a) - 2(c), the value of transverse velocity at $x = 0.25$ is more than at $x = 0.50$ and $x = 0.75$. That is, the transverse velocity decreases as x increases at different cross-sections. In the downstream of the flow, though there is no significant change in the behaviour of transverse velocity, the quantity of the velocity decreases.

5.2. Mean pressure drop $\Delta\bar{p}$:

The values of the mean pressure drop over the length of the tube is calculated for different values of H and α. It can be observed, from figure 3(a) and 3(b), that the mean pressure drop increases nonlinearly as H increases. Further, as the reabsorption coefficient α increases the mean pressure drop decreases quantitatively.

5.3. Wall shear stress $|\tau_w|$:

The effects of H and α on the magnitude of wall shear stress ($|\tau_w|$) at various values of the axial distance are studied and presented graphically in Figures 4(a) and 4(b). As the reabsorption coefficient α increases, the magnitude of wall shear stress decreases. Also noted that, as constriction parameter increases, the magnitude of the wall shear stress increases.

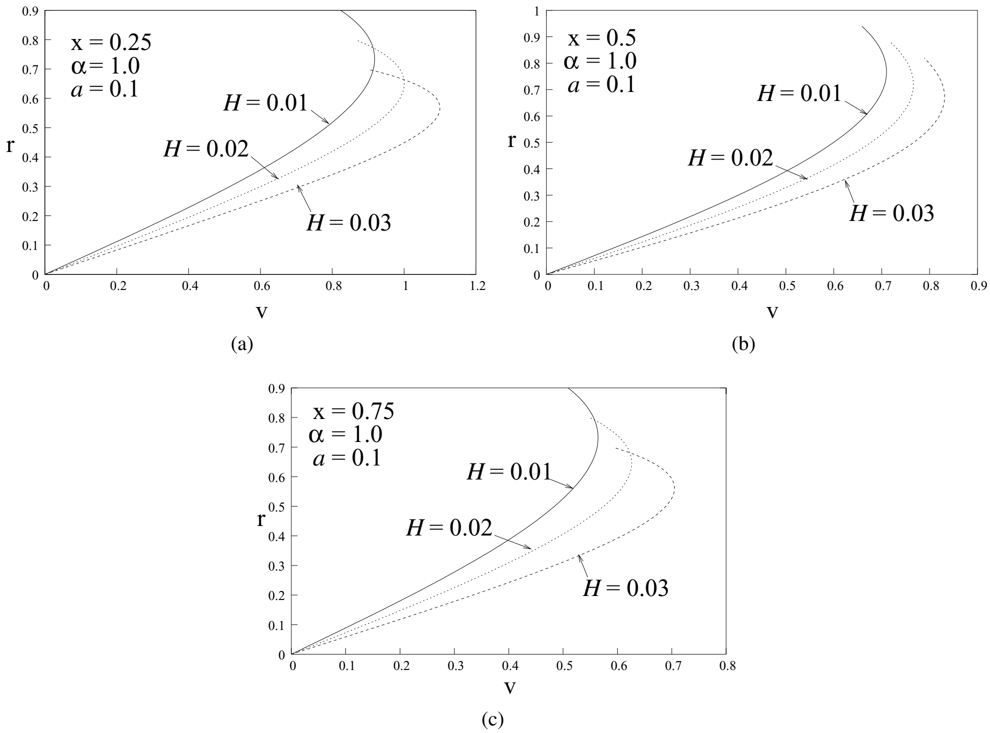

Fig. 1. Distribution of transverse velocity (v) with r, at various cross-sections of the tube.

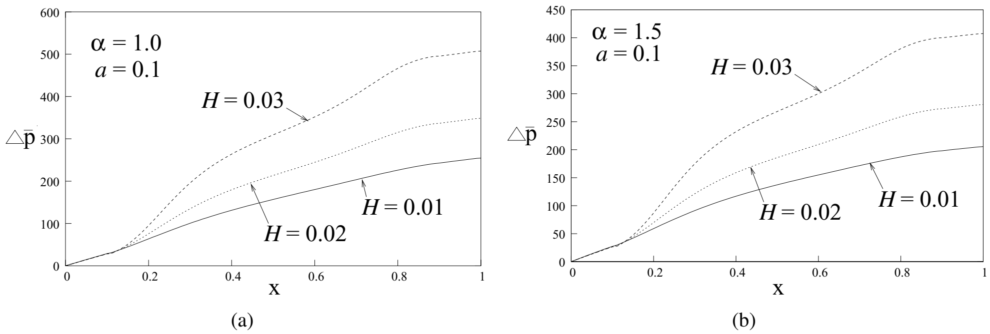

Fig. 2. Distribution of mean pressure drop $\triangle\bar{p}$ with x.

5.4. Streamlines :

We can observe the flow behaviour of the fluid by looking at the contour drawing of the stream function in the overlapping constricted tube for different values of H and α. As H increases, the narrowness of the constriction increases and the length of the boundary curve also increases. Figures 5(a)-5(c) are showing the flow pattern for different constricted tube boundary wall. From Figures 5(a) and 5(d), we can observe that as α increases the curvature of streamlines move towards the overlapping boundary because of reabsorption at the wall. Further, the flux at the cross-section $x = 1.0$ is decreased.

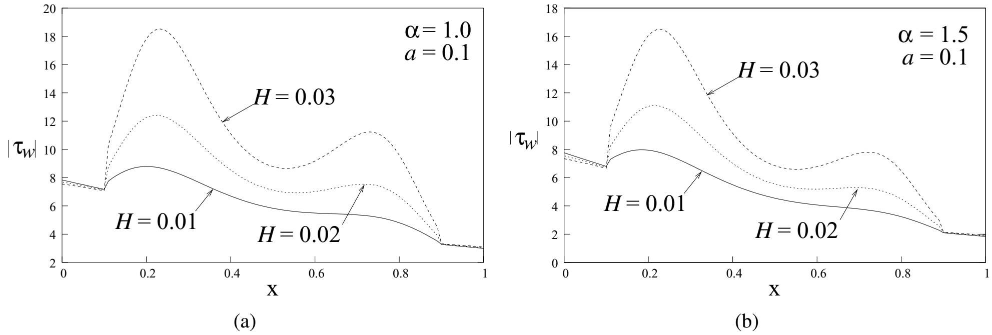

Fig. 3. Distribution of wall shear stress $|\tau_w|$ with x.

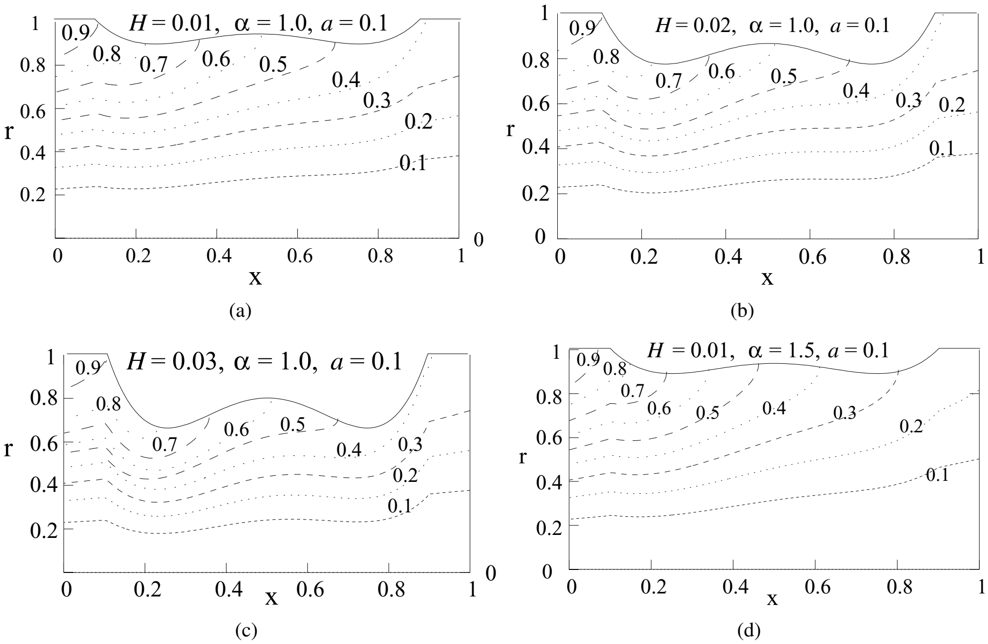

Fig. 4. Streamlines for different values of H and α when $a = 0.1$.

6. Conclusions

The present investigation reported here is an attempt to understand theoretically the effect of overlapping constriction and permeability of wall on the flow of Newtonian fluid in a tube. Perturbation solution is obtained for transverse velocity, mean pressure drop, wall shear stress and streamlines and are shown graphically to observe the effects of overlapping constriction. The following are the observations in the present analysis:

1. The transverse velocity at the overlapping constricted boundary decreases along the axial length.
2. The mean pressure drop increases due to increase in the parameter H but decreases with permeability parameter α.
3. The wall shear stress decreases like a sinusoidal wave with increase in H.
4. The streamline pattern shows that there exists a change in the direction of flow due to change in H as well as a.

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