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Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behaviour

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Abstract In this paper, we have presented a computational method for solving singularly perturbed delay differential equations with twin layers or oscillatory behaviour. In this method, the original second order singularly perturbed delay differential equation is replaced by an asymptotically equivalent first order neutral type delay differential equation. Then, we have employed numerical integration and linear interpolation to get tridiagonal system. This tridiagonal system is solved efficiently by using discrete invariant imbedding algorithm. Several model examples are solved, and computational results are presented by taking various values of the delay parameter and perturbation parameter. We have also discussed the convergence of the method.

1. Introduction

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term. In these problems, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. In the recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modelling of processes in various application fields, for e.g., the first exit time problem in the modelling of the activation of neuronal variability [1], in the study of bistable devices [2], and variational problems in control theory [3] where they provide the best and in many cases the only realistic simulation of the observed.

In [4], the authors Amiraliyev and Cimen presented an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. File and Reddy [5] presented a numerical integration of a class of singularly perturbed delay differential equations with small shift, where delay is in differentiated term. In [6],

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the authors Mohapatra and Natesan constructed a numerical method for a class of singularly perturbed differential-difference equations with small delay. The numerical method comprises of upwind finite difference operator on an adaptive grid, which is formed by equidistributing the arc-length monitor function. Kadalbajoo and Sharma [7] presented a numerical approach to solve singularly perturbed differential-difference equation, which contains negative shift in the function but not in the derivative term. Lange and Miura [8,9] gave an asymptotic approach for a class of boundary-value problems for linear second-order singularly perturbed differential-difference equations.

In this paper, we have presented a computational technique for solving singularly perturbed delay differential equations with twin layer or oscillatory behaviour. Here, the delay term is not present in the differentiated term. In this method, we have replaced the original second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employed the Trapezoidal rule. Then, linear interpolation is used to get three term recurrence relation which is solved easily by discrete invariant imbedding algorithm. Several model examples for various values of the delay parameter and perturbation parameter are solved, and computational results are presented. We have also discussed the convergence of the method.

2. Description of the method

We consider singularly perturbed delay differential equation of the standard form

$$\varepsilon y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x), 0 < x < 1, \tag{1}$$

with boundary conditions

$$y(x) = \phi(x), -\delta \leqslant x \leqslant 0 \tag{2a}$$

and

$$y(1) = \beta \tag{2b}$$

where ε is small parameter, $0 < \varepsilon$ 1 and δ is also small delay parameter, $0 < \delta < 1$; $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are bounded continuous functions in (0, 1) and β is a given constant. For $\delta = 0$, the solution of the boundary value problem (1) and (2) exhibits layer or oscillatory behaviour depending on the sign of $(a(x) + b(x))$. If $(a(x) + b(x)) < 0$, the solution of the problem (1) and (2) exhibits layer behaviour, and if $(a(x) + b(x)) > 0$, it exhibits oscillatory behaviour. The boundary value problem considered here is of the reaction–diffusion type, therefore, if the solution exhibits layer behaviour, there will be two boundary layers which will be at both the end points i.e., at $x = 0$ and $x = 1$. In this paper, we present both the cases, i.e., when the solution of the problem exhibits layer as well as oscillatory behaviour and shows the effect of the delay on the layer and oscillatory behaviour. In particular, as delay increases then the layer behaviour of the solution is destroyed and the solution begins to exhibit oscillatory behaviour across the interval.

We divide the interval [0, 1] into an even number of subintervals N with constant mesh size h. Let $0 = x_0, x_1, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$ for $i = 0,1,\ldots,N$. We choose n such that $x_n = \frac{1}{2}$. In the interval $\left[0,\frac{1}{2}\right]$, the boundary layer will be in the left hand side i.e., at $x = 0$, and in the interval $\left[\frac{1}{2},1\right]$, the boundary layer will be in the right hand side i.e., $x = 1$. Hence, we derive the numerical method by approximating $\varepsilon y''$ using Taylor series expansion of retarded terms $y'(x+\varepsilon)$ and $y'(x-\varepsilon)$, then we get

$$\varepsilon y'' \approx \frac{y'(x+\varepsilon) - y'(x-\varepsilon)}{2}$$

Using the above approximation in Eq. (1), it is replaced by an asymptotically equivalent first order differential equation as follows:

$$y'(x+\varepsilon) - y'(x-\varepsilon) \approx -2a(x)y(x-\delta) - 2b(x)y(x) + 2f(x) \tag{3}$$

This replacement is significant from the computational point of view El'sgol'ts and Norkin [10]. The above equation can be written as

$$y'(x+\varepsilon) - y'(x-\varepsilon) \approx p(x)y(x-\delta) + q(x)y(x) + r(x) \tag{4}$$

where $p(x) = -2a(x)$, $q(x) = -2b(x)$, $r(x) = 2f(x)$.

Integrating Eq. (4) in $\left[0,\frac{1}{2}\right]$ with respect to x from x_i to x_{i+1}, we get

$$\int_{x_i}^{x_{i+1}} [y'(x+\varepsilon) - y'(x-\varepsilon)]dx \approx \int_{x_i}^{x_{i+1}} [p(x)y(x-\delta) + q(x)y(x) + r(x)]dx$$

$$y(x_{i+1}+\varepsilon) - y(x_i+\varepsilon) - y(x_{i+1}-\varepsilon) + y(x_i-\varepsilon) \approx \int_{x_i}^{x_{i+1}} [p(x)y(x-\delta) + q(x)y(x) + r(x)]dx \tag{5}$$

By using Taylor series, we have

$$y(x_i+\varepsilon) \approx y(x_i) + \varepsilon y'(x_i) \approx y_i + \varepsilon y'_i$$
$$y(x_i-\varepsilon) \approx y(x_i) - \varepsilon y'(x_i) \approx y_i - \varepsilon y'_i$$
$$y(x_{i+1}-\varepsilon) \approx y(x_{i+1}) - \varepsilon y'(x_{i+1}) \approx y_{i+1} - \varepsilon y'_{i+1}$$
$$y(x_{i+1}+\varepsilon) \approx y(x_{i+1}) + \varepsilon y'(x_{i+1}) \approx y_{i+1} + \varepsilon y'_{i+1}$$

Here, we denote $y(x_i) = y_i$.

Substituting the above approximations in Eq. (5), we get

$$2\varepsilon y'_{i+1} - 2\varepsilon y'_i \approx \int_{x_i}^{x_{i+1}} [p(x)y(x-\delta) + q(x)y(x) + r(x)]dx$$

By using the Trapezoidal rule to evaluate the integral on the right of the above equation, we get

$$2\varepsilon y'_{i+1} - 2\varepsilon y'_i \approx \frac{h}{2}(p_{i+1}y(x_{i+1}-\delta) + p_i y(x_i-\delta)) + \frac{h}{2} \times (q_{i+1}y_{i+1} + q_i y_i) + \frac{h}{2}(r_{i+1} + r_i) \tag{6}$$

By means of Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we get

$$y(x_{i+1}-\delta) \approx y(x_{i+1}) - \delta y'(x_{i+1}) \approx y_{i+1} - \delta\left(\frac{y_{i+1} - y_i}{h}\right) \approx \left(1 - \frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i$$

$$y(x_i-\delta) \approx y(x_i) - \delta y'(x_i) \approx y_i - \delta\left(\frac{y_i - y_{i-1}}{h}\right) \approx \left(1 - \frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1}$$

Substituting the above approximations in Eq. (6), we get

$$2\varepsilon y'_{i+1} - 2\varepsilon y'_i = \frac{h}{2}p_{i+1}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i\right] + \frac{h}{2}p_i\left[\left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1}\right] + \frac{h}{2}[q_{i+1}y_{i+1}] + \frac{h}{2}[q_iy_i] + \frac{h}{2}r_i + \frac{h}{2}r_{i+1}$$

$$2\varepsilon\left(\frac{y_{i+1}-y_i}{h}\right) - 2\varepsilon\left(\frac{y_i-y_{i-1}}{h}\right) = \left[\frac{h}{2}p_{i+1}\left(1-\frac{\delta}{h}\right) + \frac{h}{2}q_{i+1}\right]y_{i+1} + \left[\frac{h}{2}p_{i+1}\left(\frac{\delta}{h}\right) + \frac{h}{2}p_i\left(1-\frac{\delta}{h}\right) + \frac{h}{2}q_i\right]y_i + \left[\frac{h}{2}p_i\frac{\delta}{h}\right]y_{i-1} + \frac{h}{2}r_i + \frac{h}{2}r_{i+1} \tag{7}$$

Rewriting the Eq. (7) into three term recurrence relation, we get

$$\left[\frac{2\varepsilon}{h} - \frac{\delta}{2}p_i\right]y_{i-1} - \left[\frac{4\varepsilon}{h} + \frac{\delta}{2}p_{i+1} + \frac{h}{2}p_i\left(1-\frac{\delta}{h}\right) + \frac{h}{2}q_i\right]y_i + \left[\frac{2\varepsilon}{h} - \frac{h}{2}p_{i+1}\left(1-\frac{\delta}{h}\right) - \frac{h}{2}q_{i+1}\right]y_{i+1} = \frac{h}{2}r_i + \frac{h}{2}r_{i+1}$$

This tridiagonal system can be written as follows:

$$E_iy_{i-1} - F_iy_i + G_iy_{i+1} = H_i, \text{for } i = 1, 2, \ldots, n-1 \tag{8}$$

where

$$E_i = \frac{2\varepsilon}{h} - \frac{\delta}{2}p_i$$

$$F_i = \frac{4\varepsilon}{h} + \frac{\delta}{2}p_{i+1} + \frac{h}{2}p_i\left(1-\frac{\delta}{h}\right) + \frac{h}{2}q_i$$

$$G_i = \frac{2\varepsilon}{h} - \frac{h}{2}p_{i+1}\left(1-\frac{\delta}{h}\right) - \frac{h}{2}q_{i+1}$$

$$H_i = \frac{h}{2}r_{i+1} + \frac{h}{2}r_i$$

Now in the interval $\left[\frac{1}{2}, 1\right]$ i.e., for right – end boundary layer, integrating Eq. (4) with respect to x from x_{i-1} to x_i, we get

$$\int_{x_{i-1}}^{x_i}[y'(x+\varepsilon) - y'(x-\varepsilon)]dx - = \int_{x_{i-1}}^{x_i}[p(x)y(x-\delta)dx + q(x)y(x) + r(x)]dx$$

$$[y(x_i+\varepsilon) - y(x_{i-1}+\varepsilon)] - [y(x_i-\varepsilon) + y(x_{i-1}-\varepsilon)] = \int_{x_{i-1}}^{x_i}[p(x)y(x-\delta) + q(x)y(x) + r(x)]dx$$

The above equation simplified as

$$2\varepsilon y'_i - 2\varepsilon y'_{i-1} = \int_{x_{i-1}}^{x_i}[p(x)y(x-\delta) + q(x)y(x) + r(x)]dx \tag{9}$$

Using the Trapezoidal rule to evaluate the integral on the right of Eq. (9), we get

$$\frac{2\varepsilon}{h}(y_{i+1} - y_i) - \frac{2\varepsilon}{h}(y_i - y_{i-1}) = \frac{h}{2}(p_{i-1}y(x_{i-1}-\delta) + p_iy(x_i-\delta)) + \frac{h}{2}(q_iy_i + q_{i-1}y_{i-1}) + \frac{h}{2}(r_i + r_{i-1}) \tag{10}$$

By means of Taylor series expansion of $y(x_{i-1}-\delta)$ and $y(x_i - \delta)$, then approximating $y'(x)$ by linear interpolation, we get

$$y(x_{i-1}-\delta) \approx y(x_{i-1}) - \delta y'(x_{i-1}) \approx y_{i-1} - \delta\left(\frac{y_i - y_{i-1}}{h}\right) \approx \left(1+\frac{\delta}{h}\right)y_{i-1} - \frac{\delta}{h}y_i$$

$$y(x_i-\delta) \approx y(x_i) - \delta y'(x_i) \approx y_i - \delta\left(\frac{y_{i+1} - y_i}{h}\right) \approx \left(1+\frac{\delta}{h}\right)y_i - \frac{\delta}{h}y_{i+1}$$

Substituting the above approximations in Eq. (10) and rearranging the terms, we get the tridiagonal system as follows:

$$E_iy_{i-1} - F_iy_i + G_iy_{i+1} = H_i \text{ for } i = n+1, n+2, \ldots, N-1. \tag{11}$$

where

$$E_i = \frac{2\varepsilon}{h} - \frac{h}{2}p_{i-1}\left(1+\frac{\delta}{h}\right) - \frac{h}{2}q_{i-1}$$

$$F_i = \frac{4\varepsilon}{h} + \frac{h}{2}p'_i\left(1+\frac{\delta}{h}\right) + p_i\left(1+\frac{\delta}{h}\right) - p_{i-1}\frac{\delta}{2} + \frac{h}{2}q_i$$

$$G_i = \frac{2\varepsilon}{h} + \frac{\delta}{2}P_i$$

$$H_i = \frac{h}{2}[r_i + r_{i-1}]$$

Now, from Eq. (8) in $\left[0, \frac{1}{2}\right]$ for $i = 1, 2, \ldots, n-1$; and from Eq. (11) in $\left[\frac{1}{2}, 1\right]$ for $i = n+1, n+2, \ldots, N-1$; we have a system of ($N$-2) equations with ($N$+1) unknowns. From the given boundary conditions Eq. (2), we have two more equations.

We need one more equation to solve for the unknowns $y_0, y_1, \ldots, y_N$. To get this equation, we consider the reduced problem of Eq. (1) by setting $\varepsilon = 0$, then we have

$$a(x)y(x-\delta) + b(x)y(x) = f(x)$$

which does not satisfy both the boundary conditions.

At $x = x_n = \frac{1}{2}$, above equation becomes

$$a(x_n)y(x_n-\delta) + b(x_n)y(x_n) = f(x_n) \tag{12}$$

Using Taylor series expansion, we have

$$y(x_n-\delta) \approx y(x_n) - \delta y'(x_n) = y_n - \delta\left(\frac{y_{n+1} - y_{n-1}}{2h}\right)$$

Substituting this in Eq. (12) and by simplification, we have

$$\frac{\delta}{2h}y_{n-1} - (-a_n - b_n)y_n - \frac{\delta}{2h}y_{n+1} = f_n \tag{13}$$

With Eq. (13), we now have (N + 1) equations to solve for the (N + 1) unknowns $y_0, y_1, \ldots, y_N$. We solve this tridiagonal algebraic system by using an efficient and stable method of invariant imbedding [11].

3. Convergence analysis

Writing the tridiagonal system (8) in matrix-vector form, we get

$$AY = C \tag{14}$$

in which $A = (m_{i\,j})$, $1 \leqslant i, j \leqslant n-1$ is a tridiagonal matrix of order $n-1$, with

$$m_{ii+1} = \frac{2\varepsilon}{h} + ha_{i+1}\left(1-\frac{\delta}{h}\right) + hb_{i+1},$$
$$m_{ii} = -\frac{4\varepsilon}{h} + a_{i+1}\delta + ha_i\left(1-\frac{\delta}{h}\right) + hb_i,$$
$$m_{ii-1} = \frac{2\varepsilon}{h} + \delta a_i$$

and $C = (d_i)$ is a column vector with $d_i = h(f_{i+1} + f_i)$, where $i = 1, 2, \ldots, \; n-1$ with local truncation error

$$T_i(h_i) = h^2\left[\frac{1}{2}(a_i' + b_i')y_i + \frac{1}{2}(a_i + b_i - \delta a_i')y_i' - \frac{1}{2}f_i'\right] + O(h^3) \quad (15)$$

Writing the tridiagonal system (11) in matrix–vector form, we get

$$AY = C \quad (16)$$

in which $A = (m_{i\,j})$, $n+1 \leqslant i, j \leqslant N-1$ is a tridiagonal matrix of order $N-1$, with

$$m_{ii+1} = \frac{2\varepsilon}{h} - \delta a_i$$
$$m_{ii} = \frac{4\varepsilon}{h} - ha_i\left(1+\frac{\delta}{h}\right) + \delta a_{i-1} - hb_i$$
$$m_{ii-1} = \frac{2\varepsilon}{h} + ha_{i-1}\left(1+\frac{\delta}{h}\right) + ha_{i-1}$$

and $C = (d_i)$ is a column vector with $d_i = h\,(f_i + f_{i-1})$, where $i = n+1, n+2, \ldots, N-1$ with local truncation error

$$T_i(h_i) = h^2\left[\left(-b_i' + \frac{a_i'}{2} - b_i + \frac{b_i'}{2} - a_i\right)y' + f_i'\right] + O(h^3) \quad (17)$$

and $Y = (y_1, y_2, \ldots, y_{N-1})^t$.

We also have

$$A\overline{Y} - T(h) = C \quad (18)$$

where $\overline{Y} = (\overline{y_1}, \overline{y_2}, \ldots, \overline{y_{N-1}})^t$ denotes the actual solution, and $T(h) = (T_1(h_1), T_2(h_2), \ldots, T_{N-1}(h_{N-1}))^t$ is the local truncation error.

From (14), (16) and (18), we get

$$A(\overline{Y} - Y) = T(h) \quad (19)$$

Thus, the error equation is

$$AE = T(h) \quad (20)$$

where $E = \overline{Y} - Y = (e_1, e_2, \ldots, e_{N-1})^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{ij} = h(a_i + b_i) + O(h^2) = hB_i', i = 1, 2, \ldots, n-1$$

where $B_i' = (a_i + b_i)$

$$S_i = h(a_n + b_n) = hB_i'', i = n \text{ where } B_i'' = (a_n + b_n)$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = h[2(-a_i - b_i)] + O(h^2) = hB_i''', i = n+1, \ldots, N-1$$

where $B_i''' = 2(-a_i - b_i)$

We can choose h sufficiently small, so that the matrix A is irreducible and monotone. It follows that A^{-1} exists and its elements are non negative.

Hence from Eq. (20), we get

$$E = A^{-1}T(h) \quad (21)$$

Also from the theory of matrices, we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i}\, S_i = 1, \quad k = 1, 2, \ldots, N-1 \quad (22)$$

where $\bar{m}_{k,i}$ is (k, i) element of the matrix A^{-1},

Therefore,

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} \leqslant \frac{1}{\min\limits_{1\leqslant i\leqslant N-1} S_i} = \frac{1}{hB_{i_o}} \leqslant \frac{1}{h|B_{i_o}|} \quad (23)$$

for some i_0 between 1 and $N-1$ and

$$B_{i_0} = \begin{bmatrix} B_i', & i = 1(1)n-1 \\ B_i'', & i = n \\ B_i''', & i = n+1(1)N-1 \end{bmatrix} \quad (24)$$

From Eqs. (15), (17), (21) and (23), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k.i} T_i(h), \quad j = 1, 2, \ldots, N-1$$

which implies

$$e_j \leqslant \frac{kh}{|B_{i_0}|}, \quad j = 1, 2, \ldots, N-1 \quad (25)$$

where k is a constant independent of h.

Therefore,

$$\|E\| = O(h)$$

i.e., our method reduces to a first order convergent for uniform mesh.

4. Numerical experiments

To demonstrate the applicability of the method, we implement the method on four numerical experiments, two with twin boundary layers and two with oscillatory behaviour. We present the graphs of the computed solution of the problem for different values of ε and for different values of δ of $o(\varepsilon)$. The maximum absolute errors for the examples are calculated using the double mesh principle [12], $E^N = \max_{0\leqslant i\leqslant N}\left|y_i^N - y_{2i}^{2N}\right|$.

Example 1. Consider an example of singularly perturbed delay differential equation with layer behaviour

$\varepsilon y''(x) - 2y(x-\delta) - y(x) = 1$ under the interval with boundary conditions

$y(x) = 1, -\delta \leqslant x \leqslant 0$ and $y(1) = 0$.

The maximum absolute errors are presented in Tables 1 and 2 for different values of ε and for different values of δ. Also, we present the graph of the computed solution of the problem for $\varepsilon = 0.1$ for different values of δ shown in Fig. 1.

Table 1 The maximum absolute error of the examples for different values of δ with $\varepsilon = 0.1$.

δ	N				
	100	200	300	400	500
Example 1					
0.03	3.1674e−003	1.6058e−003	1.0754e−003	8.0837e−004	6.4760e−004
0.05	3.1437e−003	1.5949e−003	1.0685e−003	8.0338e−004	6.4367e−004
0.09	3.0784e−003	1.5660e−003	1.0502e−003	7.9000e−004	6.3310e−004
Results in Phaneendra et.al. [13]					
0.03	9.3352e−003	4.9360e−003	3.3540e−003	2.5398e−003	2.0438e−003
0.05	8.7514e−003	4.7344e−003	3.2355e−003	2.4561e−003	1.9803e−003
0.09	7.2037e−003	4.1449e−003	2.8840e−003	2.2111e−003	1.7913e−003
Example 2					
0.03	2.1999e−003	1.1041e−003	7.3705e−004	5.5315e−004	4.4269e−004
0.05	2.2012e−003	1.1049e−003	7.3749e−004	5.5345e−004	4.4293e−004
0.09	2.1999e−003	1.1038e−003	7.3676e−004	5.5289e−004	4.4247e−004
Results in Phaneendra et.al. [13]					
0.03	8.9194e−003	4.5468e−003	3.0511e−003	2.2959e−003	1.8404e−003
0.05	8.9177e−003	4.5440e−003	3.0482e−003	2.2934e−003	1.8382e−003
0.09	8.8966e−003	4.5252e−003	3.0345e−003	2.2825e−003	1.8292e−003
Example 3					
0.03	2.5991e−003	1.2872e−003	8.5528e−004	6.4039e−004	5.1179e−004
0.05	2.6270e−003	1.3013e−003	8.6474e−004	6.4750e−004	5.1749e−004
0.09	2.6813e−003	1.3289e−003	8.8320e−004	6.6139e−004	5.2863e−004
Results in Phaneendra et.al. [13]					
0.03	7.1024e−002	3.5558e−002	2.3661e−002	1.7721e−002	1.4163e−002
0.05	6.9203e−002	3.4790e−002	2.3181e−002	1.7373e−002	1.3890e−002
0.09	6.6055e−002	3.3490e−002	2.2377e−002	1.6794e−002	1.3439e−002
Example 4					
0.03	1.5929e−002	7.4850e−003	4.8816e−003	3.6202e−003	2.8764e−003
0.05	1.5470e−002	7.2782e−003	4.7473e−003	3.5209e−003	2.7975e−003
0.09	2.1396e−002	1.0097e−002	6.5922e−003	4.8916e−003	3.8879e−003
Results in Phaneendra et.al. [13]					
0.03	1.9740e−001	1.0467e−001	7.0844e−002	5.3521e−002	4.2985e−002
0.05	2.5749e−001	1.3585e−001	9.2035e−002	6.9554e−002	5.5884e−002
0.09	1.5004e−000	7.1504e−001	4.6444e−001	3.4319e−001	2.7196e−001

Table 2 The maximum absolute error of the examples for different values of ε for $\delta = 0.5\varepsilon$.

ε	N				
	2^4	2^5	2^6	2^7	2^8
Example 1					
2^{-4}	2.1118e−002	1.1692e−002	6.1941e−003	3.1887e−003	1.6178e−003
2^{-5}	2.7872e−002	1.6023e−002	8.6367e−003	4.4957e−003	2.2948e−003
2^{-6}	3.5711e−002	2.1293e−002	1.1869e−002	6.2731e−003	3.2240e−003
2^{-7}	4.6679e−002	2.8350e−002	1.6107e−002	8.6728e−003	4.5120e−003
2^{-8}	5.4895e−002	3.6018e−002	2.1373e−002	1.1929e−002	6.2847e−003
2^{-9}	5.7371e−002	4.7254e−002	2.8581e−002	1.6140e−002	8.6961e−003
2^{-10}	5.7878e−002	5.5695e−002	3.6153e−002	2.1406e−002	1.1956e−002
Example 2					
2^{-4}	1.8632e−002	9.6189e−003	4.8865e−003	2.4643e−003	1.2376e−003
2^{-5}	2.8161e−002	1.4818e−002	7.6255e−003	3.8713e−003	1.9509e−003
2^{-6}	3.7958e−002	2.0967e−002	1.0977e−002	5.6273e−003	2.8498e−003
2^{-7}	5.0640e−002	2.8316e−002	1.5267e−002	7.9105e−003	4.0287e−003
2^{-8}	6.3580e−002	3.7706e−002	2.0984e−002	1.1012e−002	5.6555e−003
2^{-9}	8.3843e−002	5.0477e−002	2.8297e−002	1.5261e−002	7.9111e−003
2^{-10}	9.9137e−002	6.3529e−002	3.7660e−002	2.0974e−002	1.1011e−002

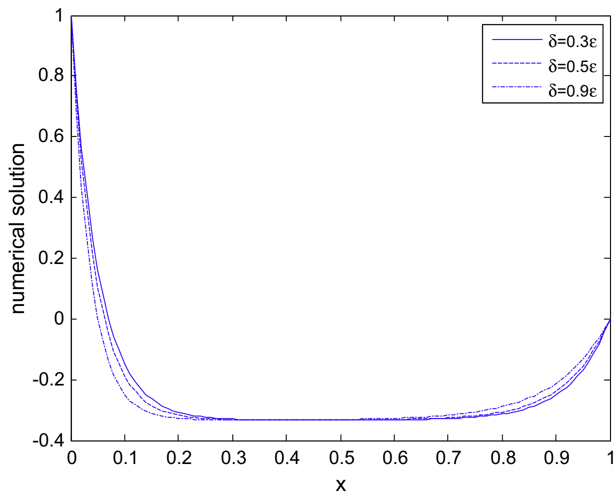

Figure 1 The numerical solution of Example 1 with $\varepsilon = 0.01$.

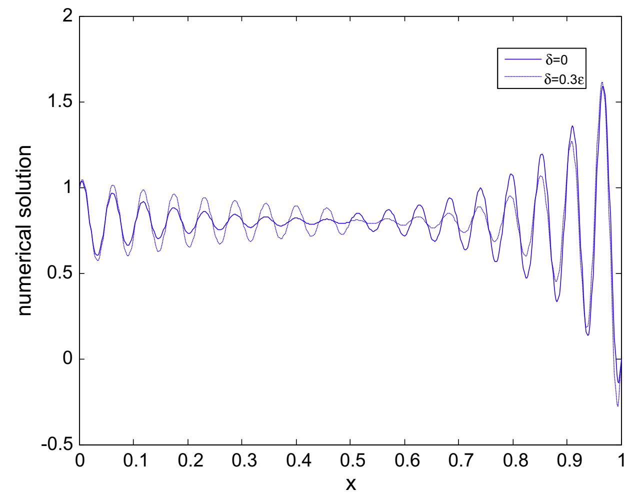

Figure 3 The numerical solution of Example 3 with $\varepsilon = 0.01$.

Example 2. Consider singularly perturbed delay differential equation with layer behaviour

$\varepsilon y''(x) + 0.25y(x - \delta) - y(x) = 1$ with boundary conditions $y(x) = 1, -\delta \leqslant x \leqslant 0$ and $y(1) = 0$.

The maximum absolute errors are presented in Tables 1 and 2 for different values of ε and for different values of δ. Also, we present the graph of the computed solution of the problem for $\varepsilon = 0.01$ for different values of δ shown in Fig. 2.

Example 3. Consider a singularly perturbed delay differential equation with oscillatory behaviour

$\varepsilon y''(x) + 0.25y(x - \delta) + y(x) = 1$ with boundary conditions $y(x) = 1, -\delta \leqslant x \leqslant 0, y(1) = 0$.

The maximum absolute errors are presented in Table 1 for $\varepsilon = 0.1$ for different values of δ. Also, we present the graph of the computed solution of the problem for $\varepsilon = 0.01$ for different values of δ shown in Fig. 3.

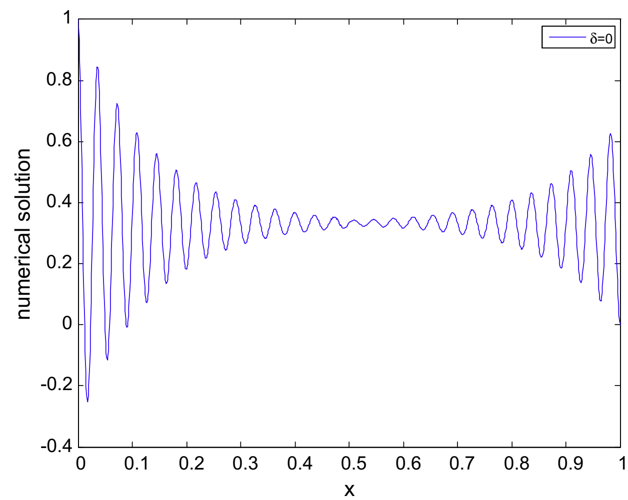

Figure 4 The numerical solution of Example 4 with $\varepsilon = 0.01$.

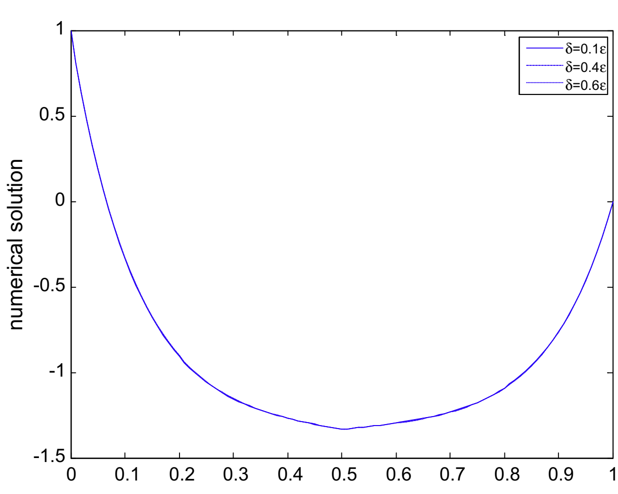

Figure 2 The numerical solution of Example 2 with $\varepsilon = 0.01$.

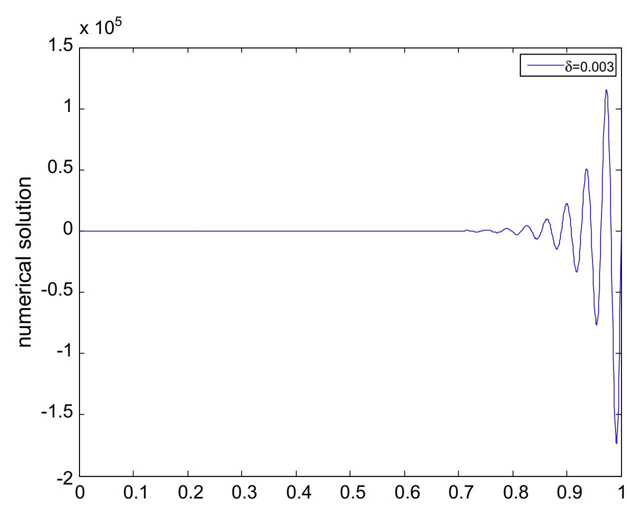

Figure 5 The numerical solution of Example 4 with $\varepsilon = 0.01$.

Example 4. Consider the singularly perturbed delay differential equation with oscillatory behaviour

$\varepsilon y''(x) + y(x - \delta) + 2y(x) = 1$, with boundary conditions $y(x) = 1, -\delta \leqslant x \leqslant 0, y(1) = 0$.

The maximum absolute errors are presented in Table 1 for $\varepsilon = 0.1$ for different values of δ. Also the graphs of the computed solution of the problem presented for $\varepsilon = 0.01$ for different values of δ in Figs. 4 and 5.

For $\delta = 0$, the solution to the boundary value problem (1) and (2) exhibits layer or oscillatory behaviour according to the sign of the coefficient of the reaction term. In Examples 1 and 4, the coefficient of the delay term is of $O(1)$ while that of $o(1)$ in Examples 2 and 3.

5. Discussions and conclusion

We have discussed a computational technique for singularly perturbed delay differential equations with twin layer or oscillatory behaviour. In this problem, the delay is not in the differentiated term. In this method, we have transformed the second order singularly perturbed delay differential equation into an asymptotically equivalent first order neutral type delay differential equation and employed the Trapezoidal rule. Then, linear interpolation is used to get tridiagonal relation which is solved easily by method of invariant imbedding algorithm. The method is demonstrated by implementing on several model examples by taking various values for the delay parameter and perturbation parameter.

To show the effect of delay on the twin boundary layer or oscillatory behaviour of the solution, several numerical examples are carried out in Section 3. We observed that when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour. From Fig. 1, we observed that when the delay is $o(\varepsilon)$, the solution maintains layer behaviour although the coefficients in the equation are of $O(1)$ and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases.

To demonstrate the effect on the oscillatory behaviour, we have considered Examples 3 and 4. From Fig. 3 of Example 3, we observed that if the coefficient of the delay is of $o(1)$, the amplitude of the oscillations increases slowly as the delay increases provided the delay is of $o(\varepsilon)$. From Figs. 4 and 5 of Example 4, we observed that if the coefficient of the delay is of $O(1)$, the amplitude of the oscillations increases slowly from the left end $x = 0$ to right end $x = 1$ as the delay increases provided the delay is of $o(\varepsilon)$.

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