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Axisymmetric creeping flow past a porous approximate sphere with an impermeable core

D. Srinivasacharya[a] and M. Krishna Prasad[b]

Department of Mathematics, National Institute of Technology, Warangal - 506 004, A.P., India

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Abstract. The creeping flow of an incompressible viscous liquid past and through a porous approximate sphere with an impermeable core is considered. The flow in the free-fluid region outside the sphere is governed by the Stokes equation. The flow inside the porous sphere is governed by Brinkman's model. The boundary conditions used at the porous-liquid interface of the clear fluid and porous region are continuity of the velocity, continuity of the pressure and Ochoa-Tapia and Whitaker's stress jump condition. On the surface of the impermeable core no slip condition is used. An exact solution for the problem is obtained. An expression for the drag on the porous approximate spherical particle is obtained. A comparison is made with our earlier results on the particle with the fluid core. The variation of drag is studied with respect to permeability and stress jump coefficient. It is observed that the stress jump condition, characterized by a stress jump coefficient, has a significant influence on the drag acting on a particle.

1 Introduction

Several axi-symmetric flow problems past and within porous particles of different geometries have been considered in the last few decades using Stokes' version of the Navier-Stokes equations for the flow outside the porous particles and Darcy's law or Brinkman's equation to describe the flow within the porous particles. While working with the problems of flow past porous particles, the specification of boundary conditions at a porous-liquid interface is not trivial and different boundary conditions are proposed in the literature [1–6]. When the Brinkman equation is used to model flow in the porous region, Neale and Nader [4] suggested to impose continuity of both stress and velocity at the interface. However, by applying volume average technique, Ochoa-Tapia and Whitaker [5,6] developed a much different interfacial boundary condition, which accounts for a jump in the stress at the interface. They derived the stress jump boundary condition as

$$\epsilon^{-1}\,\frac{\partial u^p}{\partial y} - \frac{\partial u^l}{\partial y} = \frac{\sigma}{\sqrt{k}}\,u^p,$$

where u^p, u^l are tangential velocity components in the porous region and the liquid region, respectively, y is the ordinate normal to surface, ϵ is the porosity, k is the permeability of the homogeneous portion of the porous region, and σ is the stress jump coefficient.

Srivastava and Srivastava [7] studied the Stokes flow through a porous sphere with a solid core using the stress jump condition at the fluid-porous interface and matched Stoke's and Oseen's solutions far away from the sphere. They concluded that drag on a porous sphere decreases with increasing permeability of the medium. Recently, Srinivasacharya and Krishna Prasad discussed the creeping flow past a porous approximate sphere [8] and spherical shell [9]using Ochoa-Tapia and Whitaker's boundary conditions.

In this paper, we consider creeping flow past a porous approximate sphere with an impermeable core using a slip condition at the interface proposed by Ochoa-Tapia [5,6]. The analysis is an extension of that presented by Srinivasacharya and Kishna Prasad [9]. The difference between the problems considered in [9] and the present paper is the flow in the core region. The flow examined is axially symmetric in nature. The flow equations are based on the Stokesian version of the Navier-Stokes equations in the general viscous flow regime and the use of Brinkman's model in

[a] e-mail: dsc@nitw.ac.in; e-mail: dsrinivasacharya@yahoo.com
[b] e-mail: kpm973.nitw@gmail.com

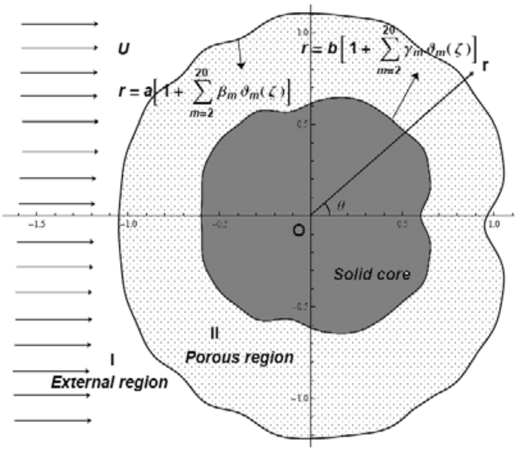

Fig. 1. The physical situation and the coordinate system ($m = 20$, $a = 1$, $b = 0.6$).

the porous region. The drag experienced by the porous approximate spherical shell is evaluated. The variation of drag is studied with respect to geometric and permeability parameters. The reason for choosing this particular problem is two fold: First, it includes two different interfaces of interest, namely, an interface between porous and clear media at the external surface of the approximate sphere and an interface between porous and impermeable media at the core's boundary. Second, it is of significance in the processing of solid particles in a chemical reactor such as new catalytic solid-gas reaction: particularly limestone or dolomite sulphuring, and oil shale combustion.

2 Formulation of the problem

Consider the creeping flow of an incompressible Newtonian viscous fluid past a porous approximate sphere with an impermeable core of average radius a having a solid concentric approximate spherical core of average radius b ($a > b$) inside it (see fig. 1). Let (r, θ, ϕ) denote a spherical polar coordinate system with $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ unit basis vectors. Assume that there is a uniform velocity U far away from the shell along the axis of symmetry $\theta = 0$. Let the equation of the porous approximate sphere be $r = a[1 + \sum_{m=2}^{\infty} \beta_m \vartheta_m(\zeta)] \equiv r_a$ and the equation of the impermeable core be $r = b[1 + \sum_{m=2}^{\infty} \gamma_m \vartheta_m(\zeta)] \equiv r_b$ where β_m and γ_m are small, $\zeta = \cos\theta$ and $\vartheta_n(\zeta)$ is the Gegenbauer function [10] of the first kind of order n and degree $-1/2$. If all the β_m and γ_m are zero, the approximate spherical shell reduces to a spherical shell of external and internal radii a and b. The formulation follows closely that in [9].

2.1 Governing equations

Following [9], the governing equations of motion for the region outside the approximate sphere ($r \geq r_a$) are

$$\nabla \cdot \boldsymbol{q}^{(1)} = 0, \tag{1}$$

$$\nabla p^{(1)} + \mu \nabla \times \nabla \times \boldsymbol{q}^{(1)} = 0, \tag{2}$$

where $\boldsymbol{q}^{(1)}$ is the volumetric average of the velocity, μ is the coefficient of viscosity, and $p^{(1)}$ is the average of the pressure.

For the porous region ($r_b \leq r \leq r_a$), the equations of motion are

$$\nabla \cdot \boldsymbol{q}^{(2)} = 0, \tag{3}$$

$$\nabla p^{(2)} + \frac{\mu}{k} \boldsymbol{q}^{(2)} + \mu \nabla \times \nabla \times \boldsymbol{q}^{(2)} = 0, \tag{4}$$

where $\boldsymbol{q}^{(2)}$ is the volumetric average of the velocity, $p^{(2)}$ is the average of the pressure and k is the permeability of the porous medium.

Since the flow is in the meridian plane and is axially symmetric, all the physical quantities are independent of ϕ. Hence we assume that

$$\boldsymbol{q}^{(i)} = u^{(i)}(r,\theta)\, \boldsymbol{e}_r + v^{(i)}(r,\theta)\, \boldsymbol{e}_\theta, \quad i = 1, 2. \tag{5}$$

In view of the incompressibility condition $\nabla \cdot \boldsymbol{q}^{(i)} = 0$, $i = 1, 2$, we introduce the stream function $\psi^{(i)}(r, \theta)$, $i = 1, 2$ through

$$u^{(i)} = -\frac{1}{r^2 \sin\theta} \frac{\partial \psi^{(i)}}{\partial \theta}, \quad v^{(i)} = \frac{1}{r \sin\theta} \frac{\partial \psi^{(i)}}{\partial r}, \quad i = 1, 2. \tag{6}$$

Eliminating the pressure from (2) and (4), and substituting (6) in the resulting equations, we get the following dimensionless equations for $\psi^{(i)}$:

$$E^4 \psi^{(1)} = 0, \tag{7}$$

$$E^2 (E^2 - \alpha^2) \psi^{(2)} = 0, \tag{8}$$

where $\alpha^2 = a^2/k$ and $E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\zeta^2)}{r^2} \frac{\partial^2}{\partial \zeta^2}$ is the Stokesian stream function operator.

2.2 Boundary conditions

To determine the flow velocity and pressure outside and within the porous approximate sphere, we use the following boundary conditions at the surface of a porous body to link the different flow regimes. The continuity of velocity components, continuity of pressure and stress jump conditions at the interface $r = r_b$ were considered in [9] whereas no slip condition at $r = r_b$ is considered in this problem because of the impermeable core.

i) Continuity of velocity components on the boundary of the approximate sphere,

$$u^{(1)}(r, \theta) = u^{(2)}(r, \theta) \quad \text{and} \quad v^{(1)}(r, \theta) = v^{(2)}(r, \theta) \quad \text{on} \quad r = r_a. \tag{9}$$

ii) Continuity of pressure on the boundary,

$$p^{(1)}(r, \theta) = p^{(2)}(r, \theta) \quad \text{on} \quad r = r_a. \tag{10}$$

iii) Ochoa-Tapia's stress jump boundary condition for the tangential stress,

$$\frac{\partial v^{(2)}}{\partial r} - \frac{\partial v^{(1)}}{\partial r} = \frac{\sigma}{\sqrt{k}} v^{(2)} \quad \text{on} \quad r = r_a, \tag{11}$$

where σ is the stress jump coefficient.

iv) On the impermeable core,

$$u^{(2)}(r, \theta) = 0 \quad \text{and} \quad v^{(2)}(r, \theta) = 0 \quad \text{on} \quad r = r_b. \tag{12}$$

Additionally, we have the regularity conditions at infinity,

$$\lim_{r \to \infty} u^{(1)}(r, \theta) = U \cos\theta, \quad \lim_{r \to \infty} v^{(1)}(r, \theta) = -U \sin\theta \tag{13}$$

and the condition that velocity and pressure must be nonsingular everywhere in the flow field.

3 Solution of the problem

Using the separation of variables, the solution of (7) (as given in [9]) which is regular at infinity, *i.e.*, far away from the shell and on the axis, is

$$\psi^{(1)} = \left[r^2 + \frac{A_2}{r} + B_2 r \right] \vartheta_2(\zeta) + \sum_{n=3}^{\infty} \left[A_n r^{-n+1} + B_n r^{-n+3} \right] \vartheta_n(\zeta), \tag{14}$$

and the solution of (8) (as given in [9]) is

$$\begin{aligned} \psi^{(2)} = &\left[C_2 r^2 + \frac{D_2}{r} + E_2 \sqrt{r} K_{3/2}(\alpha r) + F_2 \sqrt{r} I_{3/2}(\alpha r) \right] \vartheta_2(\zeta) \\ &+ \sum_{n=3}^{\infty} \left[C_n r^n + D_n r^{-n+1} + E_n \sqrt{r} K_{n-1/2}(\alpha r) + F_n \sqrt{r} I_{n-1/2}(\alpha r) \right] \vartheta_n(\zeta), \end{aligned} \tag{15}$$

where $I_{n-1/2}(\alpha r)$ and $K_{n-1/2}(\alpha r)$ denote the modified Bessel functions of the first kind and second kind of order $n - 1/2$, respectively. The solutions are general solutions and are equivalent to those in [9].

Using eqs. (14) and (15), the expressions for the pressure in the both flow regions are

$$p^{(1)} = -\frac{B_2}{r^2}\, P_1(\zeta) + \sum_{n=3}^{\infty} B_n \left(\frac{6-4\,n}{n}\right) r^{-n}\, P_{n-1}(\zeta), \tag{16}$$

$$p^{(2)} = \alpha^2 \left[C_2\, r - \frac{D_2}{2\,r^2}\right] P_1(\zeta) + \alpha^2 \sum_{n=3}^{\infty} \left[C_n\, \frac{r^{n-1}}{n-1} - D_n\, \frac{r^{-n}}{n}\right] P_{n-1}(\zeta). \tag{17}$$

The expression for stream functions ($\psi^{(1)}$, $\psi^{(2)}$) and pressure distributions ($p^{(1)}$, $p^{(2)}$) given in (14)–(17) are general and equivalent to those in [9]. The difference in the two problems (present problem and [9])will be reflected in the values of the arbitrary constants in those formulas.

3.1 Determination of arbitrary constants

The boundary conditions from eqs. (9)–(12) in terms of the stream function in dimensionless form are

$$\left.\begin{aligned} &\psi^{(1)}(r,\theta) = \psi^{(2)}(r,\theta), \quad \psi_r^{(1)}(r,\theta) = \psi_r^{(2)}(r,\theta),\\ &\psi_{rr}^{(2)} - \psi_{rr}^{(1)} = \alpha\,\sigma\,\psi_r^{(2)}, \quad p^{(1)}(r,\theta) = p^{(2)}(r,\theta), \end{aligned}\right\} \quad \text{on} \quad r = 1 + \sum_{m=2}^{\infty} \beta_m\, \vartheta_m(\zeta), \tag{18}$$

$$\psi^{(2)}(r,\theta) = 0, \quad \psi_r^{(2)}(r,\theta) = 0 \quad \text{on} \quad r = \eta \left[1 + \sum_{m=2}^{\infty} \gamma_m\, \vartheta_m(\zeta)\right], \tag{19}$$

where $\eta = b/a$.

We develop the solution corresponding to the boundaries $r = 1 + \beta_m \vartheta_m(\zeta)$ and $r = \eta[1 + \gamma_m \vartheta_m(\zeta)]$. Assume that the coefficients β_m and γ_m are sufficiently small so that squares and higher powers of β_m and γ_m can be neglected [10]. The comparison of the stream functions (14) and (15) with those obtained in case of flow of an incompressible viscous fluid past a porous sphere [11], indicates that the terms involving A_n, B_n, C_n, D_n, E_n and F_n for $n > 2$ are the extra terms which are not present in the case of a perfect sphere. The body that we are considering is an approximate sphere and the flow generated is not expected to be much different from the one generated by flow past a porous sphere. Also the coefficients A_n, B_n, C_n, D_n, E_n and F_n for $n > 2$ are of order β_m and the coefficients C_n, D_n, E_n and F_n for $n > 2$ are of order γ_m. Therefore, as in [12], while implementing the boundary conditions, we ignore the departure from the spherical form and set in (18) $r = 1$ in the terms involving A_n, B_n, C_n, D_n, E_n and F_n for $n > 2$ and in (19) $r = \eta$ in the terms involving C_n, D_n, E_n and F_n for $n > 2$.

Using the observations made above and the boundary conditions (18) and (19) in the expressions (14) and (15) and equating leading coefficients to zero in the resulting equations, we get a system of equations in A_2, B_2, C_2, D_2, E_2, and F_2. The expressions for these constants are given in appendix A. Using the identities given in [10] (p. 142), we get $A_n = B_n = C_n = D_n = E_n = F_n = 0$ for $n \neq m-2, m, m+2$ and another system of equations in A_n, B_n, C_n, D_n, E_n, and F_n for $n = m-2, m, m+2$. Solving this system of equations, we get the expressions for the arbitrary constants A_n, B_n, C_n, D_n, E_n and F_n for $n = m-2, m, m+2$. As the expressions for these constants are lengthy, they have not been presented here.

In the case where the approximate spherical shell is given by $r = a\,[1 + \sum_{m=2}^{\infty} \beta_m\, \vartheta_m(\zeta)]$ and $r = b\,[1 + \sum_{m=2}^{\infty} \gamma_m\, \vartheta_m(\zeta)]$, we employ the above technique for each m and obtain the expressions for the stream functions for the regions $r \geq r_a$ and $r_b \leq r \leq r_a$ by superposition of the expressions thus obtained. Thus, the stream functions for the regions $r \geq r_a$ and $r_b \leq r \leq r_a$ are

$$\begin{aligned} \psi^{(1)} = {} & \left[r^2 + \frac{A_2}{r} + B_2\, r\right] \vartheta_2(\zeta) + \sum_{m=2}^{\infty} \Big\{ \left[A_{m-2}\, r^{-m+3} + B_{m-2}\, r^{-m+5}\right] \vartheta_{m-2}(\zeta)\\ & + \left[A_m\, r^{-m+1} + B_m\, r^{-m+3}\right] \vartheta_m(\zeta) + \left[A_{m+2}\, r^{-m-1} + B_{m+2}\, r^{-m+1}\right] \vartheta_{m+2}(\zeta) \Big\}, \end{aligned} \tag{20}$$

$$\begin{aligned} \psi^{(2)} = {} & \left[C_2\, r^2 + \frac{D_2}{r} + E_2 \sqrt{r}\, K_{3/2}(\alpha r) + F_2 \sqrt{r}\, I_{3/2}(\alpha r)\right] \vartheta_2(\zeta)\\ & + \sum_{m=2}^{\infty} \Big\{ \big[C_{m-2}\, r^{m-2} + D_{m-2}\, r^{-m+3} + E_{m-2} \sqrt{r}\, K_{m-5/2}(\alpha r)\\ & + F_{m-2} \sqrt{r}\, I_{m-5/2}(\alpha r)\big] \vartheta_{m-2}(\zeta) + \big[C_m\, r^m + D_m\, r^{-m+1} + E_m \sqrt{r}\, K_{m-1/2}(\alpha r)\\ & + F_m \sqrt{r}\, I_{m-1/2}(\alpha r)\big] \vartheta_m(\zeta) + \big[C_{m+2}\, r^{m+2} + D_{m+2}\, r^{-m-1} + E_{m+2} \sqrt{r}\, K_{m+3/2}(\alpha r)\\ & + F_{m+2} \sqrt{r}\, I_{m+3/2}(\alpha r)\big] \vartheta_{m+2}(\zeta) \Big\}. \end{aligned} \tag{21}$$

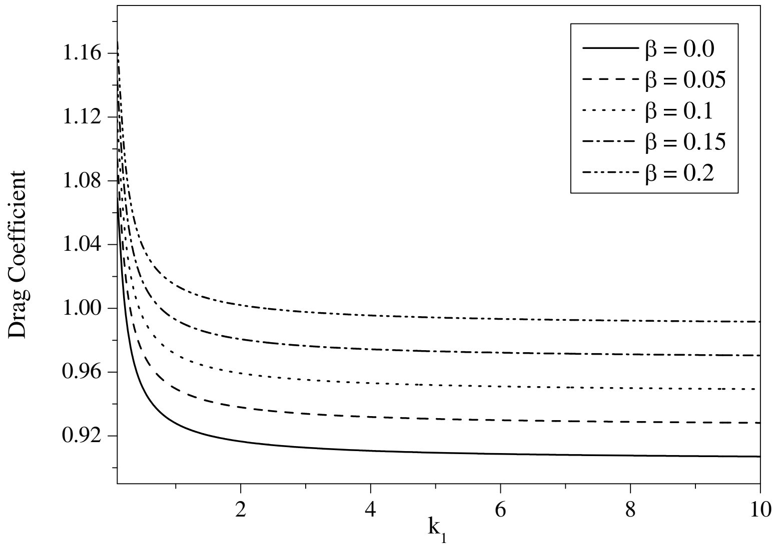

Fig. 2. Variation of the drag coefficient with permeability k_1 with varying β; $\sigma = 0$, $\eta = 0.6$, and $\gamma = 0.1$ are fixed.

4 Drag on the body

The drag force acting on the porous approximate sphere in terms of the stream function is given by

$$D = 2\pi a^2 \int_0^{\pi} r^3 \sin^3\theta \frac{\partial}{\partial r}\left(\frac{1}{r^2 \sin^2\theta} E^2 \psi^{(1)}\right) r \mathrm{d}\theta. \tag{22}$$

Using eq. (14) and carrying out the integration, it is found to be

$$D = 4\pi\mu U a \left[B_2 + \frac{1}{5}\left(B_2^{(1)}\beta_2 + B_2^{(2)}\gamma_2\right) + \frac{2}{35}\left(B_2^{(3)}\beta_4 + B_2^{(4)}\gamma_4\right)\right], \tag{23}$$

where $B_2^{(1)}$, $B_2^{(2)}$, $B_2^{(3)}$ and $B_2^{(4)}$ are given in appendix A. The same expression is found in [9] with the change in the values of $B_2^{(1)}$, $B_2^{(2)}$, $B_2^{(3)}$ and $B_2^{(4)}$.

It is interesting to note that although the boundary surface is given by $r = a\left[1 + \sum_{m=2}^{\infty} \beta_m \vartheta_m(\zeta)\right]$ and $r = b\left[1 + \sum_{m=2}^{\infty} \gamma_m \vartheta_m(\zeta)\right]$, only the coefficients β_2, β_4, γ_2 and γ_4, contribute to the drag. This implies that the drag on the porous approximate sphere with impermeable core is relatively insensitive to the details of the surface geometry. This is similar to the observations made by Srinivasacharya [12].

If $\beta_m = 0$ and $\gamma_m = 0$ for $m > 2$, the approximate spherical shell reduces to a spherical shell and the drag is

$$D = 4\pi\mu U a B_2. \tag{24}$$

The expression for the drag experienced by a porous sphere ($\eta = 0$) with continuity of tangential stress ($\sigma = 0$) is

$$D = \frac{12\pi\mu U a \alpha^2(-\alpha\cosh\alpha + \sinh\alpha)}{\alpha(3 + 2\alpha^2)\cosh\alpha - 3\sinh\alpha}, \tag{25}$$

which agrees with the porous sphere case derived by Brinkman [13], Neale *et al.* [14] and Qin and Kaloni [15]. The agreement with (25) and [13], [14] and [15] was pointed out in the flow past a porous approximate spherical shell [9].

The variation of the drag coefficient $D_N = D/(4\pi\mu U a)$ *versus* permeability $k_1 (= 1/\alpha^2)$ for fixed values of the outer deformation parameter β ($\beta_2 = \beta_4 = \beta$), and inner deformation parameter γ ($\gamma_2 = \gamma_4 = \gamma$) with continuity of the stress ($\sigma = 0$) is shown in figs. 2 and 3. Figure 2 shows the variation of the D_N with k_1 for fixed values of β when the inner sphere is perturbed $\gamma = 0.1$. It is observed that the drag coefficient is decreasing as the permeability k_1 is increasing. There is an increase in the D_N as the deformation parameter of the outer sphere β is increasing. It is interesting to note that the D_N on the porous sphere with an impermeable core is less than that of the D_N on the porous approximate sphere with an impermeable core. Figure 3 presents the variation of the D_N *versus* k_1 with the deformation of the inner sphere γ when the outer sphere is deformed $\beta = 0.15$. As γ increases, there is a slight decrease

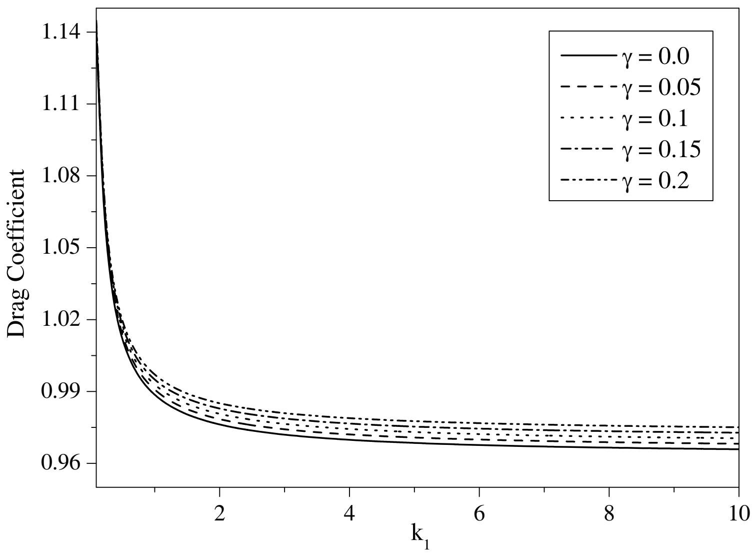

Fig. 3. Variation of the drag coefficient with permeability k_1 with varying γ; $\sigma = 0$, $\eta = 0.6$, and $\beta = 0.15$ are fixed.

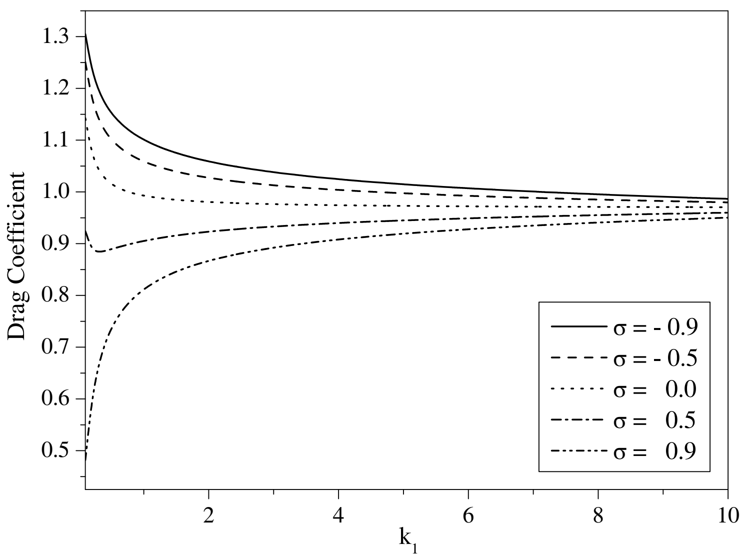

Fig. 4. Variation of the drag coefficient with permeability k_1 with varying σ; $\beta = 0.15$, $\gamma = 0.1$, and $\eta = 0.6$ are fixed.

in the D_N. These are in tune with the observations made in ref.[9]. Therefore, the effects of permeability parameter k_1 and deformation of outer sphere β on the drag coefficient are relatively insensitive to the core region (fluid core or impermeable core) of the approximate sphere.

The effect of the stress jump coefficient σ and permeability k_1 on the drag coefficient D_N at the porous-liquid interface is plotted in fig. 4. The values of σ are taken in the range -1 to 1 as proposed by Ochoa-Tapia and Whitaker [5,6]. The validity of these values of σ along with the combination of other parameters and permeability is examined for the present problem so that the drag experienced by the porous particle give a physical significance. It is observed that the drag coefficient D_N is decreasing as the σ is increasing. The D_N is decreasing as k_1 is increasing when there is a jump in the stress at the boundary. There is no change in the behavior of the drag on the approximate sphere with impermeable core for different values of σ unlike the drag on the approximate sphere with fluid core [9].

Figures 5 and 6 depict the variation of the drag coefficient D_N with permeability k_1 for various values of the separation parameter η for fixed values $\sigma = -0.9$ and $\sigma = 0.9$, respectively. As η tends to 1, *i.e.*, the distance between the solid core and the outer boundary of the porous envelope (or annulus) decreases, the porous particle with solid core becomes as a solid particle except at the deformations of the solid core and porous envelope. As η tends to 0, the porous particle with solid core becomes equivalent to a particle that is porous throughout. When the stress jump coefficient σ is negative, it is observed that increasing η (*i.e.*, decreasing the thickness of the porous region) increases the drag coefficient and the drag coefficient decreases as the permeability k_1 increases. But, when the stress jump coefficient σ

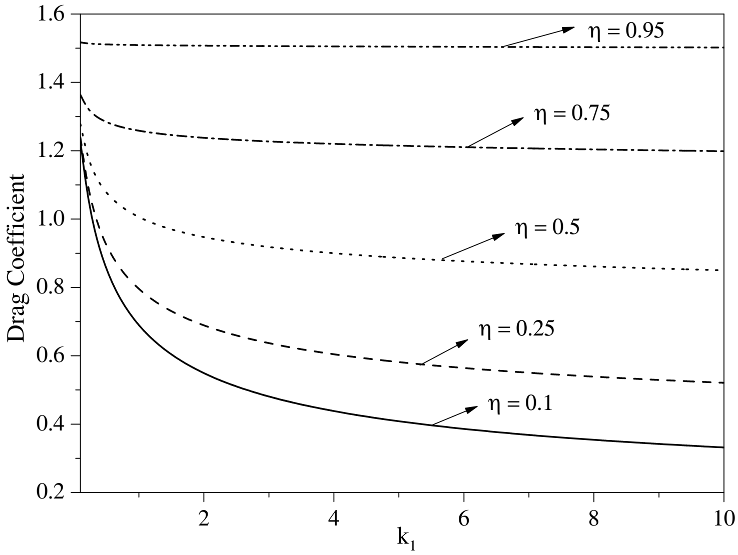

Fig. 5. Variation of the drag coefficient with permeability k_1 with varying η; $\beta = 0.15,\ \gamma = 0.1$, and $\sigma = -0.9$ are fixed.

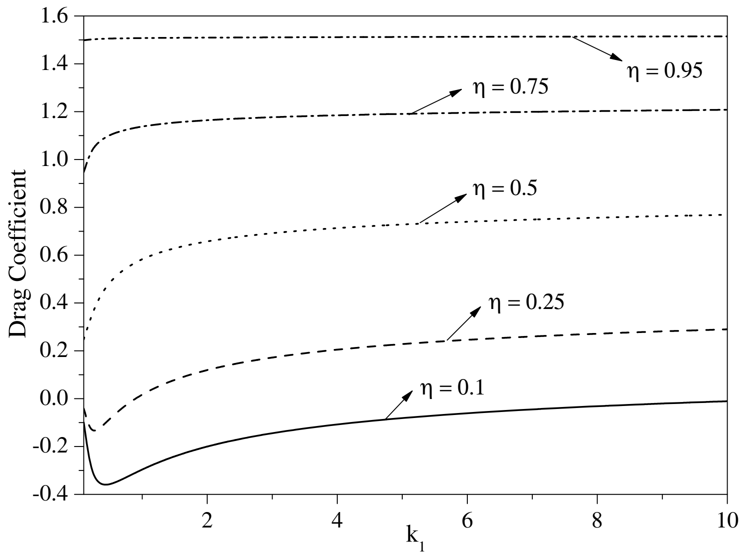

Fig. 6. Variation of the drag coefficient with permeability k_1 with varying η; $\beta = 0.15,\ \gamma = 0.1$, and $\sigma = 0.9$ are fixed.

is positive, the drag becomes negative for low values of the separation parameter $\eta < 0.3$ and the drag increases as the permeability increases. Therefore, it is found that the variation of drag not only depends on permeability but also on the stress jump coefficient and separation parameter.

5 Conclusions

An exact solution for the problem of the creeping flow of an incompressible viscous liquid past a porous approximate sphere with an impermeable core is obtained by considering Brinkman's model in the porous region and Stokes' equations in the liquid region. At the porous-liquid interface Ochoa-Tapia's stress jump boundary condition, the continuity of the normal velocity and the continuity of the pressure have been used. An expression for the drag on the porous approximate spherical shell is obtained. It is observed that the drag coefficient on the porous sphere with an impermeable core is less than that on the porous approximate sphere with an impermeable core as in the case of drag on approximate sphere with fluid core [9]. The drag is decreasing as the permeability is increasing. The drag is decreasing as the stress jump coefficient σ is increasing. The drag on approximate sphere with impermeable core is always positive for all the values of k_1 and σ, but drag on approximate sphere with fluid core is negative for positive values of σ and beyond certain value of k_1 [9]. It is found that there is a significant effect of the stress jump coefficient σ on the hydrodynamic drag. Therefore, one has to consider the stress jump in the tangential stress components while studying viscous flow problems involving the Brinkman equation in porous media and Stokes' equation in a free-flow region, which has a significant impact on the physical problem.

Appendix A.

The expressions for the constants A_2, B_2, C_2, D_2 and F_2 are

$$\begin{aligned}
A_2 &= \Big(-3\,\eta^2\,(2\,\alpha + (-4+\alpha^2)\sigma)\,z_1 + 6\,\sqrt{\eta}\,(\alpha - 2\,\sigma)\,z_2 + \big(-12\,\eta^{3/2}\,z_3 + (12+\alpha^2(2+\eta^3))\,z_4 + 3\,\alpha\,\eta^2\,z_6\big)(\alpha+\sigma) \\
&\quad + \alpha\,\big(-\alpha\,(2+\eta^3)\,(2+\alpha\,\sigma) + 4\,(-1+\eta^3)\,\sigma\big)\,z_5\Big)/(2\,\Delta), \\
B_2 &= -3\,\alpha\,\left(3\,\eta^2\,\omega_1 + \alpha\,(2+\eta^3)\,\omega_2\right)/(2\,\Delta), \\
C_2 &= 3\,\left(\omega_2 - \eta^{3/2}\,(\alpha+\sigma)\,z_3\right)/\Delta, \\
D_2 &= -3\,\eta^2\,\left(3\,\omega_1 + \alpha\,\eta\,\omega_2 + 2\,\alpha\,(\alpha+\sigma)\,\eta^{-1/2}\,z_3\right)/(\alpha\,\Delta), \\
E_2 &= \left(3\,(\alpha+\sigma)\,\left(\alpha\,(2+\eta^3)\,I_{1/2}(\alpha\eta) + 3\,\sqrt{\eta}\,(I_{3/2}(\alpha) - \eta^{3/2}\,I_{3/2}(\alpha\eta))\right) - 9\,\alpha\,\sigma\,\sqrt{\eta}\,I_{1/2}(\alpha)\right)/(\alpha\,\Delta), \\
F_2 &= \left(3\,(\alpha+\sigma)\,\left(\alpha\,(2+\eta^3)\,K_{1/2}(\alpha\eta) - 3\,\sqrt{\eta}\,(K_{3/2}(\alpha) - \eta^{3/2}\,K_{3/2}(\alpha\eta))\right) - 9\,\alpha\,\sigma\,\sqrt{\eta}\,K_{1/2}(\alpha)\right)/(\alpha\,\Delta),
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
&z_1 = I_{1/2}(\alpha)\,K_{3/2}(\alpha\eta) + I_{3/2}(\alpha\eta)\,K_{1/2}(\alpha), \quad z_2 = I_{1/2}(\alpha)\,K_{3/2}(\alpha) + I_{3/2}(\alpha)\,K_{1/2}(\alpha), \\
&z_3 = I_{3/2}(\alpha\eta)\,K_{1/2}(\alpha\eta) + I_{1/2}(\alpha\eta)\,K_{3/2}(\alpha\eta), \quad z_4 = I_{3/2}(\alpha)\,K_{1/2}(\alpha\eta) + I_{1/2}(\alpha\eta)\,K_{3/2}(\alpha) \\
&z_5 = I_{1/2}(\alpha)\,K_{1/2}(\alpha\eta) - I_{1/2}(\alpha\eta)\,K_{1/2}(\alpha), \quad z_6 = I_{3/2}(\alpha)\,K_{3/2}(\alpha\eta) - I_{3/2}(\alpha\eta)\,K_{3/2}(\alpha), \\
&\omega_1 = -z_1\,\alpha\,\sigma + z_6\,(\alpha+\sigma), \quad \omega_2 = -z_5\,\alpha\,\sigma + z_4\,(\alpha+\sigma), \\
&\Delta = 3\,\eta^2\,\left(\alpha + \sigma\,(1-\alpha^2)\right)\,z_1 + \left(-3\,\sqrt{\eta}\,(z_2 + \eta\,z_3) + (3+\alpha^2\,(2+\eta^3))\,z_4 + 3\,\alpha\,\eta^2\,z_6\right)\,(\alpha+\sigma) \\
&\qquad + \left(\alpha^2\,(2+\eta^3)\,(1-\alpha\,\sigma) + \alpha\,\sigma\,(-1+\eta^3)\right)\,z_5.
\end{aligned} \tag{A.2}$$

The expressions for the constants $B_2^{(1)}$, $B_2^{(2)}$, $B_2^{(3)}$ and $B_2^{(4)}$ appearing in eq. (23) are

$$\left.\begin{aligned}
&B_2^{(1)} = (\alpha\,(S_1+S_2+S_3) + S_4)/(\Delta), \quad B_2^{(2)} = \alpha\,(S_5+S_6)/(\eta^{3/2}\,\Delta), \\
&B_2^{(3)} = -(\alpha\,(S_1+S_2+S_3) - 4\,S_4)/(4\,\Delta), \quad B_2^{(4)} = \alpha\,(S_5+S_6)/(4\,\eta^{3/2}\,\Delta),
\end{aligned}\right\} \tag{A.3}$$

$$\begin{aligned}
S_1 &= 3\Big(\left(3\,\eta^2\,(-\alpha\,z_1 + z_6) + \alpha\,(2+\eta^3)\,(z_4 - \alpha\,z_5)\right)\,(\alpha+\sigma) + 3\,\alpha^2\,\sqrt{\eta}\,z_2\Big)\Big(\alpha\,(2+\eta^3)\,(\omega_2 + 2\,z_5) \\
&\quad + 3\,\eta^2\,(\omega_1 + 2\,z_1) - 6\,\eta^{3/2}\,z_3\Big)/(\alpha\,\Delta), \\
S_2 &= \Big(3\,\left(3\,\alpha^3\,\eta^2\,z_1 + 3\,\alpha\,\sqrt{\eta}\,z_2 + \alpha^4\,(2+\eta^3)\,z_5\right)\sigma - 3\,\alpha\,(1+\alpha^2)\,(2+\eta^3)\,(\alpha+\sigma)\,z_4 \\
&\quad - 9\,\eta^2\,(1+\alpha^2)\,(\alpha+\sigma)\,z_6\Big)\Big(3\,\eta^2\,\omega_1 + \alpha\,(2+\eta^3)\,\omega_2\Big)/(\alpha\Delta), \\
S_3 &= 3\,\Big(3\,\alpha\,\eta^2\,(2\,\sigma + \alpha\,(2+\sigma(\alpha+2\,\sigma)))\,z_1 - 3\,\alpha^2\,\sqrt{\eta}\,(2+\sigma^2)\,z_2 - 6\,\alpha^2\,\eta^{3/2}\,\sigma\,(\alpha+\sigma)\,z_3 - \alpha\,(\alpha+\sigma)\,\big((2+\alpha^2)\,(2+\eta^3) \\
&\quad + \alpha\,\sigma\,(-4+\eta^3)\big)\,z_4 + \alpha^2\,\left((2\,\alpha + (2+\alpha^2)\,\sigma)\,(2+\eta^3) + 2\,\alpha\,(-1+\eta^3)\,\sigma^2\right)\,z_5 - 3\,\eta^2\,(\alpha+\sigma)\,(2+\alpha(\alpha+\sigma))\,z_6\Big) \\
&\quad \Big(3\,\eta^2\,(z_1 - \eta^{-1/2}\,z_3) + \alpha\,(2+\eta^3)\,z_5\Big)/(\alpha\Delta), \\
S_4 &= 9\,\alpha^2\,\Big(\omega_2 - \eta^{3/2}\,(\alpha+\sigma)z_3\Big)\Big(-3\,\eta^2\,\sigma\,(z_1 - \eta^{-1/2}\,z_3) + 3\,(\alpha+\sigma)\,(\sqrt{\eta}\,z_2 - z_4) - \alpha\,\sigma\,(-1+\eta^3)\,z_5\Big)/(\Delta), \\
S_5 &= -3\,\Big(3\,\eta\,(\omega_1 + \alpha\,\eta\,\omega_2) - 2\,\alpha\,\sqrt{\eta}\,(-1+\eta^3)\,(\alpha+\sigma)\,z_3\Big)\Big(3\,\eta^{3/2}\,(\omega_1 + \alpha\,\eta\,\omega_2) - 2\,\alpha\,(-1+\eta^3)(\alpha+\sigma)z_2\Big)/(\alpha\,\Delta), \\
S_6 &= 3\,\Big(3\,\sqrt{\eta}\,\omega_1 + \alpha\,(2+\eta^3)\,(\alpha+\sigma)\,z_3\Big)\Big(3\,\eta^{5/2}\,\omega_1 + \alpha\,\eta\,(2+\eta^3)\,(\alpha+\sigma)\,z_2\Big)/(\alpha\,\sqrt{\eta}\,\Delta).
\end{aligned}$$

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