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# Resonance type flow due to rectilinear oscillations of a sphere in a micropolar fluid

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**Abstract.** The flow of an incompressible micro-polar fluid generated due to rectilinear oscillations of a sphere about a diameter of the sphere is considered. The flow is so slow that oscillations Reynolds number is less than unity and hence nonlinear convective terms in the equations of motion are neglected. A rare but distinct special case in which material constants satisfy a resonance condition is considered. The stream function and drag acting on cylinder are obtained. The effect of physical parameters like micro-polarity and couple stress parameter on the drag due to oscillations is shown through graphs.

## 1. Introduction

Several Stokes flow problems concerning micro-polar fluids have been studied by researchers over the past a half a century ever since A.C.Eringen introduced the micro-polar fluid theory [1]. S.K.Lakshmana Rao et al in [2–5] studied the oscillatory flows of circular cylinder, sphere, spheroid and elliptic cylinder in incompressible micro-polar fluids, the main thrust of the investigation being the determination of the drag or couple as the case may be on the oscillating body. The drag or couple was expressed in terms of two parameters  $K$  and  $K'$  and their numerical variation was extensively studied for a spectrum of micro-polarity parameters and oscillation parameters. However in all these problems, a special case, which are branded as oscillatory flows of “Resonance” type that arise when the material parameters of the fluids are constrained in a particular form (to be stated later) have not been investigated. The rare but distinct possibility of resonance flows has been noticed in [4, 5] and the investigation in this case is mathematically more complicated than in the usual non-resonance type flows. This type flows arise whenever oscillations take place in any non-Newtonian fluids. For example this case arises in the papers of [6–8], but these cases were not attempted by the authors. As far as the authors know, these cases were not studied by any researcher. In this paper we propose to investigate this case of resonance type flows in micropolar fluids.

## 2. Basic equations

The basic equations of an incompressible micro-polar fluid introduced by A.C. Eringen[1] are given by:

$$\frac{\partial \rho}{\partial \tau} + \operatorname{div}(\rho \bar{Q}) = 0, \quad (1)$$



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$$\rho \left( \frac{\partial \bar{Q}}{\partial \tau} + \bar{Q} \cdot \nabla \bar{Q} \right) = -\nabla P + k \nabla \times \bar{l} - (\mu + k) \nabla \times \nabla \times \bar{Q}, \quad (2)$$

$$\rho J \left( \frac{\partial \bar{l}}{\partial \tau} + \bar{Q} \cdot \nabla \bar{l} \right) = -2k\bar{l} + k \nabla \times \bar{Q} - \gamma \nabla \times \nabla \times \bar{l} + (\alpha + \beta + \gamma) \nabla (\nabla \cdot \bar{l}), \quad (3)$$

where  $\bar{Q}, \bar{l}$  fluid velocity and micro-rotation vectors,  $\rho$  is density,  $\tau$  is time,  $J$  is gyration coefficient,  $\mu$  is coefficient of viscosity,  $k$  is micro viscosity coefficient and  $\alpha, \beta, \gamma$  are couple stress viscosity coefficients. For micro-polar fluids, the stress components  $T_{ij}$  and couple stress components  $M_{ij}$  satisfy the following constitutive equations.

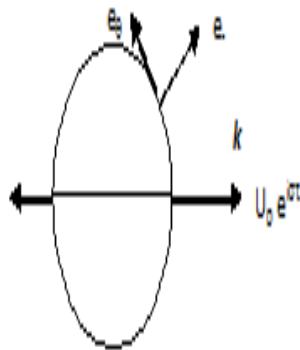
$$T_{ij} = -P \delta_{ij} + \frac{1}{2}(2\mu + k)(u_{i,j} + u_{j,i}) + k e_{ijr}(w_r - l_r), \quad (4)$$

$$M_{ij} = \alpha l_{i,i} \delta_{i,j} + \beta l_{i,j} + \gamma l_{j,i}, \quad (5)$$

where  $w_r = r^{th}$  component of  $\frac{1}{2}(\text{curl } \bar{Q})$  and  $e_{ijr}$  is permutation tensor= 0 if any two indices are equal and =1 if i,j,r are cyclic and -1 if i,j,r are acyclic.

### 3. Statement and formulation of the problem

A sphere of radius  $a$  is performing rectilinear oscillations with velocity  $U_0 e^{i\sigma\tau}$  about its diameter in an infinite vat containing incompressible micro-polar fluid. Spherical coordinate system  $(R, \theta, \phi)$  with base vectors  $(e_r, e_\theta, e_\phi)$  with origin at the centre of the sphere and Z axis along direction of oscillations of the sphere is considered. The flow is axially symmetric, hence the velocity field is independent of toroidal coordinate  $\phi$  and the flow will be in cross sectional plane of the sphere containing the base vectors  $(e_r, e_\theta)$ . The velocity and micro-rotation are assumed in the form:



**Figure 1.** Geometry of the oscillating cylinder.

$$Q = e^{i\sigma\tau} U(R, \theta) e_r + V(R, \theta) e_\theta \quad \text{and} \quad l = e^{i\sigma\tau} C(R, \theta) e_z. \quad (6)$$

The following non-dimensional scheme is introduced. Capitals and LHS terms indicate physical quantities and small letters and RHS terms indicate corresponding non-dimensional quantities.

$$\begin{aligned} R &= ar, \quad U = U_0 u, \quad V = V_0 v, \quad Q = U_0 q, \quad C = C \frac{U_0}{a}, \\ l &= \nu \frac{U_0}{a}, \quad \Psi = a U_0 \psi, \quad P = \rho U_0^2, \quad \tau = U_0 \frac{t}{a}. \end{aligned} \quad (7)$$

The following are non-dimensional parameters viz,  $j$  gyration parameter,  $\omega$  frequency parameter,  $s$  couple stress parameter,  $c$  cross viscosity or micro-polarity parameter and  $Re$  oscillations Reynolds number for micro-polar fluids.

$$j = \frac{J\rho\sigma a^2}{\gamma}, \quad \omega = \frac{\sigma U_0}{a}, \quad s = \frac{ka^2}{\gamma}, \quad c = \frac{k}{\mu + k}, \quad Re = \frac{\rho U_0 a}{\mu + k}. \quad (8)$$

By the choice of velocity field in (6) and incompressibility condition in (1), we notice that stream function can be introduced as

$$u = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \text{ and } v = -\frac{\partial \psi}{\partial r} \text{ i.e. } q = \nabla \times (\psi e_z). \quad (9)$$

Using (6), (7), (8) in (2) and (3) we get

$$\omega Re q = -Re \cdot \nabla p + c\nabla \times v - \nabla \times \nabla \times q, \quad (10)$$

$$ijv = -2sv + s\nabla \times q - \nabla \times \nabla \times v + \frac{1}{\epsilon} \nabla (\nabla \cdot v). \quad (11)$$

By taking curl to equation (10) pressure  $p$  can be eliminated and then using (6) and (9) we get,

$$i \omega Re \cdot E^2 \psi = cE^2 C + E^4 \psi, \quad (12)$$

$$c(2s + ij)C = E^2 C - sE^2 \psi, \quad (13)$$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta}.$$

Taking  $E^2$  operation to (13) and then eliminating  $E^2 C$  using (12) we get,

$$E^2(E^2 - \lambda_1^2)(E^2 - \lambda_2^2)\psi = 0, \quad (14)$$

$$c(2s + ij)C = -E^4 \psi + (i\omega \cdot Re - sc)E^2 \psi, \quad (15)$$

where

$$\lambda_1^2 + \lambda_2^2 = (2 - c)s + i(j + \omega Re) \text{ and } \lambda_1^2 \lambda_2^2 = i\omega Re(2s + ij). \quad (16)$$

The equation for  $C$  can be re-written as

$$cC = -\frac{i\omega Re}{\lambda_1^2 \lambda_2^2} E^2(E^2 - \lambda_1^2 - \lambda_2^2)\psi - E^2 \psi. \quad (17)$$

The solution for  $\psi$  if  $\lambda_1 \neq \lambda_2$  in (14) is given in [3]. The solution for  $\psi$  for the case,  $\lambda_1 = \lambda_2$  can not be obtained as a limiting case of  $\lambda_1 \rightarrow \lambda_2$ . This case is referred to as "Resonance". This resonance occurs if the material coefficients follow the following relation:

$$\frac{\gamma}{J} = \frac{(2\mu + k)(\mu + k)}{2\mu + 3k} \text{ and } \rho\sigma = \frac{(2\mu + k)k + \gamma\rho\sigma}{J(\mu + k)}.$$

In non-dimensional form these conditions are given by

$$(2 - c)s = j - Re \cdot \psi \text{ and } (2 - c)j = \omega Re(2 + c). \quad (18)$$

In this paper we are interested in the solution for  $\psi$  for the case of resonance. In this case the equations for  $\psi$  and  $C$  are given by

$$E^2(E^2 - \lambda^2)^2 \psi = 0 \text{ and } cC = -\frac{i\omega Re}{\lambda^4} E^2(E^2 - 2\lambda^2)\psi - E^2 \psi. \quad (19)$$

The following are the boundary conditions

The sphere is oscillating in the direction of Z-axis. Hence the non-dimensional velocity of the sphere  $\Gamma$  after removing  $e^{i\omega t}$  is given by

$$q_\Gamma = k = \cos \theta e_\Gamma - \sin \theta e_\theta,$$

which implies by no-slip condition

$$u = \cos \theta \text{ and } v = -\sin \theta \text{ on } r = 1. \quad (20)$$

By hyper-stick condition given by

$$\nu_\Gamma = \frac{1}{2}(\operatorname{curl} q)_\Gamma,$$

reduces to

$$C = 0 \text{ on } r = 1. \quad (21)$$

#### 4. Solution of the problem

Stream function  $\psi$ , micro-rotation component C are assumed in the form

$$\psi = f(r) \sin^2 \theta \text{ and } C = g(r) \cos^2 \theta. \quad (22)$$

Substituting (22) in (19) we get

$$D^2(D^2 - \lambda^2)^2 f = 0 \text{ and } cg = -\frac{i\omega Re}{\lambda^4} D^2(D^2 - 2\lambda^2)f - D^2f, \quad (23)$$

where

$$D^2 = \frac{d^2}{dr^2} - \frac{2}{r^2}.$$

Substituting (22) in (20) and (21), the conditions on  $f$  and  $g$  are obtained as:

$$f(1) = \frac{1}{2}, \quad f'(1) = 1 \text{ and } g(1) = 0. \quad (24)$$

Since the equation (23) for  $f$  is linear,  $f$  can be taken as

$$f = A_0 f_0 + A_1 f_1 + A_2 f_2,$$

$$\text{with } D^2 f_0 = 0, \quad D^2(D^2 - \lambda^2)^2 f_1 = 0 \text{ and } D^2(D^2 - \lambda^2)^2 f_2 = 0. \quad (25)$$

On solving (25), the solution for  $f$  is obtained as

$$f(r) = \frac{A_0}{r} + A_1 \sqrt{r} K_{\frac{3}{2}}(\lambda r) + A_2 r^{\frac{3}{2}} K_{\frac{3}{2}}'(\lambda r). \quad (26)$$

The following results are useful to note.

$$D^2 f_1 = \lambda^2 f_1 \text{ and } D^2 f_2 = (-2\lambda f_1 + \lambda^2 f_2). \quad (27)$$

Using (27) in (23), we get

$$cg = A_1(i\omega \cdot Re - \lambda^2)f_1 + A_2(i\omega \cdot Re - \lambda^2)f_2 + 2A_2\lambda^2 f_1. \quad (28)$$

The constants  $A_0, A_1, A_2$  are obtained from the boundary conditions (24) as follows:

$$\begin{bmatrix} 1 & k_{\frac{3}{2}}(\lambda) & k_{\frac{1}{2}}(\lambda) \\ -1 & \frac{1}{2}k_{\frac{3}{2}}(\lambda) + \lambda k_{\frac{3}{2}}'(\lambda) & \frac{3}{2}k_{\frac{1}{2}}(\lambda) + \lambda k_{\frac{1}{2}}'(\lambda) \\ 0 & (i\omega Re - \lambda^2)k_{\frac{3}{2}}(\lambda) & (i\omega Re - \lambda^2)k_{\frac{1}{2}}(\lambda) + 2\lambda k_{\frac{3}{2}}(\lambda) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}. \quad (29)$$

Hence from (26), (28) and (29),  $f$  and  $g$  are completely known and hence  $\psi$  and C are known. Pressure: From equation (10) pressure, after cancelling  $e^{-i\omega t}$ , is obtained as follows.

$$\begin{aligned}
dp &= \nabla p \cdot dr = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta \\
&= \frac{1}{Re} \left[ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (-i\omega Re \psi + cC + E^2 \psi) dr + \frac{1}{\sin \theta} \frac{\partial}{\partial r} (-i\omega Re \psi + cC + E^2 \psi) d\theta \right], \\
p &= \frac{1}{Re} \int (-i\omega Re f + D^2 f + cg) 2 \cos \theta \frac{dr}{r^2} + \frac{d}{dr} (-i\omega Re f + D^2 f + cg) \sin \theta d\theta, \\
&= \frac{1}{Re} \int (-i\omega Re \cdot A_0 f_0) 2 \cos \theta \frac{dr}{r^2} + \frac{d}{dr} (-i\omega Re \cdot A_0 f_0) \sin \theta d\theta, \\
p &= \frac{i\omega A_0}{r^2} \cos \theta, \tag{30}
\end{aligned}$$

Drag acting on the cylinder per length L:

$$Drag = D^* = aL \int_0^{2\pi} (T_{rr}^* \cos \theta - T_{r\theta}^* \sin \theta) |_{R=a} d\theta,$$

Multiplying  $D^*$  by  $\frac{a^2 L}{(\mu+k)} U_0$  the non-dimensional drag D is obtained a

$$De^{-i\omega t} = aL \int_0^{2\pi} (T_{rr} \cos \theta - T_{r\theta} \sin \theta) |_{r=1} d\theta, \tag{31}$$

$$T_{rr}^* = -P + (2\mu + k)e_{rr} = -P + (2\mu + k) \frac{\partial U}{\partial R} = -P + \frac{(2\mu + k)U_0}{ar^2} (f' - \frac{2f}{r}) 2 \cos \theta.$$

Hence on  $r = 1$ ,  $T_{rr}^* = -p$ .

Again,

$$\begin{aligned}
T_{r\theta}^* &= (2\mu + k)e_{r\theta} + ke_{123}(w_3 - \vartheta_3), \\
&= -\frac{(2\mu+k)U_0}{2ar} \sin \theta (D^2 f + \frac{4f}{r^2} - \frac{2f'}{r}) - \frac{kU_0}{ar} \sin \theta (\frac{1}{2} D^2 f + g).
\end{aligned}$$

Hence drag in non-dimensional form is given by

$$\begin{aligned}
D &= \int_0^{2\pi} [-Re \cdot p \cdot \cos \theta + Re D^2 (D^2 - 2\lambda^2) f \sin \theta] |_{r=1} d\theta, \\
&= \int_0^{2\pi} -i\omega Re A_0 \cos^2 \theta + i\omega Re (\frac{1}{2} - A_0) \sin^2 \theta d\theta, \\
&= \pi i\omega Re (1 - 2A_0).
\end{aligned}$$

Hence the non-dimensional drag D is given by

$$D = Real[i\omega Re \cdot \pi (1 - 2A_0) e^{i\omega t}]. \tag{32}$$

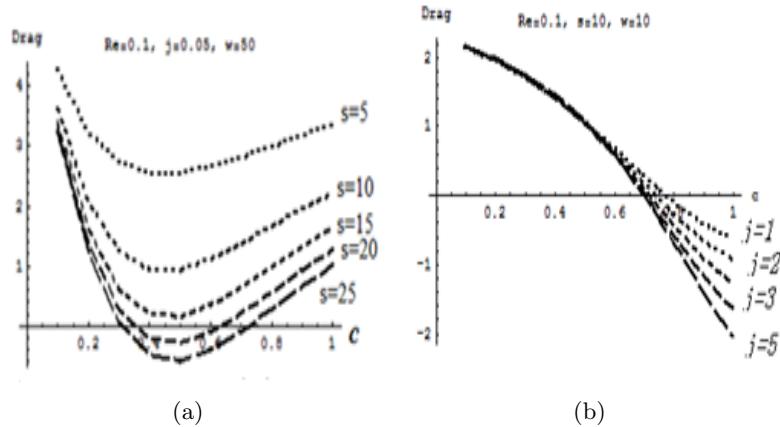
## 5. Results and discussions

The values of  $\lambda$  are obtained from (16) by solving  $x^2 - [(2-c)s + i(j + \omega Re)]x + i\omega Re(2s + ij) = 0$  for  $x$ . Then for resonance case, the values of  $\lambda$  are given by

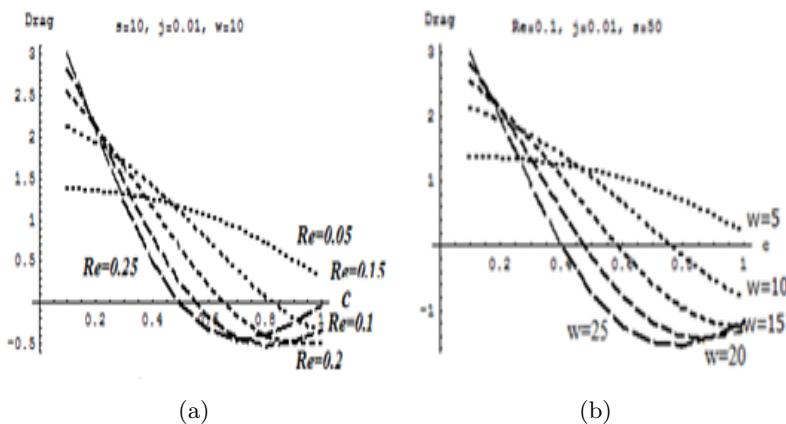
$$\lambda = \sqrt{x} = \sqrt{\frac{[(2-c)s + i(j + \omega Re)]}{2}}. \tag{33}$$

This equation involves 5 parameters which are related by two equations in (18). Hence we choose three parameters as independent. Here  $\omega$ ,  $Re$  and  $c$  are selected independently, with  $0 \leq c \leq 1$ ,  $Re \ll 1$  and  $\omega \gg 1$  such that  $\omega \cdot Re$  is not negligibly small (say  $> 1$ ). For this range of values of  $Re$ , the nonlinear convective terms can be neglected but local derivative is retained. After selecting  $c$ ,  $Re$  and  $\omega$ , the values of  $s$  and  $j$  are obtained from (18) and then  $\lambda$  is obtained from (33). The values of  $\lambda$  are complex. These values for  $\lambda$  are substituted in (29) and the constants  $A_0$ ,  $A_1$  and  $A_2$  are obtained. Then the stream function  $\psi$  and drag  $D$  are obtained from (26) and (32) respectively. Thus obtained  $\psi$  will have complex values. To get physical picture, these

values are multiplied by  $e^{i\omega t}$  and its real part is taken, if the oscillations are in cosine functions. Drag: From figure 2, it is observed that as  $s$  the couple stress parameter and  $j$  the gyration parameter are increasing, the drag  $D$  decreases. The variation drag for different values of  $j$  is not observed for small values of  $c < 0.5$ . From figure 3, we observe that as  $Re$ , the Reynolds number and  $\omega$ , the frequency parameter are increasing, the drag increases for small values of  $c$  but as  $c$  is nearing unity, the drag decreases. Since  $Re\omega$  appears in the equations as a unit, the behavior of drag with respect to  $Re$  and  $\omega$  is exactly similar. This is clear in figure 3.



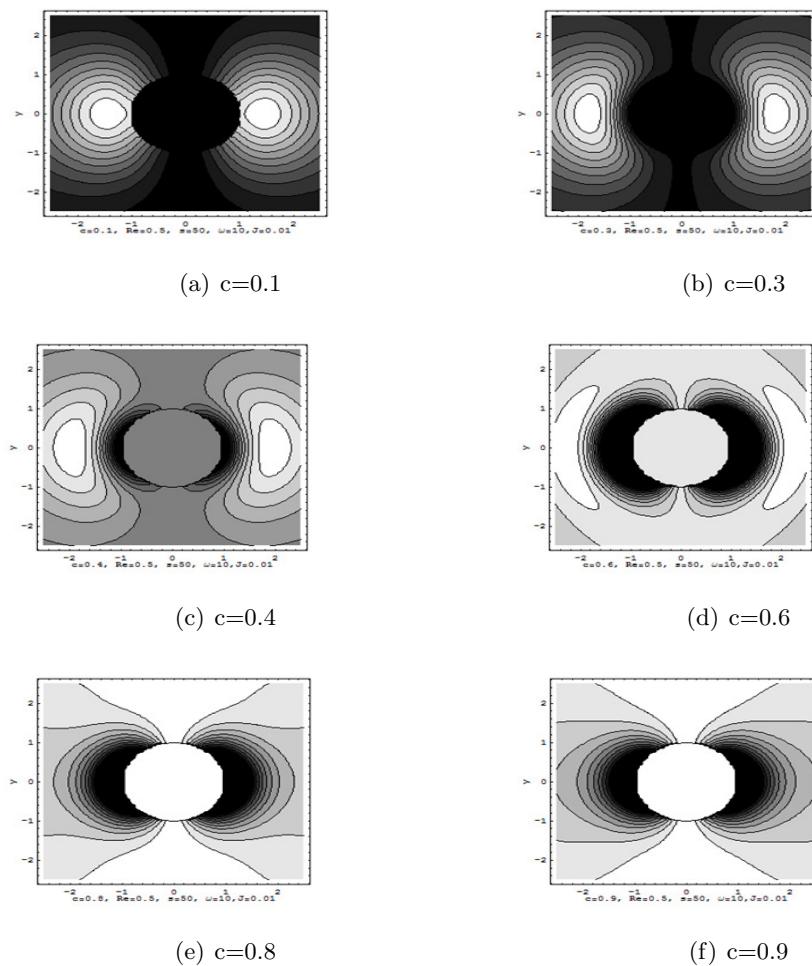
**Figure 2.** (a) Drag Vs micro-polarity parameter  $c$  at different  $s$ . (b) Drag Vs micro-polarity parameter  $c$  at different  $j$ .



**Figure 3.** (a) Drag Vs micro-polarity parameter  $c$  at different  $Re$ . (b) Drag Vs micro-polarity parameter  $c$  at different  $\omega$ .

### 5.1. Stream function

By looking the stream lines in the six figures given below, it is observed that the entire flow region is symmetric about axis of symmetry and streamlines form closed loops about the equatorial region of the sphere. For small values of cross viscosity( micro-polarity) parameter  $c$ , the stream lines are having negative values along the axis of symmetry and positive values near the equatorial region of the sphere. (Black shade represents negative values and white region represents positive values). As  $c$  value increases, we can find a drastic change in the pattern of the stream lines. The region along the axis of symmetry becomes white (i.e, positive) and the region about the equatorial line of the sphere becomes black (i.e, negative). This indicates the strong effect of micro-polarity parameter on the rotation of the sphere. It can be seen that as  $c$  is having values near to 0.5, the stream lines are forming two closed regions within which the entire flow is contained in. This is an interesting phenomena to note. This may be due to the fact that drag is highest when  $c$  is near to 0.1 and decreases as  $c$  increases and reaches a minimum value when  $c$  is about 0.5 and then the drag increases as  $c$  increases again. Since the case  $c \rightarrow 0, s \rightarrow \infty$ , represents the case of viscous fluids, we can conclude that the drag for viscous fluids is more than the drag of micro-polar fluids. Hence it can be inferred that micro-polar fluids offer less resistance to flow and hence less drag.



**Figure 4.** Stream lines at different values of micro-polarity parameter  $c$ .

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