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Terminal boundary-value technique for solving singularly perturbed delay differential equations

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Abstract

A terminal boundary-value technique is presented for solving singularly perturbed delay differential equations, the solutions of which exhibit layer behaviour. By introducing a terminal point, the original problem is divided into inner and outer region problems. An implicit terminal boundary condition at the terminal point was determined. The outer region problem with the implicit boundary condition was solved and produces an explicit boundary condition for the inner region problem. Then, the modified inner region problem (using the stretching transformation) is solved as a two-point boundary value problem. The second-order finite difference scheme was used to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. To validate the efficiency of the method, some model examples were solved. The stability and convergence of the scheme was also investigated.

MSC: 65L03; 65L10; 65L11; 65L12

Keywords: Singular perturbation; Delay differential equations; Finite differences; Terminal boundary condition; Boundary layer

1. Introduction

Singular perturbation problems containing a small parameter, ε, multiplying to their highest derivative term arise in many fields, such as fluid mechanics, fluid dynamics, chemical reactor theory and elasticity, which have received significant attention. The solution of these types of problems shows a multi-scale character, with a narrow region called the boundary layer, in which their solution changes rapidly, and an outer region in which the solution changes smoothly. Thus, the treatment of such problems is not trivial because of the boundary layer behaviour of their solutions. Detailed theory and analytical discussions of solving singular perturbation problems have been published [1–9], and have the details of numerical and asymptotic solutions [10–15].

Boundary value problems involving delay differential equations arise in studying the mathematical modelling of various practical phenomena, like micro-scale heat transfer [16], the hydrodynamics of liquid helium [17], second-sound theory [18], optically bi-stable devices [19], diffusion in polymers [20], reaction–diffusion equations [21], stability [22], control including control of chaotic systems [23] and a variety of models for physiological processes and diseases [24,25]. For example, Lange and Miura [26–30] treated the singular perturbation analysis of the problem under consideration in a series of papers. A numerical method based on the fitted mesh approach to approximate the solution

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of these types of boundary value problems was published by Kadalbajoo and Sharma [31]. The authors constructed piecewise-uniform meshes and fitted them to the boundary layer regions to adapt the singular behaviour of the operator in the narrow regions. The same authors [32] approximated the terms with delay by first-order Taylor series expansions to analyze boundary value problems of singularly perturbed differential difference equations with negative shift or delay. They used the invariant embedding technique and central and upwind finite difference discretization for the second- and first-order derivatives, respectively, and proved the stability and convergence of their method. An exponentially fitted difference scheme on a uniform mesh is accomplished by the method of integral identities to solve problems of the same type [33]. In this method, the authors used exponential basis functions and interpolating quadrature rules with weight and remainder terms in integral form. Various numerical methods have also been presented for solving singularly perturbed boundary value problems involving small shifts, including exponential methods based on piecewise analytical solutions of advection–reaction–diffusion operators [34], a fitted mesh B-spline collocation method [35], parameter–uniform numerical methods comprising a standard implicit finite difference scheme [36], ε-uniformly convergent non-standard finite differences [37] and ε-uniformly convergent fitted methods [38].

We devised a terminal boundary value technique for solving singularly perturbed delay differential equations with a boundary layer at the left end of the interval. By introducing a terminal point into the domain, the original problem is divided into inner and outer region problems. A terminal boundary condition in the implicit form is determined from the reduced problem, and the outer region problem with the implicit boundary condition is then solved as a two-point boundary-value problem. From the solution of the outer region problem, an explicit terminal boundary condition is obtained. The inner region problem is modified and solved as a two-point boundary value problem using the obtained explicit terminal boundary condition. Finally, we combined the solutions of the inner region and outer region problems to obtain the approximate solution of the original problem. The method is iterative on the terminal point. We repeated the numerical scheme for various choices of the terminal point until the solution profiles did not differ materially from iteration to iteration. To validate the efficiency of the method, some model examples are solved. The stability and convergence of the scheme was also investigated.

2. Description of the method

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{1}$$

Subject to the interval and boundary conditions

$$y(x) = \phi(x), \quad -\delta \le x \le 0 \tag{2}$$

$$y(1) = \beta, \tag{3}$$

where ε is a small positive parameter ($0<\varepsilon \ll 1$), δ is delay parameter, $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are sufficiently smooth functions, and β is a known constant.

For $\delta = 0$, the problem (1)–(3) becomes a boundary value problem for singularly perturbed ordinary differential equations. The layer behaviour of the problem under consideration is maintained only for $\delta \neq 0$ but is sufficiently small (i.e. δ is of $o(\varepsilon)$). When the delay parameter δ exceeds the perturbation parameter, ε (i.e. δ is of $O(\varepsilon)$), the layer behaviour of the solution is no longer maintained; rather, the solution exhibits an oscillatory behaviour or is diminished.

We considered the cases in which δ is of $o(\varepsilon)$ ($i.e. \delta < \varepsilon$). Now, we assume that $a(x) \ge M > 0$ and $\varepsilon - \delta a(x) > 0$ throughout the interval [0,1], where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$, which in general displays a boundary layer in the neighbourhood of $x = 0$.

Since δ is of $o(\varepsilon)$ and the solution $y(x)$ of the BVP (1)–(3) is sufficiently differentiable, by using Taylor's series expansion, we obtain

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{4}$$

Substituting (4) into (1), we obtain an asymptotically equivalent two-point boundary value problem

$$\varepsilon(1-\xi a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{5}$$

with

$$y(0) = \phi(0) \tag{6}$$

$$y(1) = \beta \tag{7}$$

where $\delta = \xi\varepsilon$ with $\xi = O(1)$.

The transition from Eq. (1) to Eq. (5) is admitted because of the condition that the delay parameter $0 < \delta \ll 1$ is sufficiently small and is of $o(\varepsilon)$.

Let $x_p (0 < x_p \ll 1)$ be the terminal point or width or thickness of the boundary layer. It is well known from the singular perturbation theory (see [1,3]) that the reduced equation (i.e. Eq. (5) with $\varepsilon = 0$) is valid in the outer region. Hence, by putting $\varepsilon = 0$ in (5), we have the reduced equation

$$a(x)y'(x) + b(x)y(x) = r(x) \quad \text{for} \quad x_p \leq x \leq 1 \tag{8}$$

Now, evaluating (8) at $x = x_p$ and denoting $c_1 = a(x_p)$, $c_2 = b(x_p)$ and $c_3 = f(x_p)$ gives us

$$c_1 y'(x_p) + c_2 y(x_p) = c_3 \tag{9}$$

We take (9) as the terminal boundary condition in implicit form. As the terminal point x_p is common to both the inner and outer regions, then the inner and outer region problems are defined on $0 \leq x \leq x_p$ and $x_p \leq x \leq 1$, respectively.

That is, the outer region problem:

$$(\varepsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x_p \leq x \leq 1 \tag{10}$$

$$\text{with} \quad c_1 y'(x_p) + c_2 y(x_p) = c_3 \quad \text{and} \quad y(1) = \beta \tag{11}$$

and the inner region problem:

$$(\varepsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad 0 \leq x \leq x_p \tag{12}$$

$$\text{with} \quad y(0) = \phi(0), \quad \text{and} \quad y(x_p) = \gamma \ \text{(is obtained as described below)} \tag{13}$$

First, solving the outer region problem

$$(\varepsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x_p \leq x \leq 1 \tag{14}$$

$$\text{with} \quad c_1 y'(x_p) + c_2 y(x_p) = c_3 \tag{15}$$

$$\text{and} \quad y(1) = \beta \tag{16}$$

we obtain solution $y(x)$ over $[x_p, 1]$. From the solution $y(x)$ of the outer region problem (14)–(16) on the interval $x_p \leq x \leq 1$, we obtain the value of $y(x_p)$ that is the explicit terminal boundary condition and denote it by $y(x_p) = \gamma$.

Now, in order to solve the inner region problem, consider inner region problem (12) and (13) with the assumption $\delta = \xi\varepsilon$, $\delta = o(\varepsilon)$ and choose the transformation

$$t = \frac{x}{\varepsilon} \tag{17}$$

to form a new differential equation for the inner region solution. By using (17), we transform Eq. (12) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{18}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{19}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{20}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{21}$$

$$b(x) = b(t\varepsilon) = B(t) \tag{22}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{23}$$

to the new inner region problem of the form:

$$(1 - \xi A(t))Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t), \quad 0 \leq t \leq t_p \tag{24}$$

and the boundary condition for (24) is determined by (13) and (18).

$$Y(0) = \phi(0) = \phi_0 \tag{25}$$

$$\text{and} \quad Y(t_p) = y(x_p) = \gamma \tag{26}$$

where $t_p = x_p/\varepsilon$. We solve the new inner region problem (24)–(26) to obtain the solutions over the interval $0 \leq t \leq t_p$.

To solve the two-point boundary value problems given in Eqs. (14)–(16) (outer region problem) and (24)–(26) (inner region problem), we make use of a second-order classical finite difference scheme. In fact, any standard analytical or numerical method can be used. Finally, we combine the solutions of both the inner region defined in $0 \leq x \leq x_p$ and the outer region defined in $x_p \leq x \leq 1$ problems to obtain the approximate solution of the original problem (1)–(3) over the interval $0 \leq x \leq 1$. We repeat the process (numerical scheme) for various choices of x_p (the terminal point), until the solution profiles do not differ materially from iteration to iteration. For the computational point of view, we use an absolute error criterion, namely

$$\left| y^{m+1}(x) - y^m(x) \right| \leq \sigma, \quad 0 \leq x \leq x_p$$

where $y^m(x)$ = the solution for the mth iterate of x_p and σ = the prescribed tolerance bound.

To describe the scheme for the outer region problem (14)–(16), we divide $[x_p, 1]$ into N equal parts, each of length h, $x_p = x_0 < x_1 < x_2 < \cdots < x_N = 1$ and we have $x_i = x_p + ih$, $i = 0, 1, 2, \ldots, N$. For convenience, let $a(x_i) = a_i$, $b(x_i) = b_i$, $f(x_i) = f_i$, $y(x_p) = y_0$, $y(x_i) = y_i$. Now, applying the classical central finite difference scheme to

(14)–(16), we obtain the three-term recurrence relation

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 0, 1, \ldots, N-1 \tag{27}$$

where

$$\begin{aligned} E_i &= \frac{(\varepsilon - \delta a_i)}{h^2} - \frac{a_i}{2h} \\ F_i &= \frac{2(\varepsilon - \delta a_i)}{h^2} - b_i \\ G_i &= \frac{(\varepsilon - \delta a_i)}{h^2} + \frac{a_i}{2h} \\ H_i &= f_i \end{aligned} \tag{28}$$

Eq. (27) gives a system of N equations with $N+1$ unknown y_{-1} to y_{N-1}. To eliminate the unknown y_{-1}, we use the implicit boundary condition (15); then, by employing the second-order central difference approximation in it, we obtain

$$y_{-1} = \frac{2hc_2}{c_1} y_0 + y_1 - \frac{2hc_3}{c_1} \tag{29}$$

where c_1, c_2 and c_3 are defined in (9). Using (29) in the first equation of the recurrence relation (27) at $i = 0$, we obtain

$$\begin{aligned} &-\left(F_0 + \frac{2hc_2}{c_1} E_0\right) y_0 + (E_0 + G_0) y_1 \\ &\quad = H_0 + \frac{2hc_3}{c_1} E_0 \end{aligned} \tag{30}$$

Now, Eqs. (27) and (30) give N by an N tri-diagonal system, which can be easily solved with the Thomas algorithm.

Similarly, to set up the difference equation for the inner region problem (24)–(26), we divide the interval $0 \le t \le t_p$ into N subintervals of equal mesh length $h = (t_p - 0)/N$ with mesh points $0 = t_0 < t_1 < t_2, \ldots, < t_N = t_p$. Again applying the second-order classical finite difference scheme to (24)–(26), we obtain the three-term recurrence relation

$$E'_i y_{i-1} - F'_i y_i + G'_i y_{i+1} = H'_i, \quad i = 1, 2, \ldots, N-1 \tag{31}$$

where

$$\begin{aligned} E'_i &= \frac{(1 - \xi A_i)}{h^2} - \frac{A_i}{2h} \\ F'_i &= \frac{(1 - \xi A_i)}{h^2} - \varepsilon B_i \\ G'_i &= \frac{(1 - \xi A_i)}{h^2} + \frac{A_i}{2h} \\ H'_i &= \varepsilon F_i \end{aligned} \tag{32}$$

The Thomas algorithm is used to solve the tri-diagonal system (31).

3. Lower bound of the terminal point t_p

To gain further insight into the choice of t_p, the terminal point of the boundary layer region, consider the problem (1)–(3). Further, choose t_p such that $t_p \ll 1/\varepsilon$. Now, using Eq. (5) and assuming that $\delta \ll \varepsilon$ and $\varepsilon - \delta a(x) > 0$, we get

$$(\varepsilon - \delta a(x)) y''(x) + a(x) y'(x) + b(x) y(x) = f(x) \tag{33}$$

Using the stretching transformation (17) and taking the limit as $\varepsilon \to 0$, we obtain

$$\left(1 - \frac{\delta a(0)}{\varepsilon}\right) Y''(t) + a(0) Y'(t) = 0 \tag{34}$$

where $Y(t) = y(t\varepsilon)$ and the boundary conditions are

$$Y(0) = \phi(0) \quad \text{and} \quad Y(t_p) = \gamma \tag{35}$$

Now, solving the two-point boundary value problem (34) and (35) analytically, we obtain the solution

$$Y(t) = m_1 + m_2 e^{mt} \tag{36}$$

where $m = (-\varepsilon a(0))/(\varepsilon - \delta a(0))$, $m_2 = (\gamma - \phi(0))/(e^{(-\varepsilon a(0) t_p)/(\varepsilon - \delta a(0))} - 1)$, and $m_1 = \phi(0) - m_2$.

As suggested by Hsiao and Jordan [39] and Lorenz [40], t_p can be determined by taking the inequality

$$\exp\left(\frac{-\varepsilon a(0)}{\varepsilon - \delta a(0)} t_p\right) < \varepsilon \tag{37}$$

Taking the natural logarithm (ln) of both sides of (37), we get

$$\frac{-\varepsilon a(0) t_p}{\varepsilon - \delta a(0)} < \ln \varepsilon \tag{38}$$

Now, rearranging (38) gives

$$t_p \ge \frac{\delta a(0) - \varepsilon}{\varepsilon a(0)} \ln \varepsilon \tag{39}$$

For $\varepsilon = 10^{-\mu}$, we can obtain a crude estimate of the lower bound of t_p from

$$t_p \geq \mu\left(\frac{1-\delta a(0)10^{\mu}}{a(0)}\right)\ln(10) \approx 2.3\mu\left(\frac{1-\delta a(0)10^{\mu}}{a(0)}\right). \qquad (40)$$

4. Stability and convergence analysis

Theorem 1. *Under the assumptions* $(\varepsilon - \delta a(x)) > 0$, $a(x) \geq M > 0$ *and* $b(x) < 0$, $\forall x \in [0, 1]$, *the solution to the system of the difference equations* (27), *together with the given boundary conditions, is unique and satisfies*

$$\|y\|_{h,\infty} \leq 2M^{-1}\|f\|_{h,\infty} + (|\phi_0| + |\beta|) \qquad (41)$$

where $\|\cdot\|_{h,\infty}$ *is the discrete* $l_\infty -$*norm, given by* $\|x\|_{h,\infty} = \max_{0\leq i\leq N}\{|x_i|\}$.

Proof. Let $L_h(.)$ denote the difference operator on the left-hand side of Eq. (27) and w_i be any mesh function satisfying

$$L_h(w_i) = f$$

By rearranging the difference scheme (27) and using non-negativity of the coefficients E_i, F_i and G_i, we obtain

$$F_i\,|w_i| \leq |H_i| + E_i\,|w_{i-1}| + G_i\,|w_{i+1}|$$

Now, on the assumption $(\varepsilon - \delta a(x)) > 0$ and $a(x) \geq M > 0$, the definition of l_∞-norm and manipulating the above inequality, we obtain

$$(\varepsilon - \delta M)\left(\frac{|w_{i+1}| - 2\,|w_i| + |w_{i-1}|}{h^2}\right) + \|a\|_{h,\infty}\left(\frac{|w_{i+1}| - |w_{i-1}|}{2h}\right) + b_i\,|w_i| + |f_i| \geq 0 \qquad (42)$$

To prove the uniqueness and existence, let $\{u_i\}$, $\{v_i\}$ be two sets of solution of the difference equation (27) satisfying boundary conditions. Then, $w_i = u_i - v_i$ satisfies

$$L_h(w_i) = f_i$$

where $f_i = 0$ and $w_0 = w_N = 0$.

Summing (42) over $i = 1, 2, \ldots, N-1$, we obtain

$$-\frac{(\varepsilon - \delta M)}{h^2}|w_1| - \frac{(\varepsilon - \delta M)}{h^2}|w_{N-1}| - \|a\|_{h,\infty}\left(\frac{|w_1| - |w_{N-1}|}{2h}\right) + \sum_{i=1}^{N-1} b_i\,|w_i| \geq 0 \qquad (43)$$

Since $(\varepsilon - \delta M) > 0$, $\|a\|_{h,\infty} \geq 0$, $b_i < 0$ and $|w_i| \geq 0$ $\forall i$, $i = 1, 2, \ldots, N-1$, for inequality (43) to hold, we must have $w_i = 0$ $\forall i$, $i = 1, 2, \ldots, N-1$.

This implies the uniqueness of the solution of the tri-diagonal system of difference equations (27). For linear equations, the existence is implied by uniqueness. Now, to establish the estimate, let

$$w_i = y_i - l_i,$$

where y_i satisfies difference equations (27), the boundary conditions and

$$l_i = (1 - ih)\phi_0 + (ih)\beta,$$

then $w_0 = w_N = 0$ and w_i, $i = 1, 2, \ldots, N-1$ satisfies

$$L_h(w_i) = f_i$$

Now, let

$$|w_n| = \|w\|_{h,\infty} \geq |w_i|, \quad i = 0, 1, \ldots, N.$$

Then, summing (42) from $i = n$ to $N-1$ and using the assumption on $a(x)$, gives

$$-\frac{(\varepsilon - \delta M)}{h^2}(|w_n| - |w_{n-1}| + |w_{N-1}|) - \frac{M}{2h}(|w_{n-1}| - |w_{N-1}|) - \frac{M}{2h}|w_n| + \sum_{i=n}^{N-1} b_i\,|w_i| + \sum_{i=n}^{N-1}|f_i| \geq 0 \qquad (44)$$

Inequality (44), together with the condition on $b(x)$, implies that

$$\frac{M}{2}|w_n| \leq h\sum_{i=n}^{N-1}|f_i| \leq h\sum_{i=0}^{N}|f_i| \leq \|f\|_{h,\infty},$$

i.e., we have

$$|w_n| \leq 2M^{-1}\|f\|_{h,\infty} \qquad (45)$$

We also have

$$y_i = w_i + l_i,$$

$$\|y\|_{h,\infty} = \max_{0\leq i\leq N}\{|y_i|\} \leq \|w\|_{h,\infty} + \|l\|_{h,\infty} \leq |w_n| + \|l\|_{h,\infty}. \qquad (46)$$

Table 1
Numerical results for Example 5.1, $\varepsilon=10^{-3}$, $\delta=0.5\varepsilon$.

x	$t_p=20$ $y(x)$	$t_p=15$ $y(x)$	$t_p=10$ $y(x)$	$t_p=5$ $y(x)$	Exact solution
0.0000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.0010	0.3171426	0.4528503	0.4533947	0.4538989	0.4538692
0.0020	0.3603879	0.3792739	0.3798923	0.3804652	0.3803509
0.0030	0.3666117	0.3697083	0.3703373	0.3709202	0.3707303
0.0040	0.3678324	0.3687953	0.3694263	0.3700112	0.3697488
0.0050	0.3683816	0.3690566	0.3696884	0.3702742	0.3699358
0.0100	0.3708603	0.3714947	0.3721293		0.3717605
0.0150	0.3733556	0.3739937			0.3736230
0.0200	0.3758674				0.3754949
0.1000	0.4071563	0.4071563	0.4071563	0.4071563	0.4067525
0.2000	0.4499552	0.4499552	0.4499552	0.4499552	0.4495085
0.4000	0.5495225	0.5495225	0.5495225	0.5495225	0.5489761
0.6000	0.6711221	0.6711221	0.6711221	0.6711221	0.6704540
0.8000	0.8196300	0.8196300	0.8196300	0.8196300	0.8188125
0.9000	0.9057869	0.9057869	0.9057869	0.9057869	0.9048827
1.0000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Underline is used to indicate the numerical solutions obtained at the terminal points.

To complete the estimate, we have to find the bound on l_i

$$\|l\|_{h,\infty}=\max_{0\le i\le N}\{|l_i|\}\le\max_{0\le i\le N}\{|(1-ih)|\,|\phi_0|+|ih|\,|\beta|\}\le\max_{0\le i\le N}\{(1-ih)\,|\phi_0|+(ih)\,|\beta|\},$$

i.e., we have

$$\|l\|_{h,\infty}\le|\phi_0|+|\beta|\,. \tag{47}$$

From Eqs. (45)–(47), we obtain the estimate

$$\|y\|_{h,\infty}\le 2M^{-1}\|f\|_{h,\infty}+(|\phi_0|+|\beta|)$$

This theorem implies that the solution to the system of difference equations (27) is uniformly bounded, independently of mesh size h and the perturbation parameter ε. Thus, the scheme is stable for all step sizes.

Corollary 1. *Under the conditions of Theorem 1, the error $e_i=y(x_i)-y_i$ between the solution $y(x)$ of the*

Table 2
Numerical results for Example 5.1, $\varepsilon=10^{-4}$, $\delta=0.5\varepsilon$.

x	$t_p=20$ $y(x)$	$t_p=15$ $y(x)$	$t_p=10$ $y(x)$	$t_p=5$ $y(x)$	Exact solution
0.00000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.00010	0.3181581	0.4709184	0.4707524	0.4705484	0.4534718
0.00020	0.3612116	0.3873736	0.3871895	0.3869631	0.3795465
0.00030	0.3670360	0.3711528	0.3709686	0.3707419	0.3695746
0.00040	0.3678237	0.3689013	0.3687171	0.3684904	0.3682570
0.00050	0.3679338	0.3685967	0.3684126	0.3681859	0.3681105
0.00100	0.3682020	0.3685541	0.3683699		0.3682659
0.00150	0.3684702	0.3685541			0.3684501
0.00200	0.3687384				0.3686343
0.10000	0.4066914	0.4066914	0.4066914	0.4066914	0.4065880
0.20000	0.4494485	0.4494485	0.4494485	0.4494485	0.4493469
0.40000	0.5489215	0.5489215	0.5489215	0.5489215	0.5488281
0.60000	0.6704094	0.6704094	0.6704094	0.6704094	0.6703334
0.80000	0.8187855	0.8187855	0.8187855	0.8187855	0.8187389
0.90000	0.9048674	0.9048674	0.9048674	0.9048674	0.9048420
1.00000	1.0000000	0.0000000	0.0000000	0.0000000	1.0000000

Underline is used to indicate the numerical solutions obtained at the terminal points.

Table 3
Numerical results for Example 5.2, $\varepsilon = 10^{-3}$, $\delta = 0.5\varepsilon$.

x	$t_p = 20$ $y(x)$	$t_p = 15$ $y(x)$	$t_p = 10$ $y(x)$	$t_p = 5$ $y(x)$
0.0000	1.0000000	1.0000000	1.0000000	1.0000000
0.0010	0.3716997	0.3719800	0.3723023	0.3725916
0.0020	0.2869179	0.2872367	0.2876031	0.2879317
0.0030	0.2756975	0.2760220	0.2763947	0.2767291
0.0040	0.2744568	0.2747825	0.2751564	0.2754918
0.0050	0.2745652	0.2748914	0.2752659	<u>0.2756018</u>
0.0100	0.2762854	0.2766135	<u>0.2769902</u>	
0.0150	0.2780575	<u>0.2783891</u>		
0.0200	<u>0.2797985</u>			
0.1000	0.3038107	0.3038107	0.3038107	0.3038107
0.2000	0.3383586	0.3383586	0.3383586	0.3383587
0.4000	0.4267997	0.4267997	0.4267997	0.4267997
0.6000	0.5516247	0.5516247	0.5516247	0.5516248
0.8000	0.7323793	0.7323793	0.7323793	0.7323793
0.9000	0.8531665	0.8531665	0.8531665	0.8531665
1.0000	1.0000000	1.0000000	1.0000000	1.0000000

Underline is used to indicate the numerical solutions obtained at the terminal points.

continues problem and the solution y_i of the discretized problem, with boundary conditions, satisfies the estimate

$$\|e\|_{h,\infty} \le 2M^{-1}\|\tau\|_{h,\infty} \tag{48}$$

where

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{h^2 a(x)}{6} \left| y^{(3)}(x) \right| \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{\varepsilon h^2}{12} (\varepsilon - \delta a(x)) \left| y^{(4)}(x) \right| \right\} \tag{49}$$

Table 4
Numerical results for Example 5.2, $\varepsilon = 10^{-4}$, $\delta = 0.5\varepsilon$.

x	$t_p = 20$ $y(x)$	$t_p = 15$ $y(x)$	$t_p = 10$ $y(x)$	$t_p = 5$ $y(x)$
0.00000	1.0000000	1.0000000	1.0000000	1.0000000
0.00010	0.3313265	0.3425767	0.3544611	0.3668457
0.00020	0.2408234	0.2535924	0.2670812	0.2811385
0.00030	0.2290035	0.2420007	0.2557299	0.2700383
0.00040	0.2292476	0.2423802	0.2562526	0.2707104
0.00050	0.2315017	0.2447798	0.2588054	<u>0.2734225</u>
0.00100	0.2446982	0.2587352	<u>0.2735594</u>	
0.00150	0.2588526	<u>0.2736963</u>		
0.00200	<u>0.2738332</u>			
0.10000	0.3027937	0.3027937	0.3027937	0.3027937
0.20000	0.3372552	0.3372553	0.3372553	0.3372553
0.40000	0.4255062	0.4255062	0.4255062	0.4255062
0.60000	0.5501304	0.5501304	0.5501304	0.5501304
0.80000	0.7307238	0.7307238	0.7307238	0.7307238
0.90000	0.8514816	0.8514816	0.8514816	0.8514816
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Underline is used to indicate the numerical solutions obtained at the terminal points.

Table 5
Numerical results for Example 6.1, $\varepsilon = 10^{-3}$, $\delta = 0.5\varepsilon$.

x	$t_p = 20$ $y(x)$	$t_p = 15$ $y(x)$	$t_p = 10$ $y(x)$	$t_p = 5$ $y(x)$
0.00000	1.0000000	1.0000000	1.0000000	1.0000000
0.00100	0.4596118	0.4582375	0.4568711	0.4554691
0.00200	0.3864810	0.3849208	0.3833696	0.3817777
0.00300	0.3765825	0.3749970	0.3734207	0.3718032
0.00400	0.3752426	0.3736535	0.3720739	0.3704530
0.00500	0.3750651	0.3734757	0.3718956	<u>0.3702742</u>
0.01000	0.3753012	0.3737106	<u>0.3721293</u>	
0.01500	0.3755843	<u>0.3739937</u>		
0.02000	<u>0.3758674</u>			
0.10000	0.4071563	0.4071563	0.4071563	0.4071563
0.20000	0.4499552	0.4499552	0.4499552	0.4499552
0.40000	0.5495225	0.5495225	0.5495225	0.5495225
0.60000	0.6711221	0.6711221	0.6711221	0.6711221
0.80000	0.8196300	0.8196300	0.8196300	0.8196300
0.90000	0.9057869	0.9057869	0.9057869	0.9057869
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Underline is used to indicate the numerical solutions obtained at the terminal points.

Proof. Truncation error τ_i is given by

$$\tau_i = (\varepsilon - \delta a(x)) \left\{ \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) - y_i'' \right\} + a(x) \left\{ \left(\frac{y_{i+1} - y_{i-1}}{2h} \right) - y_i' \right\}$$

Table 6
Numerical results for Example 6.1, $\varepsilon = 10^{-4}$, $\delta = 0.5\varepsilon$.

x	$t_p = 20$ $y(x)$	$t_p = 15$ $y(x)$	$t_p = 10$ $y(x)$	$t_p = 5$ $y(x)$
0.00000	1.0000000	1.0000000	1.0000000	1.0000000
0.00010	0.4710844	0.4709184	0.4707524	0.4705484
0.00020	0.3875577	0.3873736	0.3871895	0.3869631
0.00030	0.3713371	0.3711528	0.3709686	0.3707419
0.00040	0.3690856	0.3689013	0.3687171	0.3684904
0.00050	0.3687811	0.3685967	0.3684126	<u>0.3681859</u>
0.00100	0.3687384	0.3685541	<u>0.3683699</u>	
0.00150	0.3687384	<u>0.3685541</u>		
0.00200	<u>0.3687384</u>			
0.10000	0.4066914	0.4066914	0.4066914	0.4066914
0.20000	0.4494485	0.4494485	0.4494485	0.4494485
0.40000	0.5489215	0.5489215	0.5489215	0.5489215
0.60000	0.6704094	0.6704094	0.6704094	0.6704094
0.80000	0.8187855	0.8187855	0.8187855	0.8187855
0.90000	0.9048674	0.9048674	0.9048674	0.9048674
1.00000	0.0000000	1.0000000	1.0000000	1.0000000

Underline is used to indicate the numerical solutions obtained at the terminal points.

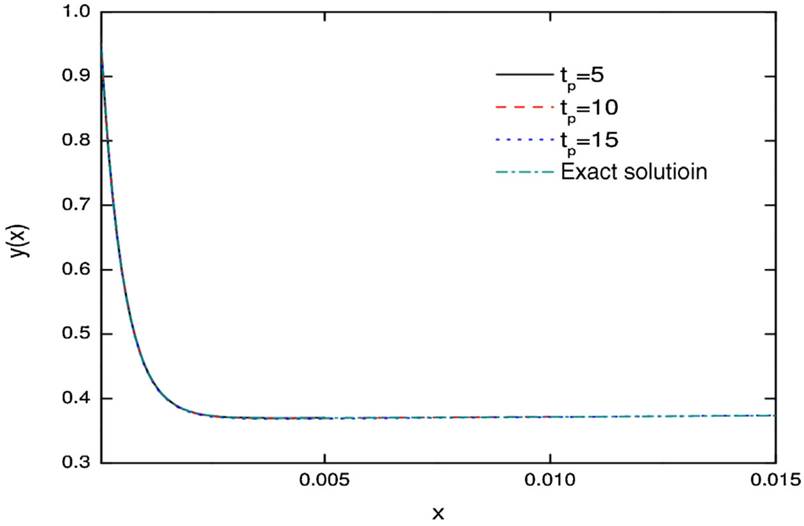

Fig. 1. Inner layer solutions of Example 5.1 for $\varepsilon = 0.001$, $\delta = 0.1\varepsilon$ and different terminal points.

$$|\tau_i| \leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left\{ \frac{h^2 a(x)}{6} \left| y^{(3)}(x) \right| \right\} + \max_{x_{i-1} \leq x \leq x_{i+1}} \left\{ \frac{\varepsilon h^2}{12} (\varepsilon - \delta a(x)) \left| y^{(4)}(x) \right| \right\} \quad (50)$$

One can easily show that the error e_i satisfies

$$L_h(e(x_i)) = L_h(y(x_i)) - L_h(y_i) = \tau_i, \quad i = 1, 2, \ldots, N-1 \quad (51)$$

and $e_0 = e_N = 0$.

Then, Theorem 1 implies that

$$\|e\|_{h,\infty} \leq 2M^{-1} \|\tau\|_{h,\infty}. \quad (52)$$

Estimate (48) establishes the convergence of the difference scheme for the fixed values of the parameter ε.

5. Numerical examples

To validate the efficiency of the method, we applied it to two linear examples and one nonlinear example.

Example 5.1. Consider the singularly perturbed delay differential equation with left layer $\varepsilon y''(x) + y'(x - \delta) - y(x) = 0$; $x \in [0, 1]$, with $y(0) = 1$ and $y(1) = 1$.

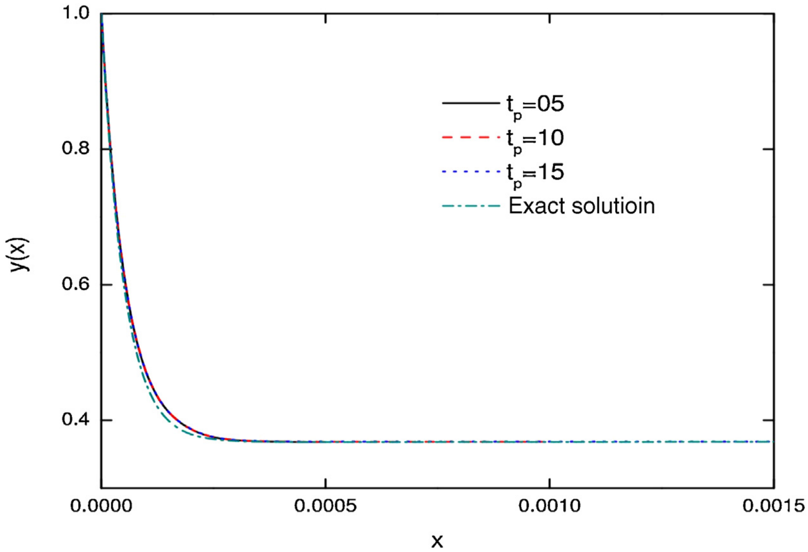

Fig. 2. Inner layer solutions of Example 5.1 for $\varepsilon = 0.0001$, $\delta = 0.1\varepsilon$ and different terminal points.

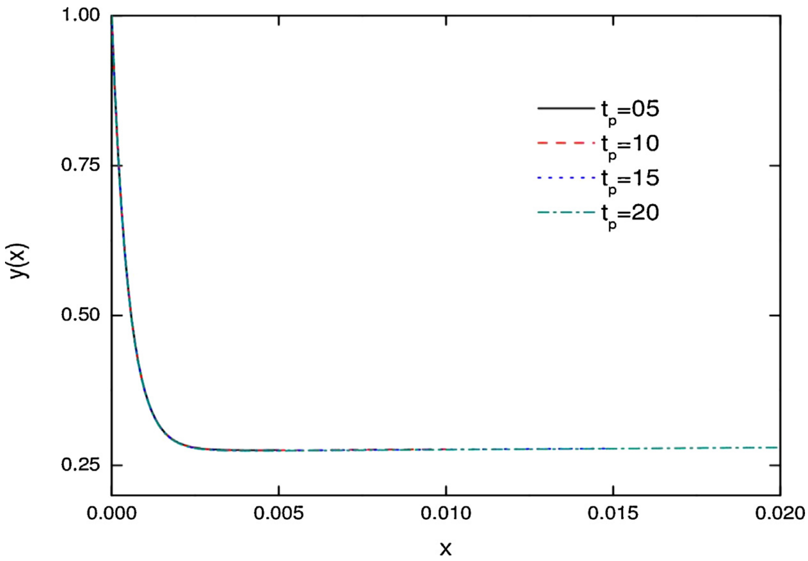

Fig. 3. Inner layer solutions of Example 5.2 for $\varepsilon = 0.001$, $\delta = 0.1\varepsilon$ and different terminal points.

The exact solution is given by

$$y(x) = \frac{(1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x}}{e^{m_1} - e^{m_2}}$$

where $m_1 = (-1 - \sqrt{1 + 4(\varepsilon - \delta)})/(2(\varepsilon - \delta))$ and $m_2 = (-1 + \sqrt{1 + 4(\varepsilon - \delta)})/(2(\varepsilon - \delta))$.

Numerical results are presented in Tables 1 and 2 for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively.

Example 5.2. Consider the following singularly perturbed variable coefficient and non-homogenous delay differential equation:

$$\varepsilon y''(x) + e^{-0.5x} y'(x - \delta) - y(x) = 0, \quad 0 \le x \le 1$$

with boundary conditions $y(0) = 1, -\delta \le x \le 0, y(1) = 1$.

The exact solution of the problem is not known. Numerical results are presented in Tables 3 and 4 for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively.

6. Nonlinear problems

To solve nonlinear singular perturbation problems, we used the method of quasi-linearization.

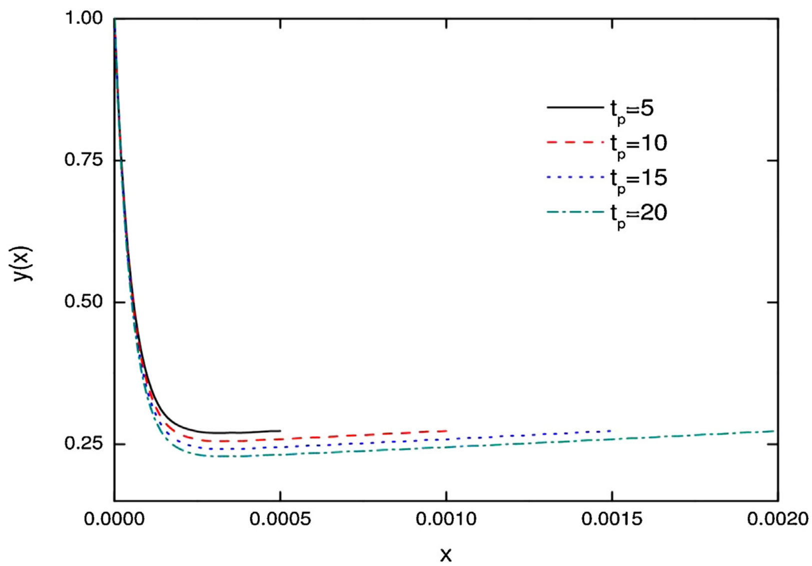

Fig. 4. Inner layer solutions of Example 5.2 for $\varepsilon = 0.0001$, $\delta = 0.1\varepsilon$ and different terminal points.

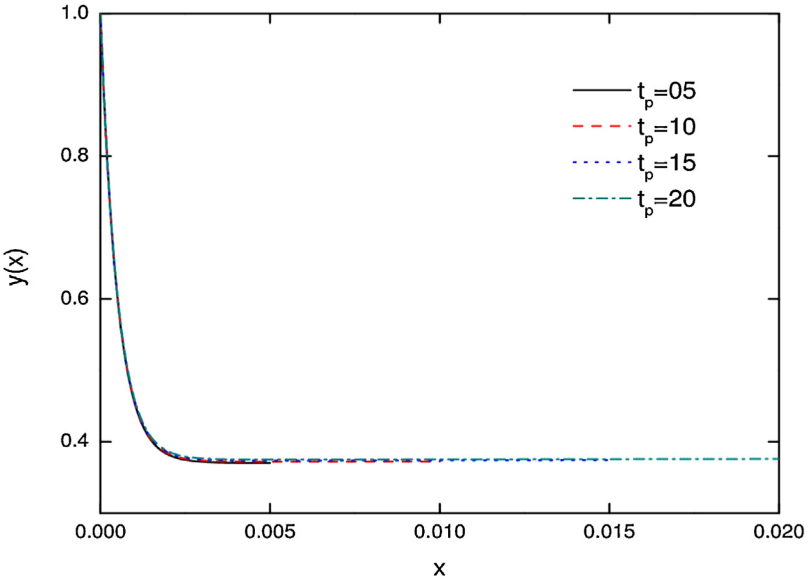

Fig. 5. Inner layer solutions of Example 6.1 for ε =0.001, $\delta = 0.1\varepsilon$ and different terminal points.

Example 6.1. Consider the following non-linear singularly perturbed delay differential equation:

$$\varepsilon y''(x) + y(x)y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1, \quad -\delta \le x \le 0, \quad y(1) = 1$$

The quasilinear form of this example is

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0;\ y(x) = 1,$$
$$-\delta \le x \le 0,\ y(1) = 1$$

The exact solution of the problem is not known. Numerical results are presented in Tables 5 and 6 for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively.

7. Discussion and conclusions

A terminal boundary-value technique has been presented for solving singularly perturbed delay differential equations whose solutions exhibits boundary layer behaviour. The method is iterative on the terminal point x_p and the process is to be repeated for different values of x_p (the terminal point which is not unique), until the

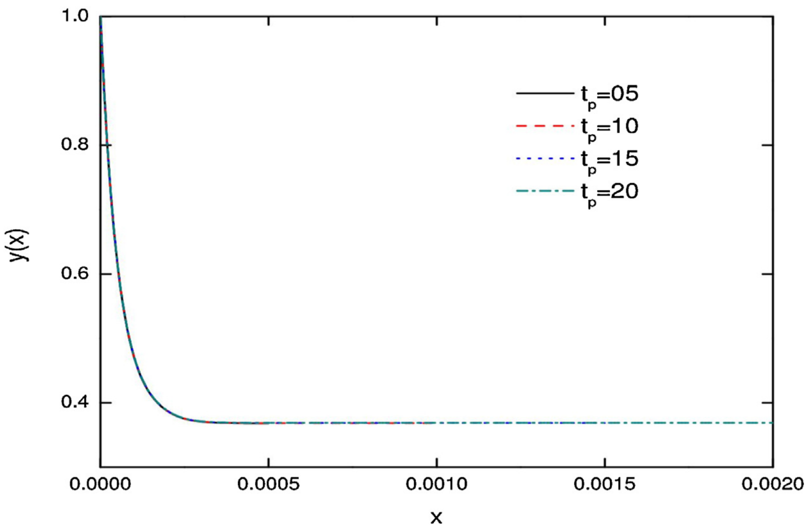

Fig. 6. Inner layer solutions of Example 6.1 for $\varepsilon = 0.0001$, $\delta = 0.1\varepsilon$ and different terminal points.

solution profile stabilizes in both the inner and outer regions. The present method has been implemented on two linear and one nonlinear problem with left-end boundary layer, by taking $\delta = 0.1\varepsilon$, $\delta = 0.5\varepsilon$ and different values of ε. The numerical results have been tabulated and compared with the exact solutions. Although the solutions are computed at all the points with mesh size h only a few values have been reported. It can be observed from the tables (Tables 1–6) and figures (Figs. 1–6) that the present method approximates the exact solution very well. In fact, the method helps us not only to get good results but also to know the behaviour of the solution in the boundary layer/inner region with $h \geq \varepsilon$ where the existing numerical methods fail to give good results (See Figs. 1–6). The method is simple, easy and efficient technique for solving singularly perturbed delay differential equations.

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