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Exact solution for flow over a contaminated fluid sphere for stokes flow

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Abstract. Stokes viscous flow past a partially contaminated fluid sphere with no slip condition is considered. No mass transfer for the entire sphere, no slip condition on contaminated part, shear stress continuity on clear part and regularity condition at far away from the body are considered for boundary conditions for evaluation of stream function.

1. Introduction

Study of flow in drops has wide applications in natural and engineering such as nuclear reactors, internal combustion engines, sediment and pollutant transport processes. Stokes [1] solved the creeping flow problem past a sphere neglecting inertial terms in the Navier Stokes equations. Basset [2] calculated drag over a sphere in terms of slip parameter s (Trostel number), $s = \frac{\beta a}{\mu}$ (defined as slip parameter). Creeping flow over a fluid sphere was studied analytically independently by Rybczynski [3] and Hadamard [4]. Happel and Brenner [5] discussed creeping flow past a sphere with no slip boundary condition. Clift et al. [6] and Michaelides [7] in their monographs discussed about viscous flow past a fluid sphere with no slip boundary condition. Sadhal and Johnson [8] derived exact solution of drag force for a fluid sphere in terms of cap angle with stagnation cap over its boundary along the rear of the fluid drop. Stagnation cap is the collection of surfactant at the rear side the fluid sphere. Saboni [9] numerically discussed about the contamination effects on a fluid sphere for Reynolds numbers from 0.1 to 400 and viscosity ratio ranging from 0 to 10 at different stagnation cap angle. Feng et al. [10] has discussed about the viscous flow past a viscous drop with interfacial condition at small but finite Reynolds numbers. Feng derived a formula for drag coefficient. As special cases he derived an expression for drag for solid sphere with interfacial slip condition, solid sphere with no slip boundary condition.

In this paper, stokes viscous flow over a contaminated fluid sphere is considered with the continuous shear stresses over the clear part. Velocity is represented in terms of stream function. The stream lines are drawn for different values of viscosity ratio μ, contamination cap lengths.

2. Formulation of the problem

A fluid sphere of radius a with contamination at rear end, which is held fixed in an uniform viscous flow, is considered. The fluid within the sphere and fluid flowing past the sphere are

assumed to be immiscible. The flow is steady, incompressible, axi symmetric, with uniform velocity U_0. A spherical polar coordinate system is considered with origin at the center of the sphere and Z-axis along the direction of uniform flow. μ_i, μ_e, ρ_i, ρ_e are viscosity, density of interior and exterior fluids. The viscosity ratio is taken as $\mu = \frac{\mu_i}{\mu_e}$, $\rho = \frac{\rho_i}{\rho_e}$.
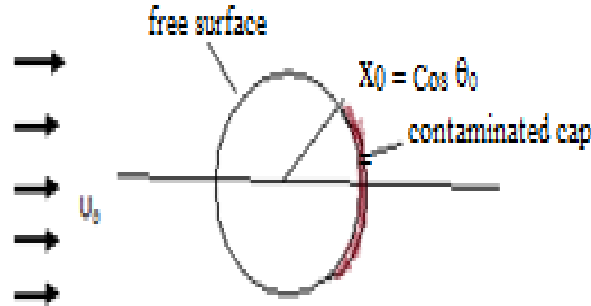

Figure 1. Geometry of the problem.

The viscous fluid is assumed to flow from left to right. In the fluid sphere the clear part (no cap region) is considered for $-1 < x < x_0$ and contaminated part (cap region) is for $x_0 < x < 1$, where $x = \cos\theta$. x_0 is the position of cap or cosine angle of contamination. The velocity components in radial direction U and transverse direction V can be expressed in terms of stream function as

$$U(R,\theta) = \frac{1}{R^2 \sin\theta} \frac{\partial \Psi}{\partial \theta}, \quad V(R,\theta) = \frac{-1}{R \sin\theta} \frac{\partial \Psi}{\partial R}, \tag{2.1}$$

Any physical (dimensional or non dimensional) quantity for internal flow is represented by fi and external flow by fe. The following non-dimensional scheme is used to obtain equations of motion.

$$R = ar, \quad \bar{Q} = U_0 \bar{q}, \quad U = U_0 u, \quad V = V_0 v, \quad \Psi = a^2 U_0 \psi.$$

Reynolds number for external flow is $Re = \frac{2a\rho_e U_0}{\mu_e}$ and for internal flow $Rei = \frac{Re\rho}{\mu}$.
The equation of motion in terms of stream function for the internal flow is

$$E^4 \psi i = 0. \tag{2.2}$$

The equation of motion in terms of stream function for the external flow is

$$E^4 \psi e = 0, \tag{2.3}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-x^2)}{r^2} \frac{\partial^2}{\partial x^2} = \text{Stokes stream function operator.}$$

Equations (2.2), (2.3) are solved for ψ_e, ψ_i using the following boundary conditions (2.4) - (2.9):

$$at \ \ \theta = 0, \ \ \pi(x = 1, x = -1) \ \ \text{on} \ \ \text{r=1}, \ \ \psi e = \psi i = 0, \tag{2.4}$$

Tangential velocity is zero along the contaminated part

$$\frac{\partial \psi e(1)}{\partial r} = \frac{\partial \psi i(1)}{\partial r} = 0 \ \ \text{for} \ \ x_0 < x < 1. \tag{2.5}$$

The shear stress is continuous along the clear surface at the interface, which implies

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi e}{\partial r}\right) = \mu \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi i}{\partial r}\right) \quad \text{for} \quad -1 < x < x_0, \tag{2.6}$$

Far away from the sphere, regularity condition is taken

$$\lim_{r \to \infty} \psi e = \frac{1}{2}r^2(1 - x^2), \tag{2.7}$$

The physical condition, that velocity at the origin is finite, can be taken as

$$\lim_{r \to \infty} \psi i < \infty (\text{it is taken as zero}).$$

3. Solution of the problem

The Stream function ψ for stokes flow is taken in the form

$$\psi = \sum_{n=1}^{\infty} F_n(r)G_n(r).$$

Hence (2.2) or (2.3) takes the form given by

$$E^4\psi = \sum_{n=1}^{\infty} D_n^4 F_n(r)G_n(r), \tag{3.1}$$

where

$$D_n^2 F_n = F_n'' - \frac{n(n-1)}{r^2}F_n.$$

For external flow the solution for ψ of (3.1) which satisfies the regularity condition (2.7) at infinity is given by

$$\psi_e(r,x) = \left(A_2 r^2 + \frac{B_2}{r} + D_2(r)\right)G_2(x) + \sum_{n=3}^{\infty}\left(\frac{B_n}{r^{n-1}} + \frac{D_n}{r^{n-3}}\right)G_n(x), \tag{3.2}$$

Using the boundary condition $\psi_e(1,x) = 0$, stream function in (3.2) reduces to

$$\psi_e(r,x) = (r^2 - r + B_2(\frac{1}{r} - r))G_2(x) + \sum_{n=3}^{\infty} B_n\left(\frac{1}{r^{n-1}} - \frac{1}{r^{n-3}}\right)G_n(x), \tag{3.3}$$

Similarly for internal flow, the solution for ψ of (3.1) that vanishes at r = 1 (i.e $\psi_i(1,x) = 0$) is given by

$$\psi_i(r,x) = \sum_{n=2}^{\infty} A_n(r^{n+2} - r^n)G_2(x). \tag{3.4}$$

The solution of (3.1) for cap and free regions is assumed to be in the form

$$\psi_e(r,x) = \begin{cases} fe_n(r)\,G_2(x), & \text{for} -1 < x \le x_0 \\ fe_c(r)\,G_2(x), & \text{for} \quad x_0 < x \le 1, \end{cases}$$

$$\psi_e(r,x) = \begin{cases} fi_n(r)\,G_2(x), & \text{for} -1 < x \le x_0 \\ fi_c(r)\,G_2(x), & \text{for} \quad x_0 < x \le 1. \end{cases} \tag{3.5}$$

Again solving (3.1) for the region of free flow ($x_0 < x < 1$), the solutions for the coefficient of $G_2(x)$ which satisfy the continuity of interfacial stress for the external and internal flows are given by

$$fe_n(r) = r^2 - \frac{2 + 3\mu}{2 + 2\mu}r + \frac{\mu}{2 + 2\mu}\frac{1}{r}, fi_n(r) = \frac{r^4 - r^2}{2 + 2\mu}. \tag{3.6}$$

Similarly for the region of contaminated cap ($-1 < x < x_0$), the solutions for the coefficient of $G_2(x)$ which satisfy the no-slip condition for the external and internal flows are given by

$$fe_c(r) = r^2 - \frac{3}{2}r + \frac{1}{2r}, fi_c(r) = 0. \tag{3.7}$$

Hence, for external flow, LHS of condition (2.6) takes the form

$$\psi_e''(1) - 2\psi_e'(1) = \begin{cases} (fe_{n,2}''(r) - 2fe_{n,2}'(r))G_2(x), & \text{for } -1 < x \le x_0 \\ fe_c''(r)\ G_2(x), & \text{for } \ \ x_0 < x \le 1. \end{cases} \tag{3.8}$$

Substituting (3.6) and (3.7) in (3.8) we get

$$\psi_e''(1) - 2\psi_e'(1) = \begin{cases} (\frac{3\mu}{1+\mu})G_2(x), & \text{for } -1 < x \le x_0 \\ 3G_2(x), & \text{for } \ \ x_0 < x \le 1. \end{cases} \tag{3.9}$$

Substituting (3.3) for ψ_e in (3.9) we get

$$6B_2G_2(x) + \sum(4n-2)B_nG_n(x) = \begin{cases} (\frac{3\mu}{1+\mu})G_2(x), & \text{for } -1 < x \le x_0 \\ 3G_2(x), & \text{for } \ \ x_0 < x \le 1. \end{cases} \tag{3.10}$$

By using the orthogonal property of Gegenbauer polynomials, the constants in (3.10) are obtained as below

$$(4m-2)B_m = \frac{(2m-1)m(m-1)}{2}\left[\int_{-1}^{x_0}\frac{3\mu}{1+\mu}\frac{G_2(x)G_m(x)}{1-x^2}dx + \int_{x_0}^{1}\frac{3G_2(x)G_m(x)}{1-x^2}dx\right]. \tag{3.11}$$

This on simplification gives

$$B_m = \frac{m(m-1)}{4}\left\{\frac{\mu}{1+\mu}\delta_{2,m} + \frac{3}{1+\mu}\int_{x_0}^{1}\frac{G_2(x)G_m(x)}{1-x^2}dx\right\}, \tag{3.12}$$

where $\delta_{2,m}$ is kronecker delta which takes value 1 if m=2 and 0 if m$\ne$ 2.

For internal flow, by substituting (3.4) in RHS of (2.6), the condition of continuity of interfacial stress is expressed as:

$$\mu\left(\frac{\partial^2\psi_i}{\partial r^2} - 2\frac{\partial\psi_i}{\partial r}\right)\Big|_{r=1} = \mu\sum_{n=2}^{\infty}(4n-2)A_nG_n(x), \tag{3.13}$$

$$\mu\left(\frac{\partial^2\psi_i}{\partial r^2} - 2\frac{\partial\psi_i}{\partial r}\right)\Big|_{r=1} = \sum_{n=2}^{\infty}\begin{cases} (\mu fi_{n,2}''(r) - 2fi_{n,2}'(r))G_2(x), & \text{for } -1 < x \le x_0 \\ 0, & \text{for } \ \ x_0 < x \le 1, \end{cases}$$

$$\mu\left(\frac{\partial^2\psi_i}{\partial r^2} - 2\frac{\partial\psi_i}{\partial r}\right)\Big|_{r=1} = \sum_{n=2}^{\infty}\begin{cases} (\frac{3\mu}{1+\mu})G_2(x), & \text{for } -1 < x \le x_0 \\ 0, & \text{for } \ \ x_0 < x \le 1. \end{cases} \tag{3.14}$$

Using the orthogonal property of Gegenbauer polynomials, the constants A_n are obtained as below

$$A_n = \frac{3n(n-1)}{4(1+\mu)}\int_{-1}^{x_0}\frac{G_2(x)G_n(x)}{1-x^2}dx. \tag{3.15}$$

Now substituting the values of the constants A_n and B_n from (3.15) and (3.12) in (3.3) and (3.4), the flow pattern for external and internal flows can be obtained.

4. Results and discussions

The stream function ψ for external flow and for internal flow is taken in the infinite series of Gegenbauer polynomials as in (3.3) and (3.4). This function is to satisfy the discontinuity condition at the junction point of free and contaminated cap regions. Previously this was accomplished by using double series proposed by Collins [11]. By using the present procedure, this infinite series can be obtained easily by taking solutions in the free and cap regions and matching them with the series solution. When the number of terms in the series are limited to twenty (i.e. $N = 20$) the error in the solution is about 10^{-3}. Hence we did not take $N > 20$. The following figure 2, for different values of cap lengths x_0, illustrates variation in the original curve and truncated series solution. The wavy curve represents the stream function represented by infinite series at the boundary and the smooth curve is the stream function at the boundary. We can observe that the difference between these two curves is very small i.e, the series satisfies the boundary condition, within the reasonable tolerance. Both the external and internal stream functions represented by infinite series satisfy the viscous stream function differential equation (3.1). Hence we have obtained exact solution for this contaminated cap problem.
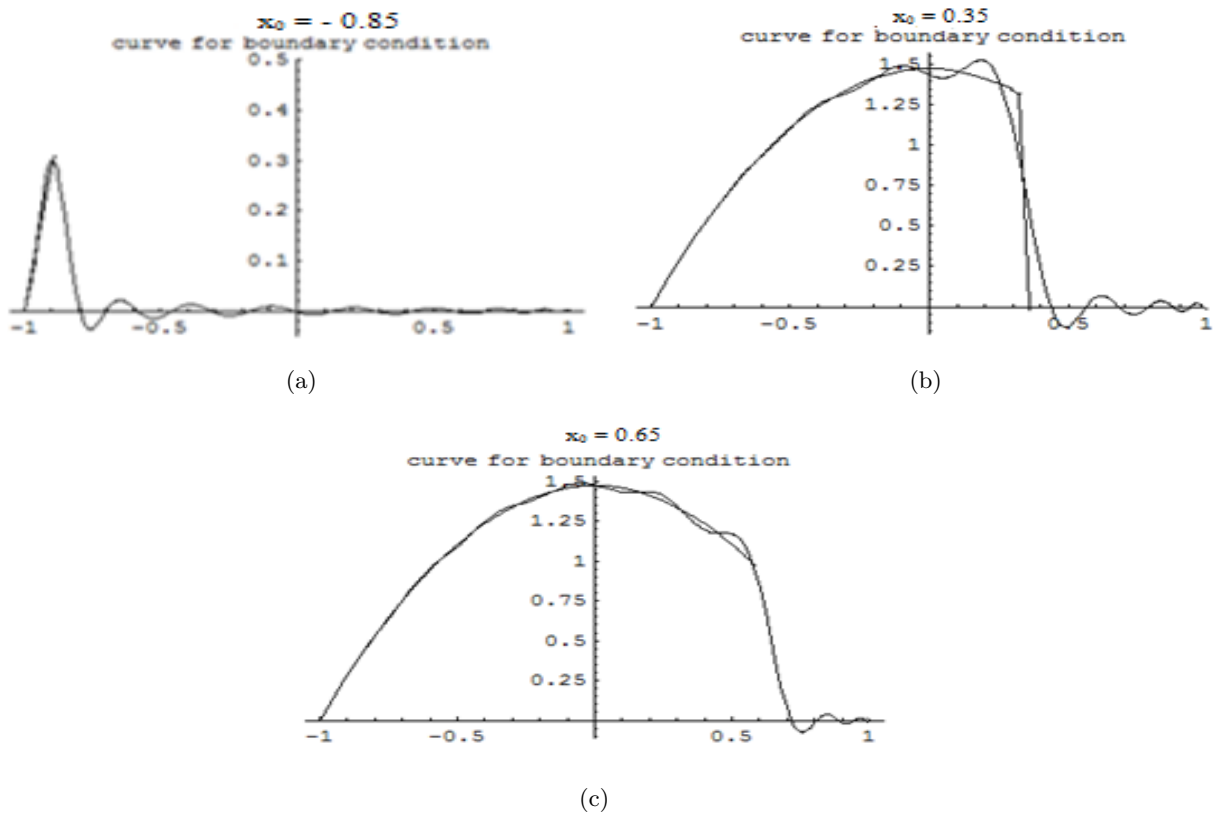

Figure 2. Stress at boundary for various values of cap length x_0.

Now by taking $N = 20$ terms in the stream function, stream line pattern for different values of cap lengths x_0 is obtained. It can be clearly observed from the figure 3 below that when x_0 is near to -1 flow in the internal region of the fluid sphere is very low. As the value of x_0 increases, internal circulation also increases. These flow patterns are similar to the results obtained by Feng and Michaelides [12] for low Reynolds numbers. These results are reasonably in very good agreement with the numerical results of Feng et al. [10]. The difference between the values of stream function at the boundary and the values of stream function obtained by infinite series is very small or insignificant ($\psi = 1 - 0.000001$) for $x_0 < -0.85$ i.e when contaminated cap is almost covering the fluid sphere. When the cap size is decreasing, i.e when x_0 is nearing 1, we can find more circulations within the sphere in the negative direction to the flow direction outside the sphere. The values of the stream function are decreasing and numerically increasing as x_0 is increasing. We can also observe that as Peclet number Pe is increasing, these circulations within the sphere are also increasing. When μ the ratio of viscosities increases, it implies that that the fluid is becoming thick and as $\mu \to \infty$, the sphere will be impermeable and the internal circulations will be absent.
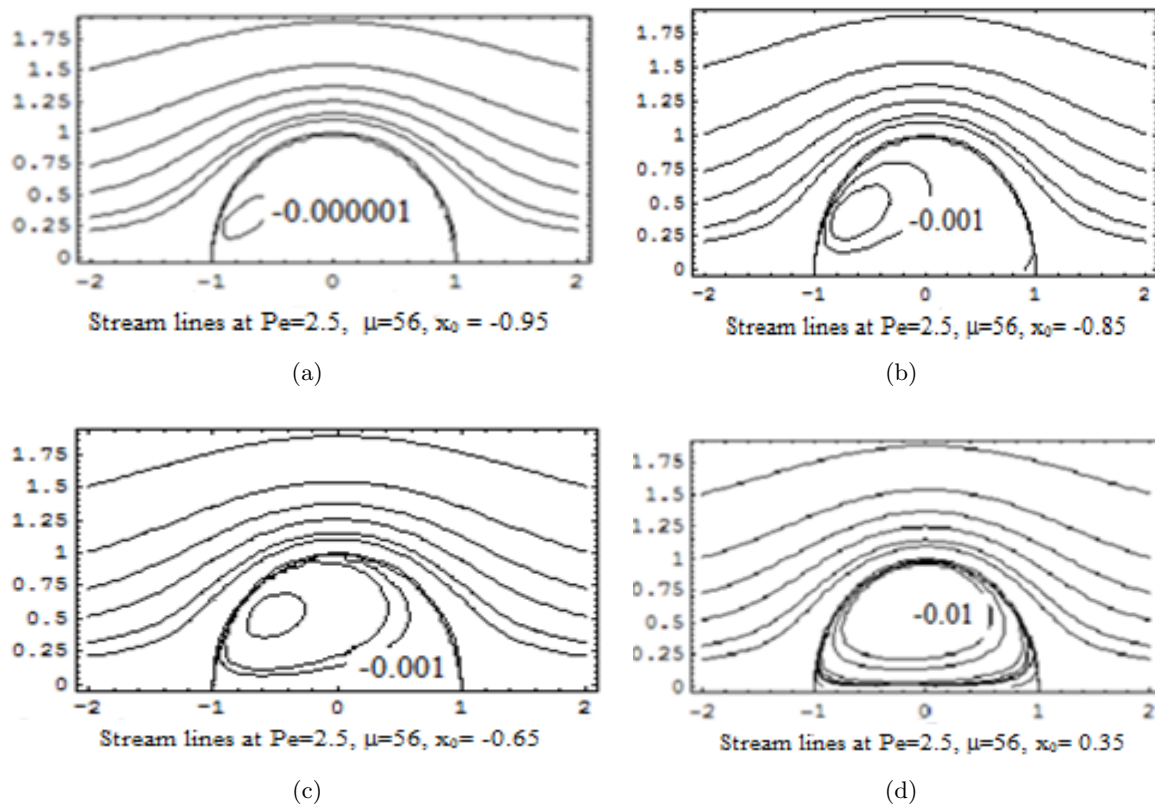

(a)

Stream lines at Pe=2.5, μ=56, x_0 = -0.95

(b)

Stream lines at Pe=2.5, μ=56, x_0= -0.85

(c)

Stream lines at Pe=2.5, μ=56, x_0= -0.65

(d)

Stream lines at Pe=2.5, μ=56, x_0= 0.35

Figure 3. Stream lines for internal and external flows for various cap lengths x_0.

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