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Unsteady flows of a micropolar fluid between parallel plates using state space approach

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Abstract. In this paper, we investigate the unsteady flow of an incompressible micropolar fluid between infinite parallel plates using state space approach when one of the plates is set to move suddenly while the other is at rest. Analytical expressions of the fluid velocity and microrotation are obtained in the Laplace transform domain. A standard numerical inversion technique is used to invert the Laplace transform of the velocity and microrotation. The effect of various material parameters on flow variables is discussed and the results are presented through graphs.

Nomenclature

(x, y)	Space coordinates
u	Velocity of the fluid along the x-direction
U	Velocity of the plate
v	Frequency of the oscillation of plate
c	Microrotation component of the fluid
ρ	Density
t	Time
p	Pressure
j	Gyration parameter
$\overline{q}$	Velocity vector of the fluid
$\overline{v}$	Microrotation vector of the fluid
μ, k	Viscosity coefficients
α, β, γ	Gyroviscosity coefficients

1 Introduction

The theory of micropolar fluids introduced by A.C. Eringen [1] in 1966 is a substantial generalization of the classical Newtonian viscous fluid theory. The micropolar fluid belongs to the class of fluids with non-symmetric stress tensor. It is a non-Newtonian fluid that can be used to analyze the behavior of lubricants, colloidal suspensions, polymeric fluids, liquid crystals and animal blood [2]. An account of the earlier developments in micropolar fluid theory can be seen in Stokes [3] and the existing state of art can be seen in the excellent treatises of Lukaszewicz [4] and Eringen [5]. Recent works of Papautsky *et al.* [6,7] on microchannel/laminar flow behavior, and Kucaba-Pietal [8] on microchannel flow modeling using micropolar fluid theory and the references therein indicate the increasing awareness in the fluid behavior through experimental studies as well.

The unsteady unidirectional flows between parallel plates have attracted the attention of numerous researchers in the past few years. Several authors have studied the unsteady flows between parallel plates arising due to the sudden

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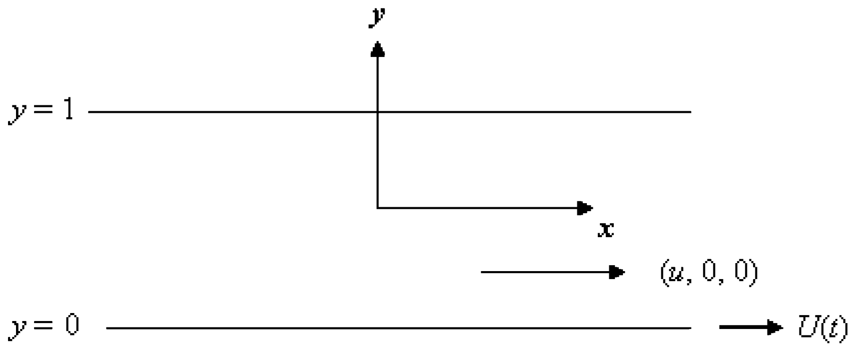

Fig. 1. Flow configuration.

motion of the plates for Newtonian and diverse non-Newtonian fluids as can be seen in the works of Erdogan [9], Erdogan and Imrak [10,11], Fetecau and Fetecau [12], Gupta and Arora [13], Hassanien and Mansour [14], Hassanien [15] and Hayat *et al.* [16,17]. In this paper, making use of the state space approach, we investigate the unsteady flow of an incompressible micropolar fluid between two infinite parallel plates when one of them is held fixed while the other is suddenly moving with some velocity. The state space approach is quite popular in modern control theory and is applicable to all linear systems that can be analyzed by integral transforms in time [18]. This has been successfully used by Helmy [19], Devakar and Iyengar [20] and few more for solving some problems in fluid dynamics [21]. It is heartening to note that the state space approach has enabled us to get the solution of the problems in an elegant way without going for the decoupling of the system of equations governing the flow. Taking the Laplace transform of the equations of motion and using the relevant boundary conditions, the flow field variables are obtained in the Laplace transform domain. The flow field variables in the space-time domain are obtained using the numerical inversion technique due to Honig and Hirdes [22]. The variation in velocity and microrotation due to variation in time, distance and material parameters is depicted through graphs.

2 Basic equations and formulation of the problem

The equations governing the flow of an incompressible micropolar fluid [1], in the absence of body force and body couple, are given by

$$\mathrm{div}(\overline{q}) = 0, \tag{1}$$

$$\rho\frac{\mathrm{d}\overline{q}}{\mathrm{d}t} = -\mathrm{grad}(p) + k\,\mathrm{curl}(\overline{v}) - (\mu+k)\,\mathrm{curlcurl}(\overline{q}), \tag{2}$$

$$\rho\, j\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} = -2k\,\overline{v} + k\,\mathrm{curl}(\overline{q}) - \gamma\,\mathrm{curlcurl}(\overline{v}) + (\alpha+\beta+\gamma)\,\mathrm{grad}(\mathrm{div}(\overline{v})), \tag{3}$$

where, the scalar quantities ρ and j are, respectively, the density and gyration parameters and are assumed constants. The vectors $\overline{q}$ and $\overline{v}$ are the velocity and micro rotation, respectively. p is the fluid pressure at any point. The material constants μ, k are the viscosity coefficients and α, β, γ are the gyroviscosity coefficients. These constants conform to the usual inequalities

$$k \geq 0,\ 2\mu + k \geq 0, \qquad \gamma \geq 0, \qquad |\beta| \leq \gamma, \qquad 3\alpha+\beta+\gamma \geq 0.$$

Let us consider an incompressible micropolar fluid between two rigid parallel plates $y = 0$ and $y = h$. Initially let us assume that both the fluid and plates are at rest. At time $t = 0+$, we allow the lower plate $y = 0$ to move in its own plane along the x-direction with velocity $(U(t), 0, 0)$ while the upper plate $y = h$, is held fixed. As the flow occurs only in the x-coordinate direction, the velocity and microrotation are expected to be in the form $\overline{q} = (u(y,t), 0, 0)$ and $\overline{v} = (0, 0, c(y,t))$, respectively. We assume that there is no applied pressure gradient pushing the flow in the x-direction; the flow arises due to the motion of the plate only. (See fig. 1.)

The above choice of velocity automatically satisfies the continuity equation (1). Now the governing equations of the flow reduce to

$$\rho\frac{\partial u}{\partial t} = k\frac{\partial c}{\partial y} + (\mu+k)\frac{\partial^2 u}{\partial y^2}\,, \tag{4}$$

$$\rho\, j\frac{\partial c}{\partial t} = -2kc - k\frac{\partial u}{\partial y} + \gamma\frac{\partial^2 c}{\partial y^2}\,. \tag{5}$$

The conditions to be satisfied are the following:

$$\text{Initial conditions:}\quad u(y,0)=0 \quad\text{and}\quad c(y,0)=0 \quad\text{for all}\quad y. \tag{6}$$
$$\text{Boundary conditions:}\quad \text{for}\quad t>0,\quad u(0,t)=U(t),\quad u(h,t)=0 \tag{7}$$
$$\text{and}\quad c(0,t)=c(h,t)=0. \tag{8}$$

Introducing the non-dimensional variables through

$$\tilde{y}=\frac{y}{h},\qquad \tilde{u}=\frac{u}{U},\qquad \tilde{t}=\frac{U}{h}t \qquad\text{and}\qquad \tilde{c}=\frac{h}{U}c, \tag{9}$$

eqs. (4) and (5) take the form

$$R\frac{\partial\tilde{u}}{\partial\tilde{t}}=m\frac{\partial\tilde{c}}{\partial\tilde{y}}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\,, \tag{10}$$
$$\frac{R}{n_2}\frac{\partial\tilde{c}}{\partial\tilde{t}}=-2n\tilde{c}-n\frac{\partial\tilde{u}}{\partial\tilde{y}}+\frac{\partial^2\tilde{c}}{\partial\tilde{y}^2}\,, \tag{11}$$

where

$$R=\frac{\rho U h}{(\mu+k)},\qquad m=\frac{k}{(\mu+k)},\qquad n=\frac{kh^2}{\gamma}\qquad\text{and}\qquad n_2=\frac{\gamma}{j(\mu+k)}\,. \tag{12}$$

Dropping $\sim$'s, we get the equations in the non-dimensional form as

$$R\frac{\partial u}{\partial t}=m\frac{\partial c}{\partial y}+\frac{\partial^2 u}{\partial y^2}\,, \tag{13}$$
$$\frac{R}{n_2}\frac{\partial c}{\partial t}=-2nc-n\frac{\partial u}{\partial y}+\frac{\partial^2 c}{\partial y^2}\,. \tag{14}$$

Equations (13) and (14) are to be solved subject to the non-dimensional boundary conditions:

$$\text{Initial conditions:}\quad u(y,0)=0 \quad\text{and}\quad c(y,0)=0 \quad\text{for all}\quad y. \tag{15}$$
$$\text{Boundary conditions:}\quad \text{for}\quad t>0,\quad u(0,t)=U(t),\quad u(1,t)=0 \tag{16}$$
$$\text{and}\quad c(0,t)=c(1,t)=0. \tag{17}$$

3 Solution of the problem

Taking the Laplace transform to eqs. (13), (14), (16) and (17) with respect to "t" and making use of the initial conditions (15), we obtain

$$\frac{\mathrm{d}^2\overline{u}}{\mathrm{d}y^2}+m\frac{\mathrm{d}\overline{c}}{\mathrm{d}y}-Rs\overline{u}=0, \tag{18}$$
$$\frac{\mathrm{d}^2\overline{c}}{\mathrm{d}y^2}-n\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}-a\overline{c}=0, \tag{19}$$

where

$$a=\left(2n+\frac{Rs}{n_2}\right), \tag{20}$$

with the conditions

$$\overline{u}(0,s)=\overline{U}(s),\qquad \overline{u}(1,s)=0,\qquad \overline{c}(0,s)=0\qquad\text{and}\qquad \overline{c}(1,s)=0. \tag{21}$$

3.1 The state space approach

Introducing the new variables $\overline{u'} = \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$ and $\overline{c'} = \frac{\mathrm{d}\overline{c}}{\mathrm{d}y}$, eqs. (18) and (19) can be written in the matrix form,

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix}\overline{u}\\ \overline{c}\\ \overline{u'}\\ \overline{c'}\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ Rs & 0 & 0 & -m\\ 0 & a & n & 0\end{pmatrix}\begin{pmatrix}\overline{u}\\ \overline{c}\\ \overline{u'}\\ \overline{c'}\end{pmatrix}, \tag{22}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}y}\overline{V}(y,\,s) = A(s)\,\overline{V}(y,\,s), \tag{23}$$

where

$$A(s) = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ Rs & 0 & 0 & -m\\ 0 & a & n & 0\end{pmatrix}, \qquad \overline{V}(y,\,s) = \begin{pmatrix}\overline{u}(y,\,s)\\ \overline{c}(y,\,s)\\ \overline{u'}(y,\,s)\\ \overline{c'}(y,\,s)\end{pmatrix}. \tag{24}$$

The formal solution of the matrix differential equation (23) is seen to be

$$\overline{V}(y,\,s) = \exp[A(s)\,y]\,\overline{V}(0,\,s). \tag{25}$$

To determine the matrix $\exp[A(s)\,y]$, we note that the characteristic equation of the matrix $A(s)$ is

$$k^4 - (Rs + a - mn)\,k^2 + Rsa = 0, \tag{26}$$

with the characteristic roots $\pm k_1$, $\pm k_2$, where

$$\left.\begin{aligned} k_1 &= \sqrt{\tfrac{(Rs+a-mn)+\sqrt{(Rs+a-mn)^2-4Rsa}}{2}}\\ k_2 &= \sqrt{\tfrac{(Rs+a-mn)-\sqrt{(Rs+a-mn)^2-4Rsa}}{2}}\end{aligned}\right\} \tag{27}$$

are with positive real parts. The Maclaurin series expansion of $\exp[A(s)\,y]$ is given by

$$\exp[A(s)\,y] = \sum_{r=0}^{\infty}\frac{[A(s)\,y]^r}{r!}\,. \tag{28}$$

Using Cayley-Hamilton theorem, the infinite series (28) can be written in the form

$$\exp[A(s)\,y] = L(y,\,s) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3, \tag{29}$$

where I is the unit matrix of order 4 and a_0, a_1, a_2 and a_3 are some parameters depending on y and s. Following the state space technique as is used in problems dealing with the modern control theory [18], we note that the characteristic roots $\pm k_1$, $\pm k_2$ satisfy eq. (29) and hence by replacing the matrix A with its characteristic roots $\pm k_1$, $\pm k_2$ therein, we get the following system of linear equations:

$$\left.\begin{aligned} \exp[k_1\,y] &= a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3\\ \exp[-k_1\,y] &= a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3\\ \exp[k_2\,y] &= a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3\\ \exp[-k_2\,y] &= a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3\end{aligned}\right\}, \tag{30}$$

for the determination of a_0, a_1, a_2 and a_3. Solving this system, we get

$$\left.\begin{aligned} a_0 &= \tfrac{1}{F}\left[k_1^2\cosh(k_2 y) - k_2^2\cosh(k_1 y)\right]\\ a_1 &= \tfrac{1}{F}\left[\tfrac{k_1^2}{k_2}\sinh(k_2 y) - \tfrac{k_2^2}{k_1}\sinh(k_1 y)\right]\\ a_2 &= \tfrac{1}{F}\left[\cosh(k_1 y) - \cosh(k_2 y)\right]\\ a_3 &= \tfrac{1}{F}\left[\tfrac{1}{k_1}\sinh(k_1 y) - \tfrac{1}{k_2}\sinh(k_2 y)\right]\end{aligned}\right\}, \tag{31}$$

where
$$F = k_1^2 - k_2^2. \tag{32}$$
Substituting A, A^2, A^3 and using eq. (31) in eq. (29), we obtain the elements ($L_{i\,j}; i,j = 1,2,3,4$) of the matrix $L(y,\,s)$ as

$$\begin{aligned}
L_{11} &= \frac{1}{F}\left\{(k_1^2 - Rs)\cosh(k_2 y) - (k_2^2 - Rs)\cosh(k_1 y)\right\},\\
L_{12} &= \frac{ma}{F}\left\{\frac{1}{k_2}\sinh(k_2 y) - \frac{1}{k_1}\sinh(k_1 y)\right\},\\
L_{13} &= \frac{1}{F}\left\{\left(\frac{a-k_2^2}{k_2}\right)\sinh(k_2 y) - \left(\frac{a-k_1^2}{k_1}\right)\sinh(k_1 y)\right\},\\
L_{14} &= \frac{m}{F}\left\{\cosh(k_2 y) - \cosh(k_1 y)\right\},\\
L_{21} &= \frac{nRs}{F}\left\{\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right\},\\
L_{22} &= \frac{1}{F}\left\{(k_1^2 - a)\cosh(k_2 y) - (k_2^2 - a)\cosh(k_1 y)\right\}, \qquad L_{23} = -\frac{n}{m}L_{14},\\
L_{24} &= \frac{1}{F}\left\{\left(\frac{Rs-k_2^2}{k_2}\right)\sinh(k_2 y) - \left(\frac{Rs-k_1^2}{k_1}\right)\sinh(k_1 y)\right\},\\
L_{31} &= RsL_{13}, \qquad L_{32} = aL_{14}, \qquad L_{33} = L_{11} + nL_{14},\\
L_{34} &= \frac{m}{F}\left\{k_2\sinh(k_2 y) - k_1\sinh(k_1 y)\right\}, \qquad L_{41} = -\frac{nRs}{m}L_{14},\\
L_{42} &= aL_{24}, \qquad L_{43} = -\frac{n}{m}L_{34}, \qquad L_{44} = L_{22} + nL_{14}.
\end{aligned} \tag{33}$$

With these, the solution (25) is obtained in the form
$$\overline{V}(y,\,s) = L(y,\,s)\,\overline{V}(0,\,s), \tag{34}$$
where $L_{i\,j}$'s are explicitly given in (33).

The first two components $\overline{u}(0,\,s)$ and $\overline{c}(0,\,s)$ of the vector $\overline{V}(0,\,s)$ are given in (21).

From eq. (34) and the boundary conditions $\overline{u}(0,s) = \overline{U}(s)$ and $\overline{c}(0,s) = 0$ of eq. (21), we notice that
$$\overline{u}(y,s) = \overline{U}(s)L_{11} + L_{13}\overline{u'}(0,s) + L_{14}\overline{c'}(0,s) \tag{35}$$
and
$$\overline{c}(y,s) = \overline{U}(s)L_{21} + L_{23}\overline{u'}(0,s) + L_{24}\overline{c'}(0,s). \tag{36}$$
Here we note that $\overline{u}(y,\,s)$ and $\overline{c}(y,\,s)$ are not yet completely determined as we do not have the expressions for $\overline{u'}(0,\,s)$ and $\overline{c'}(0,\,s)$. These can be obtained by invoking the conditions $\overline{u}(1,s) = 0$ and $\overline{c}(1,s) = 0$ of eq. (21) in eqs. (35) and (36). With this we have the equations
$$L_{1\,3}^1\overline{u'}(0,s) + L_{14}^1\overline{c'}(0,s) + \overline{U}(s)L_{11}^1 = 0, \tag{37}$$
$$L_{23}^1\overline{u'}(0,s) + L_{24}^1\overline{c'}(0,s) + \overline{U}(s)L_{21}^1 = 0, \tag{38}$$
where L_{ij}^1's are the values of L_{ij}'s at $y = 1$.

These two equations lead to
$$\left.\begin{aligned}
\overline{u'}(0,\,s) &= \overline{U}(s)\left\{\frac{L_{14}^1L_{21}^1 - L_{24}^1L_{11}^1}{L_{13}^1L_{24}^1 - L_{23}^1L_{14}^1}\right\}\\
\overline{c'}(0,\,s) &= \overline{U}(s)\left\{\frac{L_{11}^1L_{23}^1 - L_{21}^1L_{13}^1}{L_{13}^1L_{24}^1 - L_{23}^1L_{14}^1}\right\}
\end{aligned}\right\}. \tag{39}$$
The expressions of the velocity and microrotation, in the Laplace transform domain, can be obtained by using (35), (36) and taking into account (39). These are seen to be
$$\overline{u}(y,s) = \overline{U}(s)\left\{L_{11} + \frac{L_{21}^1(L_{14}^1L_{13} - L_{13}^1L_{14}) - L_{11}^1(L_{24}^1L_{13} - L_{23}^1L_{14})}{L_{13}^1L_{24}^1 - L_{23}^1L_{14}^1}\right\}, \tag{40}$$
$$\overline{c}(y,s) = \overline{U}(s)\left\{L_{21} + \frac{L_{21}^1(L_{14}^1L_{23} - L_{13}^1L_{24}) - L_{11}^1(L_{24}^1L_{23} - L_{23}^1L_{24})}{L_{13}^1L_{24}^1 - L_{23}^1L_{14}^1}\right\}, \tag{41}$$

where $L_{i\,j}$'s $(i,\,j=1,2,3,4)$ are the elements of the matrix $L(y,\,s)$ given in eq. (33) earlier and $L^1_{i\,j}$'s $(i,\,j=1,2,3,4)$ are the values of the elements of $L(y,\,s)$ at $y=1$. Since $L_{i\,j}$, L^1_{ij} $(i,\,j=1,2,3,4)$ contain terms involving k_1, k_2 and a and each one of k_1, k_2 and a depends on s, the analytical inversion of the expression $\overline{u}(y,\,s)$ and $\overline{c}(y,\,s)$ is thorny. Hence, we have chosen a numerical inversion procedure suggested by Honig and Hirdes [22] to determine $u(y,\,t)$ and $c(y,\,t)$ numerically.

3.2 The numerical inversion of Laplace transforms

In order to invert $\overline{u}(y,s)$ and $\overline{c}(y,s)$, we adopt a numerical inversion technique due to Honig and Hirdes [22]. Using this method the inverse $f(t)$ of the Laplace transform $\overline{f}(s)$ is approximated by

$$f(t)=\frac{e^{bt}}{t_1}\left[\frac{1}{2}\bar{f}(b)+\mathrm{Re}\left(\sum_{k=1}^{N}\bar{f}\left(b+\frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right)\right],\quad 0<t_1\leq 2t,$$

where N is sufficiently large integer chosen such that,

$$e^{bt}\mathrm{Re}\left[\bar{f}\left(b+\frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right]<\varepsilon,$$

where ε is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter b is a positive free parameter that must be greater than the real part of all the singularities of $\overline{f}(s)$. The optimal choice of b was obtained according to the criteria described in Honig and Hirdes [22].

3.3 Flow due to the sudden constant motion of the lower plate

Suppose the lower plate $y=0$ starts moving with velocity $U(t)=U$ for $t>0$ (in non-dimensional form, $U(t)=1$ with $\overline{U}(s)=\frac{1}{s}$) while the upper plate is held fixed.

The solution of the problem in the (y,s) space in this case is given by

$$\overline{u}(y,s)=\frac{1}{s}\left\{L_{11}+\frac{L^1_{21}(L^1_{14}L_{13}-L^1_{13}L_{14})-L^1_{11}(L^1_{24}L_{13}-L^1_{23}L_{14})}{L^1_{13}L^1_{24}-L^1_{23}L^1_{14}}\right\},\tag{42}$$

$$\overline{c}(y,s)=\frac{1}{s}\left\{L_{21}+\frac{L^1_{21}(L^1_{14}L_{23}-L^1_{13}L_{24})-L^1_{11}(L^1_{24}L_{23}-L^1_{23}L_{24})}{L^1_{13}L^1_{24}-L^1_{23}L^1_{14}}\right\},\tag{43}$$

where $L_{i\,j}$'s $(i,\,j=1,2,3,4)$ are the elements of the matrix $L(y,\,s)$ given earlier and $L^1_{i\,j}$'s $(i,\,j=1,2,3,4)$ are the values of the components of $L(y,\,s)$ at $y=1$. We numerically invert $\overline{u}(y,s)$ and $\overline{c}(y,s)$ for various values of time t, distance y and material parameters.

3.3.1 Discussion of results

The numerical inversion technique proposed by Honig and Hirdes [22] is used to compute the velocity component $u(y,\,t)$ and the microrotation component $c(y,\,t)$for various values of y and t. The results are displayed through graphs.

From fig. 2 we note that velocity at any y increases with increase in time t for any given set of material parameter values and Reynolds number. As expected, as time progresses there is an increase in the velocity component. The variation of each of the parameters m, n, n_2 and R while others are fixed, have no significant influence on the velocity in this case. For any given set of values of m, n, n_2 and R, the microrotation increases from 0 at $y=0$ to a peak and then decreases to 0 as y tends to 1. Figure 3 shows the variation of microrotation with time. It is noticed that, during the initial period the gradient of velocity is higher near moving plate and hence will get a sharp increase in microrotation near moving plate as compared to the stationary plate. At later time, the velocity gradient is evenly distributed hence the microrotation profile between parallel plates is symmetric.

Increase in n or n_2 significantly increases the microrotation. This can be noticed from figs. 4 and 5. From [23], we have $\gamma=(\mu+\frac{k}{2})j$ that implies $n=\frac{kh^2}{(\mu+\frac{k}{2})j}$. As the gyration parameter (microinertia per unit mass [23]) j decreases the parameter n increases which leads to an increase in the microrotation. That is to say decreasing of microinertia increases the microrotation.

As $\gamma=(\mu+\frac{k}{2})j$, n_2 becomes $1-\frac{m}{2}$. As m decreases, n_2 increases which has an increasing effect on the velocity. That is decreasing of microrotation m has an increasing effect on the velocity.

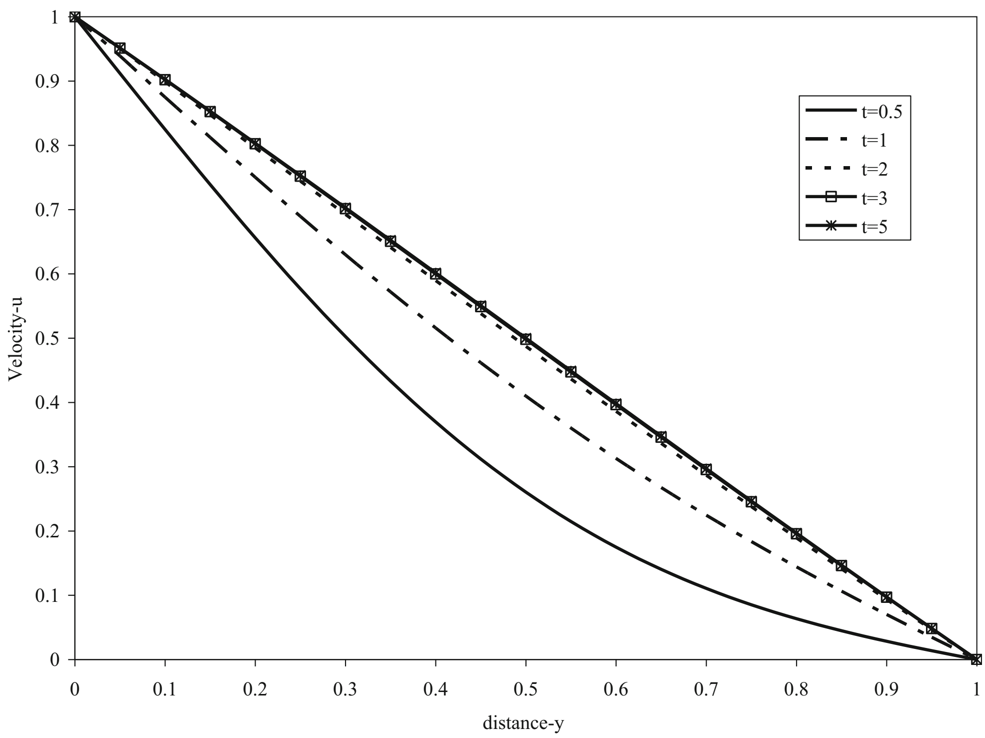

Fig. 2. Variation of velocity with distance for different times at $m = 0.5$, $n = 1$, $n_2 = 2$, $R = 2$.

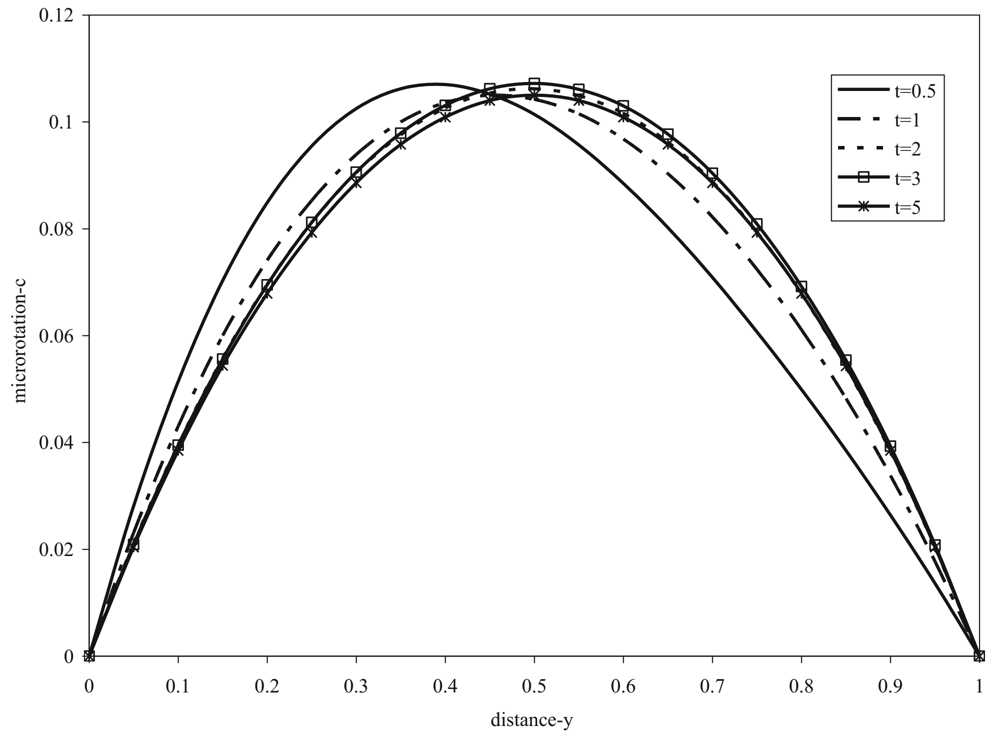

Fig. 3. Variation of microrotation with distance for different times at $m = 0.5$, $n = 1$, $n_2 = 2$, $R = 2$.

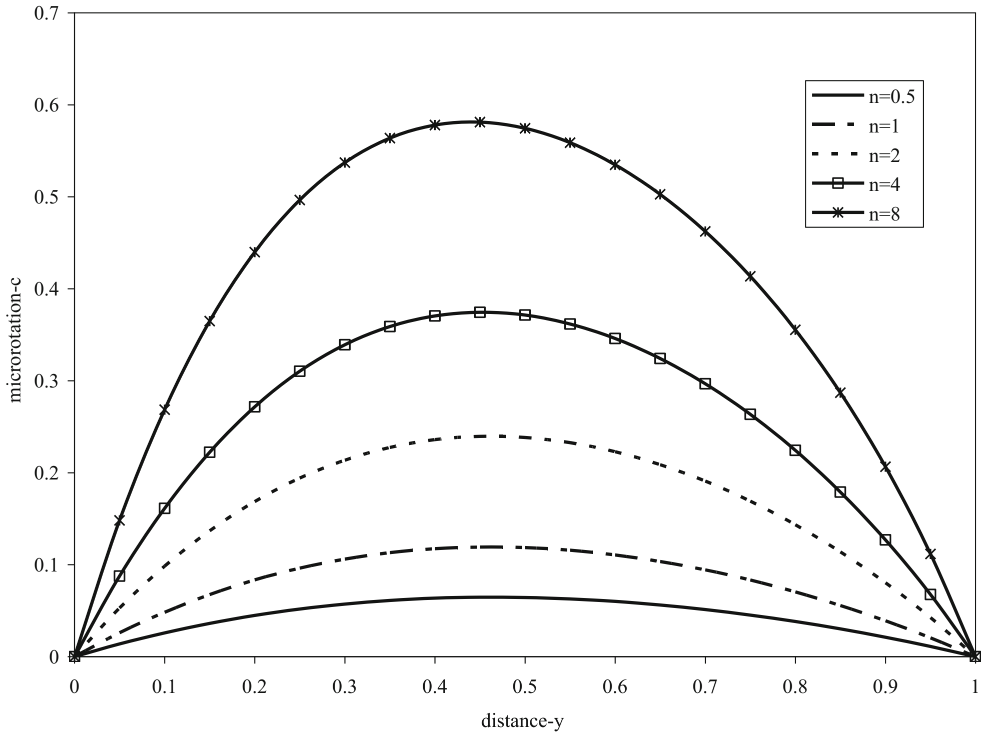

Fig. 4. Variation of microrotation with distance for different values of n at $t = 1$, $m = 0.5$, $n_2 = 2$, $R = 2$.

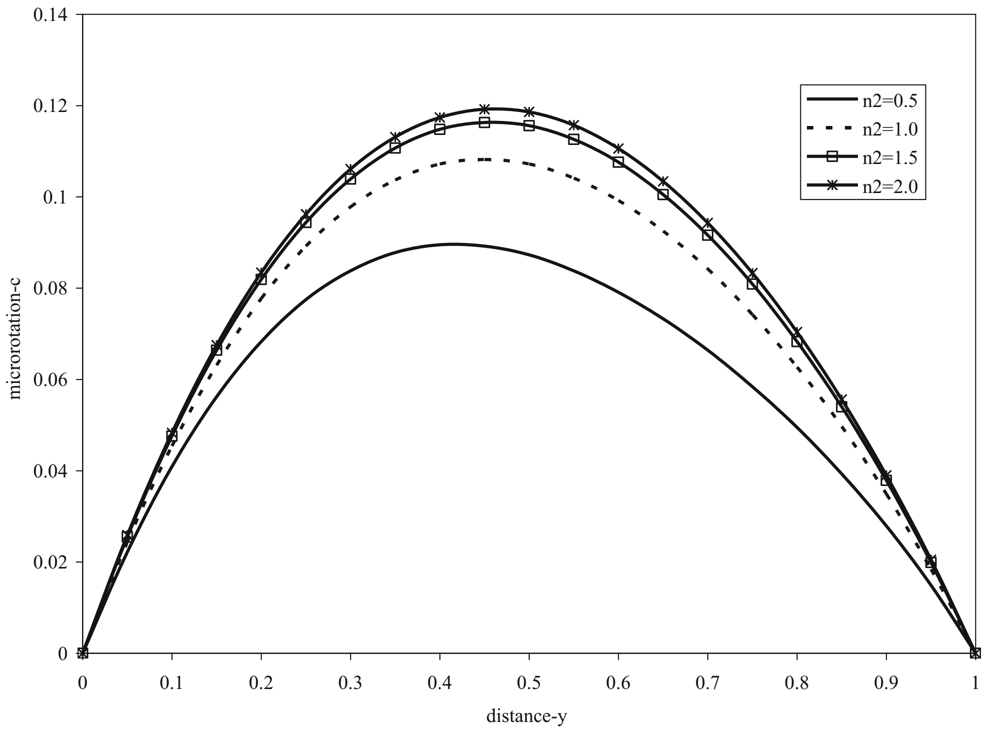

Fig. 5. Variation of microrotation with distance for different values of n_2 at $t = 1$, $m = 0.5$, $n = 1$, $R = 2$.

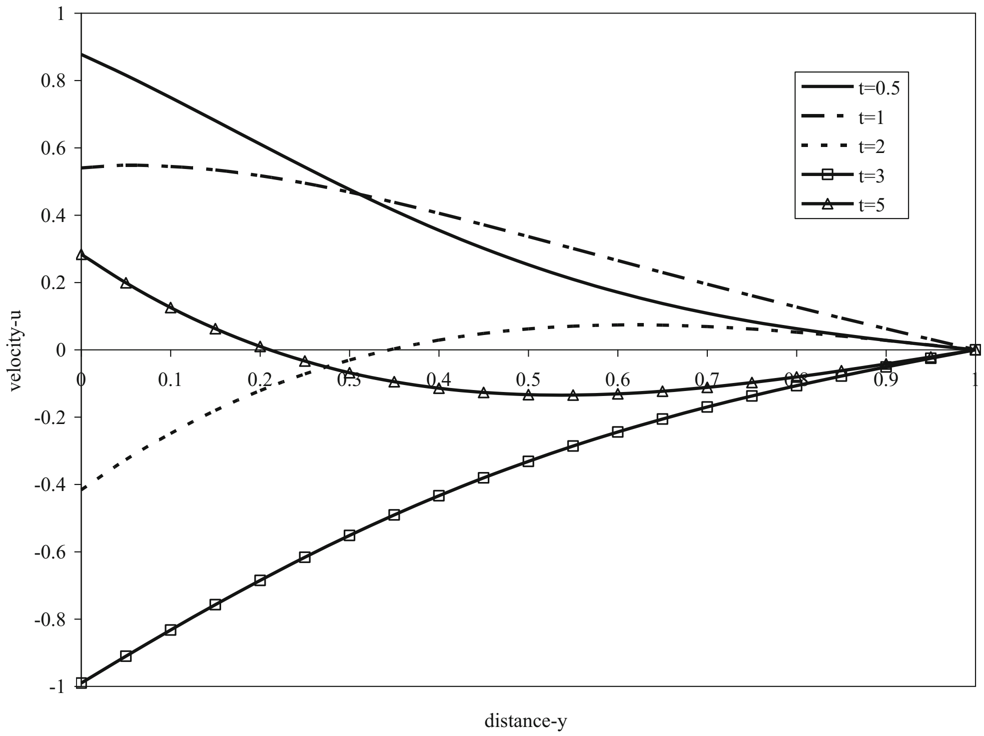

Fig. 6. Variation of velocity with distance for different times at $m = 0.5$, $n = 1$, $n_2 = 2$, $R = 2$.

3.4 Flow due to the sudden oscillatory motion of the lower plate

Suppose the lower plate $y = 0$ starts oscillating with velocity $U\cos(vt)$ at time $t = 0+$ (in non-dimensional form $U(t) = \cos(vt)$ with $\overline{U}(s) = \frac{s}{s^2+v^2}$).

Then the solution of the problem in (y, s) space in this case is seen to be

$$\overline{u}(y,s) = \frac{s}{s^2+v^2}\left\{L_{11} + \frac{L_{21}^1(L_{14}^1L_{13} - L_{13}^1L_{14}) - L_{11}^1(L_{24}^1L_{13} - L_{23}^1L_{14})}{L_{13}^1L_{24}^1 - L_{23}^1L_{14}^1}\right\}, \tag{44}$$

$$\overline{c}(y,s) = \frac{s}{s^2+v^2}\left\{L_{21} + \frac{L_{21}^1(L_{14}^1L_{23} - L_{13}^1L_{24}) - L_{11}^1(L_{24}^1L_{23} - L_{23}^1L_{24})}{L_{13}^1L_{24}^1 - L_{23}^1L_{14}^1}\right\}, \tag{45}$$

where L_{ij} 's $(i,\, j = 1,2,3,4)$ are the elements of the matrix $L(y,\, s)$ given earlier and L_{ij}^1 's $(i,\, j = 1,2,3,4)$ are the values of the components of $L(y,\, s)$ at $y = 1$. Since L_{ij}, $L_{ij}^1(i,\, j = 1,2,3,4)$ contain terms involving k_1, k_2 and a and each one of k_1, k_2 and a depends on s, as in the earlier case, we resort to the numerical inversion of the expressions of $\overline{u}(y,s)$ and $\overline{c}(y,s)$.

3.4.1 Discussion of results

We have computed $u(y,\, t)$ and $c(y,\, t)$ for various values of y and t for different values of the material parameters and their variation is displayed through graphs.

Figure 6 shows the variation of the velocity with distance for different times for given values of m, n, n_2, R. The variation is qualitatively similar to that observed by the authors in the case of oscillation of a single plate in Stokes' second problem [24]. There is no appreciable change in the velocity while each of the parameters m, n, n_2 is changed. It can be seen from fig. 7 that as R increases, as expected, the velocity increases for the given time t and any y.

Figures 8 to 11 show the variation of microrotation for any y as one of the parameters m, n, n_2 and R is varied while others are kept fixed. The behavior of microrotation is qualitatively similar to that observed in [24]. The microrotation reaches a maximum nearer to the moving plate and trails down to zero as we approach the plate $y = 1$. As R increases for fixed t, m, n, n_2, the microrotation also increases for any y. Increasing the value of Reynolds number R increases the linear momentum and hence the gradient of velocity which causes an increase in the angular momentum and hence an increase in the microrotation. (See fig. 11.) The effect of the parameters n, n_2 on the microrotation in this case is similar to the earlier one.

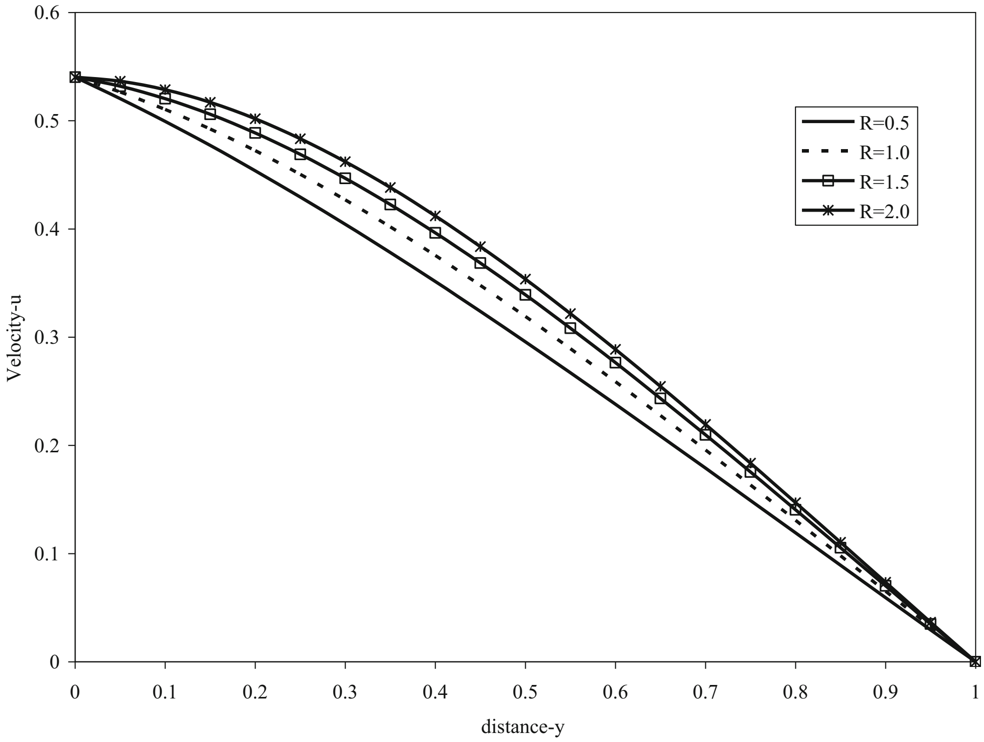

Fig. 7. Variation of velocity with distance for different values of R at $t = 1$, $m = 0.5$, $n = 1$, $n_2 = 2$.

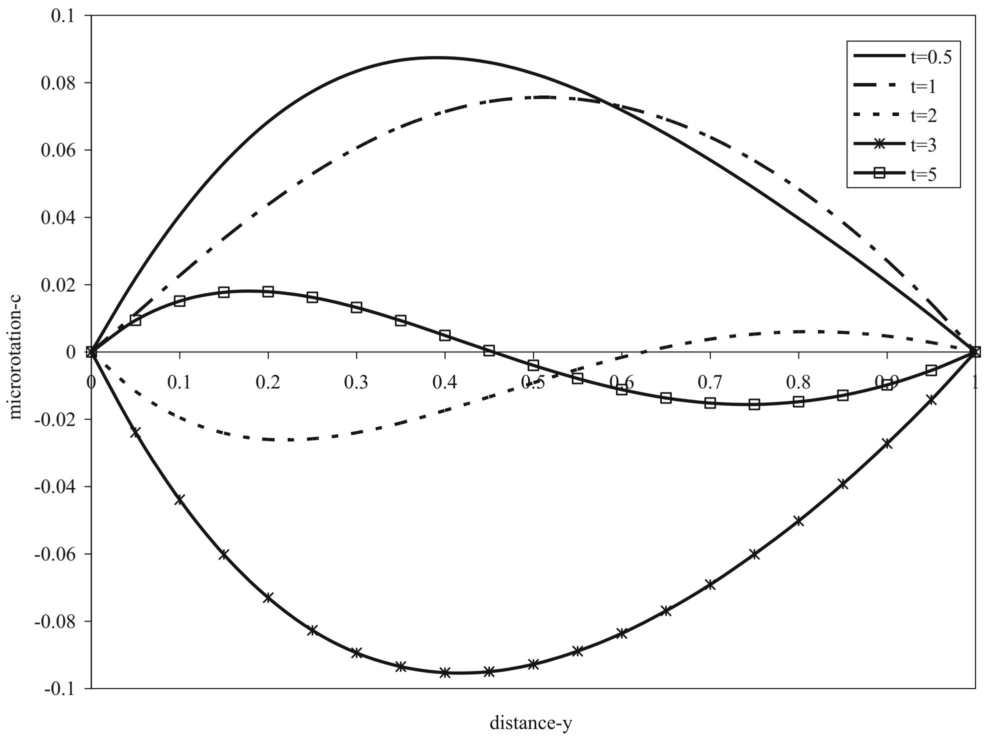

Fig. 8. Variation of microrotation with distance for different times at $m = 0.5$, $n = 1$, $n_2 = 2$, $R = 2$.

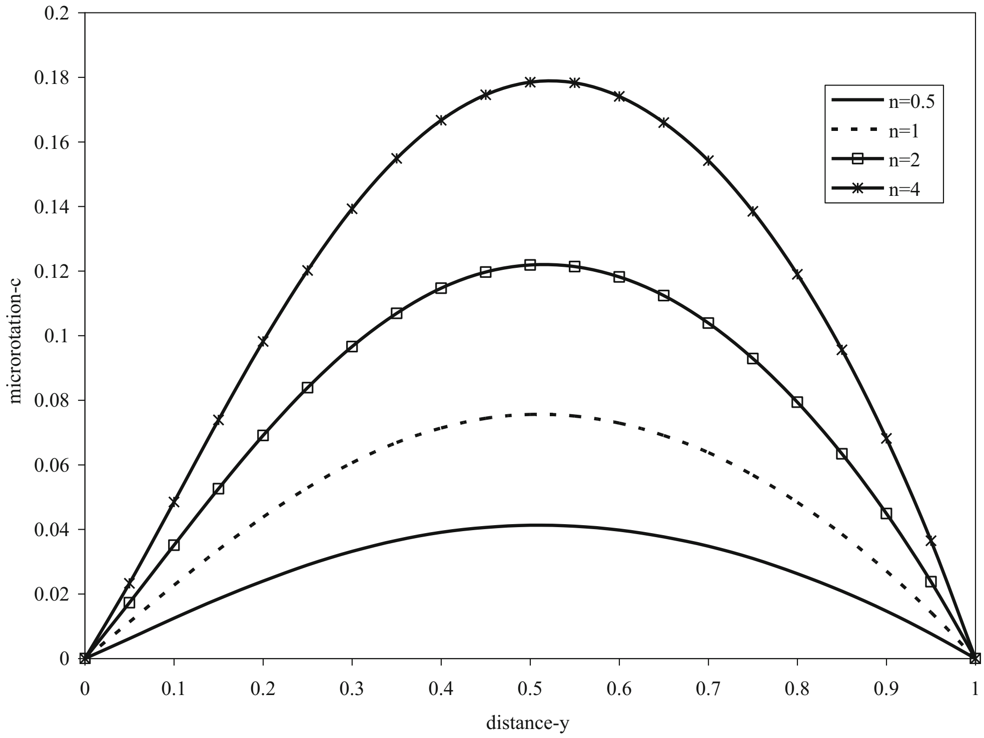

Fig. 9. Variation of microrotation with distance for different values of n at $t = 1$, $m = 0.5$, $n_2 = 2$, $R = 2$.

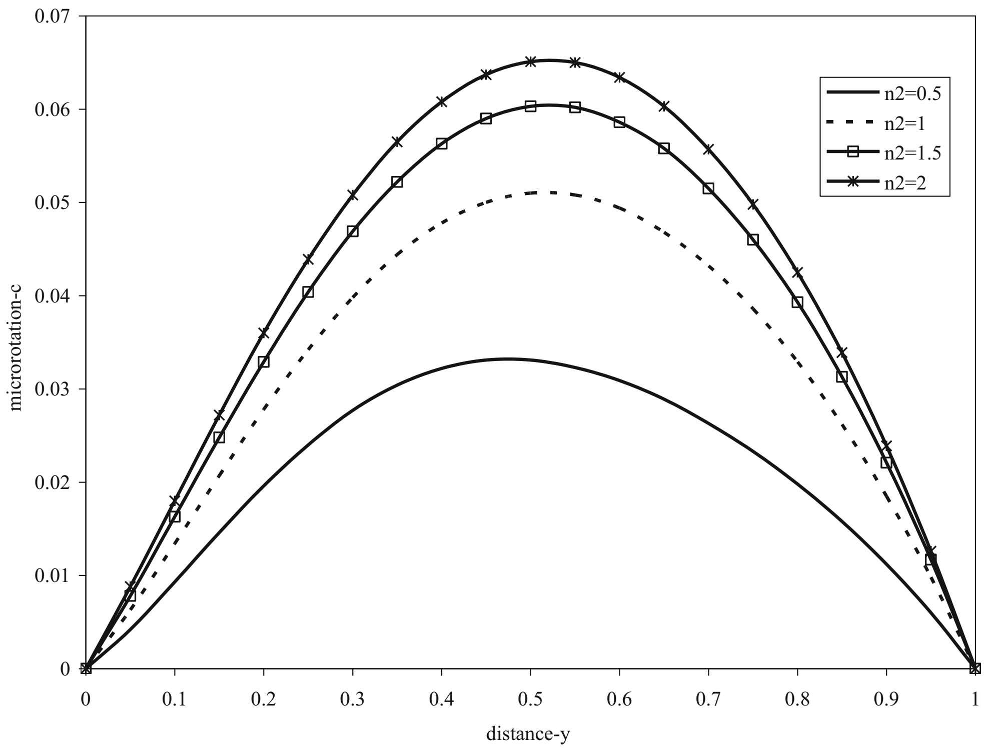

Fig. 10. Variation of microrotation with distance for different values of n_2 at $t = 1$, $m = 0.5$, $n = 1$, $R = 2$.

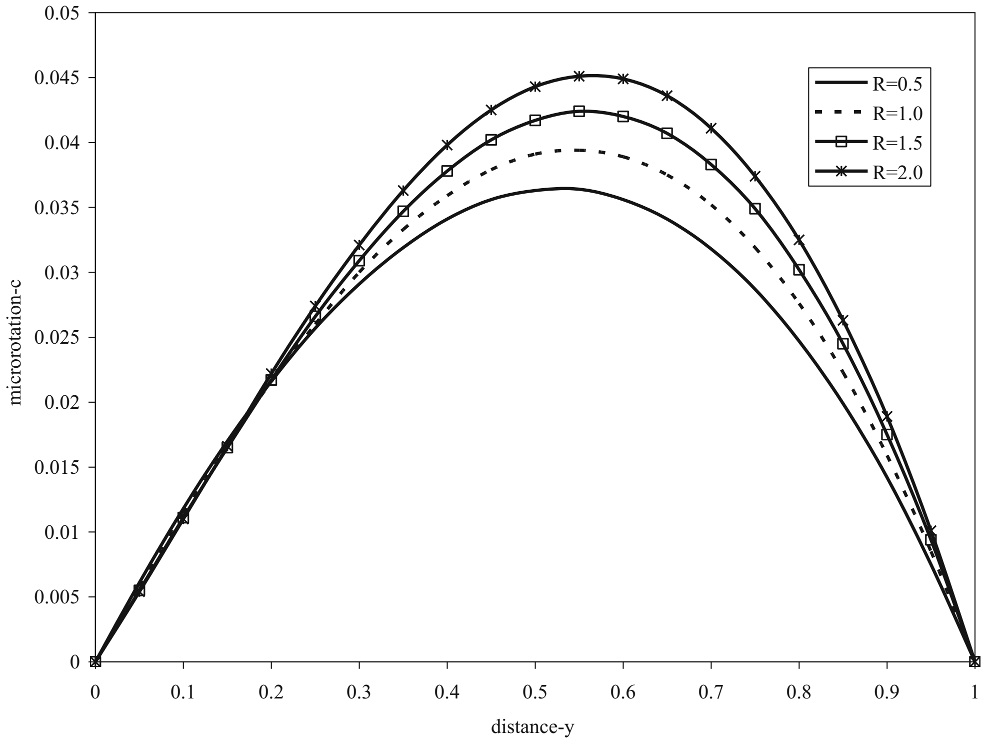

Fig. 11. Variation of microrotation with distance for different values of R at $t = 1$, $m = 0.5$, $n = 1$, $n_2 = 2$.

4 Conclusions

Unsteady flows between parallel plates arising due to the sudden motion of the lower plate are studied for an incompressible micropolar fluid making use of the state space approach. Two different cases namely, flow due to the sudden constant motion of the lower plate and the flow due to the sudden oscillatory motion of the lower plate are considered. The flow variables are obtained in the Laplace transform domain and these are numerically inverted by the numerical inversion procedure due to Honig and Hirdes [22] to estimate them in space-time domain. In both the cases, the velocity and microrotation are plotted for different values of time t, distance from the moving/oscillating plate y and material parameters. The observations made in the work are the following:

- It is noticed that, during the initial period, the gradient of velocity is higher near the moving plate and hence it will get a sharp increase in microrotation near the moving plate as compared to the stationary plate. At a later time, the velocity gradient is evenly distributed hence the microrotation profile between parallel plates is symmetric.
- As expected, as the Reynolds number R increases, there is an increase in the velocity of the fluid.
- Increasing of the Reynolds number R increases the microrotation. Thus, increasing the value of the Reynolds number R increases the linear momentum and hence the gradient of velocity which causes an increase in the angular momentum and hence the observed increase in the microrotation.
- Increasing of microinertia has a decreasing effect on the microrotation.

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