J. Numer. Math., Vol. 16, No. 3, pp. 217–235 (2008)
DOI 10.1515/ JNUM.2008.010

Uniformly convergent numerical method for solving modified Burgers' equations on a non-uniform mesh

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Received January 28, 2008
Received in revised form April 13, 2008

Abstract — In this paper, we consider the one-dimensional modified Burgers' equation in the finite domain. This type of problem arises in the field of sonic boom and explosions theory. At the high Reynolds' number there is a boundary layer in the right side of the domain. From the numerical point of view, one of the difficulties in dealing with this problem is that even smooth initial data can give rise to solution varying regions, i.e., boundary layer regions. To tackle this situation, we propose a numerical method on non-uniform mesh of Shishkin type, which works well at high as well as low Reynolds number. The proposed numerical method comprises of Euler implicit and upwind finite difference scheme. First we discretize in the temporal direction by means of Euler implicit method which yields the set of ordinary differential equations at each time level. The resulting set of differential equations are approximated by upwind scheme on Shishkin mesh. The proposed method has been shown to be parameter uniform and of almost first order accurate in the space and time. An extensive amount of analysis has been carried out in order to prove parameter uniform convergence of the method. some test examples have been solved to verify the theoretical results.

Keywords: Modified Burgers' equation, Euler implicit method, Shishkin mesh, upwind scheme and uniform convergence

1. Introduction

In this paper, we consider one-dimensional modified Burgers' turbulence model,

$$\frac{\partial u}{\partial t} + kt^{-n/2} u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \qquad (x,t) \in \Omega \times (0,T] \tag{1.1a}$$

where $n = 1$ or 2 and $\Omega = [0,1]$

$$u(x,0) = f(x) \tag{1.1b}$$

$$u(0,t) = 0 = u(1,t) \tag{1.1c}$$

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which has many applications, among others, in explosions and sonic boom theory [2,7]. In (1.1), $n = 1$ or 2 corresponds to cylindrical or spherical waves, respectively, k is related to the speed of propagation and ε is the reciprocal of an effective Reynolds number of the problem.

In general, the solutions of this class of problems posses a boundary layer on the right side of the rectangular domain, when the singular perturbation parameter ε is small, i.e., $\varepsilon \ll 1$ (see [12]). Due to the presence of singular perturbation parameter ε, wild oscillations occur in the computed solutions using classical finite difference schemes, unless the mesh discretization used is very fine [6]. To tackle such situations we need to derive a method using a class of special piecewise uniform meshes introduced in [11], which are constructed *a priori* as a function of parameter ε, coefficient of convection term and number of points N used in the spatial mesh.

In the case of plane waves, $n = 0$, equation (1.1) admits a transformation which reduces it to the simple diffusion equation [3,8]. A particularly interesting analytical solution is the so-called *N-waves solution* [10] where the effect of diffusion is confined to two thin boundary layers, corresponding to head and tail shocks in explosion or sonic boom theory. In cases where flows occur in a nonuniform atmosphere, the problem becomes even more complicated because both k and ε are now functions of x and t [14]. Although an inviscid analysis of (1.1) can provide attention insight into the qualitative description of the flow field, such as approximate shock speed and decay rates, to asses the long term effect of diffusion on the whole flow field.

In [13], Sachdev and Seebass study the finite difference solution of (1.1) with $\varepsilon = 10^{-2}$. By choosing the initial conditions which have $\max(|u_x|), |u_{xx}|)$ not much greater than $O(1)$, they show that a uniform mesh size (in both x and t) of $O(10^{-2})$ is adequate for the predictor–corrector finite difference scheme of Douglas and Jones [4]. However in application, much smaller values of ε occur. Hence the use of a uniform mesh would be impractical. In 1978, Chong [1] used the variable mesh finite difference method for the this class of of parabolic differential equations. His method gave more accurate result in comparison to [13] in the boundary layer region specially when ε is very small. But still, his method is not uniformly convergent with respect to the singular perturbation parameter ε.

The derivation of ε*-uniform* convergence based on fitted mesh for ordinary differential equations has been given in [5,11] which also contain numerical experiments for such meshes. We construct a numerical method based on Euler implicit and with upwind finite difference operator on piecewise uniform mesh of Shishkin type, which is first order accurate in time and almost of first order accurate in space. In particular we analyze the accuracy of proposed method on piecewise uniform mesh by reducing it to a system of ordinary differential equations. We prove that the numerical solution generated by the proposed method converges uniformly to the solution of the continuous problem with respect to singular perturbation parameter.

A description of the contents of the paper is as follows. In Section 2, we describe the discretization in temporal direction by means of Euler method and linearize the resulting system of ordinary differential equations at each time level. The error in temporal direction are shown to be of first order and free from parameter ε. In

Section 3, in order to prove the ε-uniform convergence, the sharper bounds on the derivatives are obtained by means of decomposition of the solution into smooth and singular components. In Section 4, the formulation of numerical method comprising a discrete operator on Shishkin mesh is given. In Section 5, it is shown that the discrete operator satisfies the discrete maximum principle. The numerical solution is decomposed in smooth and singular components and the error estimates for the smooth and singular solutions have been obtained separately. The ε-uniform convergence of the numerical solution (generated by the proposed method) to the solution of the continuous problem is shown. Section 6, contains the numerical experiments to corroborate the results predicted by the theory.

Throughout this paper the constant C (sometimes subscripted) will be the positive generic constant, independent of mesh parameters, i.e., Δx and Δt the singular perturbation parameter ε and the norm $\|\cdot\|$ (sometimes subscripted) used is the pointwise maximum norm.

2. Temporal semi-discretization

We discretized the given problem in the time direction by means of Euler Implicit method with uniform step size Δt

$$u_0(x) = f(x) \tag{2.1a}$$

$$\frac{u_{j+1} - u_j}{\Delta t} + k(t_{j+1})^{-n/2} u_{j+1}(u_x)_{j+1} = \varepsilon(u_{xx})_{j+1} \tag{2.1b}$$

$$u_{j+1}(0) = 0, \qquad u_{j+1}(1) = 0. \tag{2.1c}$$

Simplified form of the above equation is

$$u_0(x) = f(x) \tag{2.2a}$$

$$-\varepsilon(u_{xx})_{j+1} + k(t_{j+1})^{-n/2} u_{j+1}(u_x)_{j+1} + \frac{u_{j+1}}{\Delta t} = \frac{u_j}{\Delta t} \tag{2.2b}$$

$$u_{j+1}(0) = 0, \qquad u_{j+1}(1) = 0. \tag{2.2c}$$

This is the system of nonlinear ordinary differential equations. To tackle the nonlinearity we use the following linearization.

2.1. Linearization

We linearize (2.1) by the following way

$$u_0^{(l+1)}(x) = f(x) \tag{2.3a}$$

$$-\varepsilon(u_{j+1}^{(l+1)})_{xx} + k(t_{j+1})^{-n/2} u_{j+1}^{(l)} (u_{j+1}^{(l+1)})_x + \frac{u_{j+1}^{(l+1)}}{\Delta t} = \frac{u_j^{(l+1)}}{\Delta t} \tag{2.3b}$$

$$u_{j+1}^{(l+1)}(0) = 0, \qquad u_{j+1}^{(l+1)}(1) = 0 \tag{2.3c}$$

where l is the iteration index. Rewriting the above equation as

$$u_0^{(l+1)}(x) = f(x) \tag{2.4a}$$

$$-\varepsilon\Delta t(u_{j+1}^{(l+1)})_{xx} + \Delta t k(t_{j+1})^{-n/2} u_{j+1}^{(l)}(u_{j+1}^{(l+1)})_x + u_{j+1}^{(l+1)} = u_j^{(l+1)} \tag{2.4b}$$

$$u_{j+1}^{(l+1)}(0) = 0, \qquad u_{j+1}^{(l+1)}(1) = 0. \tag{2.4c}$$

For simplicity we use the following notation

$$u_{j+1}^{(l+1)} = U_{j+1}, \quad ku_{j+1}^{(l)} = a_{j+1}(x) = a(x,t_{j+1}), \quad u_j^{(l+1)} = g_j(x) = g(x,t_j).$$

Subject to above notations, the equation (2.4) becomes,

$$U_0(x) = f(x) \tag{2.5a}$$

$$-\varepsilon\Delta t(U_{j+1})_{xx} + \Delta t(t_{j+1})^{-n/2} a_{j+1}(x)(U_{j+1})_x + U_{j+1} = g_j(x) \tag{2.5b}$$

$$U_{j+1} = 0, \qquad U_{j+1} = 0. \tag{2.5c}$$

Writing in the operator form,

$$U_0(x) = f(x) \tag{2.6a}$$

$$L_\varepsilon U_{j+1} = g_j(x) \tag{2.6b}$$

$$U_{j+1} = 0, \qquad U_{j+1} = 0 \tag{2.6c}$$

where

$$L_\varepsilon \equiv -\varepsilon\Delta t\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Delta t(t_{j+1})^{-n/2} a_{j+1}(x)\frac{\mathrm{d}}{\mathrm{d}x} + I$$

is a linear operator. Since $t^{-n/2} \geqslant T^{-n/2}, t \leqslant T$ and we assume that $a_{j+1}(x) \geqslant \beta \, \forall x \in \bar{\Omega}$, now therefore $(t_{j+1})^{-n/2} a_{j+1}(x) \geqslant T^{-n/2}\beta$, put $T^{-n/2}\beta = \alpha$ for fixed value of n.

2.2. Error in temporal direction

The local truncation error in the time of the semidiscretization method (2.6) is given by $e_{j+1} \equiv U(t_{j+1}) - \hat{U}_{j+1}$, where $\hat{U}_{j+1}$ is the computed solution of the boundary value problem

$$\hat{U}_0 = f(x) \tag{2.7a}$$

$$-\varepsilon\Delta t(\hat{U}_{j+1})_{xx} + \Delta t(t_{j+1})^{-n/2} a_{j+1}(x)(\hat{U}_{j+1})_x + \hat{U}_{j+1} = g_j(x), \quad 0 < x < 1, \quad t > 0 \tag{2.7b}$$

$$\hat{U}_{j+1}(0) = 0, \quad \hat{U}_{j+1}(1) = 0, \quad t \geqslant 0. \tag{2.7c}$$

Local error estimates of each time step contribute to the global error in the temporal discretization which is defined, at t_j, as $E_j \equiv u(x,t_j) - U_j(x)$.

Lemma 2.1 (local error estimates). *Suppose that*

$$\left|\frac{\partial^i}{\partial t^i}u(x,t)\right| \leqslant C, \quad (x,t)\in\bar{\Omega}\times[0,T], \quad 0\leqslant i\leqslant 2.$$

The local error estimates in the temporal direction is given by

$$\|e_{j+1}\| \leqslant C_1(\Delta t)^2. \tag{2.8}$$

Proof. Linearize the original problem and after simplification, we have

$$U_t = \varepsilon U_{xx} - t^{-n/2}u^{(l)}U_x \tag{2.9}$$

where $U = u^{l+1}$. Since the solution of (2.9) is smooth enough, it holds

$$\begin{aligned} U(t_j) &= U(t_{j+1}) - \Delta t(U(t_{j+1}))_t + \int_{t_{j+1}}^{t_j}(t-s)(U(t_{j+1}))_{tt}\ \mathrm{d}s \\ &= U(t_{j+1}) - \Delta t(\varepsilon U_{xx} - t^{-n/2}u^l U_x)(t_{j+1}) \\ &\quad + \int_{t_{j+1}}^{t_j}(t-s)(U(t_{j+1}))_{tt}\ \mathrm{d}s \quad \text{(using (2.9))} \\ &= L_\varepsilon U(t_{j+1}) + O(\Delta t)^2. \end{aligned} \tag{2.10}$$

Subtracting (2.7) from (2.10), we get

$$L_\varepsilon e_{j+1} \leqslant C(\Delta t)^2.$$

Since L_ε satisfies the maximum principle, therefore

$$\left\|e_{j+1}\right\| \leqslant C(\Delta t)^2.$$ □

Theorem 2.1 (global error estimates). *With the help of Lemma* 2.1*, we have*

$$\|E_j\| \leqslant C\Delta t \quad \forall j\leqslant T/\Delta t. \tag{2.11}$$

Proof. Using the local error estimates up to the nth time step, which is given by Lemma 2.1, we get the following global error estimate at $(j+1)$th time step

$$\begin{aligned} \|E_{j+1}\| &= \|\sum_{l=1}^{j} e_l\|, \quad j\leqslant T/\Delta t \qquad (2.12) \\ &\leqslant \|e_1\| + \|e_2\| + \cdots + \|e_j\| \\ &\leqslant C_1(j\Delta t)\Delta t \quad \text{(using the equation (2.8))} \\ &\leqslant C_1 T\Delta t \quad (\text{since } j\Delta t\leqslant T) \\ &= C\Delta t \quad (\text{with } C = C_1 T) \end{aligned}$$

where C is a positive constant independent of ε and Δt. □

3. Asymptotic behavior of semi-discretized problem

Lemma 3.1 (maximum principle). *Let $\psi_{j+1}(x) \in C^2\bar{\Omega}$ be the mesh function such that $L_\varepsilon \psi_{j+1} \geqslant 0$ and $\psi_{j+1}(0) \geqslant 0$, then $\psi_{j+1} \geqslant 0 \ \forall x$.*

Proof. Proof is given by contradiction. Let there exist a point x^* such that $\psi_{j+1}(x^*) \leqslant 0$. Clearly $x^* \notin \Omega$. $(\psi_{j+1})_x = 0$ and $(\psi_{j+1})_{xx} \geqslant 0$,

$$\begin{aligned} L_\varepsilon(\psi_{j+1})(x^*) &= -\Delta t(\psi_{j+1})_{xx}(x^*) + \Delta t(t_{j+1})^{-n/2} a_{j+1}(x)(\psi_{j+1})_x(x^*) + \psi_{j+1}(x^*) \\ &\leqslant 0 \quad (\text{since } (t_{j+1})^{-n/2} a_{j+1}(x) \geqslant 0) \end{aligned}$$

which is a contradiction. Therefor $\psi_{j+1}(x) \geqslant 0 \ \forall x$. □

In order to define the local error bounds for the finite difference scheme in next section, we need the error bounds to the exact solutions of the previous semidiscrete problems.

Theorem 3.1. *The exact solution $U_{j+1}(x)$ of* (2.6) *satisfies*

$$\left| \frac{\mathrm{d}^i U_{j+1}(x)}{\mathrm{d}x^i} \right| \leqslant C(1 + \varepsilon^{-i} \exp(-\alpha(1-x)/\varepsilon)), \qquad i = 0, 1, \ldots, 4. \tag{3.1}$$

Proof. The maximum principle for L_ε together with the smoothness requirements imposed on $g_j(x)$ and on U_{j+1} gives $|U_{j+1}| \leqslant C$. The proof for the bounds of its derivatives can be derived similarly as in [11]. □

3.1. Decomposition of solution

In order to get the sharper bound on the solution of the equation we decompose the solution in the smooth and singular component.

$$U_{j+1}(x) = V_{j+1}(x) + W_{j+1}(x)$$

where V_{j+1} is smooth component, can be written

$$V_{j+1} = V_0 + \varepsilon V_1 + \varepsilon^2 V_2$$

and V_0, V_1, and V_2, are defined respectively to be the solutions of the problems

$$\Delta t(t_{j+1})^{-n/2} a_{j+1}(x) V_0^{(1)} + V_0 = g_j(x), \qquad V_0(0) = U_{j+1}(0) \tag{3.2}$$

$$\Delta t(t_{j+1})^{-n/2} a_{j+1}(x) V_1^{(1)} + V_1 = -V_0^{(2)}, \qquad V_1(0) = 0 \tag{3.3}$$

$$-\varepsilon\Delta t V_2^{(2)} + \Delta t (t_{j+1})^{-n/2} a_{j+1}(x) V_2^{(1)} + V_2 = -V_1^{(2)} \tag{3.4}$$
$$V_2(0) = 0, \qquad V_2(1) = 0.$$

thus the smooth component V_{j+1} is the solution of

$$L_\varepsilon V_{j+1} = g_j(x), \quad V_{j+1}(0) = V_0(0) + \varepsilon V_1(0), \quad V_{j+1}(1) = U_{j+1}(1) \tag{3.5}$$

where superscripts denote the order of the derivatives, i.e., $V^{(i)} = \mathrm{d}^i V/\mathrm{d}x^i$, $i = 1,2,3$, and consequently the singular component W_{j+1} is the solution of the homogeneous problem

$$L_\varepsilon W_{j+1} = 0, \quad W_{j+1}(0) = 0, \quad W_{j+1}(1) = U_{j+1}(1) - V_{j+1}(1). \tag{3.6}$$

The bounds of V_{j+1}, W_{j+1} and their derivatives are given as follows.

Theorem 3.2. *The smooth component V_{j+1} and its derivatives satisfy the following estimate*

$$|V_{j+1}^{(i)}(x)| \leqslant C\left(1 + \varepsilon^{-(i-2)} \mathrm{e}^{-\alpha(1-x)/\varepsilon}\right), \quad i = 0,1,2,3$$

where C is a positive constant.

Proof. Since V_0 is the solution of the reduced problem which is the first order linear differential equation (3.2), with bounded coefficients, i.e., $(t_{j+1})^{-n/2} a_{j+1}(x)$ and $g_j(x)$, therefore V_0 is the bounded and the bound is independent of the ε.

Using the bounds of $V_0, (t_{j+1})^{-n/2} a_{j+1}(x)$ and $g_j(x)$ in equation (3.2) we get $V_0^{(1)}$ is bounded and independent of ε.

Differentiating the equation (3.2) w.r.t. x we get the second order differential equation, by using the bounds of V_0, $V_0^{(1)}$, $(t_{j+1})^{-n/2} a_{j+1}(x)$ and $g_j(x)$ we get the estimate for $V_0^{(2)}$.

Similarly twice differentiating of the equation (3.2) gives the third order differential equation, by using the bounds of V_0, $V_0^{(1)}$ and $V_0^{(2)}$ we get the estimate for $V_0^{(3)}$.

Thus we have

$$|V_0^{(i)}| \leqslant C_1, \quad i = 0,1,2,3 \tag{3.7}$$

where C_1 is a positive constant independent of the ε.

In the same fashion we get the bounds on V_1 and its derivatives $V_1^{(i)}$ for $i = 1,2,3$. Thus we have

$$|V_1^{(i)}| \leqslant C_2, \quad i = 0,1,2,3 \tag{3.8}$$

where C_2 is the positive constant independent of the ε.

Since V_2 is the solution of the equation (3.4), which is similar to the equation (2.6) therefore the estimates for the bounds on V_2 and its derivatives are given by Theorem 3.1,

$$|V_2^{(i)}| \leqslant C_3 \varepsilon^{-i} \mathrm{e}^{-\alpha(1-x)/\varepsilon}, \quad i = 0, 1, 2, 3 \tag{3.9}$$

where C_3 is positive constant independent of ε.

$$|V_{j+1}^{(i)}| = V_0^{(i)} + V_1^{(i)} + V_2^{(i)}, \quad i = 0, 1, 2, 3 \tag{3.10}$$

with the help of the equations (3.7), (3.8) and (3.9) the above equation can be written as

$$|V_{j+1}^{(i)}| \leqslant C(1 + \varepsilon^{-(i-2)} \mathrm{e}^{-\alpha(1-x)/\varepsilon}), \quad i = 0, 1, 2, 3. \tag{3.11}$$

This completes the proof of the theorem. □

Theorem 3.3. *The singular component $W_{j+1}(x)$ and its derivatives satisfy the following error estimate*

$$|W_{j+1}(x)| \leqslant C \mathrm{e}^{-\alpha(1-x)/\varepsilon} \tag{3.12}$$

$$|W_{j+1}^{(i)}(x)| \leqslant C \varepsilon^{-(i)}, \quad i = 1, 2, 3. \tag{3.13}$$

Proof. We consider the barrier functions defined as

$$\psi^{\pm}(x) = C \mathrm{e}^{-\alpha(1-x)/\varepsilon} \pm W_{j+1}(x) \tag{3.14}$$

now we have

$$\psi^{\pm}(0) = C \mathrm{e}^{-\alpha/\varepsilon} \pm W_{j+1}(0) \tag{3.15}$$

$$= C \mathrm{e}^{-\alpha/\varepsilon} \quad (\text{since } W_{j+1}(0) = 0) \tag{3.16}$$

$$\geqslant 0 \tag{3.17}$$

$$\psi^{\pm}(1) = C \pm W_{j+1}(1). \tag{3.18}$$

We can choose the value C such that

$$\psi^{\pm}(1) = C \pm W_{j+1}(1) \geqslant 0 \tag{3.19}$$

and

$$\begin{aligned} L_\varepsilon \psi^{\pm}(x) &= -\varepsilon \Delta t \psi_{xx}^{\pm}(x) + \Delta t (t_{j+1})^{-n/2} a_{j+1}(x) \psi_x^{\pm}(x) + \psi^{\pm}(x) \qquad (3.20) \\ &= -\frac{\Delta t \alpha^2}{\varepsilon} \mathrm{e}^{-\alpha(1-x)/\varepsilon} + \Delta t (t_{j+1})^{-n/2} a_{j+1}(x) \frac{\alpha}{\varepsilon} \mathrm{e}^{-\alpha(1-x)/\varepsilon} + \mathrm{e}^{-\alpha(1-x)/\varepsilon} \\ &= \frac{\alpha \Delta t}{\varepsilon} \big[-\alpha + (t_{j+1})^{-n/2} a_{j+1}(x) \big] \mathrm{e}^{-\alpha(1-x)/\varepsilon} + \mathrm{e}^{-\alpha(1-x)/\varepsilon} \end{aligned}$$

since α is the minimum of $(t_{j+1})^{-n/2}a_{j+1}(x)$. Therefore

$$-\alpha + (t_{j+1})^{-n/2}a_{j+1}(x) \geqslant 0.$$

Thus we have

$$L_\varepsilon \psi^{\pm}(x) \geqslant 0.$$

Since L_ε satisfies the maximum principle and we have shown that $L_\varepsilon \psi^{\pm}(x) \geqslant 0$, therefore

$$\psi^{\pm}(x) = C\mathrm{e}^{-\alpha(1-x)/\varepsilon} \pm W_{j+1}(x) \geqslant 0 \qquad \forall x \in \bar{\Omega}$$

or

$$|W_{j+1}(x)| \leqslant C\mathrm{e}^{-\alpha(1-x)/\varepsilon} \qquad \forall x \in \bar{\Omega}.$$

For $i = 1, 2, 3$ the detailed proof can be seen in [9]. □

4. Spatial discretization

Shishkin mesh. Shishkin meshes are piecewise-uniform meshes which condense approximately in the boundary layer regions as $\varepsilon \to 0$. This is accomplished by the use of transition parameter τ, which depends naturally on ε, and crucially on N.

Thus for a given N and ε, the interval [0,1] is divided into parts, $[0, 1-\tau]$, $[1-\tau, 1]$ where the transition point τ is given by

$$\tau \equiv \min\left\{\frac{1}{2}, m\varepsilon \log N\right\}$$

where m is a constant which we choose such that $m \geqslant 1/\alpha$. It is clear that when $\tau = 1/2$ the mesh is uniform otherwise the mesh condenses near the right boundary. The value of the constant C depends on which scheme is used.

Define the fitted piecewise-uniform mesh (Shishkin mesh) that discretizes the interval $[0, 1]$ with N piecewise uniform subintervals as

$$h_i = x_i - x_{i-1} = \begin{cases} H = 2(1-\tau)/N, & 0 \leqslant i \leqslant N/2 \\ h = 2\tau/N, & N/2 < i \leqslant N \end{cases}$$

and the piecewise-uniform mesh $\bar{\Omega}^N$ with the spatial nodal values x_i, $i = 0, 1, \ldots, N$, is given as

$$\bar{\Omega}^N = \left\{x_i : x_i = \begin{cases} 2(1-\tau)\,i/N, & 0 \leqslant i \leqslant N/2 \\ (1-\tau) + 2\tau(i - N/2)/N, & N/2 < i \leqslant N \end{cases}\right\}.$$

Difference scheme. The approximation of the semi-discrete equation (2.5) in the spatial direction by upwind scheme on non-uniform mesh of Shishkin type, i.e., $\bar{\Omega}^N$ is given as,

$$U_0^N = f(x) \tag{4.1a}$$

$$\begin{aligned} -\varepsilon\Delta t\delta^2(U_{j+1}^N)(x_i) \ + \ \Delta t(t_{j+1})^{-n/2}a_{j+1}(x_i)D^-(U_{j+1}^N)(x_i) \\ + \ U_{j+1}^N(x_i) = g_j(x_i) \qquad \forall x_i \in \Omega^N \end{aligned} \tag{4.1b}$$

$$U_{j+1}^N(0) = 0, \qquad U_{j+1}^N(1) = 0 \tag{4.1c}$$

where $U_{j+1}^N(x_i)$ is the the approximate solution of the $U_{j+1}(x)$ at the point x_i and at the $(j+1)$th time level and

$$D^-U_{j+1}^N(x_i) = \frac{U_{j+1}^N(x_i) - U_{j+1}^N(x_{i-1})}{h_i}, \qquad D^+U_{j+1}^N(x_i) = \frac{U_{j+1}^N(x_{i+1}) - U_{j+1}^N(x_i)}{h_{i+1}}$$

$$\delta^2 = \frac{D^+ - D^-}{\bar{h}_i}, \qquad \bar{h}_i = \frac{h_{i+1}+h_i}{2}.$$

Writing (4.1) in the operator form,

$$U_0^N = f(x) \tag{4.2a}$$

$$L_\varepsilon^N U_{j+1}^N = g_j(x_i) \tag{4.2b}$$

$$U_{j+1}^N(0) = 0, \quad U_{j+1}^N(1) = 0 \tag{4.2c}$$

where $L_\varepsilon^N = -\varepsilon\Delta t\delta^2 + \Delta t(t_{j+1})^{-n/2}a_{j+1}(x_i)D^- + I$.

4.1. Decomposition of numerical solution

In order to get the sharper bound on the numerical solution of the fully discrete problem, we decompose the solution in the smooth and singular component.

$$U_{j+1}^N(x_i) = V_{j+1}^N(x_i) + W_{j+1}^N(x_i)$$

where V_{j+1}^N is smooth component and W_{j+1}^N is the singular component of the solution U_{j+1}^N.

Smooth component V_{j+1}^N satisfies the following non-homogeneous equation

$$\begin{aligned} L_\varepsilon^N V_{j+1}^N(x_i) &= g_j(x_i) \\ V_{j+1}^N(0) &= V_{j+1}(0) \\ V_{j+1}^N(1) &= V_{j+1}(1) \end{aligned} \tag{4.3}$$

and singular component W_{j+1}^N satisfies the following homogeneous problem

$$\begin{aligned} L_\varepsilon^N W_{j+1}^N(x_i) &= 0 \\ W_{j+1}^N(0) &= W_{j+1}(0) \\ W_{j+1}^N(1) &= W_{j+1}(1). \end{aligned} \tag{4.4}$$

The error can be written as

$$U_{j+1}^N - U_{j+1} = (V_{j+1}^N - V_{j+1}) + (W_{j+1}^N - W_{j+1}).$$

The error in the smooth and singular component is estimated separately.

5. Stability and convergence analysis

Lemma 5.1 (discrete maximum principle). *Let $\psi_{j+1}^N(x_i)$ is any discrete mesh function on $\bar{\Omega}^N$ such that $\psi_{j+1}^N(0) \geqslant 0$, $\psi_{j+1}^N(N) \geqslant 0$ and $L_\varepsilon^N \psi_{j+1}^N(x_i) \geqslant 0\, \forall x_i \in \Omega^N$, then $\psi_{j+1}^N(x_i) \geqslant 0\, \forall x_i \in \bar{\Omega}^N$.*

Proof. Let there exist a point x_p such that $\psi_{j+1}(x_p) \leqslant 0$ and $\psi_{j+1}(x_p) = \min_{\bar{\Omega}^N} \psi_{j+1}(x_i)$. It is clear that $p \notin \{1, N\}$. Since $\psi_{j+1}(x_{p+1}) - \psi(x_p) \geqslant 0$ and $\psi_{j+1}(x_p) - \psi(x_{p-1}) \leqslant 0$, we have

$$\begin{aligned} D^-\psi_{j+1}^N(x_p) &= \frac{\psi_{j+1}^N(x_p) - \psi_{j+1}^N(x_{p-1})}{h_p} \leqslant 0 \\ D^+\psi_{j+1}^N(x_p) &= \frac{\psi_{j+1}^N(x_{p+1}) - \psi_{j+1}^N(x_p)}{h_p} \geqslant 0 \\ \delta^2\psi_{j+1}^N(x_p) &= \frac{D^+\psi_{j+1}^N(x_p) - D^-\psi_{j+1}^N(x_p)}{\bar{h}_p} \\ &= \frac{1}{\bar{h}_p}\left(\frac{\psi_{j+1}^N(x_{p+1}) - \psi_{j+1}^N(x_p)}{h_{p+1}} - \frac{\psi_{j+1}^N(x_p) - \psi_{j+1}^N(x_{p-1})}{h_p}\right) \geqslant 0. \end{aligned}$$

Now

$$\begin{aligned} L_\varepsilon^N \psi_{j+1}^N(x_p) &= -\varepsilon\Delta t\,\delta^2\psi_{j+1}^N(x_p) + \Delta t(t_{j+1})^{-n/2} a_{j+1}(x_i) D^-\psi_{j+1}^N(x_p) + \psi_{j+1}^N(x_p) \\ &\leqslant 0 \quad \text{(using above inequalities)} \end{aligned}$$

which is as contradiction. So $\psi_{j+1}^N(x_p) \geqslant 0$, since p is an arbitrarily chosen point, therefore we have $\psi_{j+1}^N(x_i) \geqslant 0\ \ \forall x_i \in \bar{\Omega}^N$. □

Lemma 5.2. *Let $Z(x_i)$ is the mesh function such that $Z(x_0) = Z(x_N) = 0$. Then*

$$|Z(x_i)| \leqslant \frac{1}{(\Delta t\alpha)} \max_{0\leqslant j\leqslant N} |L_\varepsilon^N Z(x_j)|. \tag{5.1}$$

Proof. Let

$$M \leqslant \frac{1}{(\Delta t\alpha)} \max_{0\leqslant i\leqslant N} |L_\varepsilon^N Z(x_i)|.$$

Construct the barrier functions $\psi_{j+1}^{\pm}(x_i)$ as

$$\psi_{j+1}^{\pm}(x_i) = Mx_i \pm Z(x_i)$$

$$\psi_{j+1}^{\pm}(x_0) = 0, \qquad \psi_{j+1}^{\pm}(x_N) \geqslant 0$$

$$\begin{aligned} L_\varepsilon^N \psi_{j+1}^{\pm}(x_i) &= \left(\Delta t(t_{j+1}^{-n/2})a_{j+1}(x_i) + x_i\right) M \pm L_\varepsilon^N Z(x_i) \\ &\geqslant (\Delta t\alpha)M \pm L_\varepsilon^N Z(x_i) \geqslant 0. \end{aligned}$$

The Discrete maximum principle implies that $\psi_{j+1}^{\pm}(x_i) \geqslant 0$ for $0 \leqslant i \leqslant N$. □

Theorem 5.1 (error in smooth component). *The error in the smooth component satisfies the following estimate*

$$\left|V_{j+1}^N(x_i) - V_{j+1}(x_i)\right| \leqslant CN^{-1}, \quad x_i \in \bar{\Omega}^N \tag{5.2}$$

where C is a constant independent of ε and mesh parameters.

Proof. The truncation error in the smooth solution is given by

$$\begin{aligned} \left|L_\varepsilon^N(V_{j+1}^N - V_{j+1})(x_i)\right| &= \left|g_j(x_i) - L_\varepsilon^N V_{j+1}\right| \\ &= \left|L_\varepsilon V_{j+1}(x_i) - L_\varepsilon^N V_{j+1}(x_i)\right| \\ &= \left|(L_\varepsilon - L_\varepsilon^N)V_{j+1}(x_i)\right| \\ &= \Bigg|-\varepsilon\Delta t\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \delta^2\right)V_{j+1}(x_i) \\ &\quad + (\Delta t)(t_{j+1})^{-n/2}a_{j+1}(x_i)\left(\frac{\mathrm{d}}{\mathrm{d}x} - D^-\right)V_{j+1}(x_i)\Bigg|. \end{aligned} \tag{5.3}$$

Let $x_i \in \Omega^N$. Then for any $\psi_{j+1} \in C^2(\bar{\Omega})$,

$$\left|\left(D^- - \frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{j+1}(x_i)\right| \leqslant \frac{(x_{i+1} - x_i)}{2}\left\|\psi_{j+1}^{(2)}\right\|$$

and for any $\psi_{j+1}(x) \in \bar{\Omega}$,

$$\left| \left(\delta^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \psi_{j+1}(x_i) \right| \leqslant \frac{(x_{i+1} - x_{i-1})}{3} \| \psi_{j+1}^{(3)} \|.$$

For the proof of these results one can seen the Lemma 4.1 (see [11]). Using these results in (5.3) followed by a simplification yields

$$\begin{aligned} \left| L_\varepsilon^N (V_{j+1} - V_{j+1})(x_i) \right| &\leqslant \Delta t C\,(x_{i+1} - x_{i-1}) \left(\varepsilon \| V_{j+1}^{(3)} \| + \| V_{j+1}^{(3)} \| \right) \\ &\leqslant \Delta t C (x_{i+1} - x_{i-1})(\varepsilon + \mathrm{e}^{-\alpha(1-x_i)/\varepsilon}) + (1 + \mathrm{e}^{-\alpha(1-x_i)/\varepsilon})) \\ &\quad \text{(by using Theorem 3.1)} \\ &\leqslant \Delta t\, C (x_{i+1} - x_{i-1}) \quad \text{(since } \mathrm{e}^{-\alpha(1-x_i)/\varepsilon} \leqslant 1) \\ &= \Delta t\, C N^{-1} \quad \text{(since } x_{i+1} - x_{i-1} \leqslant 2N^{-1}). \end{aligned}$$

An application of Lemma 5.2 to the mesh function $(V_{j+1}^N - V_{j+1})(x_i)$ yields the estimate

$$\left| (V_{j+1}^N - V_{j+1})(x_i) \right| \leqslant CN^{-1}, \quad x_i \in \bar{\Omega}^N. \tag{5.4}$$

This completes the proof of the theorem. □

Theorem 5.2 (error in singular component). *The error in singular component satisfies the following estimate*

$$\left| W_{j+1}^N(x_i) - W_{j+1}(x_i) \right| \leqslant CN^{-1} (\ln N)^2, \quad x_i \in \bar{\Omega}^N \tag{5.5}$$

where C is a constant independent of ε *and mesh parameters.*

Proof. To estimate the singular component of the local truncation error $L_\varepsilon^N \times (W_{j+1}^N - W_{j+1})$, the argument depends on whether $\tau = 1/2$ or $\tau = (\varepsilon \ln N)/\alpha$.

1. Consider the case when $\tau = 1/2$, i.e., $\varepsilon \ln N \geqslant 1/2$. The classical argument is used above in the Theorem 5.1, leads to the estimate

$$\left| L_\varepsilon^N (W_{j+1}^N - W_{j+1})(x_i) \right| \leqslant \Delta t C (x_{i+1} - x_{i-1}) \left(\varepsilon \| W_{j+1}^{(3)} \| + \| W_{j+1}^{(2)} \| \right).$$

Since $x_{i+1} - x_{i-1} = 2N^{-1}$ and estimates for $W_{j+1}^{(3)}$ and $W_{j+1}^{(2)}$ from the Theorem 3.3 lead to

$$\left| L_\varepsilon^N (W_{j+1}^N - W_{j+1}) \right| \leqslant \Delta t C \varepsilon^{-2} N^{-1}$$

but, in this case $\varepsilon^{-1} \leqslant 2 \ln N/\alpha$ and so

$$\left| L_\varepsilon^N (W_{j+1}^N - W_{j+1})(x_i) \right| \leqslant \Delta t C N^{-1} (\ln N)^2$$

an application of Lemma 5.2, leads to estimates

$$\left|(W_{j+1}^N - W_{j+1})(x_i)\right| \leqslant CN^{-1}(\ln N)^2. \tag{5.6}$$

2. In this case mesh is non-uniform or piecewise uniform with the mesh spacing $2(1-\tau)/N$ in the subinterval $[0, 1-\tau]$ and τ/N in the subinterval $[1-\tau, 1]$.

Consider the interval $[0, 1-\tau]$ with no boundary layer:

$$\left|(W_{j+1}^N - W_{j+1})(x_i)\right| \leqslant \left|W_{j+1}^N(x_i)\right| + \left|W_{j+1}(x_i)\right|.$$

From the Theorem 3.2, we have

$$\begin{aligned}\left|W_{j+1}(x_i)\right| &\leqslant C\mathrm{e}^{-\alpha(1-x_i)/\varepsilon}\\ &\leqslant C\mathrm{e}^{-\alpha(1-1+\tau)/\varepsilon} \quad \text{(since } \mathrm{e}^{-\alpha(1-x)/\varepsilon} \text{ is an increasing function)}\\ &\leqslant C\mathrm{e}^{-\alpha(\tau)/\varepsilon}\\ &= CN^{-1} \quad (\text{since } \tau = \varepsilon \ln N/\alpha).\end{aligned}$$

To obtain the similar bound on W_{j+1}^N we have to construct an auxiliary function $\tilde{W}_{j+1}^N$, which is the solution of the difference equation (4.4) except the coefficient $t_{j+1}^{-n/2} a_{j+1}(x)$ is replaced by α. Then by the Lemma 7.5 of [11], we have

$$W_{j+1}^N(x_i) \leqslant \tilde{W}_{j+1}^N(x_i), \quad 0 \leqslant i \leqslant N.$$

With the help of Lemma 7.3 in [11], we have

$$|W_{j+1}^N(x_i)| \leqslant CN^{-1}, \quad 0 \leqslant i \leqslant N/2.$$

The above estimates for $W_{j+1}^N(x_i)$ and $W_{j+1}(x_i)$ for $0 \leqslant i \leqslant N/2$ show that in the interval $[0, 1-\tau]$

$$\left|W_{j+1}^N(x_i) - W_{j+1}(x_i)\right| \leqslant CN^{-1}.$$

Consider the interval $[1-\tau, 1]$ and applying the classical argument as before to the following error estimate of the local truncation error for $N/2+1 \leqslant i \leqslant N-1$

$$\begin{aligned}\left|L_\varepsilon^N(W_{j+1}^N - W_{j+1})(x_i)\right| &\leqslant \Delta t\, C\varepsilon^{-2}\,|x_{i+1} - x_{i-1}|\\ &= 2\Delta t\, C\varepsilon^{-2}\tau N^{-1}\\ &\leqslant 2TC\varepsilon^{-2}\tau N^{-1} \quad (\text{since } \Delta t \leqslant T)\\ &= C_1\varepsilon^{-2}\tau N^{-1} \quad (\text{with } C_1 = TC)\end{aligned}$$

and

$$\left|W_{j+1}^N(x_N) - W_{j+1}(x_N)\right| = 0$$

and

$$\left|W_{j+1}^N(x_{N/2}) - W_{j+1}(x_{N/2})\right| \leqslant \left|W_{j+1}^N(x_{N/2})\right| + \left|W_{j+1}(x_{N/2})\right| \leqslant CN^{-1}$$

from the results obtained in the subinterval $[0, 1-\tau]$.

Introducing the barrier function

$$\Phi_{j+1}(x_i) = (x_i - (1-\tau))C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1}$$

again construct the barrier functions

$$\psi^{\pm}_{j+1}(x_i) = \Phi_{j+1}(x_i) \pm \left(W^N_{j+1} - W_{j+1}\right)(x_i)$$

satisfy the inequalities

$$\psi^{\pm}_{j+1}(x_{N/2}) \geqslant 0, \qquad \psi^{\pm}_{j+1}(x_{N/2}) = 0$$

and

$$L^N_\varepsilon \psi^{\pm}_{j+1}(x_i) \geqslant 0, \quad N/2+1 \leqslant i \leqslant N-1.$$

The discrete maximum principle on the interval $[1-\tau, 1]$ then gives

$$\psi^{\pm}_{j+1}(x_i) \geqslant 0, \quad N/2 \leqslant i \leqslant N$$

and it follows

$$\left|(W^N_{j+1} - W_{j+1})(x_i)\right| \leqslant \Phi_{j+1}(x_i) \leqslant C_1\varepsilon^{-2}\tau^2N^{-1} + C_2N^{-1}.$$

Since $\tau = \varepsilon \ln N/\alpha$, therefore

$$\begin{aligned}\left|(W^N_{j+1} - W_{j+1})(x_i)\right| &\leqslant C_1N^{-1}(\ln N)^2 + C_2N^{-1}\\ &\leqslant CN^{-1}(\ln N)^2 \quad (\text{with } C = \max\{C_1, C_2\}).\end{aligned} \tag{5.7}$$

Combining the separate estimates in two subintervals, we have

$$\left|(W^N_{j+1} - W_{j+1})(x_i)\right| \leqslant CN^{-1}(\ln N)^2, \quad x_i \in \bar{\Omega}^N.$$

□

Theorem 5.3 (error in semi-discrete solution). *The error in the semi-discrete solution $U_{j+1}(x)$ satisfies the following estimate*

$$\left|(U^N_{j+1} - U_{j+1})(x_i)\right| \leqslant CN^{-1}(\ln N)^2, \quad x_i \in \bar{\Omega}^N \tag{5.8}$$

where C is a positive constant independent of ε and mesh parameters.

Proof. Since we have

$$\left|(U^N_{j+1} - U_{j+1})(x_i)\right| \leqslant \left|(W^N_{j+1} - W_{j+1})(x_i)\right| + \left|(V^N_{j+1} - V_{j+1})(x_i)\right|, \quad x_i \in \bar{\Omega}^N.$$

Combining the inequalities (5.4) and (5.7) we have

$$\left|(U^N_{j+1} - U_{j+1})(x_i)\right| \leqslant CN^{-1}(\ln N)^2, \quad x_i \in \bar{\Omega}^N. \qquad \square$$

Theorem 5.4 (error in fully-discrete solution). *Let $u(x_i, t_{j+1})$ be the solution of the problem* (1.1) *at the point (x_i, t_{j+1}), $U_{j+1}(x_i)$ be the solution of the differential equation* (2.6) *and $U^N_{j+1}(x_i)$ be the solution of totally discrete problem* (4.2). *Then error satisfies the following estimate*

$$\max_{0\leqslant\varepsilon\leqslant1} \left\|U^N_{j+1}(x_i) - u(x_i, t_{j+1})\right\| \leqslant C(\Delta t + N^{-1}(\ln N)^2) \tag{5.9}$$

where C is a positive constant independent of ε and mesh parameters.

Proof. The proof is directly follows from Theorem 2.1 and Theorem 5.3,

$$\begin{aligned} \|U^N_{j+1}(x_i) - u(x_i,t_{j+1})\| &\leqslant \|U^N_{j+1}(x_i) - U_{j+1}(x_i)\| + \|U_{j+1}(x_i) - u(x_i,t_{j+1})\| \\ &\leqslant C_1\Delta t + C_2N^{-1}(\log N)^2 \\ &\qquad \text{(by using Theorem 5.3 and Theorem 2.1)} \\ &\leqslant C(\Delta t + N^{-1}(\log N)^2) \quad (\text{with } C = \max\{C_1, C_2\}). \qquad \square \end{aligned}$$

5.1. Numerical Results

In this section we present the numerical results which validate the theoretical result. Nevertheless it is seen that the numerical behavior of the proposed method using piecewise uniform of Shishkin type mesh is ε-uniform. The problem is solved using proposed method comprising Euler implicit and upwind difference operator on piecewise uniform mesh of Shishkin type with N points. The piecewise uniform Shishkin mesh used in these computations are of the form described in Section 4 and so condensed on the right side boundary $x = 1$.

We will show computationally that the numerical solutions given by proposed method converge uniformly with respect to ε.

Modified Burgers' equation. In the case of modified Burgers' equation

$$u_t + t^{-n/2}uu_x = \varepsilon u_{xx}.$$

With initial condition

$$u(x,0) = \frac{x}{1 + \exp(x^2/4\varepsilon - 1)}$$

and boundary conditions

$$u(0,t) = 0 = u(1,t)$$

Table 1.
Maximum pointwise errors E_ε^N for modified Burgers' equation at $n = 1$ for different values of ε and N.

$\varepsilon \setminus N$	16	32	64	128	256
2^{-8}	4.003148E-2	2.550326E-2	1.662987E-2	1.027287E-2	1.009326E-2
2^{-10}	4.626916E-2	2.968019E-2	1.919389E-2	1.179892E-2	6.967773E-3
2^{-12}	4.764934E-2	3.059739E-2	1.969282E-2	1.207915E-2	7.130987E-3
2^{-14}	4.799736E-2	3.081504E-2	1.981298E-2	1.214585E-2	7.168962E-3
2^{-16}	4.807754E-2	3.086872E-2	1.984286E-2	1.216244E-2	7.178402E-3
2^{-18}	4.809760E-2	3.088209E-2	1.985031E-2	1.216659E-2	7.180761E-3
2^{-20}	4.810261E-2	3.089028E-2	1.985608E-2	1.217152E-2	7.185273E-3
2^{-22}	4.810386E-2	3.089112E-2	1.985655E-2	1.217178E-2	7.185420E-3
2^{-24}	4.810418E-2	3.089133E-2	1.985666E-2	1.217184E-2	7.185457E-3
2^{-26}	4.810426E-2	3.089138E-2	1.985669E-2	1.217186E-2	7.185466E-3
E^N	4.810426E-2	3.089138E-2	1.985669E-2	1.217186E-2	7.185466E-3

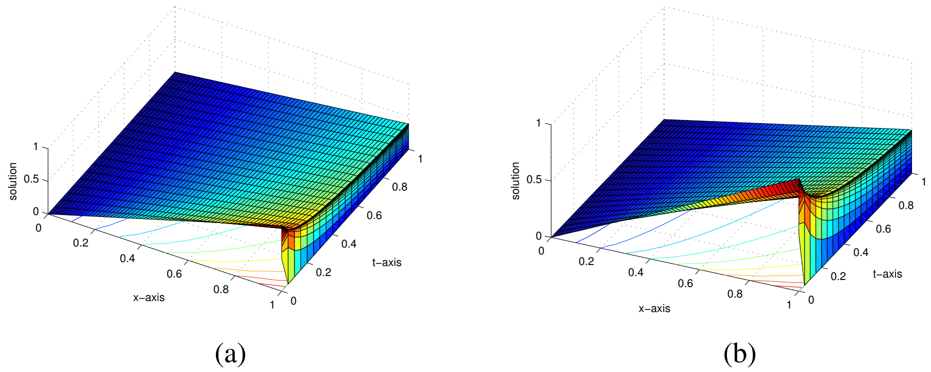

Figure 1. Numerical solutions for modified Burgers' equation $n = 1$, i.e., cylindrical wave at $N = 128$, $\Delta t = 0.1$ (a) for $\varepsilon = 2^{-14}$ and (b) for $\varepsilon = 2^{-20}$.

no exact solution is available for $n = 1,2$, therefore, we estimate the maximum pointwise errors E_ε^N by

$$E_\varepsilon^N = \max_{\Omega^N} |U^N(x_i,t_j) - U^{2N}(x_i,t_j)|$$

where $U^{2N}(x_i,t_j)$ is the computed solution corresponding with $2N$ points, and

$$E^N = \max_{\varepsilon} E_\varepsilon^N.$$

The maximum pointwise computed errors at $T = 2$ and $n = 1$ for this example are given in Table 1 by using proposed method on fitted piecewise uniform mesh for different values of ε and N.

Table 2.
Maximum pointwise errors E_ε^N for modified Burgers' equation at $n = 2$ for different values of ε and N.

$\varepsilon \setminus N$	16	32	64	128	256
2^{-8}	2.685989E-2	1.702507E-2	1.049890E-2	6.288740E-3	2.687786E-3
2^{-10}	3.043733E-2	1.920504E-2	1.190741E-2	7.076528E-3	3.036208E-3
2^{-12}	3.162580E-2	1.990117E-2	1.228574E-2	7.282884E-3	4.197863E-3
2^{-14}	3.192748E-2	2.012732E-2	1.238445E-2	7.346255E-3	4.229487E-3
2^{-16}	3.200277E-2	2.017911E-2	1.240944E-2	7.362305E-3	4.237483E-3
2^{-18}	3.202158E-2	2.019209E-2	1.241572E-2	7.366331E-3	4.239489E-3
2^{-20}	3.202645E-2	2.019534E-2	1.241729E-2	7.367338E-3	4.239990E-3
2^{-22}	3.202762E-2	2.019615E-2	1.241768E-2	7.367590E-3	4.240116E-3
2^{-24}	3.202791E-2	2.019636E-2	1.241778E-2	7.367653E-3	4.240147E-3
2^{-26}	3.202798E-2	2.019641E-2	1.241780E-2	7.367669E-3	4.240155E-3
E^N	3.202798E-2	2.019641E-2	1.241780E-2	7.367669E-3	4.240155E-3

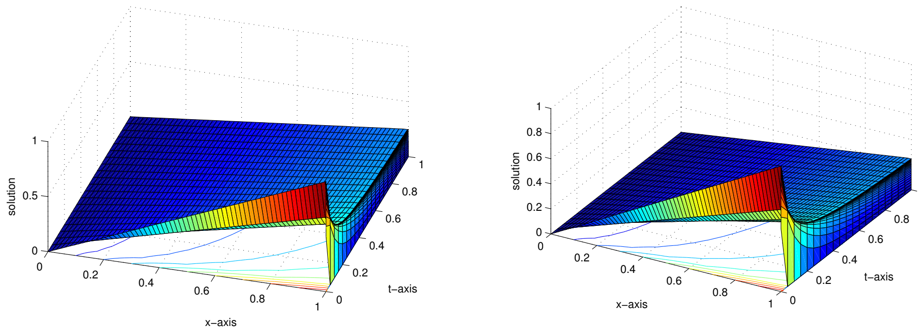

Figure 2. Numerical solutions for modified Burgers' equation $n = 2$, i.e., spherical wave at $N = 128$, $\Delta t = 0.1$ (a) for $\varepsilon = 2^{-14}$ and (b) for $\varepsilon = 2^{-20}$.

Table 2 gives the maximum pointwise computed errors at $T = 2$ and $n = 2$ for this example by using proposed method on Shishkin mesh for different values of ε and N.

Conclusions

In this paper we propose a numerical scheme for solving modified Burgers' equations (cylindrical and spherical waves) at high Reynolds number. The solutions of this equation changes rapidly at high Reynolds number due to the occurrence of the boundary on the right side of the domain. To capture the boundary layer we used a non-uniform mesh of Shishkin type. The proposed method comprises Euler discretization in time and upwind discretization on Shishkin mesh in space. The method has been shown to be almost-first order accurate in the spatial direction and first order in the temporal direction. An extensive amount of analysis has been carried out to obtain the parameter uniform error estimates.

In support of the predicted theory some test examples are solved using the proposed method. To illustrate the performance of the proposed method, the maximum errors is given in Tables 1 and 2 for cylindrical and spherical waves, respectively. Since the exact solution for modified Burgers' equation is not known, therefore maximum pointwise errors are computed by *half mesh principle*. The errors specified in Tables 1 and 2 show that the proposed method converges uniformly with respect to perturbation parameter and convergence behavior matched with the theoretical result. Figures 1 and 2 show the physical behavior of the computed solution.

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