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Short Communication

Creeping flow past a porous approximate sphere

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This paper concerns the creeping flow of incompressible viscous fluid past and through a porous approximate sphere. The Brinkman model for the flow inside the porous particle and Stokes model for the flow outside the particle in their stream function formulations are used. The stream function and the pressure distribution, both for the flow inside and outside are obtained in terms of Bessel and Gegenbauer functions of the first kind. The drag force experienced by the particle is determined and its variation with respect to permeability parameter is studied numerically. The special cases of flow past a porous sphere and spheroid are obtained from the present analysis.

Introduction

The flow of fluid around porous bodies has many industrial and engineering applications, such as flow through porous beds (fixed or fluidized), sedimentation of fine particulate suspensions, modeling of polymer macromolecule coils in a solvent, catalytic reactions where porous pallets are used, floc settling processes, the flow of oil in oil fields or reservoirs during oil recovery etc. Several researchers have considered the flow of Newtonian fluids past a porous body with different models. Joseph and Tao [1] considered the creeping flow past a porous spherical shell immersed in a uniform viscous incompressible fluid using Darcy's law for the flow inside the porous region and Stokes equations for the fluid outside the sphere with continuity of normal velocity and pressure at the surface of the porous sphere and no-slip of tangential velocity component of the free fluid. They found that the drag on the porous sphere is same as that of a rigid sphere with reduced radius. The same problem, with Saffman's boundary conditions at the surface of the sphere was studied by Padmavathi et al. [2] and it was shown therein that the torque on a porous sphere is always less than that on a rigid sphere, where as the drag in general is not.

However Darcy's law appears to be inadequate for the flows with high porosity, and large shear rates and for flows near the surface of the bounded porous medium. To model such flows a modification of Darcy's law was proposed by Brinkman [3] and Debye and Bueche [4] independently. The validity of this equation was confirmed by experimental verification of Oomes et al. [5] and Matsumoto and Suganuma [6] and theoretically justified by Tam [7] and Lundgren [8]. Using Brinkman model for the flow inside the porous sphere and Navier-Stokes equations for the free fluid region, Qin and Kaloni [9] obtained a Cartesian tensor solution for the creeping flow past a porous sphere. Higdon and Kojima [10] have studied Stokes flow past porous particles using Brinkman's equations for the flow inside. They derived some asymptotic results for small and large permeability by using Green's function formulation of the Brinkman's equation. Recently, Zlatanovski [11] has considered the axisymmetric flow past a porous prolate spheroidal particle using the Brinkman model for the flow inside the spheroidal particle and Stokes model for the free flow region.

In the present paper, we consider the creeping flow past a porous approximate sphere. We have used the Brinkman's equation for the flow inside the porous region and the Stokes equation for the free flow region using the stream function formulation. As boundary conditions, continuity of the velocity, pressure and tangential stresses across the interface are employed. The stream function (and thus the velocity) and pressure (both for the flow inside and outside) are calculated. The drag force experienced by the particle is determined. The cases of sphere and oblate spheroid are obtained as special cases.

Statement of the problem

Let (r, θ, ϕ) denote a spherical polar co-ordinate system with $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ unit base vectors and $h_1 = 1, h_2 = r$, and $h_3 = r \sin\theta$ as the corresponding scale factors.

Consider an incompressible viscous fluid flow past a porous approximate sphere with a uniform velocity U far away from the body along the axis of symmetry $\theta = 0$. Consider the body $r = a[1 + f(\theta)]$, where $f(\theta)$ is a function of θ which can be

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expressed as $f(\zeta) = \sum_{m=2}^{\infty} \beta_m \vartheta_m(\zeta)$, where $\vartheta_m(\zeta) = \left[P_{m-2}(\zeta) - P_m(\zeta)\right]/(2m-1), \zeta = \cos\theta$ in which $P_m(\zeta)$ is Legendre function of the first kind. For small β_ms we refer to this body as an approximate sphere.

We assume that the flow outside the porous approximate sphere to be Stokesian and inside to be governed by Brinkman model. The equations of motion for the region outside the approximate sphere are

$$\operatorname{div} \vec{q}^{(1)} = 0, \tag{1}$$

$$\operatorname{grad} p^{(1)} + \mu \operatorname{curl} \operatorname{curl} \vec{q}^{(1)} = 0, \tag{2}$$

where $\vec{q}^{(1)}$ is the velocity, μ is the coefficient of viscosity, and $p^{(1)}$ is the pressure.

For the region inside the approximate sphere the equations of the motion are

$$\operatorname{div} \vec{q}^{(2)} = 0, \tag{3}$$

$$\operatorname{grad} p^{(2)} + \frac{\mu}{k}\vec{q}^{(2)} + \mu \operatorname{curl} \operatorname{curl} \vec{q}^{(2)} = 0, \tag{4}$$

where $\vec{q}^{(2)}$ is the velocity, $p^{(2)}$ is the pressure, μ is the viscosity, and k is the permeability of the porous medium.

Since the flow of the fluid is in the meridian plane and the flow is axially symmetric, all the physical quantities are independent of ϕ. We choose the velocity vectors $\vec{q}^{(1)}$ and $\vec{q}^{(2)}$ in the form

$$\vec{q}^{(i)} = \left[u^{(i)}(r,\theta)\vec{e}_r + v^{(i)}(r,\theta)\vec{e}_\theta\right], \quad i = 1,2. \tag{5}$$

In view of the incompressibility condition $\operatorname{div} \vec{q}^{(i)} = 0, i = 1,2$, we introduce the stream function $\psi^{(i)}(r,\theta), i = 1,2$, through

$$u^{(i)} = \frac{-1}{r^2\sin\theta}\frac{\partial\psi^{(i)}}{\partial\theta}; \quad v^{(i)} = \frac{1}{r\sin\theta}\frac{\partial\psi^{(i)}}{\partial r}, \quad i = 1,2. \tag{6}$$

Eliminating pressure from (2) and (4), and substituting (6) in the resulting equations, we get the following dimensionless equations for $\psi^{(i)}, i = 1,2$:

$$E^4\psi^{(1)} = 0, \tag{7}$$

$$E^2(E^2 - \alpha^2)\psi^{(2)} = 0, \tag{8}$$

where $\alpha^2 = a^2/k$ and $E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\zeta^2)}{r^2}\frac{\partial^2}{\partial\zeta^2}$ is the Stokesian stream function operator.

Boundary conditions

The boundary conditions are [9,11]:

(i) Continuity of velocity components on the boundary of the approximate sphere, i.e.,

$$u^{(1)}(r,\theta) = u^{(2)}(r,\theta) \quad \text{and} \quad v^{(1)}(r,\theta) = v^{(2)}(r,\theta) \quad \text{on} \quad r = a\left[1 + \sum \beta_m\vartheta_m(\zeta)\right]. \tag{9}$$

(ii) Continuity of pressure on the boundary, i.e.,

$$p^{(1)}(r,\theta) = p^{(2)}(r,\theta) \quad \text{on} \quad r = a\left[1 + \sum \beta_m\vartheta_m(\zeta)\right]. \tag{10}$$

(iii) Continuity of tangential stress components on the boundary, i.e.,

$$\tau^{(1)}{}_{r\theta}(r,\theta) = \tau^{(2)}{}_{r\theta}(r,\theta) \quad \text{on} \quad r = a\left[1 + \sum \beta_m\vartheta_m(\zeta)\right]. \tag{11}$$

Additionally, we have the regularity conditions at infinity, i.e.,

$$\underset{r\to\infty}{Lt}\, u^{(1)}(r,\theta) = U\cos\theta, \quad \underset{r\to\infty}{Lt}\, v^{(1)}(r,\theta) = -U\sin\theta \tag{12}$$

and the condition that velocity and pressure must be nonsingular everywhere in the flow field.

Solution of the problem

The solution of (7) which is regular at infinity is

$$\psi^{(1)} = \left[r^2 + \frac{A_2}{r} + B_2 r\right]\vartheta_2(\zeta) + \sum_{n=3}^{\infty}\left[A_n r^{1-n} + B_n r^{3-n}\right]\vartheta_n(\zeta) \tag{13}$$

and the solution of (8), which is finite as $r \to 0$ is

$$\psi^{(2)} = \left[C_2 r^2 + D_2\sqrt{r} I_{3/2}(\alpha r)\right]\vartheta_2(\zeta) + \sum_{n=3}^{\infty}\left[C_n r^{-n} + D_n\sqrt{r} I_{n-1/2}(\alpha r)\right]\vartheta_n(\zeta), \tag{14}$$

where $I_{n-1/2}(\alpha r)$ denotes the modified Bessel function of the first kind and $\vartheta_n(\zeta)$ is the Gegenbauer function of the first kind of order n and degree $-1/2$.

Using the eqs. (13) and (14), the expressions for the pressure in both flow regions are

$$p^{(1)} = \mu\frac{B_2}{r^2}P_1(\zeta) + \mu\sum_{n=3}^{\infty} B_n\left(\frac{6-4n}{n}\right) r^{-n} P_{n-1}(\zeta), \tag{15}$$

$$p^{(2)} = \mu\left[\alpha^2 C_2 r P_1(\zeta) + \sum_{n=3}^{\infty} C_n\alpha^2\frac{r^{n-1}}{n-1}P_{n-1}(\zeta)\right]. \tag{16}$$

Determination of arbitrary constants

We first propose to develop the solution corresponding to the boundary $r = a[1 + \beta_m\vartheta_m(\zeta)]$ and assume that the coefficient β_m is sufficiently small so that squares and higher powers of β_m can be neglected, i.e., $(r/a)^k \approx 1 + k\beta_m\vartheta_m(\zeta)$, where k is positive or negative.

Comparison of the above solution with those obtained in case of flow of an incompressible viscous fluid past a porous sphere, indicates that the terms involving A_n, B_n, C_n, and D_n for $n > 2$ are the extra terms here which are not present in the case of sphere. The body that we are considering now is an approximate sphere and the flow generated is not expected to be far different from the one generated by flow past a porous sphere. Also the coefficients A_n, B_n, C_n, and D_n for $n > 2$ are of order β_m. Therefore as in [12] in the terms involving A_n, B_n, C_n, and D_n for $n > 2$, we ignore the departure from the spherical form and set $r = 1$ while implementing the boundary conditions.

The boundary conditions (9) to (11) in terms of stream function are

$$\begin{aligned} &\psi^{(1)}(r,\theta) = \psi^{(2)}(r,\theta), \qquad \psi_r^{(1)}(r,\theta) = \psi_r^{(2)}(r,\theta),\\ &\psi_{rr}^{(2)}(r,\theta) = \psi_{rr}^{(2)}(r,\theta), \qquad p^{(1)}(r,\theta) = p^{(2)}(r,\theta). \end{aligned} \tag{17}$$

Using the above boundary conditions and the observations made above, the constants appearing in the solutions of the problem are seen to be

$$\begin{aligned} A_2 &= \frac{\alpha(6+\alpha^2)\cosh(\alpha) - 3(2+\alpha^2)\sinh(\alpha)}{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)}, \qquad & B_2 &= \frac{3\alpha^2(-\alpha\cosh(\alpha) + \sinh(\alpha))}{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)},\\ C_2 &= \frac{3\alpha\sqrt{2\pi\alpha}}{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)}, & D_2 &= \frac{3\alpha\cosh(\alpha) - 3\sinh(\alpha)}{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)}. \end{aligned} \tag{18}$$

For $n \neq m-2, m, m+2$,

$$A_n = B_n = C_n = D_n = 0, \tag{19}$$

and for $n = m-2, m, m+2$, we have the following system of equations:

$$A_n + B_n - C_n - D_n I_{n-1/2}(\alpha) = (-2 + A_2 - B_2 + 2C_2)b_n, \tag{20}$$

$$(1-n)A_n + (3-n)B_n - nC_n - D_n\left\{(n-1)I_{n-1/2}(\alpha) - (a\alpha)I_{n-3/2}(\alpha)\right\} = (-2 - 2A_2 + 2C_2 + D_2 I_{3/2}(\alpha))b_n, \tag{21}$$

$$n(n-1)A_n + (n-2)(n-3)B_n - n(n-1)C_n - D_n\left\{n(n-1) + \alpha^2\right\} I_{n-1/2}(\alpha) = (6A_2 - 4D_2 I_{3/2}(\alpha))b_n, \tag{22}$$

$$(n-1)(6-4n)B_n - n\alpha^2 C_n = n(n-1)\left[-2B_2 + \alpha^2 C_2\right] a_n, \tag{23}$$

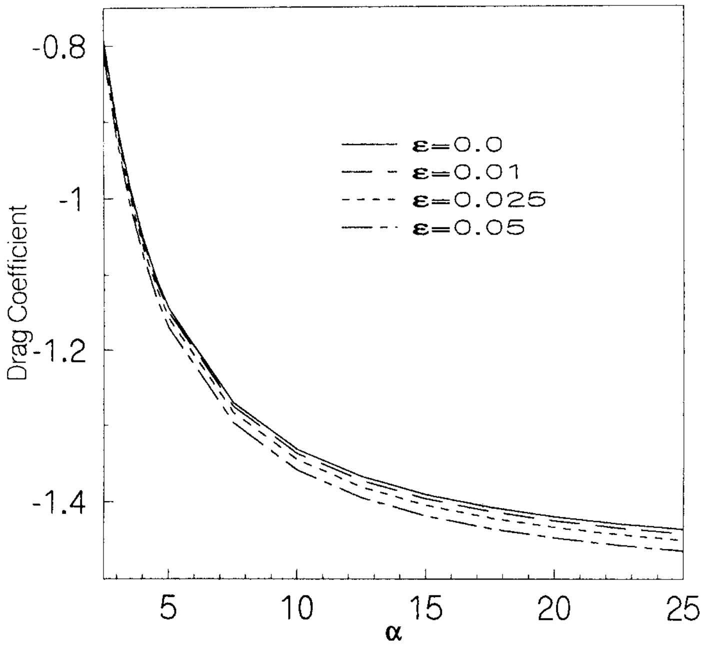

Fig. 1 Variation of drag coefficient with α.

where

$$b_{m-2} = -\frac{(m-2)(m-3)}{2(2m-1)(2m-3)}, \qquad b_m = \frac{m(m-1)}{(2m+1)(2m-3)}, \qquad b_{m+2} = -\frac{(m+1)(m+2)}{2(2m-1)(2m+1)},$$
$$a_{m-2} = \frac{(m-2)}{2(2m-1)(2m-3)}, \qquad a_m = \frac{1}{(2m+1)(2m-3)}, \qquad a_{m+2} = -\frac{(m+1)}{2(2m-1)(2m+1)}. \tag{24}$$

Solving these equations, we obtain the values of A_n, B_n, C_n, and D_n. Substituting these values in (13) and (14), we get the expressions for the stream functions $\psi^{(1)}(r,\theta)$ and $\psi^{(2)}(r,\theta)$. Hence the velocity components are determined. In case the equation of the approximate sphere is $r = a[1 + \sum \beta_m \vartheta_m(\zeta)]$, we employ the same technique as above and determine the corresponding arbitrary constants.

Drag on the body

The drag force acting on the porous approximate sphere is given by

$$D = 2\pi a^2 \int_0^{\pi} \left[\tau_{rr}^{(1)} \cos\theta - \tau_{r\theta}^{(1)} \sin\theta\right]_{r=a[1+\sum \beta_m \vartheta_m(\zeta)]} \sin\theta d\theta. \tag{25}$$

On carrying out the integration it is found to be

$$D = 4\pi\mu U a \left[\frac{3\alpha^2 \{-\alpha\cosh(\alpha) + \sinh(\alpha)\}}{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)} + \frac{3\alpha^2 \left\{-35 - \alpha^2 - 4\alpha^4 + \left(35 + 7\alpha^2 - 4\alpha^4\right)\cosh(2\alpha) - 38\alpha\sinh(2\alpha)\right\}}{10\left\{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)\right\}^2}\beta_2 + \frac{3\alpha^2 \left\{10 + 8\alpha^2 + 2\alpha^4 - \left(10 - 4\alpha^2 - 2\alpha^4\right)\cosh(2\alpha) + 4\alpha\sinh(2\alpha)\right\}}{70\left\{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)\right\}^2}\beta_4 \right]. \tag{26}$$

It is interesting to note that though the boundary surface is $r = a\left[1 + \sum \beta_m \vartheta_m(\zeta)\right]$ the coefficients β_2 and β_4 only, contribute to the drag. This implies that the drag on the approximate sphere is relatively insensitive to the details of the surface geometry. This is similar to the observations made by Iyengar and Srinivasacharya [12] in case of micropolar fluids.

The variation of drag coefficient $D_N = D/(4\pi\mu U a)$ for various values of α and $\beta_2 = \beta_4 = \varepsilon$ is shown in Fig. 1. From Fig. 1 it can be observed that the drag coefficient is decreasing as the permeability parameter (α) is increasing. There is slight decrease in the drag coefficient as the deformation parameter (ε) is increasing. It is interesting to note that the drag on the sphere is more than that of the drag on the approximate sphere.

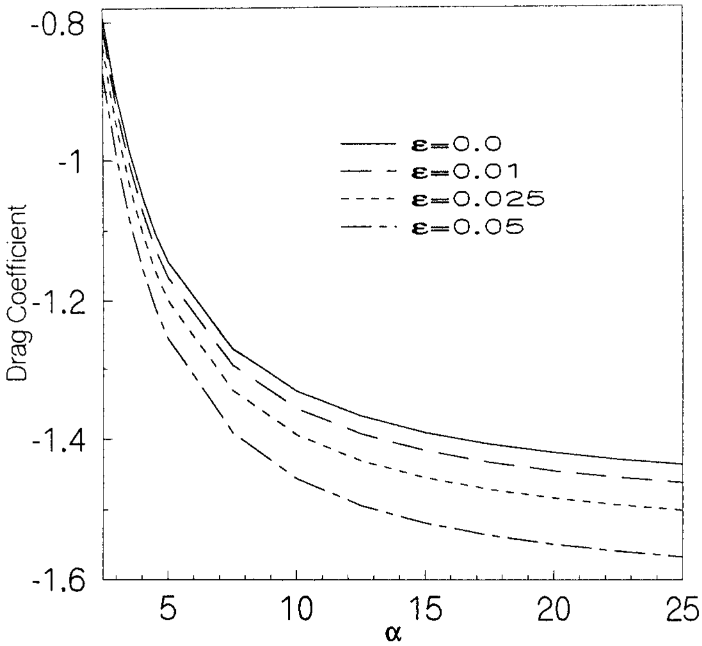

Fig. 2 Variation of drag coefficient with α (oblate spheroid).

Special cases

Case (i). Sphere
If $\beta_m = 0$, for $m > 2$, the approximate sphere reduces to a sphere and the drag is

$$\frac{12\pi\mu U a\alpha^2 \{-\alpha\cosh(\alpha) + \sinh(\alpha)\}}{\alpha(3+2\alpha^2)\cosh(\alpha) - 3\sinh(\alpha)} \tag{27}$$

which can be simplified to

$$\frac{-12\pi\mu U a\alpha^2 \{1 - \tanh(\alpha)/\alpha\}}{2\alpha^2 + 3\{1 - \tanh(\alpha)/\alpha\}} \tag{28}$$

which agrees with the porous sphere case derived Brinkman [3], Neal et al. [13] and Qin and Kaloni [9].

Case (ii). Oblate spheroid
Consider the oblate spheroid given by

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{a^2(1-\varepsilon)^2} = 1 \tag{29}$$

whose equatorial radius is a in which ε is so small that ε^2 and higher powers may be neglected.

Following Happel and Brenner [14] its polar equation can be put in the form $r = a\,[1 + 2\varepsilon\vartheta_m(\zeta)]$ where $c = a(1-\varepsilon)$ (see [14], p. 144). This is like $r = a\,[1 + \beta_2\vartheta_2(\zeta)]$, where $a = c$ and $\beta_2 = 2\varepsilon$.

Using (13) and (14), the expressions for stream functions can be determined. The drag on oblate spheroidal particle is seen to be

$$D = 4\pi\mu U a\left[\frac{3\alpha^2\{-\alpha\cosh(\alpha)+\sinh(\alpha)\}}{\alpha(3+2\alpha^2)\cosh(\alpha)-3\sinh(\alpha)} + \left\{\frac{3\alpha^2\{-\alpha\cosh(\alpha)+\sinh(\alpha)\}}{\alpha(3+2\alpha^2)\cosh(\alpha)-3\sinh(\alpha)} + \frac{3\alpha^2\{-35-\alpha^2-4\alpha^4+(35+7\alpha^2-4\alpha^4)\cosh(2\alpha)-38\alpha\sinh(2\alpha)\}}{5\{\alpha(3+2\alpha^2)\cosh(\alpha)-3\sinh(\alpha)\}^2}\right\}\varepsilon\right]. \tag{30}$$

The variation of the drag coefficient $D_N = D/(4\pi\mu U a)$ for various values of α and ε is shown in Fig. 2. From Fig. 2 it can be observed that the drag coefficient is decreasing as the permeability parameter (α) and ε is increasing. Also the drag on the porous sphere is more than that of the drag on the porous oblate spheroid.

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