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ELSEVIER Applied Mathematics and Computation 154 (2004) 83–101
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

An exponentially fitted finite difference method for singular perturbation problems

Y.N. Reddy [a,*], P. Pramod Chakravarthy [b]

[a] *Department of Mathematics, National Institute of Technology, Warangal 506 004, India*
[b] *Department of Mathematics, Kakatiya Institute of Technology & Science, Warangal 506 015, India*

Abstract

In this paper an exponentially fitted finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. A fitting factor is introduced in a tridiagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the system. The stability of the algorithm is investigated. Several linear and nonlinear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems; Two-point boundary value problems; Boundary layer; Exponentially fitted method

1. Introduction

Singularly perturbed second-order two-point boundary value problems arise very frequently in fluid mechanics and other branches of Applied Mathematics. These problems have been received a significant amount of attention in past and recent years. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some

* Corresponding author.
E-mail addresses: ynreddy@recw.ernet.in, ynreddy@nitw.ernet.in (Y.N. Reddy).

0096-3003/$ - see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/S0096-3003(03)00693-3

other parts. So, typically there are thin transition layers where the solutions can jump abruptly, while away from the layers the solution behaves regularly and vary slowly. Thus more efficient, simpler computational techniques are required to solve singular perturbation problems. For a good analytical discussion one may refer to: Bender and Orszag [1], Kevorkian and Cole [5], O' Malley [8], Nayfeh [7]. In the paper, Kadalbajoo and Reddy [4] gives an erudite outline of the singular perturbation problems. For some numerical methods one may refer to recent books: Miller [6], Hemker and Miller [3], Doolan et al. [2].

In this paper an exponentially fitted finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. A fitting factor is introduced in a tridiagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the system. The stability of the algorithm is investigated. Several linear and nonlinear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

2. Exponentially fitted finite difference method

A difference scheme with a fitting factor containing exponential functions is known as exponentially fitted difference scheme.

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1], \tag{1}$$

$$\text{with } y(0) = \alpha, \tag{2a}$$

$$\text{and } y(1) = \beta, \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0,1]. Further more, we assume that $b(x) \leqslant 0$, $a(x) \geqslant M > 0$ throughout the interval [0,1], where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$ which in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

From the theory of singular perturbations it is known that the solution of (1) and (2) is of the form (cf. [8, pp. 22–26])

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))\mathrm{e}^{-\int_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)\mathrm{d}x} + O(\varepsilon), \tag{3}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(1) = \beta. \tag{4}$$

By taking the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '0' and restricting to their first terms, (3) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))\mathrm{e}^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x} + \mathrm{O}(\varepsilon). \tag{5}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, 2, \ldots, N$.

From (5) we have

$$y(x_i) = y_0(x_i) + (\alpha - y_0(0))\mathrm{e}^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x_i} + \mathrm{O}(\varepsilon),$$

$$\text{i.e.,}\ \ y(ih) = y_0(ih) + (\alpha - y_0(0))\mathrm{e}^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)ih} + \mathrm{O}(\varepsilon).$$

Therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))\mathrm{e}^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)i\rho}, \tag{6}$$

where $\rho = h/\varepsilon$.

Now, we consider the second-order finite difference scheme

$$\varepsilon\sigma(\rho)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + b(x_i)y_i = f(x_i),$$
$$1 \leqslant i \leqslant N-1, \tag{7}$$

$y_0 = \alpha$; $y_N = \beta$; where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (7) converges uniformly to the solution of (1) and (2).

Multiplying (7) by h and taking the limit as $h \to 0$; we get

$$\lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y_{i+1} - 2y_i + y_{i-1}) + \frac{1}{2}a(ih)(y_{i+1} - y_{i-1})\right] = 0$$

if $f(x_i) - b(x_i)y_i$ is bounded.

$$\therefore \lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y(ih+h) - 2y(ih) + y(ih-h)) + \frac{1}{2}a(ih)(y(ih+h) - y(ih-h))\right] = 0, \tag{8}$$

substituting (6) in (8) and simplifying, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho} = \frac{1}{2}a(0)\coth\left[\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right], \tag{9}$$

$$\therefore \text{ We have } \sigma = \frac{\rho}{2} a(0) \coth\left[\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right], \tag{10}$$

which is a constant fitting factor.

From (7) we have

$$\left(\frac{\varepsilon\sigma}{h^2} - \frac{a(x_i)}{2h}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma}{h^2} - b(x_i)\right)y_i + \left(\frac{\varepsilon\sigma}{h^2} + \frac{a(x_i)}{2h}\right)y_{i+1} = f(x_i),$$
$$i = 1, 2, \ldots, N-1, \tag{11}$$

where the fitting factor σ is given by (10).

Eq. (11) can be written as a three term recurrence relation:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, 3, \ldots, N-1, \tag{12}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} - \frac{a(x_i)}{2h},$$
$$F_i = \frac{2\varepsilon\sigma}{h^2} - b(x_i),$$
$$G_i = \frac{\varepsilon\sigma}{h^2} + \frac{a(x_i)}{2h},$$
$$H_i = f(x_i),$$

This gives us the tridiagonal system which can be solved easily by Thomas algorithm described in Section 3.

Remark. For $b(0) = 0$ we get the exponentially fitted method developed by Doolan et al. [2, pp. 93–94], as

$$\left(\frac{\varepsilon\sigma}{h^2} - \frac{a(x_i)}{2h}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma}{h^2} - b(x_i)\right)y_i + \left(\frac{\varepsilon\sigma}{h^2} + \frac{a(x_i)}{2h}\right)y_{i+1} = f(x_i),$$
$$i = 1, 2, \ldots, N-1,$$

with the fitting factor $\sigma = \frac{\rho}{2} a(0) \coth\left(a(0)\frac{\rho}{2}\right)$.

3. Thomas algorithm

We briefly discuss the Thomas algorithm to solve the tridiagonal system:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, 3, \ldots, N-1 \tag{13}$$

subject to the boundary conditions

$$y_0 = y(0) = \alpha, \tag{14a}$$

$$y_N = y(1) = \beta. \tag{14b}$$

We set

$$y_i = W_i y_{i+1} + T_i \quad \text{for } i = N-1, N-2, \ldots, 2, 1, \tag{15}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

From (15), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}, \tag{16}$$

substituting (16) in (13), we have

$$E_i(W_{i-1}y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i.$$

$$\therefore y_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) y_{i+1} + \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right). \tag{17}$$

By comparing (17) and (15), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right), \tag{18a}$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right). \tag{18b}$$

To solve these recurrence relations for $i = 1, 2, 3, \ldots, N-1$, we need the initial conditions for W_0 and T_0. For this we have $y_0 = \alpha = W_0 y_1 + T_0$. If we choose $W_0 = 0$, then we get $T_0 = \alpha$. With these initial values, we compute W_i and T_i for $i = 1, 2, 3, \ldots, N-1$ from (17) in forward process, and then obtain y_i in the backward process from (15) and (14b).

4. Stability analysis

We will now show that the algorithm is computationally stable. By stability, we mean that the effect of an error made in one stage of the calculation is not propagated into larger errors at later stages of the calculations. Let us now examine the recurrence relation given by (18a). Suppose that a small error e_{i-1} has been made in the calculation of W_{i-1}; then, we have

$\overline{W}_{i-1} = W_{i-1} + e_{i-1}$ and we are actually calculating,

$$\overline{W}_i = \left(\frac{G_i}{F_i - E_i \overline{W}_{i-1}}\right). \tag{19}$$

From (19) and (18a), we have

$$\begin{aligned} e_i &= \left(\frac{G_i}{F_i - E_i(W_{i-1} + e_{i-1})}\right) - \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) \\ &= \left(\frac{G_i E_i e_{i-1}}{(F_i - E_i(W_{i-1} + e_{i-1}))(F_i - E_i W_{i-1})}\right) = \left(\frac{W_i^2 E_i}{G_i}\right) e_{i-1}, \end{aligned} \tag{20}$$

under the assumption that the error is small initially. From the assumptions made earlier that $a(x) > 0$ and $b(x) \leqslant 0$, we have

$$F_i \geqslant E_i + G_i, \quad i = 1, 2, 3, \ldots, N-1.$$

Form (18a) we have

$$\begin{aligned} W_1 &= \frac{G_1}{F_1} < 1, \text{ since } F_1 > G_1, \\ W_2 &= \frac{G_2}{F_2 - E_2 W_1} < \frac{G_2}{F_2 - E_2}, \text{ since } W_1 < 1, \\ &< \frac{G_2}{E_2 + G_2 - E_2} = 1, \text{ since } F_2 \geqslant E_2 + G_2, \end{aligned}$$

successively, it follows that

$$\begin{aligned} |e_i| &= |W_i|^2 \left|\frac{E_i}{G_i}\right| |e_{i-1}| \\ &< |e_{i-1}| \text{ since } |E_i| \leqslant |G_i|. \end{aligned}$$

Therefore the recurrence relation (18a) is stable. Similarly we can prove that the recurrence relation (18b) is also stable. Finally the convergence of the Thomas algorithm is ensured by the condition $|W_i| < 1$, $i = 1, 2, 3, \ldots, N-1$.

5. Numerical examples

To demonstrate the applicability of the method we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The approximate solution is compared with the exact solution.

Example 5.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag ([1], p. 480; problem 9.17 with $\alpha = 0$)

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]},$$

where $m_1 = \left(-1 + \sqrt{1+4\varepsilon}\right)/(2\varepsilon)$ and $m_2 = \left(-1 - \sqrt{1+4\varepsilon}\right)/(2\varepsilon)$.

The numerical results are given in Table 1(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 1
Numerical results of Example 5.1, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	1.0000000	1.0000000
0.01	0.3734365	0.3719724
0.02	0.3771425	0.3756784
0.04	0.3847230	0.3832599
0.06	0.3924560	0.3909945
0.08	0.4003443	0.3988851
0.10	0.4083913	0.4069350
0.20	0.4511180	0.4496879
0.30	0.4983149	0.4969323
0.40	0.5504497	0.5491403
0.50	0.6080388	0.6068334
0.60	0.6716532	0.6705877
0.70	0.7419230	0.7410401
0.80	0.8195445	0.8188941
0.90	0.9052870	0.9049277
1.00	1.0000000	1.0000000
Panel B		
0.00	1.0000000	1.0000000
0.01	0.3734084	0.3716135
0.02	0.3771425	0.3753479
0.04	0.3847230	0.3829296
0.06	0.3924560	0.3906645
0.08	0.4003443	0.3985557
0.10	0.4083913	0.4066062
0.20	0.4511180	0.4493649
0.30	0.4983149	0.4966200
0.40	0.5504497	0.5488445
0.50	0.6080388	0.6065609
0.60	0.6716532	0.6703468
0.70	0.7419230	0.7408404
0.80	0.8195445	0.8187471
0.90	0.9052870	0.9048464
1.00	1.0000000	1.0000000

Example 5.2. Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, ([9], Example 2)

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$.

The numerical results are given in Table 2(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 2
Numerical results of Example 5.2, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	0.0000000	0.0000000
0.01	−0.9799542	−0.9878747
0.02	−0.9697992	−0.9776400
0.04	−0.9487992	−0.9564800
0.06	−0.9269992	−0.9345200
0.08	−0.9043992	−0.9117600
0.10	−0.8809992	−0.8882000
0.20	−0.7519993	−0.7584000
0.30	−0.6029994	−0.6086000
0.40	−0.4339994	−0.4388000
0.50	−0.2449995	−0.2490000
0.60	−0.0359996	−0.0392001
0.70	0.1930003	0.1906000
0.80	0.4420002	0.4403999
0.90	0.7110001	0.7102000
1.00	1.0000000	1.0000000
Panel B		
0.00	0.0000000	0.0000000
0.01	−0.9799999	−0.9897020
0.02	−0.9697999	−0.9794040
0.04	−0.9487999	−0.9582080
0.06	−0.9269999	−0.9362120
0.08	−0.9043999	−0.9134160
0.10	−0.8809999	−0.8898200
0.20	−0.7519999	−0.7598400
0.30	−0.6029999	−0.6098600
0.40	−0.4339999	−0.4398800
0.50	−0.2450000	−0.2499000
0.60	−0.0360000	−0.0399201
0.70	0.1930000	0.1900600
0.80	0.4420000	0.4400399
0.90	0.7110000	0.7100199
1.00	1.0000000	1.0000000

Example 5.3. Finally we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole ([5], p. 33; Eqs. (2.3.26) and (2.3.27) with $\alpha = -1/2$)

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0, \; x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [7], p. 148; Eq. (4.2.32)) as our 'exact' solution:

$$y(x) = \frac{1}{2 - x} - \frac{1}{2}\mathrm{e}^{-(x - x^2/4)/\varepsilon}.$$

Table 3
Numerical results of Example 5.3, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	0.0000000	0.0000000
0.01	0.5049049	0.5024893
0.02	0.5087239	0.5050505
0.04	0.5138949	0.5102041
0.06	0.5191650	0.5154639
0.08	0.5245441	0.5208333
0.10	0.5300354	0.5263158
0.20	0.5593066	0.5555555
0.30	0.5919883	0.5882353
0.40	0.6287109	0.6250000
0.50	0.6702697	0.6666667
0.60	0.7176828	0.7142857
0.70	0.7722733	0.7692308
0.80	0.8357928	0.8333333
0.90	0.9106092	0.9090909
1.00	1.0000000	1.0000000
Panel B		
0.00	0.0000000	0.0000000
0.01	0.5049284	0.5025126
0.02	0.5087248	0.5050505
0.04	0.5138956	0.5102041
0.06	0.5191657	0.5154639
0.08	0.5245448	0.5208333
0.10	0.5300362	0.5263158
0.20	0.5593073	0.5555555
0.30	0.5919890	0.5882353
0.40	0.6287115	0.6250000
0.50	0.6702704	0.6666667
0.60	0.7176834	0.7142857
0.70	0.7722737	0.7692308
0.80	0.8357933	0.8333333
0.90	0.9106096	0.9090909
1.00	1.0000000	1.0000000

The numerical results are given in Table 3(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

6. Nonlinear problems

Nonlinear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using quasi-linearization method. The outer solution (the solution of the given problem by putting $\varepsilon = 0$) is taken to be the initial approximation.

The approximate solution is compared with the exact solution.

7. Nonlinear examples

Again to demonstrate the applicability of the method, we have applied it to three nonlinear singular perturbation problems with left-end boundary layer.

Example 7.1. Consider the following singular perturbation problem from Bender and Orszag ([1], p. 463; Eq. (9.7.1))

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0, \; x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 0$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right].$$

We have chosen to use Bender and Orszag's uniformly valid approximation ([1], p. 463; Eq. (9.7.6)) for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)e^{-2x/\varepsilon}.$$

For this example, we have boundary layer of thickness $O(\varepsilon)$ at $x = 0$ (cf. [1]).

The numerical results are given in Table 4(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 7.2. Now consider the following singular perturbation problem from Kevorkian and Cole ([5], p. 56; Eq. (2.5.1))

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = -1$ and $y(1) = 3.9995$.

Table 4
Numerical results of Example 7.1, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	0.0000000	0.0000000
0.01	0.6866083	0.6831968
0.02	0.6766735	0.6733446
0.04	0.6570967	0.6539265
0.06	0.6378976	0.6348783
0.08	0.6190617	0.6161861
0.10	0.6005758	0.5978370
0.20	0.5129694	0.5108256
0.30	0.4324508	0.4307829
0.40	0.3579566	0.3566750
0.50	0.2886465	0.2876821
0.60	0.2238445	0.2231436
0.70	0.1629993	0.1625189
0.80	0.1056546	0.1053605
0.90	0.0514289	0.0512933
1.00	0.0000000	0.0000000
Panel B		
0.00	0.0000000	0.0000000
0.01	0.6866081	0.6831968
0.02	0.6766733	0.6733446
0.04	0.6570964	0.6539265
0.06	0.6378974	0.6348783
0.08	0.6190615	0.6161861
0.10	0.6005756	0.5978370
0.20	0.5129692	0.5108256
0.30	0.4324506	0.4307829
0.40	0.3579564	0.3566750
0.50	0.2886464	0.2876821
0.60	0.2238446	0.2231436
0.70	0.1629993	0.1625189
0.80	0.1056545	0.1053605
0.90	0.0514289	0.0512933
1.00	0.0000000	0.0000000

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995.$$

We have chosen to use the Kivorkian and Cole's uniformly valid approximation ([5], pp. 57–58; Eq. (2.5.5), (2.5.11) and (2.5.14)) for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right),$$

where $c_1 = 2.9995$ and $c_2 = (1/c_1)\log_e[(c_1 - 1)/(c_1 + 1)]$.

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. [5], pp. 56–66).

The numerical results are given in Table 5(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 7.3. Finally we consider the following singular perturbation problem from O' Malley ([8], p. 9; Eq. (1.10) Case 2):

$$\varepsilon y''(x) - y(x)y'(x) = 0, \quad x \in [-1, 1]$$

with $y(-1) = 0$ and $y(1) = -1$.

Table 5
Numerical results of Example 7.2, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	−1.0000000	−1.0000000
0.01	3.0161460	3.0095000
0.02	3.0194790	3.0195000
0.04	3.0395010	3.0395000
0.06	3.0595010	3.0595000
0.08	3.0795010	3.0795000
0.10	3.0995010	3.0995000
0.20	3.1995010	3.1995000
0.30	3.2995010	3.2995000
0.40	3.3995010	3.3995000
0.50	3.4995010	3.4995000
0.60	3.5995000	3.5995000
0.70	3.6995000	3.6995000
0.80	3.7995000	3.7995000
0.90	3.8995000	3.8995000
1.00	3.9995000	3.9995000
Panel B		
0.00	−1.0000000	−1.0000000
0.01	3.0161450	3.0095000
0.02	3.0194790	3.0195000
0.04	3.0395010	3.0395000
0.06	3.0595010	3.0595000
0.08	3.0795010	3.0795000
0.10	3.0995010	3.0995000
0.20	3.1995010	3.1995000
0.30	3.2995010	3.2995000
0.40	3.3995010	3.3995000
0.50	3.4995010	3.4995000
0.60	3.5995000	3.5995000
0.70	3.6995000	3.6995000
0.80	3.7995000	3.7995000
0.90	3.8995000	3.8995000
1.00	3.9995000	3.9995000

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0.$$

We have chosen to use O' Malley's approximate solution ([8], pp. 9–10; Eqs. (1.13) and (1.14)) for comparison,

$$y(x) = -\frac{\left(1 - e^{-(x+1)/\varepsilon}\right)}{\left(1 + e^{-(x+1)/\varepsilon}\right)}.$$

Table 6
Numerical results of Example 7.3, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
−1.00	0.0000000	0.0000000
−0.98	−1.0000000	−1.0000000
−0.96	−1.0000000	−1.0000000
−0.92	−1.0000000	−1.0000000
−0.88	−1.0000000	−1.0000000
−0.84	−1.0000000	−1.0000000
−0.80	−1.0000000	−1.0000000
−0.60	−1.0000000	−1.0000000
−0.40	−1.0000000	−1.0000000
−0.20	−1.0000000	−1.0000000
0.00	−1.0000000	−1.0000000
0.20	−1.0000000	−1.0000000
0.40	−1.0000000	−1.0000000
0.60	−1.0000000	−1.0000000
0.80	−1.0000000	−1.0000000
1.00	−1.0000000	−1.0000000
Panel B		
−1.00	0.0000000	0.0000000
−0.98	−1.0000000	−1.0000000
−0.96	−1.0000000	−1.0000000
−0.92	−1.0000000	−1.0000000
−0.88	−1.0000000	−1.0000000
−0.84	−1.0000000	−1.0000000
−0.80	−1.0000000	−1.0000000
−0.60	−1.0000000	−1.0000000
−0.40	−1.0000000	−1.0000000
−0.20	−1.0000000	−1.0000000
0.00	−1.0000000	−1.0000000
0.20	−1.0000000	−1.0000000
0.40	−1.0000000	−1.0000000
0.60	−1.0000000	−1.0000000
0.80	−1.0000000	−1.0000000
1.00	−1.0000000	−1.0000000

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = -1$ (cf. [8], p. 9–10, Eqs. (1.10), (1.13) and (1.14), Case 2).

The numerical results are given in Table 6(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

8. Right-end boundary layer problems

Finally, we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1], \tag{21}$$

$$\text{with } y(0) = \alpha, \tag{22a}$$

$$\text{and } y(1) = \beta, \tag{22b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \leqslant M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

From the theory of singular perturbations the solution of (21), and (22) is of the form

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon), \tag{23}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(0) = \alpha. \tag{24}$$

By taking the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '1' and restricting to their first terms, (23) becomes,

$$y(x) = y_0(x) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-x)} + O(\varepsilon). \tag{25}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, 2, \ldots, N$.

From (25) we have

$$y(ih) = y_0(ih) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-ih)} + O(\varepsilon).$$

Therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))\mathrm{e}^{\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\left(\frac{1}{\varepsilon}-i\rho\right)}, \tag{26}$$

where $\rho = h/\varepsilon$.

Now, we consider the second-order finite difference scheme

$$\varepsilon\sigma(\rho)\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{y_{i+1}-y_{i-1}}{2h}\right) + b(x_i)y_i = f(x_i),$$
$$1 \leqslant i \leqslant N-1, \tag{27}$$

$y_0 = \alpha$; $y_N = \beta$; where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (27) converges uniformly to the solution of (21) and (22).

Multiplying (27) by h and taking the limit as $h \to 0$; we get

$$\lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y_{i+1}-2y_i+y_{i-1}) + \frac{1}{2}a(ih)(y_{i+1}-y_{i-1})\right] = 0$$

if $f(x_i) + b(x_i)y_i$ is bounded.

$$\text{i.e, } \lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y(ih+h)-2y(ih)+y(ih-h)) + \frac{1}{2}a(ih)(y(ih+h)-y(ih-h))\right] = 0, \tag{28}$$

substituting (26) in (28) and simplifying, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho} = \frac{1}{2}a(0)\coth\left[\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right]. \tag{29}$$

$$\therefore \text{ We have } \sigma = \frac{\rho}{2}a(0)\coth\left[\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right]. \tag{30}$$

From (27) we have

$$\left(\frac{\varepsilon\sigma}{h^2} - \frac{a(x_i)}{2h}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma}{h^2} - b(x_i)\right)y_i + \left(\frac{\varepsilon\sigma}{h^2} + \frac{a(x_i)}{2h}\right)y_{i+1} = f(x_i),$$
$$i = 1, 2, \ldots, N-1, \tag{31}$$

where the fitting factor σ is given by (30).

Eq. (31) can be written as a three term recurrence relation:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, 3, \ldots, N-1, \tag{32}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} - \frac{a(x_i)}{2h},$$

$$F_i = \frac{2\varepsilon\sigma}{h^2} - b(x_i),$$

$$G_i = \frac{\varepsilon\sigma}{h^2} + \frac{a(x_i)}{2h},$$

$$H_i = f(x_i).$$

This gives us the tridiagonal system which can be solved easily by Thomas algorithm.

9. Examples with right-end boundary layer

To illustrate the method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval we considered two examples. The approximate solution is compared with the exact solution.

Example 9.1. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at $x = 1$, i.e., at the right end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{(e^{(x-1)/\varepsilon} - 1)}{(e^{-1/\varepsilon} - 1)}.$$

The numerical results are given in Table 7(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 9.2. Now we consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$, and $y(1) = 1 + 1/e$.

Clearly this problem has a boundary layer at $x = 1$. The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$.

The numerical results are given in Table 8(Panel A) and (Panel B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 7
Numerical results of Example 9.1, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	1.0000000	1.0000000
0.10	0.9999999	1.0000000
0.20	0.9999999	1.0000000
0.30	0.9999999	1.0000000
0.40	0.9999999	1.0000000
0.50	0.9999999	1.0000000
0.60	0.9999999	1.0000000
0.70	0.9999999	1.0000000
0.80	0.9999999	1.0000000
0.90	0.9999999	1.0000000
0.92	0.9999999	1.0000000
0.94	0.9999999	1.0000000
0.96	0.9999999	1.0000000
0.98	0.9999999	1.0000000
0.99	0.9999546	0.9999546
1.00	0.0000000	0.0000000
Panel B		
0.00	1.0000000	1.0000000
0.10	1.0000000	1.0000000
0.20	1.0000000	1.0000000
0.30	1.0000000	1.0000000
0.40	1.0000000	1.0000000
0.50	1.0000000	1.0000000
0.60	1.0000000	1.0000000
0.70	1.0000000	1.0000000
0.80	1.0000000	1.0000000
0.90	1.0000000	1.0000000
0.92	1.0000000	1.0000000
0.94	1.0000000	1.0000000
0.96	1.0000000	1.0000000
0.98	1.0000000	1.0000000
0.99	1.0000000	1.0000000
1.00	0.0000000	0.0000000

10. Discussion and conclusions

We have presented an exponentially fitted finite difference method for solving singularly perturbed two-point boundary value problems. We have implemented the present method on three linear examples, three nonlinear examples, with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. Numerical results are presented in tables and compared with the exact solutions. It can be observed from the tables that the present method approximates the exact solution very well.

Table 8
Numerical results of Example 9.2, (Panel A) $\varepsilon = 10^{-3}$, $h = 10^{-2}$ and (Panel B) $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
Panel A		
0.00	1.0000000	1.0000000
0.10	0.9051976	0.9048374
0.20	0.8193827	0.8187308
0.30	0.7417031	0.7408183
0.40	0.6713879	0.6703200
0.50	0.6077387	0.6065307
0.60	0.5501237	0.5488117
0.70	0.4979706	0.4965853
0.80	0.4507618	0.4493290
0.90	0.4080285	0.4065697
0.92	0.3999808	0.3985191
0.94	0.3920919	0.3906278
0.96	0.3843585	0.3828929
0.98	0.3767777	0.3753111
0.99	0.3730880	0.3716217
1.00	1.3678790	1.3678790
Panel B		
0.00	1.0000000	1.0000000
0.10	0.9052780	0.9048374
0.20	0.8195282	0.8187308
0.30	0.7419009	0.7408183
0.40	0.6716266	0.6703200
0.50	0.6080087	0.6065307
0.60	0.5504170	0.5488117
0.70	0.4982805	0.4965853
0.80	0.4510824	0.4493290
0.90	0.4083550	0.4065697
0.92	0.4003080	0.3985191
0.94	0.3924196	0.3906278
0.96	0.3846866	0.382929
0.98	0.3771060	0.3753111
0.99	0.3733720	0.3715767
1.00	1.3678790	1.3678790

References

[1] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.
[2] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.
[3] P.W. Hemker, J.J.H. Miller (Eds.), Numerical Analysis of Singular Perturbation Problems, Academic Press, New York, 1979.
[4] M.K. Kadalbajoo, Y.N. Reddy, Asymptotic and numerical analysis of singular perturbations; a survey, Applied Mathematics and Computation 30 (1) (1989) 223–259.

[5] J. Kevorkian, J.D. Cole, Perturbation Methods in Applied Mathematics, Springer-Verlag, New York, 1981.
[6] J.J.H. Miller (Ed.), Boundary and Interior Layers—Computational and Asymptotic Methods, Proceedings of BAIL I Conference, Boole Press, Dublin, 1980.
[7] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1973.
[8] R.E. O' Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.
[9] H.J. Reinhardt, Singular Perturbations of difference methods for linear ordinary differential equations, Applicable Analysis 10 (1980) 53–70.