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Numerical patching method for singularly perturbed two-point boundary value problems using cubic splines

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Abstract

A numerical patching method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The method is distinguished by the following fact: The original singularly perturbed two-point boundary value problem is divided into two problems, namely inner and outer region problems. The terminal boundary condition is obtained from the solution of the reduced problem. Using general stretching transformation, a modified inner region problem is constructed. Then, both inner region problem and outer region problems are solved as two-point boundary value problems by employing cubic splines. Several linear and non-linear problems are solved to demonstrate the applicability of the method.

Keywords: Ordinary differential equations; Two-point boundary value problems; Singular perturbation problems; Boundary layer; Cubic splines

1. Introduction

Singularly perturbed second order two-point boundary value problems arise very frequently in fluid mechanics and other branches of Applied Mathematics.

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These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. So, typically there are thin transition layers where the solutions can jump abruptly, while away from the layers the solution behaves regularly and vary slowly. The numerical treatment of the singular perturbation problems is far from the trivial because of the boundary layer behavior of the solution. There are a wide variety of asymptotic techniques for solving singular perturbation problems (cf. [2–6]).

In this paper, a numerical patching method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The method is distinguished by the following fact: The original singularly perturbed two-point boundary value problem is divided into two problems, namely inner and outer region problems. The terminal boundary condition is obtained from the solution of the reduced problem. Using general stretching transformation, a modified inner region problem is constructed. Then, both inner region problem and outer region problems are solved as two-point boundary value problems by employing cubic splines. The proposed method is iterative on the terminal point. The process is to be repeated for various choices of the terminal point, until the solution profiles do not differ materially from iteration to iteration. Several linear and non-linear problems are solved to demonstrate the applicability of the method.

2. Numerical patching method: linear problems

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1] \tag{1}$$

with

$$y(0) = \alpha \tag{2a}$$

and

$$y(1) = \beta, \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \geqslant M > 0$ throughout the interval [0, 1], where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$. Consider $x_p = O(\varepsilon)$ be the thickness of the boundary layer.

Now we divide the original problem into two problems, an inner region problem and an outer region problem. The inner region problem is defined in the interval $0 \leqslant x \leqslant x_{\mathrm{p}}$ and the outer region problem is defined in the interval $x_{\mathrm{p}} \leqslant x \leqslant 1$.

3. Terminal boundary condition

To obtain the boundary condition at terminal point x_{p}, we solve the reduced problem with an appropriate boundary condition.

The reduced problem is

$$a(x)y'(x) + b(x)y(x) = f(x) \tag{3}$$

with

$$y(1) = \beta. \tag{4}$$

Let $y_{\mathrm{R}}(x)$ be the (analytical or numerical) solution of the reduced problem. At the terminal point x_{p} we get $y(x_{\mathrm{p}}) = y_{\mathrm{R}}(x_{\mathrm{p}})$.

Let the terminal boundary condition: $y(x_{\mathrm{p}}) = \gamma$ (say).

4. Inner region problem

Since the terminal boundary condition is common to both the inner and outer regions, we define the inner region problem as a two-point boundary value problem:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad 0 \leqslant x \leqslant x_{\mathrm{p}} \tag{5}$$

with

$$y(0) = \alpha \tag{6a}$$

and

$$y(x_{\mathrm{p}}) = \gamma. \tag{6b}$$

Now we introduce a stretched variable 't' to magnify the boundary layer region and there by eliminate any rapid variation that might be exhibited by the solution when the solution is considered as a function of the stretched variable.

We can think of two stretching transformations:

$$\text{(i) } t = \frac{x}{\varepsilon} \quad \text{and} \quad \text{(ii) } t = \frac{\int_0^x a(s)\,\mathrm{d}s}{\varepsilon}.$$

The constant multiple of x/ε would be equally effective.

Since $a(x) > 0 \ \forall\, x \in [0, 1]$; we have $a(0) > 0$.

Now we consider the transformation

$$t = \frac{a(0)x}{\varepsilon}$$

to construct a new differential equation for the inner region solution:

Then we get

$$y(x) = y\left(\frac{t\varepsilon}{a(0)}\right) = Y(t),$$

$$y'(x) = \frac{a(0)}{\varepsilon}y'\left(\frac{t\varepsilon}{a(0)}\right) = \frac{a(0)}{\varepsilon}Y'(t),$$

$$y''(x) = \left(\frac{a(0)}{\varepsilon}\right)^2 y''\left(\frac{t\varepsilon}{a(0)}\right) = \left(\frac{a(0)}{\varepsilon}\right)^2 Y''(t),$$

$$a(x) = a\left(\frac{t\varepsilon}{a(0)}\right) = A(t),$$

$$b(x) = b\left(\frac{t\varepsilon}{a(0)}\right) = B(t),$$

$$f(x) = f\left(\frac{t\varepsilon}{a(0)}\right) = F(t),$$

$$y(0) = \alpha = Y(0) \quad \text{and} \quad y(x_{\mathrm{p}}) = \gamma = Y(t_P).$$

Substituting these in (5), we get

$$(a(0))^2 Y''(t) + a(0)A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \tag{7}$$

with

$$Y(0) = \alpha \tag{8a}$$

and

$$Y(t_{\mathrm{p}}) = \gamma. \tag{8b}$$

This is the modified inner region problem over $[0, t_{\mathrm{p}}]$.

5. Outer region problem

In the outer region, we have two-point boundary value problem as

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x_{\mathrm{p}} \leqslant x \leqslant 1 \tag{9}$$

with

$$y(x_{\mathrm{p}}) = \gamma \tag{10a}$$

and

$$y(1) = \beta. \tag{10b}$$

This is the outer region problem over $[x_{\mathrm{p}}, 1]$.

6. Cubic spline approximation

In order to solve the two-point boundary value problems (7)–(8) and (9)–(10), we have used cubic spline approximation [1].

By definition, cubic spline is a continuous function and has continuous first and second derivatives. The spline proves to be an effective tool in the elementary process of interpolation and approximate integration. The outstanding characteristic, however, is its effectiveness in numerical differentiation. Cubic splines are frequently used to find the solution of two-point boundary value problems. We briefly present the method of finding solution of two-point boundary value problems by cubic splines:

Consider the two-point boundary value problem

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x); \quad a \leqslant x \leqslant b \tag{11}$$

with

$$y(a) = k_1 \tag{12a}$$

and

$$y(b) = k_2. \tag{12b}$$

We divide the interval $[a, b]$ into n equal parts with mesh size h and having nodes at $a = x_0, x_1, x_2, x_3, \dots, x_n = b$. By definition: A spline function of degree m with nodes at the points x_i; $i = 0, 1, 2, \dots, n$ is a function $S(x)$ with the properties:

(i) on each interval $[x_{i-1}, x_i]$; $i = 0, 1, 2, \dots, n$; $S(x)$ is a polynomial of degree m.
(ii) $S(x)$ and its first $m-1$ are continuous on $[a, b]$.

To find the cubic spline approximation for the function $y(x)$, $x \in [x_0, x_n]$ at the nodes we have

$$S''(x) = \left(\frac{x_i - x}{h}\right)M_{i-1} + \left(\frac{x - x_{i-1}}{h}\right)M_i, \tag{13}$$

where $S''(x_i) = M_i$.

The interpolating conditions are

$$S(x_{i-1}) = y_{i-1} \tag{14}$$

and

$$S(x_i) = y_i. \tag{15}$$

Integrating (13) twice and substituting the interpolating conditions (14) and (15), we obtain the cubic spline approximation function

$$S(x) = \frac{(x_i - x)^3}{6h} M_{i-1} + \frac{(x - x_{i-1})^3}{6h} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) \left(\frac{x_i - x}{h} \right) + \left(y_i - \frac{h^2}{6} M_i \right) \left(\frac{x - x_{i-1}}{h} \right). \tag{16}$$

The continuity of the first order derivative of $S(x)$ at $x = x_i$ requires

$$S'(x_i^-) = S'(x_i^+). \tag{17}$$

We have

$$\text{(i)}\ S'(x_i^-) = \frac{h}{3} M_i + \frac{h}{6} M_{i-1} + \frac{y_i - y_{i-1}}{h}; \quad i = 1, 2, \ldots, n,$$

$$\text{(ii)}\ S'(x_i^+) = -\frac{h}{3} M_i - \frac{h}{6} M_{i+1} + \frac{y_{i+1} - y_i}{h}; \quad i = 0, 1, 2, \ldots, n-1$$

and so that the continuity of the first order derivatives implies

$$\frac{h}{6} M_{i-1} + \frac{2h}{3} M_i + \frac{h}{6} M_{i+1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h}; \quad i = 1, 2, \ldots, n-1. \tag{18}$$

From (11) we have

$$y''(x) = -p(x)y'(x) - q(x)y(x) + r(x).$$

We have $M_i = S''(x_i) = y_i''$; $i = 0, 1, 2, \ldots, n$, therefore

$$M_i = -p_i y_i' - q_i y_i + r_i; \quad i = 0, 1, 2, \ldots, n,$$

where $p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$,

$$\text{i.e.;}\ M_i = -p_i S_i' - q_i S_i + r_i. \tag{19}$$

Replacing S_i' by $S'(x_i^+)$ and S_i by y_i in (19) we get

$$\left(1 - \frac{hp_i}{3} \right) M_i - \frac{hp_i}{6} M_{i+1} = r_i - q_i y_i - \frac{p_i}{h} (y_{i+1} - y_i); \quad i = 0, 1, 2, \ldots, n-1. \tag{20}$$

Replacing S_i' by $S'(x_i^-)$ and S_i by y_i in (19) we get

$$\frac{hp_i}{6} M_{i-1} + \left(1 + \frac{hp_i}{3} \right) M_i = r_i - q_i y_i - \frac{p_i}{h} (y_i - y_{i-1}); \quad i = 1, 2, \ldots, n. \tag{21}$$

Adding (20) and (21), we get

$$\frac{hp_i}{6} M_{i-1} + 2M_i - \frac{hp_i}{6} M_{i+1} = 2r_i - 2q_i y_i - \frac{p_i}{h} (y_{i+1} - y_{i-1}); \quad i = 1, 2, \ldots, n-1. \tag{22}$$

Eliminating M_i from (22) and (18), we get

$$\left(1+\frac{hp_i}{3}\right)y_{i+1}-2\left(1-\frac{h^2q_i}{3}\right)y_i+\left(1-\frac{hp_i}{3}\right)y_{i-1}$$
$$=\frac{2h^2}{3}r_i+\frac{h^2}{6}\left(1-\frac{hp_i}{3}\right)M_{i-1}+\frac{h^2}{6}\left(1+\frac{hp_i}{3}\right)M_{i+1}. \tag{23}$$

Replacing i by $i-1$ in (20) we have

$$\left(1-\frac{hp_{i-1}}{3}\right)M_{i-1}-\frac{hp_{i-1}}{6}M_i=r_{i-1}-q_{i-1}y_{i-1}-\frac{p_{i-1}}{h}(y_i-y_{i-1});$$
$$i=1,2,\ldots,n. \tag{24}$$

Eliminating M_i from (21) and (24), we get

$$\left[\left(1+\frac{hp_i}{3}\right)\left(1-\frac{hp_{i-1}}{3}\right)+\frac{h^2p_{i-1}p_i}{36}\right]M_{i-1}$$
$$=\left(1+\frac{hp_i}{3}\right)(r_{i-1}-q_{i-1}y_{i-1})-\left(1+\frac{hp_i}{3}\right)\left(\frac{p_{i-1}}{h}\right)(y_i-y_{i-1})$$
$$+\frac{hp_{i-1}}{6}(r_i-q_iy_i)-\frac{p_ip_{i-1}}{6}(y_i-y_{i-1});\quad i=1,2,\ldots,n. \tag{25}$$

Consider

$$\left(1+\frac{hp_i}{3}\right)\left(1-\frac{hp_{i-1}}{3}\right)+\frac{h^2p_{i-1}p_i}{36}=a_i. \tag{26}$$

Therefore (25) becomes

$$a_iM_{i-1}=\left(1+\frac{hp_i}{3}\right)\left[r_{i-1}-q_{i-1}y_{i-1}-\frac{p_{i-1}}{h}(y_i-y_{i-1})\right]$$
$$+\frac{hp_{i-1}}{6}\left[r_i-q_iy_i-\frac{p_i}{h}(y_i-y_{i-1})\right];\quad i=1,2,\ldots,n. \tag{27}$$

Replacing i by $i+1$ in (21), we get

$$\frac{hp_{i+1}}{6}M_i+\left(1+\frac{hp_{i+1}}{3}\right)M_{i+1}=r_{i+1}-q_{i+1}y_{i+1}-\frac{p_{i+1}}{h}(y_{i+1}-y_i);$$
$$i=0,1,2,\ldots,n-1. \tag{28}$$

Eliminating M_i from (20) and (28), we get

$$\left[\left(1+\frac{hp_{i+1}}{3}\right)\left(1-\frac{hp_i}{3}\right)+\frac{h^2p_ip_{i+1}}{36}\right]M_{i+1}$$
$$=\left(1-\frac{hp_i}{3}\right)(r_{i+1}-q_{i+1}y_{i+1})-\left(1-\frac{hp_i}{3}\right)\left(\frac{p_{i+1}}{h}\right)(y_{i+1}-y_i)$$
$$-\frac{hp_{i+1}}{6}(r_i-q_iy_i)+\frac{p_ip_{i+1}}{6}(y_{i+1}-y_i);\quad i=0,1,2,\ldots,n-1.$$

Consider $b_i=a_{i+1}$; $i=0,1,2,\ldots,n-1$.

Therefore we have

$$b_iM_{i+1}=\left(1-\frac{hp_i}{3}\right)\left[r_{i+1}-q_{i+1}y_{i+1}-\frac{p_{i+1}}{h}(y_{i+1}-y_i)\right]$$
$$-\frac{hp_{i+1}}{6}\left[r_i-q_iy_i-\frac{p_i}{h}(y_{i+1}-y_i)\right];\quad i=0,1,2,\ldots,n-1. \tag{29}$$

Substituting (27) and (29) in (23), and by simplifying, we get

$$a_i\left(1+\frac{hp_{i+1}}{2}+\frac{h^2q_{i+1}}{6}\right)y_{i+1}-\left[a_i\left(1+\frac{hp_{i+1}}{2}\right)+b_i\left(1-\frac{hp_{i-1}}{2}\right)-\frac{2h^2q_ic_i}{3}\right]y_i$$
$$+b_i\left(1-\frac{hp_{i-1}}{2}+\frac{h^2q_{i-1}}{6}\right)y_{i-1}$$
$$=\frac{h^2}{6}(a_ir_{i+1}+4c_ir_i+b_ir_{i-1});\quad i=1,2,\ldots,n-1, \tag{30}$$

where

$$c_i=1+\frac{7h}{24}(p_{i+1}-p_{i-1})-\frac{h^2p_{i-1}p_{i+1}}{12}.$$

Eq. (30) constitute a tridiagonal algebraic system. The solution of this tridiagonal system can be obtained by using Thomas algorithm also called discrete invariant imbedding.

7. Solution of the original problem

After getting the solution of the inner region problem and outer region problem, we combine both to obtain the approximate solution of the original problem (1)–(2) over the interval $0\leqslant x\leqslant 1$. We repeat the process for various choices of x_{p}, until the solution profiles do not differ materially from iteration to iteration. For computational purposes we use an absolute error criterion, namely

$$\left|Y(t)^{m+1}-Y(t)^m\right|\leqslant\sigma;\quad 0\leqslant t\leqslant t_{\mathrm{p}},$$

where $Y(t)^m$ is the mth iterate of the inner region solution and σ is the prescribed tolerance bound.

8. Linear examples

To demonstrate the applicability of the method we have applied it on four linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example 8.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [2, p. 480; problem 9.17 with $\alpha = 0$]

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 1$.

For this example the stretching transformation is

$$t = \frac{x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) - \varepsilon Y(t) = 0; \quad \text{with } Y(0) = 1, \quad Y(t_p) = e^{\varepsilon t_p - 1}$$

and the outer region problem is given by

$$\varepsilon y''(x) + y'(x) - y(x) = 0 \quad \text{with } y(x_p) = e^{x_p - 1}, \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]},$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon)$.

The numerical results are given in Table 1 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 8.2. Now consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

For this example the stretching transformation is

$$t = \frac{x}{\varepsilon}$$

Table 1
Numerical results of Example 8.1

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.0000	1.0000000	1.0000000	1.0000000	1.0000000
0.0005	0.7456129	0.7471300	0.7471403	0.7514171
0.0010	0.5931332	0.5955608	0.5955774	0.6007918
0.0025	0.4143062	0.4178072	0.4178311	0.4208957
0.0050	0.3697234	0.3735048	0.3735305	0.3743263
0.0075		0.3709218	0.3709478	0.3713630
0.0100		0.3715767	0.3716028	0.3719724
0.0200			0.3753111	0.3756784
0.1000	0.4069324	0.4069451	0.4067746	0.4069350
0.2000	0.4496854	0.4496888	0.4497448	0.4496879
0.3000	0.4969299	0.4969331	0.4969121	0.4969323
0.4000	0.5491381	0.5491410	0.5491475	0.5491404
0.5000	0.6068313	0.6068341	0.6068309	0.6068334
0.6000	0.6705859	0.6705883	0.6705886	0.6705877
0.7000	0.7410386	0.7410405	0.7410397	0.7410401
0.8000	0.8188931	0.8188946	0.8188943	0.8188942
0.9000	0.9049271	0.9049280	0.9049277	0.9049277
1.0000	1.0000000	1.0000000	1.0000000	1.0000000
$\varepsilon = 10^{-4}$				
0.00000	1.0000000	1.0000000	1.0000000	1.0000000
0.00005	0.7456136	0.7471414	0.7471507	0.7512937
0.00010	0.5929970	0.5954415	0.5954563	0.6004604
0.00025	0.4135686	0.4170914	0.4171129	0.4198798
0.00050	0.3680634	0.3718610	0.3718841	0.3723570
0.00075		0.3684295	0.3684528	0.3685416
0.00100		0.3682475	0.3682709	0.3683130
0.00200			0.3686159	0.3686527
0.10000	0.4066143	0.4066097	0.4066016	0.4066062
0.20000	0.4493727	0.4493683	0.4493601	0.4493649
0.30000	0.4966276	0.4966233	0.4966155	0.4966201
0.40000	0.5488518	0.5488476	0.5488402	0.5488446
0.50000	0.6065675	0.6065637	0.6065568	0.6065609
0.60000	0.6703526	0.6703491	0.6703435	0.6703469
0.70000	0.7408450	0.7408426	0.7408376	0.7408405
0.80000	0.8187506	0.8187487	0.8187451	0.8187471
0.90000	0.9048485	0.9048474	0.9048455	0.9048465
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

and the inner region problem is given by

$$Y''(t) + Y'(t) = \varepsilon(1 + 2\varepsilon t); \quad \text{with } Y(0) = 0, \quad Y(t_p) = \varepsilon^2 t_p^2 + \varepsilon t_p - 1$$

and the outer region problem is given by

$$\varepsilon y''(x) + y'(x) = 1 + 2x \quad \text{with } y(x_p) = x_p^2 + x_p - 1, \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)\left(1 - e^{-x/\varepsilon}\right)}{(1 - e^{-1/\varepsilon})}.$$

The numerical results are given in Table 2 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 8.3. Now we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [4, p. 33; Eqs. (2.3.26) and (2.3.27) with $\alpha = -1/2$]

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

For this example the stretching transformation is

$$t = \frac{x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + \left(1 - \frac{t\varepsilon}{2}\right)Y'(t) - \frac{\varepsilon}{2}Y(t) = 0;$$

with

$$Y(0) = 0, \quad Y(t_p) = \frac{1}{2 - \varepsilon t_p}$$

and the outer region problem is given by

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0$$

with

$$y(x_p) = \frac{1}{2 - x_p}, \quad y(1) = 1.$$

We have chosen to use uniformly valid approximation (which is obtained by the method) given by Neyfah [5, p. 148; Eq. (4.2.32)] as our 'exact' solution:

$$y(x) = \frac{1}{2 - x} - \frac{1}{2}e^{-(x - x^2/4)/\varepsilon}.$$

The numerical results are given in Table 3 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 2
Numerical results of Example 8.2

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.0005	−0.4019300	−0.3995073	−0.3994845	−0.3921832
0.0010	−0.6428878	−0.6390115	−0.6389750	−0.6298573
0.0025	−0.9253398	−0.9197540	−0.9197015	−0.9135779
0.0050	−0.9949750	−0.9889548	−0.9888982	−0.9862605
0.0075		−0.9920051	−0.9919482	−0.9899068
0.0100		−0.9899000	−0.9898432	−0.9878747
0.0200			−0.9796000	−0.9776400
0.1000	−0.8881999	−0.8881485	−0.8890784	−0.8882000
0.2000	−0.7584000	−0.7583989	−0.7580781	−0.7584000
0.3000	−0.6086000	−0.6085996	−0.6087180	−0.6086000
0.4000	−0.4387999	−0.4387994	−0.4387568	−0.4388000
0.5000	−0.2490000	−0.2489994	−0.2490159	−0.2490000
0.6000	−0.0392000	−0.0391994	−0.0391942	−0.0392001
0.7000	0.1906000	0.1906006	0.1905978	0.1906000
0.8000	0.4404000	0.4404005	0.4404007	0.4404000
0.9000	0.7102000	0.7102003	0.7101997	0.7102000
1.0000	1.0000000	1.0000000	1.0000000	1.0000000
$\varepsilon = 10^{-4}$				
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00005	−0.4023833	−0.3999644	−0.3999496	−0.3933406
0.00010	−0.6437932	−0.6399230	−0.6398994	−0.6318942
0.00025	−0.9276002	−0.9220233	−0.9219892	−0.9174814
0.00050	−0.9994997	−0.9934891	−0.9934524	−0.9925632
0.00075		−0.9988156	−0.9987786	−0.9984966
0.00100		−0.9989990	−0.9989620	−0.9987538
0.00200			−0.9979960	−0.9977964
0.10000	−0.8898200	−0.8898221	−0.8898211	−0.8898200
0.20000	−0.7598400	−0.7598396	−0.7598401	−0.7598400
0.30000	−0.6098599	−0.6098576	−0.6098590	−0.6098600
0.40000	−0.4398801	−0.4398761	−0.4398782	−0.4398800
0.50000	−0.2499001	−0.2498951	−0.2498977	−0.2499000
0.60000	−0.0399201	−0.0399147	−0.0399174	−0.0399199
0.70000	0.1900599	0.1900651	0.1900625	0.1900601
0.80000	0.4400400	0.4400442	0.4400421	0.4400400
0.90000	0.7100199	0.7100226	0.7100213	0.7100201
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Example 8.4. Finally we consider the following singular perturbation problem

$$\varepsilon y''(x) + 2y'(x) + y(x) = -3; \quad x \in [-1, 1]$$

with $y(-1) = 1$ and $y(1) = 2$.

Table 3
Numerical results of Example 8.3

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.0005	0.2013328	0.2001186	0.2001121	0.1968407
0.0010	0.3221646	0.3202217	0.3202114	0.3162644
0.0025	0.4645047	0.4617034	0.4616884	0.4595191
0.0050	0.5012531	0.4982302	0.4982141	0.4978630
0.0075		0.5016630	0.5016468	0.5016016
0.0100		0.5025126	0.5024964	0.5024893
0.0200			0.5050505	0.5050505
0.1000	0.5270723	0.5270895	0.5267506	0.5263158
0.2000	0.5563190	0.5563194	0.5564258	0.5555556
0.3000	0.5889998	0.5889999	0.5889670	0.5882353
0.4000	0.6257567	0.6257567	0.6257664	0.6250000
0.5000	0.6674021	0.6674022	0.6673999	0.6666667
0.6000	0.7149801	0.7149802	0.7149810	0.7142857
0.7000	0.7698538	0.7698539	0.7698540	0.7692308
0.8000	0.8338380	0.8338382	0.8338384	0.8333333
0.9000	0.9094033	0.9094036	0.9094036	0.9090909
1.0000	1.0000000	1.0000000	1.0000000	1.0000000
$\varepsilon = 10^{-4}$				
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00005	0.2012282	0.2000183	0.2000110	0.1967453
0.00010	0.3219684	0.3200324	0.3200208	0.3160807
0.00025	0.4639831	0.4611932	0.4611764	0.4590136
0.00050	0.5001251	0.4971178	0.4970998	0.4967540
0.00075		0.4999703	0.4999521	0.4999107
0.00100		0.5002501	0.5002319	0.5002274
0.00200			0.5005005	0.5005005
0.10000	0.5263911	0.5263913	0.5263919	0.5263158
0.20000	0.5556318	0.5556319	0.5556325	0.5555556
0.30000	0.5883117	0.5883120	0.5883123	0.5882353
0.40000	0.6250751	0.6250759	0.6250764	0.6250000
0.50000	0.6667399	0.6667402	0.6667413	0.6666667
0.60000	0.7143551	0.7143552	0.7143559	0.7142857
0.70000	0.7692937	0.7692935	0.7692939	0.7692308
0.80000	0.8333846	0.8333842	0.8333845	0.8333333
0.90000	0.9091222	0.9091225	0.9091226	0.9090909
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

For this example the stretching transformation is

$$t = \frac{2(x+1)}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) + \frac{\varepsilon}{4} Y(t) = \frac{-3\varepsilon}{4}$$

with

$$Y(0) = 1, \quad Y(t_p) = -3 + 5e^{1-(t_p\varepsilon/4)}$$

and the outer region problem is given by

$$\varepsilon y''(x) + 2y'(x) + y(x) = -3; \quad \text{with } y(x_p) = -3 + 5e^{(1-x_p)/2}, \quad y(1) = 2.$$

The exact solution is given by

$$y(x) = \frac{(4e^{-m_2} - 5e^{m_2})e^{-m_1 x} - (4e^{-m_1} - 5e^{m_1})e^{-m_2 x}}{e^{(m_1-m_2)} - e^{-(m_1-m_2)}} - 3,$$

where $m_1 = (1 - \sqrt{1-\varepsilon})/\varepsilon$ and $m_2 = (1 + \sqrt{1-\varepsilon})/\varepsilon$.

The numerical results are given in Table 4 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

9. Non-linear problems

We have applied the present method on three non-linear singular perturbation problems with left-end boundary layer. Non-linear singular perturbation problems are first converted as a sequence of linear singular perturbation problems by using quasilinearization method. The solution of the reduced problem is taken as initial approximation.

Example 9.1. Consider the following singular perturbation problem from Bender and Orszag [2, p. 463; Eq. (9.7.1)]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 0$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right].$$

For this example the stretching transformation is

$$t = \frac{2x}{\varepsilon}$$

Table 4
Numerical results of Example 8.4

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
−1.00000	1.0000000	1.0000000	1.0000000	1.0000000
−0.99975	4.8575020	4.8342570	4.8341130	4.7726250
−0.99950	7.1716220	7.1344310	7.1342020	7.0609590
−0.99850	10.1891000	10.1337100	10.1333700	10.1065700
−0.99800	10.4740100	10.4169100	10.4165500	10.4053000
−0.99750	10.5744300	10.5167100	10.5163600	10.5130900
−0.99600		10.5618900	10.5615400	10.5644100
−0.99500		10.5574700	10.5571200	10.5604200
−0.99200			10.5371500	10.5405200
−0.99000			10.5236200	10.5269900
−0.80000	9.3005050	9.3007840	9.3009080	9.3007850
−0.60000	8.1297150	8.1299300	8.1299710	8.1299320
−0.40000	7.0703620	7.0705280	7.0705590	7.0705270
−0.20000	6.1118380	6.1119610	6.1119880	6.1119620
0.00000	5.2445480	5.2446370	5.2446580	5.2446380
0.20000	4.4598080	4.4598700	4.4598830	4.4598700
0.40000	3.7497590	3.7498010	3.7498100	3.7498010
0.60000	3.1072950	3.1073200	3.1073250	3.1073200
0.80000	2.5259830	2.5259930	2.5259950	2.5259930
1.00000	2.0000000	2.0000000	2.0000000	2.0000000
$\varepsilon = 10^{-4}$				
−1.000000	1.0000000	1.0000000	1.0000000	0.9999996
−0.999975	4.8596660	4.8364660	4.8363310	4.7708310
−0.999950	7.1754270	7.1383080	7.1380920	7.0632990
−0.999850	10.1985000	10.1432100	10.1428900	10.1133900
−0.999800	10.4863100	10.4292900	10.4289600	10.4146000
−0.999750	10.5897100	10.5320700	10.5317300	10.5253900
−0.999600		10.5863400	10.5860000	10.5858100
−0.999500		10.5880100	10.5876800	10.5879200
−0.999200			10.5859700	10.5863100
−0.999000			10.5846100	10.5849500
−0.800000	9.2972310	9.2983870	9.2983610	9.2982920
−0.600000	8.1270920	8.1280050	8.1279860	8.1279270
−0.400000	7.0683000	7.0690000	7.0689860	7.0689390
−0.200000	6.1102520	6.1107740	6.1107690	6.1107300
0.000000	5.2433610	5.2437410	5.2437390	5.2437090
0.200000	4.4589550	4.4592200	4.4592180	4.4591980
0.400000	3.7491870	3.7493590	3.7493580	3.7493440
0.600000	3.1069540	3.1070510	3.1070530	3.1070440
0.800000	2.5258310	2.5258710	2.5258720	2.5258680
1.000000	2.0000000	2.0000000	2.0000000	2.0000000

and the inner region problem is given by

$$Y''(t) + 2Y'(t) + \frac{\varepsilon}{2 + t\varepsilon} Y(t) = \left(\frac{\varepsilon}{2 + t\varepsilon}\right)\left[\log_e\left(\frac{4}{2 + t\varepsilon}\right) - 1\right]$$

with

$$Y(0) = 0, \quad Y(t_p) = \log_e\left(\frac{4}{2 + \varepsilon t_p}\right)$$

and the outer region problem is given by

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

with

$$y(x_p) = \log_e\left(\frac{2}{1 + x_p}\right), \quad y(1) = 0.$$

We have chosen to use Bender and Orszag's uniformly valid approximation [2, p. 463; Eq. (9.7.6)] for comparison,

$$y(x) = \log_e(2/(1+x)) - (\log_e 2)e^{-2x/\varepsilon}.$$

For this example, we have boundary layer of thickness $O(\varepsilon)$ at $x = 0$ (cf. [2]).

The numerical results are given in Table 5 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 9.2. Now consider the following singular perturbation problem from Kevorkian and Cole [4, p. 56; Eq. (2.5.1)]

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = -1$ and $y(1) = 3.9995$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995.$$

For this example the stretching transformation is

$$t = \frac{(2.9995)x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + \left[\frac{t\varepsilon}{(2.9995)^2} + 1\right] Y'(t) = \left(\frac{\varepsilon}{2.9995}\right)\left[\frac{t\varepsilon}{(2.9995)^2} + 1\right]$$

Table 5
Numerical results of Example 9.1

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00025	0.2785943	0.2769114	0.2769021	0.2724822
0.00050	0.4456941	0.4430018	0.4429868	0.4376527
0.00125	0.6418320	0.6379539	0.6379324	0.6350010
0.00250	0.6906503	0.6864750	0.6864519	0.6859799
0.00375		0.6891031	0.6890798	0.6890208
0.00500		0.6881596	0.6881363	0.6881282
0.01000			0.6831968	0.6831968
0.10000	0.5981092	0.5981095	0.5981665	0.5978370
0.20000	0.5110389	0.5110391	0.5110467	0.5108256
0.30000	0.4309491	0.4309491	0.4309499	0.4307829
0.40000	0.3568026	0.3568026	0.3568026	0.3566749
0.50000	0.2877782	0.2877781	0.2877780	0.2876821
0.60000	0.2232134	0.2232133	0.2232133	0.2231436
0.70000	0.1625668	0.1625668	0.1625668	0.1625189
0.80000	0.1053898	0.1053898	0.1053898	0.1053605
0.90000	0.0513068	0.0513068	0.0513068	0.0512933
1.00000	0.0000000	0.0000000	0.0000000	0.0000000
$\varepsilon = 10^{-4}$				
0.000000	0.0000000	0.0000000	0.0000000	0.0000000
0.000025	0.2789103	0.2772330	0.2772226	0.2727072
0.000050	0.4462509	0.4435671	0.4435506	0.4381026
0.000125	0.6430061	0.6391391	0.6391152	0.6361252
0.000250	0.6928972	0.6887299	0.6887042	0.6882268
0.000375		0.6924715	0.6924456	0.6923889
0.000500		0.6926473	0.6926214	0.6926158
0.001000			0.6921477	0.6921477
0.100000	0.5978634	0.5978650	0.5978647	0.5978370
0.200000	0.5108468	0.5108471	0.5108473	0.5108256
0.300000	0.4307997	0.4307995	0.4307997	0.4307829
0.400000	0.3566880	0.3566877	0.3566878	0.3566749
0.500000	0.2876920	0.2876918	0.2876917	0.2876821
0.600000	0.2231507	0.2231505	0.2231505	0.2231435
0.700000	0.1625238	0.1625236	0.1625237	0.1625189
0.800000	0.1053635	0.1053634	0.1053634	0.1053605
0.900000	0.0512946	0.0512946	0.0512946	0.0512933
1.000000	0.0000000	0.0000000	0.0000000	0.0000000

with

$$Y(0) = -1, \quad Y(t_p) = \frac{\varepsilon t_p}{2.9995} + 2.9995$$

and the outer region problem is given by

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995$$

with

$$y(x_p) = x_p + 2.9995, \quad y(1) = 3.9995.$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation [4, pp. 57–58; Eqs. (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right),$$

where $c_1 = 2.9995$ and $c_2 = (1/c_1)\log_e[(c_1 - 1)/(c_1 + 1)]$.

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. [4, pp. 56–66]).

The numerical results are given in Table 6 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 9.3. Finally we consider the following singular perturbation problem from O' Malley [6, p. 9; Eq. (1.10) case 2]:

$$\varepsilon y''(x) - y(x)y'(x) = 0; \quad x \in [-1, 1]$$

with $y(-1) = 0$ and $y(1) = -1$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0.$$

For this example the stretching transformation is

$$t = \frac{x + 1}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) = 0; \quad \text{with } Y(0) = 0, \quad Y(t_p) = -1$$

and the outer region problem is given by

$$\varepsilon y''(x) + y'(x) = 0$$

with

$$y(x_p) = -1, \quad y(1) = -1.$$

We have chosen to use O' Malley's approximate solution [6, pp. 9–10; Eqs. (1.13) and (1.14)] for comparison,

$$y(x) = -\frac{\left(1 - e^{-(x+1)/\varepsilon}\right)}{\left(1 + e^{-(x+1)/\varepsilon}\right)}.$$

Table 6
Numerical results of Example 9.2

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.000000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.000500	2.1554020	2.1364770	2.1363630	1.1487790
0.001000	2.8372090	2.8141980	2.8140590	2.4571870
0.100000	3.0994530	3.0994990	3.0995020	3.0995010
0.200000	3.1994600	3.1994990	3.1995020	3.1995010
0.300000	3.2994660	3.2995000	3.2995020	3.2995010
0.400000	3.3994710	3.3995000	3.3995010	3.3995010
0.500000	3.4994740	3.4995010	3.4995010	3.4995010
0.600000	3.5994790	3.5995010	3.5995000	3.5995000
0.700000	3.6994850	3.6995020	3.6995000	3.6995000
0.800000	3.7994900	3.7995010	3.7995000	3.7995000
0.900000	3.8994940	3.8995000	3.8995000	3.8995000
1.000000	3.9995000	3.9995000	3.9995000	3.9995000
$\varepsilon = 10^{-4}$				
0.000000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.000050	2.1547550	2.1357970	2.1356830	1.1483290
0.000100	2.8362060	2.8131540	2.8130140	2.4562870
0.100000	3.0994030	3.0995250	3.0995390	3.0995000
0.200000	3.1994130	3.1995290	3.1995400	3.1995000
0.300000	3.2994250	3.2995240	3.2995380	3.2995000
0.400000	3.3994370	3.3995230	3.3995370	3.3995000
0.500000	3.4994480	3.4995190	3.4995350	3.4995000
0.600000	3.5994560	3.5995150	3.5995270	3.5995000
0.700000	3.6994690	3.6995140	3.6995220	3.6995000
0.800000	3.7994820	3.7995100	3.7995160	3.7995000
0.900000	3.8994880	3.8995060	3.8995070	3.8995000
1.000000	3.9995000	3.9995000	3.9995000	3.9995000

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = -1$ (cf. [6, pp. 9–10, Eqs. (1.10), (1.13), (1.14), case 2]).

The numerical results are given in Table 7 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

10. Right-end boundary layer problems

Now we discuss our method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1] \tag{31}$$

Table 7
Numerical results of Example 9.3

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
−1.0000	0.0000000	0.0000000	0.0000000	0.0000000
−0.9995	−0.4024333	−0.4000145	−0.3999998	−0.2449296
−0.9970	−0.9591436	−0.9533787	−0.9533435	−0.9051501
−0.9960	−0.9891850	−0.9832396	−0.9832034	−0.9640279
−0.9950	−1.0000000	−0.9939896	−0.9939529	−0.9866142
−0.9920		−0.9997544	−0.9997175	−0.9993293
−0.9900		−1.0000000	−0.9999632	−0.9999092
−0.9800			−1.0000000	−1.0000000
−0.4000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
−0.2000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.0000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.2000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.4000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.6000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.8000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
1.0000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
$\varepsilon = 10^{-4}$				
−1.00000	0.0000000	0.0000000	0.0000000	0.0000000
−0.99995	−0.4024333	−0.4000145	−0.3999998	−0.2449577
−0.99970	−0.9591436	−0.9533787	−0.9533435	−0.9051394
−0.99960	−0.9891850	−0.9832396	−0.9832034	−0.9640300
−0.99950	−1.0000000	−0.9939896	−0.9939529	−0.9866174
−0.99900		−1.0000000	−0.9999632	−0.9999092
−0.99850			−0.9999994	−0.9999994
−0.99800			−1.0000000	−1.0000000
−0.40000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
−0.20000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.00000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.20000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.40000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.60000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
0.80000	−1.0000000	−1.0000000	−1.0000000	−1.0000000
1.00000	−1.0000000	−1.0000000	−1.0000000	−1.0000000

with

$$y(0) = \alpha \tag{32a}$$

and

$$y(1) = \beta, \tag{32b}$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$ and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \leqslant M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$. Consider $x_p = O(\varepsilon)$ be the thickness of the boundary layer.

Now we divide the original problem into two problems, an inner region problem and an outer region problem. The outer region problem is defined in the interval $0 \leqslant x \leqslant x_p$ and the inner region problem is defined in the interval $x_p \leqslant x \leqslant 1$.

11. Terminal boundary condition

To obtain the boundary condition at terminal point x_p, we solve the reduced problem with an appropriate boundary condition.

The reduced problem is

$$a(x)y'(x) + b(x)y(x) = f(x)$$

with $y(0) = \alpha$.

Let the terminal boundary condition: $y(x_p) = \gamma$ (say).

12. Inner region problem

Since the terminal boundary condition is common to both the inner and outer regions, we define the inner region problem as a two-point boundary value problem:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad x_p \leqslant x \leqslant 1 \tag{33}$$

with

$$y(x_p) = \gamma \tag{34a}$$

and

$$y(1) = \beta. \tag{34b}$$

In this case we take the stretching transformation

$$t = \frac{-a(1)(1-x)}{\varepsilon}$$

to construct a new differential equation for the inner region solution:

Then, we get

$$y(x) = y\left(1 + \frac{t\varepsilon}{a(1)}\right) = Y(t),$$

$$y'(x) = \frac{a(1)}{\varepsilon} y'\left(1 + \frac{t\varepsilon}{a(1)}\right) = \frac{a(1)}{\varepsilon} Y'(t),$$

$$y''(x) = \left(\frac{a(1)}{\varepsilon}\right)^2 y''\left(1 + \frac{t\varepsilon}{a(1)}\right) = \left(\frac{a(1)}{\varepsilon}\right)^2 Y''(t),$$

$$a(x) = a\left(1 + \frac{t\varepsilon}{a(1)}\right) = A(t),$$

$$b(x) = b\left(1 + \frac{t\varepsilon}{a(1)}\right) = B(t),$$

$$f(x) = f\left(1 + \frac{t\varepsilon}{a(1)}\right) = F(t)$$

and

$$y(x_{\mathrm{p}}) = \gamma = Y(t_P),$$
$$y(1) = \beta = Y(0).$$

Substituting these in (33), we get

$$(a(1))^2 Y''(t) + a(1)A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \tag{35}$$

with

$$Y(t_{\mathrm{p}}) = \gamma \tag{36a}$$

and

$$Y(0) = \beta. \tag{36b}$$

This is the modified inner region problem $[0, t_{\mathrm{p}}]$.

13. Outer region problem

In the outer region we have two-point boundary value problem as

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad 0 \leqslant x \leqslant x_{\mathrm{p}} \tag{37}$$

with

$$y(0) = \alpha \tag{38a}$$

and

$$y(x_{\mathrm{p}}) = \gamma. \tag{38b}$$

This is the outer region problem over $[0, x_{\mathrm{p}}]$.

14. Solution of the original problem

After getting the solution of the inner region problem and outer region problem, we combine both to obtain the approximate solution of the original problem (31)–(32) over the interval $0 \leqslant x \leqslant 1$. We repeat the process for various choices of x_p, until the solution profiles do not differ materially from iteration to iteration. For computational purposes we use an absolute error criterion, namely

$$\left|Y(t)^{m+1} - Y(t)^{m}\right| \leqslant \sigma; \quad 0 \leqslant t \leqslant t_p,$$

where $Y(t)^m$ is the mth iterate of the inner region solution and σ is the prescribed tolerance bound.

15. Examples with right-end boundary layer

To illustrate the method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval we have implemented on three examples.

Example 15.1. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at $x = 1$. That is; at the right end of the underlying interval.

For this example the stretching transformation is

$$t = \frac{1 - x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) = 0; \quad \text{with } Y(0) = 0, \quad Y(t_p) = 1$$

and the outer region problem is given by

$$\varepsilon y''(x) - y'(x) = 0 \quad \text{with } y(0) = 1, \quad y(x_p) = 1.$$

The exact solution is given by

$$y(x) = \frac{\left(e^{(x-1)/\varepsilon} - 1\right)}{(e^{-1/\varepsilon} - 1)}.$$

The numerical results are given in Table 8 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 15.2. Now we consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$; and $y(1) = 1 + 1/e$.

Table 8
Numerical results of Example 15.1

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.000	1.0000000	1.0000000	1.0000000	1.0000000
0.200	1.0000000	1.0000000	1.0000000	1.0000000
0.400	1.0000000	1.0000010	1.0000000	1.0000000
0.600	1.0000000	1.0000010	1.0000000	1.0000000
0.800	1.0000000	1.0000010	1.0000000	1.0000000
0.980			1.0000000	1.0000000
0.982			1.0000000	1.0000000
0.990		1.0000000	0.9999831	0.9999546
0.992		0.9998645	0.9998477	0.9996645
0.995	1.0000000	0.9959016	0.9958848	0.9932620
0.996	0.9917355	0.9876710	0.9876544	0.9816845
0.998	0.8925620	0.8889039	0.8888890	0.8646612
0.999	0.6694215	0.6666780	0.6666667	0.6321158
1.000	0.0000000	0.0000000	0.0000000	0.0000000
$\varepsilon = 10^{-4}$				
0.0000	1.0000000	1.0000000	1.0000000	1.0000000
0.2000	1.0000000	1.0000010	1.0000010	1.0000000
0.4000	1.0000000	1.0000010	1.0000010	1.0000000
0.6000	1.0000000	1.0000010	1.0000010	1.0000000
0.8000	1.0000000	1.0000010	1.0000010	1.0000000
0.9980			1.0000000	1.0000000
0.9982			1.0000000	1.0000000
0.9990		1.0000000	0.9999831	0.9999546
0.9992		0.9998645	0.9998477	0.9996646
0.9995	1.0000000	0.9959016	0.9958848	0.9932636
0.9996	0.9917355	0.9876710	0.9876544	0.9816856
0.9998	0.8925620	0.8889039	0.8888890	0.8646290
0.9999	0.6694215	0.6666780	0.6666667	0.6321816
1.0000	0.0000000	0.0000000	0.0000000	0.0000000

Clearly this problem has a boundary layer at $x = 1$.

For this example the stretching transformation is

$$t = \frac{1 - x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) - \varepsilon(1 + \varepsilon)Y(t) = 0$$

with

$$Y(0) = 1 + \frac{1}{e}, \quad Y(t_p) = e^{-(1 - t_p\varepsilon)}$$

and the outer region problem is given by

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0$$

with

$$y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon), \quad y(x_p) = e^{-x_p}.$$

The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$.

The numerical results are given in Table 9 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 15.3. Consider the following singular perturbation problem

$$\varepsilon y''(x) - 2y'(x) = 0; \quad x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 1$.

Clearly, this problem has a boundary layer at $x = 1$. That is; at the right end of the underlying interval.

For this example the stretching transformation is

$$t = \frac{2(1-x)}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) = 0; \quad \text{with } Y(0) = 1, \quad Y(t_p) = 0$$

and the outer region problem is given by

$$\varepsilon y''(x) - 2y'(x) = 0 \quad \text{with } y(0) = 0, \quad y(x_p) = 0.$$

The exact solution is given by

$$y(x) = \frac{e^{-2(1-x)/\varepsilon} - e^{-2/\varepsilon}}{1 - e^{-2/\varepsilon}}.$$

The numerical results are given in Table 10 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

16. Discussion and conclusions

We have presented a numerical patching method for solving singularly perturbed two-point boundary value problems with the boundary layer at one end point. The solution of the given singular perturbed two-point boundary value problem is computed numerically by dividing it into inner region problem and outer region problem. The terminal boundary condition is obtained from the solution of the reduced problem. A new inner region problem

Table 9
Numerical results of Example 15.2

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.000	1.0000000	1.0000000	1.0000000	1.0000000
0.100	0.9048380	0.9048371	0.9048375	0.9048374
0.200	0.8187316	0.8187301	0.8187305	0.8187308
0.300	0.7408195	0.7408174	0.7408180	0.7408183
0.400	0.6703214	0.6703191	0.6703199	0.6703200
0.500	0.6065322	0.6065295	0.6065305	0.6065307
0.600	0.5488133	0.5488105	0.5488114	0.5488117
0.700	0.4965870	0.4965841	0.4965851	0.4965853
0.800	0.4493307	0.4493277	0.4493287	0.4493290
0.900	0.4065714	0.4065683	0.4065696	0.4065697
0.980			0.3753111	0.3753111
0.982			0.3745613	0.3745613
0.990		0.3715767	0.3715936	0.3716216
0.992		0.3709686	0.3709854	0.3711671
0.995	0.3697234	0.3737971	0.3738139	0.3764278
0.996	0.3775868	0.3816230	0.3816397	0.3875963
0.998	0.4758141	0.4794410	0.4794560	0.5036843
0.999	0.6984436	0.7011626	0.7011738	0.7357640
1.000	1.3678790	1.3678790	1.3678790	1.3678790
$\varepsilon = 10^{-4}$				
0.0000	1.0000000	1.0000000	1.0000000	1.0000000
0.1000	0.9048444	0.9048371	0.9048366	0.9048374
0.2000	0.8187437	0.8187308	0.8187295	0.8187308
0.3000	0.7408355	0.7408180	0.7408167	0.7408182
0.4000	0.6703411	0.6703197	0.6703184	0.6703200
0.5000	0.6065547	0.6065307	0.6065289	0.6065307
0.6000	0.5488379	0.5488117	0.5488096	0.5488116
0.7000	0.4966132	0.4965853	0.4965832	0.4965853
0.8000	0.4493578	0.4493291	0.4493269	0.4493290
0.9000	0.4065992	0.4065699	0.4065677	0.4065697
0.9980			0.3686159	0.3686159
0.9982			0.3685422	0.3685422
0.9990		0.3682475	0.3682643	0.3682929
0.9992		0.3683092	0.3683260	0.3685090
0.9995	0.3680634	0.3721593	0.3721761	0.3747965
0.9996	0.3762879	0.3803496	0.3803662	0.3863337
0.9998	0.4753670	0.4790219	0.4790369	0.5032970
0.9999	0.6984565	0.7011976	0.7012087	0.7356979
1.0000	1.3678790	1.3678790	1.3678790	1.3678790

is constructed and solved as a two-point boundary value problem. The outer region problem is also solved as a two-point boundary value problem. The cubic spline approximation is used to solve these boundary value problems.

Table 10
Numerical results of Example 15.3

x	$t_p = 5$	$t_p = 10$	$t_p = 20$	Exact solution
$\varepsilon = 10^{-3}$				
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.2000	0.0000000	0.0000000	0.0000000	0.0000000
0.4000	0.0000000	0.0000000	0.0000000	0.0000000
0.6000	0.0000000	0.0000000	0.0000000	0.0000000
0.8000	0.0000000	0.0000000	0.0000000	0.0000000
0.9900			0.0000000	0.0000000
0.9920			0.0000000	0.0000001
0.9950		0.0000000	0.0000169	0.0000454
0.9960		0.0001355	0.0001524	0.0003355
0.9975	0.0000000	0.0040984	0.0041152	0.0067380
0.9980	0.0082645	0.0123290	0.0123457	0.0183166
0.9985	0.0330579	0.0370207	0.0370370	0.0497860
0.9990	0.1074380	0.1110961	0.1111111	0.1353388
0.9995	0.3305785	0.3333220	0.3333333	0.3678623
1.0000	1.0000000	1.0000000	1.0000000	1.0000000
$\varepsilon = 10^{-4}$				
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.20000	0.0000000	0.0000000	0.0000000	0.0000000
0.40000	0.0000000	0.0000000	0.0000000	0.0000000
0.60000	0.0000000	0.0000000	0.0000000	0.0000000
0.80000	0.0000000	0.0000000	0.0000000	0.0000000
0.99900			0.0000000	0.0000000
0.99920			0.0000000	0.0000001
0.99950		0.0000000	0.0000169	0.0000454
0.99960		0.0001355	0.0001524	0.0003354
0.99975	0.0000000	0.0040984	0.0041152	0.0067404
0.99980	0.0082645	0.0123290	0.0123457	0.0183253
0.99985	0.0330579	0.0370207	0.0370370	0.0497623
0.99990	0.1074380	0.1110961	0.1111111	0.1352904
0.99995	0.3305785	0.3333220	0.3333333	0.3678184
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Infact any standard analytical or numerical method can be used. We have implemented the present method on four linear examples, three non-linear example with left-end boundary layer and three examples with right-end boundary layer by taking different values of ε. The proposed method is iterative on the terminal point. The process is to be repeated for various choices of x_p, until the solution profiles do not differ materially from iteration to iteration. Numerical results are presented in tables. It can be observed from the tables that the present method approximates the exact solution very well.

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