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Higher order finite difference method for a class of singular boundary value problems

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Abstract

In this paper, a fourth order finite difference method for a class of singular boundary value problems is presented. The original differential equation is modified at the singular point. The fourth order finite difference method is then employed to solve the boundary value problem. Some model problems are solved, and the numerical results are compared with exact solution.

Keywords: Ordinary differential equations; Singular point; Boundary value problem; Finite differences

1. Introduction

In applied mathematics many problems lead to singular boundary value problems of the form

$$Ly \equiv y''(x) + \frac{k}{x}y'(x) + q(x)y(x) = r(x), \quad 0 < x < 1,$$

$$y'(0) = 0 \quad \text{and} \quad y(1) = \beta,$$

which occur very frequently in the theory of thermal explosions and in the study of Electro-hydrodynamics. Such problems also arise in the study of

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generalized axially symmetric potentials after separation of variables has been employed. There is considerable interest on numerical methods on singular boundary value problems. Jamet [5] considered the usual three point finite difference scheme for singular boundary value problems and showed in the maximum norm that his scheme is $O(h^{1-\alpha})$ convergent. The usual classical three-point finite difference discretization for singular boundary value problems has been studied by Russell and Shampine [7]. Iyengar and Jain [4] have discussed the spline function and the three point finite difference methods for singular boundary value problems. Eriksson and Thomee [3] have studied the Garlekin type piece wise polynomial procedure for these type problems and it provide fourth order methods for the singular boundary value problems. Chawla and Katti [2] have described finite difference methods for singular boundary value problems. Attempts by many researchers for the removal of singularity are based on using the series expansion procedures in the neighbourhood $(0, \delta)$ of singularity and then solve the regular boundary value problem in the interval $(\delta, 1)$ using any numerical method.

In this paper, we discuss a direct method for solving singular boundary value problem. The finite difference methods are always a convenient choice for solving boundary value problems, because of their simplicity. The original differential equation is modified at the singular point. The fourth order finite difference method is then employed to solve the boundary value problem. By stabilizing the classical central difference (CD) method, we develop a fourth order finite difference method. To obtain this method, we re-approximate the CD approximation by rewriting its error terms as a combination of first and second derivative terms and approximating them. Such a re-approximation process has a stabilizing effect, for details refer [6]. The matrix problem associated with this method is a tridiagonal algebraic system, which can be solved by 'Thomas Algorithm'. Some model problems are solved, and the numerical results are compared with exact solution.

2. Description of the method

We consider a singular two-point boundary value problem given by

$$Ly \equiv y''(x) + \frac{k}{x}y'(x) + q(x)y(x) = r(x), \tag{1}$$

$$y'(0) = 0, \tag{2}$$

$$y(1) = \beta. \tag{3}$$

Jamet [5] has shown that for Eq. (1) the derivative boundary condition is imposed due to nature of physical situation of the problem. Due to the singularity at $x = 0$, we modify the problem near the singular point.

To set up difference equation of (1) divide [0, 1] into n equal parts, each of the length h, we have $x_i = ih$, $i = 0, 1, \ldots, n$. For simplicity, let $q(x_i) = q_i$; $r(x_i) = r_i$; $y(x_i) = y_i$; $y'(x_i) = y'_i$ and $y''(x_i) = y''_i$.

Since $x = 0$ is singular point of Eq. (1), we first modify Eq. (1) at $x = x_0 = 0$ as follows:

$$y''(0) + \operatorname*{Lt}_{x\to 0} \frac{k}{x} y'(x) + q(0)y(0) = r(0).$$

Using L. Hospital rule, we have

$$\operatorname*{Lt}_{x\to 0} \frac{k}{x} y'(x) = ky''(0),$$

then we obtain

$$(1 + k)y''(x) + q(x)y(x) = r(x) \quad \text{at } x = 0. \tag{1*}$$

Now, we describe a fourth order finite difference method, which leads to a tridiagonal system, which can be solved by Thomas Algorithm. By Taylor series expansion we obtain the CD formulas for y'_i, y''_i assuming that y has continuous fourth order derivatives in the interval [0, 1]:

$$y''_i \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) \tag{4}$$

and

$$y'_i \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''(\eta), \tag{5}$$

where ξ, $\eta \in [x_{i-1}, x_{i+1}]$. Substituting (4) and (5) in (1) at $x = x_i$, we get the CD operator L_h, defined by

$$L_h y_i \equiv a_i y_{i+1} - b_i y_i + c_i y_{i-1} = d_i + \tau_i[y], \quad 1 \leqslant i \leqslant n - 1, \tag{6}$$

where

$$a_i = \frac{1}{h^2} + \frac{k}{2hx_i}, \qquad b_i = \frac{2}{h^2} - q_i, \qquad c_i = \frac{1}{h^2} - \frac{k}{2hx_i}, \qquad d_i = r_i \tag{7}$$

and

$$\tau_i[y] = \frac{h^2}{12} y^{(4)}(\xi) + \frac{h^2 k}{6x_i} y'''(\eta),$$

where ξ, $\eta \in [x_{i-1}, x_{i+1}]$, here $\tau_i[y]$ are local truncation errors of the CD approximation. To obtain numerical solution of (1) by the CD operator L_h, we solve the system of equations formed by the three-term recurrence relation:

$$L_h y_i \equiv a_i y_{i+1} - b_i y_i + c_i y_{i-1} = d_i, \quad 1 \leqslant i \leqslant n - 1. \tag{8}$$

By rewriting the CD formulas for y_i', y_i'' in new form as given below:

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + R_1, \tag{9}$$

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + R_2, \tag{10}$$

where

$$R_1 = -\frac{2h^4 y^{(6)}(\xi)}{6!} \quad \text{and} \quad R_2 = -\frac{h^4 y^{(5)}(\eta)}{5!},$$

for ξ, $\eta \in [x_{i-1}, x_{i+1}]$. Substituting these y_i', y_i'' from Eqs. (9) and (10) in (1) at $x = x_i$, we get the CD approximation in a form that includes all the $\mathrm{O}(h^2)$ error terms:

$$L_h y_i - \frac{h^2}{12}\left(2\frac{k}{x_i} y_i''' + y_i^{(4)}\right) + \widetilde{R} = r_i, \tag{11}$$

L_h is the CD operator given as in (5) and $\widetilde{R} = R_1 + (k/x_i)R_2$.

By writing $q(x) = q$, $r(x) = r$, in (1) we obtain

$$y'' = r - qy - \frac{k}{x} y'.$$

Differentiating above equation with respect to x, we obtain

$$y''' = r' - \left[\frac{k}{x} y'' + \left(q - \frac{k}{x^2}\right) y' + q' y\right]. \tag{12}$$

Now differentiating (12) with respect to x, we get

$$y^{(4)} = r'' - \left[\frac{k}{x} y''' + \left(q - 2\frac{k}{x^2}\right) y'' + \left(2\frac{k}{x^3} + 2q'\right) y' + q'' y\right], \tag{13}$$

then

$$2\frac{k}{x} y''' + y^{(4)} = \left[2\frac{k}{x^2} - \frac{k^2}{x^2} - q\right] y'' + \left[-\frac{k}{x}\left(q - \frac{k}{x^2}\right) - 2\frac{k}{x^3} - 2q'\right] y' - \left[q'' + \frac{k}{x} q'\right] y + \frac{k}{x} r' + r''. \tag{14}$$

Substituting (14) in (11), we get the equation

$$L_h y_i - \frac{h^2}{12}\left[\left(2\frac{k}{x_i^2} - \frac{k^2}{x_i^2} - q_i\right) y_i'' + \left(-\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) - 2\frac{k}{x_i^3} - 2q_i'\right) y_i' - \left(q_i'' + \frac{k}{x_i} q_i'\right) y_i\right] + \widetilde{R} = r_i + \frac{h^2}{12}\left(\frac{k}{x_i} r_i' + r_i''\right). \tag{15}$$

We approximate the converted error terms in Eq. (15) by using for y_i'' and y_i' from Eqs. (4) and (5). Then adding these new approximations to $L_h y_i$ defined by (6) and (7), we obtain the fourth order operator

$$L_h^* y_i \equiv a_i^* y_{i+1} - b_i^* y_i + c_i^* y_{i-1} = d_i^* + \tau_i^*[y], \quad 1 \leqslant i \leqslant n-1, \tag{16}$$

where

$$a_i^* = a_i - \frac{1}{12}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\right] + \frac{h}{24}\left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q'\right],$$

$$b_i^* = b_i - \frac{1}{6}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\right] - \frac{h^2}{12}\left[q_i'' + \frac{k}{x_i}q_i'\right],$$

$$c_i^* = c_i - \frac{1}{12}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\right] - \frac{h}{24}\left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q'\right],$$

$$d_i^* = d_i + \frac{h^2}{12}\left[\frac{k}{x_i}r_i' + r_i''\right].$$

Here a_i, b_i, c_i, d_i are given in (7) and $\tau_i^*[y]$ are the local truncation errors of the Eq. (16), given by

$$\tau_i^*[y] = -\left[2\frac{k}{x_i^2} - \frac{k^2}{x_i^2} - q_i\right]\frac{h^4}{144}y^{(4)} + \left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q_i'\right]\frac{h^4}{72}y_i''' - \widetilde{R},$$

where $\widetilde{R} = R_1 + (k/x_i)R_2 = \mathrm{O}(h^4)$. We solve the system of equations formed by the three-term recurrence relationship:

$$L_h^* y_i \equiv a_i^* y_{i+1} - b_i^* y_i + c_i^* y_{i-1} = d_i^* \quad 1 \leqslant i \leqslant n-1. \tag{17}$$

3. Modification at singularity

The difference scheme (17) cannot be used at $i = 0$, as it is not defined at $x = x_0$. Hence we have modified Eq. (1) at singular point $x = x_0 = 0$ as in Eq. (1^*).

$$(1+k)y''(x) + q(x)y(x) = r(x), \quad x = x_0. \tag{18}$$

Now we replace $y''(x)$ with CD formulae (9) at $x = x_0$ $(x = 0)$ in Eq. (18) and obtain

$$(1+k)\left(\frac{y_1 - 2y_0 + y_{-1}}{h^2} - \frac{h^2}{12}y_0^{(4)} + R_1\right) + q_0 y_0 = r_0. \tag{19}$$

Differentiating Eq. (18) twice with respect to x, we obtain

$$(1+k)y^{(4)} = r'' - 2q'y' - qy'' - q''y. \tag{20}$$

Substituting Eq. (20) in Eq. (19) and replacing y' and y'' with Eqs. (9) and (10) at $x = 0$, we obtain

$$\left(\frac{1+k}{h^2}+\frac{h}{12}q_0'+\frac{q_0}{12}\right)y_1-\left(\frac{2(1+k)}{h^2}-\frac{h^2}{12}q_0''-\frac{5}{6}q_0\right)y_0 + \left(\frac{1+k}{h^2}-\frac{h}{12}q_0'+\frac{q_0}{12}\right)y_{-1}=r_0+\frac{h^2}{12}r_0''+\widetilde{R}, \tag{21}$$

where

$$\widetilde{R}=-\frac{h^2}{12}q_0R_1-\frac{h^2}{6}q_0'R_2-(1+k)R_1+\frac{h^4}{144}q_0y^{(4)}(\xi)+\frac{h^4}{72}2q_0'y'''(\eta)$$

which can be neglected.

To eliminate y_{-1} in Eq. (21) we use the boundary condition (2) $y'(0) = 0$ and applying finite difference approximation,

$$\frac{y_1-y_{-1}}{2h}=0.$$

Hence from Eq. (21)

$$y_0=\frac{\frac{2(k+1)}{h^2}+\frac{q_0}{6}}{\frac{2(k+1)}{h^2}-\frac{5q_0}{6}-\frac{h^2}{12}q_0''}y_1-\frac{r_0+\frac{h^2}{12}r_0''}{\frac{2(k+1)}{h^2}-\frac{5q_0}{6}-\frac{h^2}{12}q_0''}. \tag{22}$$

4. Solution

Eqs. (17) and (22) form 'n' equations with $(n+1)$ unknowns $y_0, y_1, \ldots, y_n$. Using the condition given in Eq. (3) becomes $(n+1)$ equations with $(n+1)$ unknowns, which will be sufficient to solve for these unknowns. The matrix problem associated here is a tridiagonal algebraic system whose solution can easily be determined by an efficient algorithm called Thomas Algorithm. The idea of this algorithm is very simple. We shall briefly describe it in the following. In this algorithm we start with a difference relation of the form

$$y_i = W_i y_{i+1} + T_i, \tag{23}$$

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ and are to be determined from (23) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \tag{24}$$

Substituting (24) in (17),

$$a_i^* y_{i+1} - b_i^* y_i + c_i^*(W_{i-1}y_i + T_{i-1}) = d_i^*,$$

$$y_i = \frac{a_i^*}{(c_i^* W_{i-1} - b_i^*)} y_{i+1} + \frac{c_i^* T_{i-1} - d_i^*}{(c_i^* W_{i-1} - b_i^*)}. \tag{25}$$

By comparing Eq. (25) with (23), we obtain the recurrence relations

$$W_i = \frac{a_i^*}{(c_i^* W_{i-1} - b_i^*)} \quad \text{and} \quad T_i = \frac{c_i^* T_{i-1} - d_i^*}{(c_i^* W_{i-1} - b_i^*)}. \tag{26}$$

To solve these recurrence relations for $i = 1, 2, \ldots, n-1$, we need to know the initial conditions for W_0 and T_0. Eq. (22) is of the form

$$y_0 = W_0 y_1 + T_0 \tag{27}$$

then

$$W_0 = \frac{\frac{2(k+1)}{h^2} + \frac{q_0}{6}}{\frac{2(k+1)}{h^2} - \frac{5q_0}{6} - \frac{h^2}{12} q_0''}$$

and

$$T_0 = -\frac{r_0 + \frac{h^2}{12} r_0''}{\frac{2(k+1)}{h^2} - \frac{5q_0}{6} - \frac{h^2}{12} q_0''}.$$

Using these initial values, we compute W_i and T_i for $i = 1, 2, \ldots, n-1$ from (26) in the forward process and then obtain y_i in the backward process from Eq. (23) using Eq. (3).

5. Numerical experiments

To demonstrate the applicability of fourth order finite difference method, we have solved several singular boundary value problems. These problems have been chosen because they have been widely discussed in the literature and because approximate solutions are available for comparison.

Example 1. Consider the linear two-point boundary value problem [7]:

$$y''(x) + \frac{2}{x} y'(x) - 4y(x) = -2$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = 5.5.$$

The problem has a unique solution

$$y(x) = 0.5 + \frac{5 \sinh 2x}{x \sinh 2}.$$

The numerical results are presented in Table 1.

Example 2. Consider the singular boundary value problem [3]:

$$-y''(x) - \frac{2}{x} y'(x) + (1 - x^2) y(x) = x^4 - 2x^2 + 7$$

Table 1
Numerical results for Example 1 with $h = 1/20$

x	$y(x)$ HFDM	Exact solution
0.0	3.257208	3.257205
0.1	3.275625	3.275624
0.2	3.331323	3.331321
0.3	3.425642	3.425641
0.4	3.560864	3.560863
0.5	3.740272	3.740272
0.6	3.968247	3.968246
0.7	4.250394	4.250393
0.8	4.593706	4.593705
0.9	5.006766	5.006765
1.0	5.500000	5.500000

Table 2
Numerical results for Example 2 with $h = 1/20$

x	$y(x)$ HFDM	Exact solution
0.0	0.999999	1.000000
0.1	0.989999	0.990000
0.2	0.959999	0.960000
0.3	0.909999	0.910000
0.4	0.839999	0.840000
0.5	0.749999	0.750000
0.6	0.639999	0.640000
0.7	0.510000	0.510000
0.8	0.360000	0.360000
0.9	0.190000	0.190000
1.0	0.000000	0.000000

with boundary conditions

$$y'(0) = 0, \quad y(1) = 0.$$

The exact solution is $y(x) = 1 - x^2$. The numerical results are presented in Table 2.

Example 3. Consider the Bessel's equation of order zero [4]:

$$y''(x) + \frac{1}{x}y'(x) + y(x) = 0$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = 1.$$

The exact solution is $y(x) = \frac{J_0(x)}{J_0(1)}$. The numerical results are presented in Table 3.

Table 3
Numerical results for Example 3 with $h = 1/20$

x	$y(x)$ HFDM	Exact solution
0.0	1.306843	1.306852
0.1	1.303578	1.303587
0.2	1.293808	1.293816
0.3	1.277604	1.277613
0.4	1.255090	1.255098
0.5	1.226434	1.226441
0.6	1.191849	1.191855
0.7	1.151594	1.151599
0.8	1.105969	1.105972
0.9	1.055313	1.055314
1.0	1.000000	1.000000

Example 4. Finally, we consider the singular boundary value problem [1]:

$$y''(x) + \frac{1}{x}y'(x) = \left(\frac{8}{8 - x^2}\right)^2$$

with the boundary conditions

$$y'(0) = 0, \quad y(1) = 0.$$

The exact solution is $y(x) = 2\log\frac{7}{8-x^2}$. The numerical results are presented in Table 4.

Table 4
Numerical results for Example 4 with $h = 1/20$

x	$y(x)$ HFDM	Exact solution
0.0	−0.267067	−0.267063
0.1	−0.264565	−0.264561
0.2	−0.257042	−0.257038
0.3	−0.244439	−0.244435
0.4	−0.226661	−0.226657
0.5	−0.203569	−0.203565
0.6	−0.174978	−0.174975
0.7	−0.140653	−0.140651
0.8	−0.100301	−0.100300
0.9	−0.053563	−0.053562
1.0	0.000000	0.000000

6. Discussion and conclusion

We have described and demonstrated the applicability of the fourth order finite difference method by solving singular boundary value problems. First of all it is a direct method. Further it is simple, accurate, and easy to implement on computer. We have implemented this method on four examples—a homogeneous singular boundary value problem, and three non-homogeneous singular boundary value problems with mesh size $h = 1/20$. The numerical results for the examples are presented in Tables 1–4. It can be observed from these tables that the present solutions compare well with the exact solutions.

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