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A numerical method for singular two point boundary value problems via Chebyshev economizition

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Abstract

In this paper we present a numerical method for solving a two point boundary value problem in the interval $[0, 1]$ with regular singularity at $x = 0$. By employing the Chebyshev economizition on $[0, \delta]$, where δ is near the singularity, we first replace it by a regular problem on some interval $[\delta, 1]$. The stable central difference method is then employed to solve the problem over the reduced interval. Some numerical results are presented to demonstrate the applicability of the method.

Keywords: Ordinary differential equations; Singular boundary value problem; Chebyshev economizition; Finite difference method

1. Introduction

Consider a linear homogeneous differential equation of form

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

which can be rewritten in the form

$$y''(x) + \frac{a(x)}{(x-a)}y'(x) + \frac{b(x)}{(x-a)^2}y(x) = 0$$

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where $a(x)$ and $b(x)$ both possess Taylor series expansions about the point $x = a$, i.e. a is regular singular point of the differential equation.

The numerical treatment of the singular boundary value problems has always been far from trivial, because of the singularity. Jamet [4] has discussed existence and uniqueness of solutions of second-order linear singular boundary value problems and presented finite difference method for numerically solving such problems. Kadalbajoo and Raman [6] used series solution in the vicinity of the singular point to subtract singularity and then used the invariant imbedding technique to solve the regular boundary value problem in the remaining interval. Cohen and Jones [1] studied a shifted chebyshev polynomial with finite deferred correction approach for a second-order linear ordinary differential equation with a regular singular point. They considered these polynomials the whole interval where the polynomials are valid, by neglecting the effect of singularity.

In this paper we present a numerical method for solving boundary value problems with regular singularity. The singular problem over the interval $[0, 1]$ is first reduced to regular problem over $[\delta, 1]$, $\delta > 0$ is near the singularity. This is done by making use of Chebshey economizition in the vicinity of the singularity and obtaining a boundary condition at $x = \delta$. The stable central difference (SCD) method is then employed for solving the regular problem. Some numerical results are presented to demonstrate the applicability of the method.

2. Description of the method

Consider a homogeneous linear ordinary differential equation give by

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0 \tag{1}$$

$$\text{With the boundary conditions } y(0) = A \quad \text{and} \quad y(1) = B \tag{2}$$

where $x = 0$ is the regular singular point of the differential equation (1). Since $x = 0$ is a regular singularity, we make use of a series expansion in a small interval near $x = 0$ and Eq. (1) has a solution of the form

$$y(x) = \sum_{k=0}^{\infty} C_k x^{k+r}, \quad C_0 \neq 0 \tag{3}$$

The coefficients C_k and the indicial roots r are obtained by differentiating (3), substituting in (1) and comparing the coefficients of like powers of x on the two sides of the equation. The general solution can be written as

$$y(x) = \sum_{i=0}^{2} \alpha_i S_i(x) \tag{4}$$

where $S_1(x)$ and $S_2(x)$ are linearly independent solutions and α_1, α_2 are arbitrary constants. Keller [7] and Coddington and Levinson [2] have discussed the basics theoretical results of series expansion about a singular point. The series solution may be valid for the entire interval $[0, 1]$ but due to its slow convergence, to over come this situation we recommend that the series expansion be approximated by an economized expansion in $(0, \theta)$ where $0 < \theta \leqslant 1$.

Let us assume that for different indicial values the series $S_1(x)$ and $S_2(x)$ are equal to $S(x)$. So that the general solution is of the form

$$y(x) = (\alpha_1 x^{m_1} + \alpha_2 x^{m_2})S(x) \tag{5}$$

where m_1, m_2 are the indicial roots of Eq. (1). In order to approximate $S(x)$ by an economized expansion $P(x)$, we assume that

$$P(x) = \sum_{j=0}^{N} a_j x^j \tag{6}$$

satisfies the differential equation

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = \tau_N^*\left(\frac{x}{\theta}\right) \tag{7}$$

taking $\theta = 1$, and choosing τ so that $P(0) = A$, where N is an arbitrary constant. Now by substituting $P(x)$ for $y(x)$ and corresponding the like powers of x on both sides in (7) we can find the coefficients a_j and write (6) as

$$y(x) = (\alpha_1 x^{m_1} + \alpha_2 x^{m_2})P(x) \quad 0 < x \leqslant \delta \tag{8}$$

Now we reduce problems (1) and (2) to regular boundary value problem, by finding a new boundary condition at $x = \delta$. To do this we have from Eq. (8)

$$y(x) = \alpha_1 R_1(x) + \alpha_2 R_2(x) \tag{9}$$

where $R_1(x) = x^{m_1}P(x)$ and $R_2(x) = x^{m_2}P(x)$. Eq. (9) at $x = \delta$ can be written as

$$y(\delta) = \alpha_1 R_1(\delta) + \alpha_2 R_2(\delta) \tag{10}$$

We also have from Eq. (9)

$$y'(\delta) = \alpha_1 R_1'(\delta) + \alpha_2 R_2'(\delta) \tag{11}$$

solving (10) and (11) for α_1 and α_2, we get

$$\alpha_1 = \frac{y(\delta)R_2'(\delta) - y'(\delta)R_2(\delta)}{R_1(\delta)R_2'(\delta) - R_2(\delta)R_1'(\delta)} \tag{12}$$

$$\alpha_2 = \frac{y'(\delta)R_1(\delta) - y(\delta)R_1'(\delta)}{R_1(\delta)R_2'(\delta) - R_2(\delta)R_1'(\delta)} \tag{13}$$

Since $y(0) = A$, we have Eq. (9)

$$A = \alpha_1 R_1(0) + \alpha_2 R_2(0) \tag{14}$$

Using Eqs. (12)–(14) we have

$$\frac{y(\delta)R_2'(\delta) - y'(\delta)R_2(\delta)}{R_1(\delta)R_2'(\delta) - R_2(\delta)R_1'(\delta)}R_1(0) + \frac{y'(\delta)R_1(\delta) - y(\delta)R_1'(\delta)}{R_1(\delta)R_2'(\delta) - R_2(\delta)R_1'(\delta)}R_2(0) = A$$

the above equation can be conveniently written as

$$\left(R_1(0)R_2'(\delta) - R_1'(\delta)R_2(\delta)\right)y(\delta) + (R_1(\delta)R_2(0) - R_2(\delta)R_1(0))y'(\delta) = Aq(\delta)$$

where

$$Q(x) = R_1(x)R_2'(x) - R_1'(x)R_2(x)$$

This equation can be conveniently written as

$$Ky(\delta) + Ly'(\delta) = M \tag{15}$$

where

$$K = R_1(0)R_2'(\delta) - R_1'(\delta)R_2(0)$$
$$L = R_1(\delta)R_2(0) - R_2(\delta)R_1(0) \quad \text{and} \quad M = AQ(\delta)$$

Eq. (15) give the new boundary condition at $x = \delta$. Thus the regular boundary value problem over $[\delta, 1]$ is given by $p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0$ with boundary conditions $Ky(\delta) + Ly'(\delta) = M$ and $y(1) = B$.

In case of non-homogeneous equation

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = h(x)$$

The above procedure can be applied by making $P(x) = \sum_{j=0}^{N} a_j x^j$ satisfies the equation

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = \tau_N^*\left(\frac{x}{\theta}\right) + h(x) \tag{16}$$

To obtain the coefficients a_j by comparing the coefficients on both sides of the Eq. (16).

3. Stable central difference method of order two

In this section we briefly describe the SCD method. Stabilizing the classical central difference (CD) method (for details see Ref. [3]) develops this SCD method. To obtain these methods, we re-approximate the CD approximation by rewriting its error terms as a combination of first- and second-derivative terms and approximating them. Such a re-approximation process has a stabilizing effect and so we shall call our new methods the stabilized central difference methods (in short SCD methods) (for details see Ref. [5]). In particular, we have discussed the second order SCD methods using three points so that we get three-term recurrence relationship. Consider the equation

$$y''(x) + f(x)y'(x) + g(x)y(x) = 0 \tag{17}$$

$$\text{With } Ky(\delta) + Ly'(\delta) = M \quad \text{and} \quad y(1) = B \tag{17a}$$

A unique solution $y(x)$ of (17) exists $f(x)$, $g(x)$ are continuous on $[\delta, 1]$ and $g(x)$ is negative there. Since these functions are continuous on a closed and bounded interval, there must exist positive constants F^*, G^*, and G_* such that

$$|f(x)| \leqslant F^*, \;\; 0 < G_* \leqslant g(x) \leqslant G^*, \;\; \delta \leqslant x \leqslant 1$$

We now divide $[\delta, 1]$ into 'n' equal parts by $\delta = x_0 < x_1 < \cdots < x_n = 1$, with $h = x_i - x_{i-1}$ $(i = 1, 2, \ldots, n)$, where h is the mesh size. By Taylor series expansion we obtain the CD formulas for y_i', y_i'' assuming that y has continuous fourth-order derivatives in the interval $[\delta, 1]$.

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{\mathrm{iv}}(\xi) \tag{18}$$

and

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''(\eta) \tag{19}$$

where ξ, $\eta \in [x_{i-1}, x_{i+1}]$. Substituting (18) and (19) in (17) at $x = x_i$, we get the CD operator L_h, defined by

$$L_h y_i \equiv a_i y_{i-1} + b_i y_i + c_i y_{i+1} = \tau_i[y], \quad 1 \leqslant i \leqslant n-1 \tag{20}$$

where

$$\begin{aligned} a_i &= \frac{1}{h^2} - \frac{f_i}{2h} \\ b_i &= g_i - \frac{2}{h^2} \\ c_i &= \frac{1}{h^2} + \frac{f_i}{2h} \\ \tau_i[y] &= \frac{h^2}{12} y_i^{\mathrm{iv}}(\xi) + \frac{h^2 f_i}{6} y'''(\eta) \end{aligned} \tag{21}$$

where ξ, $\eta \in [x_{i-1}, x_{i+1}]$, here $\tau_i[y]$ are local truncation errors of the CD approximation.

To obtain numerical solution of (17) by the CD operator L_h, we solve the system of equations formed by the three-term recurrence relation:

$$L_h y_i \equiv a_i y_{i-1} + b_i y_i + c_i y_{i+1}, \quad 1 \leqslant i \leqslant n-1 \tag{22}$$

By rewriting the CD formulas for y_i', y_i'' in new form as given below:

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i'''' + R_1 \tag{23}$$

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + R_2 \tag{24}$$

where

$$R_1 = -\frac{2h^4 y^{(6)}(\xi)}{6!} \quad \text{and} \quad R_2 = -\frac{h^4 y^{(5)}(\eta)}{5!} \quad \text{for } \xi, \eta \in [x_{i-1}, x_{i+1}]$$

By substituting these y_i', y_i'' from Eqs. (23) and (24) in (17) at $x = x_i$, we get the CD approximation in a form that includes the $\mathrm{O}(h^2)$ error term for y':

$$L_h y_i - \frac{f_i h^2}{6} y_i''' + R = 0 \tag{25}$$

where $R = -(h^2/12) y_i^{\text{iv}} + R_1 + f_i R_2$ and L_h is the CD operator as in (22). Writing $f(x) = f$, $g(x) = g$ in (17) we get

$$y'' = -fy' - gy$$

By differentiating both sides of this equation with respect to x, we get

$$y''' = -[fy'' + (g + f')y' + g'y] \tag{26}$$

Substituting (27) in (26), we get

$$L_h y_i + \frac{h^2}{6} [f_i^2 y_i'' + g_i(g_i + f_i')y' + f_i g_i' y_i] + R = 0 \tag{27}$$

where $R = R_1 + f_i R_2 - (h^2/12) y_i^{\text{iv}}$. Note that the term $h^2 f_i^2 y_i''/6$ can reinforce the central coefficient, b_i when y_i'' approximated by (23).

Now we approximate the converted error term in Eq. (27) by using the CD formulas for y_i'', y_i'. Then adding the new approximation to $L_h y_i$, defined by (21), we obtain the second-order SCD operator (SCD-2 operator) L_h^*.

$$L_h^* y_i \equiv a_i^* y_{i-1} + b_i^* y_i + c_i^* y_{i+1} = \tau_i^*[y], \quad 1 \leqslant i \leqslant n-1 \tag{28}$$

where

$$\begin{aligned} a_i^* &= a_i + \frac{f_i^2}{6} - \frac{hf_i(g_i + f_i')}{12} \\ b_i^* &= b_i - \frac{f_i^2}{3} + \frac{h^2 f_i g_i'}{6} \\ c_i^* &= c_i + \frac{f_i^2}{6} + \frac{hf_i(g_i + f_i')}{12} \end{aligned} \tag{29}$$

where a_i, b_i, c_i are as in Eq. (21). $\tau_i^*[y]$ are the local truncation errors of SCD-2 approximation of (1) given below

$$\tau_i^*[y] = \left(\frac{h^2}{12} + \frac{h^4 f_i^2}{72}\right) y_i^{\mathrm{iv}} + \frac{h^4 f_i}{36}(g_i + f_i')y_i''' - \widetilde{R} \tag{30}$$

$$\widetilde{R} = R_1 + f_i R_2 = \mathrm{O}(h^4)$$

To obtain numerical solution of (18) by SCD-2 operator, we solve the system of equations formed by the three-term recurrence relation:

$$L_h^* y_i \equiv a_i^* y_{i-1} + b_i^* y_i + c_i^* y_{i+1} = 0, \quad 1 \leqslant i \leqslant n-1 \tag{31}$$

Eq. (31) lead to a system of $(n-1)$ equations with $(n+1)$ unknowns $y_0, y_1, \ldots, y_n$. The two boundary conditions (16) and $y(1) = B$ together with $(n-1)$ equations are then sufficient to solve for the unknowns. The matrix problem associated with (31) is a tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using an efficient algorithm called the Thomas Algorithm.

4. Numerical results

In this section, we present the numerical example to illustrate the applicability of the method described in the previous sections. This problem has earlier been studied by Cohen and Jones [1] by using finite difference and deferred correction approach.

Example. We consider the second-order differential equation

$$2x(1+x)y''(x) + (1+5x)y'(x) + y(x) = 0$$

with boundary conditions

$$y(0) = y(1.5) = 1$$

The analytical solution for the problem is given by $y(x) = (1 + \sqrt{1.5x})/(1+x)$. The computational results are presented in Tables 1 and 2. The polynomials for $N = 8$ and 5 are given by

$$\begin{aligned} P_N(x) &= 1 - 0.9998916x + 0.9975789x^2 - 0.9781525x^3 + 0.8948963x^4 \\ &\quad - 0.6913812x^5 + 0.3953593x^6 - 0.1416262x^7 + 0.0232174x^8 \\ P_N(x) &= 1 - 0.991087504x + 0.916816709x^2 - 0.679150162x^3 \\ &\quad + 0.322650342x^4 - 0.069139359x^5 \end{aligned}$$

Table 1
Computational for Example with $N = 8$

x	$y(x)$ SCD-2	$y(x)$ EXACT	x	$y(x)$ SCD-2	$y(x)$ EXACT
$N = 8,\ \delta = 0.1,\ \theta = 1,\ H = 1/50$			$N = 8,\ \delta = 0.1,\ \theta = 1,\ H = 1/100$		
0.1	1.261738	1.261180	0.1	1.261312	1.261180
0.2	1.289951	1.289769	0.2	1.289798	1.289769
0.3	1.285325	1.285246	0.3	1.285245	1.285246
0.4	1.267607	1.267569	0.4	1.267554	1.267569
0.5	1.244036	1.244017	0.5	1.243997	1.244017
0.6	1.217937	1.217927	0.6	1.217907	1.217927
0.7	1.191001	1.190997	0.7	1.190980	1.190997
0.8	1.164137	1.164136	0.8	1.164121	1.164136
0.9	1.137838	1.137840	0.9	1.137825	1.137840
1.0	1.112370	1.112373	1.0	1.112359	1.112373
1.1	1.087867	1.087868	1.1	1.087854	1.087868
1.2	1.064381	1.064382	1.2	1.064371	1.064382
1.3	1.041923	1.041924	1.3	1.041914	1.041924
1.4	1.020474	1.020474	1.4	1.020469	1.020474
1.5	1.000000	1.000000	1.5	1.000000	1.000000
$N = 8,\ \delta = 0.2,\ \theta = 1,\ H = 1/50$			$N = 8,\ \delta = 0.2,\ \theta = 1,\ H = 1/100$		
0.2	1.289925	1.289769	0.2	1.289785	1.289769
0.3	1.285304	1.285246	0.3	1.285237	1.285246
0.4	1.267588	1.267569	0.4	1.267548	1.267569
0.5	1.244020	1.244017	0.5	1.243993	1.244017
0.6	1.217923	1.217927	0.6	1.217905	1.217927
0.7	1.190989	1.190997	0.7	1.190979	1.190997
0.8	1.164127	1.164136	0.8	1.164121	1.164136
0.9	1.137829	1.137840	0.9	1.137825	1.137840
1.0	1.112362	1.112373	1.0	1.112359	1.112373
1.1	1.087860	1.087868	1.1	1.087854	1.087868
1.2	1.064376	1.064382	1.2	1.064371	1.064382
1.3	1.041920	1.041924	1.3	1.041914	1.041924
1.4	1.020472	1.020474	1.4	1.020469	1.020474
1.5	1.000000	1.000000	1.5	1.000000	1.000000
$N = 8,\ \delta = 0.5,\ \theta = 1,\ H = 1/50$			$N = 8,\ \delta = 0.5,\ \theta = 1,\ H = 1/100$		
0.5	1.244041	1.244017	0.5	1.244001	1.244017
0.6	1.217941	1.217927	0.6	1.217909	1.217927
0.7	1.191005	1.190997	0.7	1.190981	1.190997
0.8	1.164140	1.164136	0.8	1.164121	1.164136
0.9	1.137841	1.137840	0.9	1.137824	1.137840
1.0	1.112372	1.112373	1.0	1.112358	1.112373
1.1	1.087868	1.087868	1.1	1.087854	1.087868
1.2	1.064382	1.064382	1.2	1.064371	1.064382
1.3	1.041923	1.041924	1.3	1.041914	1.041924
1.4	1.020474	1.020474	1.4	1.020469	1.020474
1.5	1.000000	1.000000	1.5	1.000000	1.000000

Table 2
Computational for Example with $N=5$

x	$y(x)$ SCD-2	$y(x)$ EXACT	x	$y(x)$ SCD-2	$y(x)$ EXACT
$N=5,\ \delta=0.1,\ \theta=1,\ H=1/50$			$N=5,\ \delta=0.1,\ \theta=1,\ H=1/100$		
0.1	1.261993	1.261180	0.1	1.261566	1.261180
0.2	1.290151	1.289769	0.2	1.289999	1.289769
0.3	1.285486	1.285246	0.3	1.285406	1.285246
0.4	1.267738	1.267569	0.4	1.267685	1.267569
0.5	1.244141	1.244017	0.5	1.244103	1.244017
0.6	1.218022	1.217927	0.6	1.217993	1.217927
0.7	1.191070	1.190997	0.7	1.191050	1.190997
0.8	1.164192	1.164136	0.8	1.164177	1.164136
0.9	1.137881	1.137840	0.9	1.137868	1.137840
1.0	1.112404	1.112373	1.0	1.112392	1.112373
1.1	1.087891	1.087868	1.1	1.087879	1.087868
1.2	1.064398	1.064382	1.2	1.064389	1.064382
1.3	1.041934	1.041924	1.3	1.041925	1.041924
1.4	1.020479	1.020474	1.4	1.020474	1.020474
1.5	1.000000	1.000000	1.5	1.000000	1.000000
$N=5,\ \delta=0.2,\ \theta=1,\ H=1/50$			$N=5,\ \delta=0.2,\ \theta=1,\ H=1/100$		
0.2	1.290630	1.289769	0.2	1.290488	1.289769
0.3	1.285870	1.285246	0.3	1.285802	1.285246
0.4	1.268048	1.267569	0.4	1.268007	1.267569
0.5	1.244394	1.244017	0.5	1.244367	1.244017
0.6	1.218228	1.217927	0.6	1.218210	1.217927
0.7	1.191237	1.190997	0.7	1.191228	1.190997
0.8	1.164327	1.164136	0.8	1.164323	1.164136
0.9	1.137987	1.137840	0.9	1.137987	1.137840
1.0	1.112486	1.112373	1.0	1.112487	1.112373
1.1	1.087952	1.087868	1.1	1.087951	1.087868
1.2	1.064441	1.064382	1.2	1.064440	1.064382
1.3	1.041960	1.041924	1.3	1.041958	1.041924
1.4	1.020491	1.020474	1.4	1.020491	1.020474
1.5	1.000000	1.000000	1.5	1.000000	1.000000

5. Discussion and conclusion

The numerical results for the example at different mesh points for different mesh sizes and two different values of N and δ are presented in Tables 1 and 2. Cohen and Jones [1] solved it using finite difference deferred correction technique. They used economized series expansion in the interval $[0,1]$ and obtained finite difference solution on the remaining part of the interval. So they neglected the effect of the singularity on the solution in the immediate neighborhood of the singular point. Since computed difference solution is far away from the singularity. The present method is simple, easy to program and quite efficient for solving singular boundary value problems. It can be observed from these tables that the computed solutions compare well with the exact solutions.

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