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# Numerical integration method for general singularly perturbed two point boundary value problems

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**Abstract**

In this paper, a numerical integration method is presented for solving general singularly perturbed two-point boundary value problems. The original second-order differential equation is replaced by an approximate first-order differential equation with a small deviating argument. Then, the trapezoidal formula is used to obtain the three-term recurrence relationship. The proposed method is iterative on the deviating argument. To demonstrate the applicability of the method, we have solved several model linear and non-linear examples with left-end boundary layer or right-end boundary layer or an internal layer or two boundary layers and presented the computational results.


*Keywords:* Singular perturbations; Ordinary differential equations; Two point boundary value problems; Boundary layer

## 1. Introduction

Singular perturbation problems occur very frequently in various fields of science and engineering such as fluid dynamics, specifically the fluid flow problems involving large Reynolds number and other problems in the great world of fluid motion. The numerical treatment of singular perturbation

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problems is far from trivial because of the boundary layer behaviour of the solution. However, the area of singular perturbation problems is a field of increasing interest to applied mathematicians.

The survey paper by Kadalbajoo and Reddy [5], gives an erudite outline of the singular perturbation problems and their treatment starting from Prandtl's paper [12] on fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on singular perturbation problems.

For a detailed theory and analytical discussion on singular perturbation problems one may refer to the books and high level monographs: O'Malley [10,11], Nayfeh [7–9], Kevorkian and Cole [6], Bender and Orszag [2], Hemker and Miller [4].

In this paper, a numerical integration method is presented for solving general singularly perturbed two-point boundary value problems. This method does not depend on asymptotic expansions. The main feature of this method is that it does not require very fine mesh size. The original second-order differential equation is replaced by an approximate first-order differential equation with a small deviating argument. Then, the trapezoidal formula is used to obtain the three-term recurrence relationship. Thomas algorithm is used to solve the resulting tridiagonal algebraic system of equations. The proposed method is iterative on the deviating argument. The method is to be repeated for different choices of the deviating argument until the solution profile stabilises. To demonstrate the applicability of the method, we have solved several model linear and non-linear examples with left-end boundary layer or right-end boundary layer or an internal layer or two boundary layers and presented the computational results. It is observed that the numerical integration method approximates the exact solution very well.

## 2. Numerical integration method

For convenience we call our method the 'Numerical Integration Method'. To set the stage for the numerical integration method, we consider the following linear singularly perturbed two-point boundary value problem:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad 0 \leqslant x \leqslant 1 \tag{1}$$

$$\text{with} \quad y(0) = \alpha \quad \text{and} \quad y(1) = \beta, \tag{2}$$

where $\varepsilon$ is a small positive parameter $(0 < \varepsilon \ll 1)$; $\alpha$, $\beta$ are given constants; $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0, 1]$. Furthermore, we assume that $a(x) \geqslant M > 0$ throughout the interval $[0, 1]$, where $M$ is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

Let $\delta$ be a small positive deviating argument ($0 < \delta \ll 1$). By using Taylor series expansions in the neighbourhood of the point $x$, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x), \tag{3}$$

and consequently, Eq. (1) is replaced by the following first-order differential equation with a small deviating argument:

$$2\varepsilon y(x-\delta) - 2\varepsilon y(x) + 2\varepsilon\delta y'(x) + \delta^2 a(x) y'(x) + \delta^2 b(x) y(x) = \delta^2 f(x). \tag{4}$$

Transition from Eq. (1) to Eq. (4) is admitted, because of the condition that $\delta$ is small ($0 < \delta \ll 1$). This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in Elsgolts and Norkin [3, pp. 243 and 244]. Theory and discussion on the differential equations with a deviating argument can be found in Elsgolts and Norkin [3].

We rewrite Eq. (4) in the following convenient form:

$$y'(x) = p(x) y(x-\delta) + q(x) y(x) + r(x) \quad \text{for } \delta \leqslant x \leqslant 1, \tag{5}$$

where

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)}, \tag{6}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{2\varepsilon\delta + \delta^2 a(x)}, \tag{7}$$

$$r(x) = \frac{\delta^2 f(x)}{2\varepsilon\delta + \delta^2 a(x)}. \tag{8}$$

We now divide the interval $[0,1]$ into $N$ equal parts with mesh size $h$, i.e., $h = 1/N$ and $x_i = ih$ for $i = 0,1,2,\ldots,N$. Integrating Eq. (5) in $[x_i, x_{i+1}]$, ($i = 1,2,\ldots,N-1$), we get

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} [p(x) y(x-\delta) + q(x) y(x) + r(x)]\,\mathrm{d}x. \tag{9}$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$\begin{aligned} y(x_{i+1}) - y(x_i) = {} & \frac{h}{2}[p(x_{i+1}) y(x_{i+1}-\delta) + p(x_i) y(x_i - \delta)] \\ & + \frac{h}{2}[q(x_{i+1}) y(x_{i+1}) + q(x_i) y(x_i)] + \frac{h}{2}[r(x_{i+1}) + r(x_i)]. \end{aligned} \tag{10}$$

Again, by means of Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x),$$

and, then by approximating $y'(x)$ by linear interpolation, we get

$$y(x_i - \delta) \approx y(x_i) - \delta\left(\frac{y(x_i) - y(x_{i-1})}{h}\right) = \left(1 - \frac{\delta}{h}\right)y(x_i) + \frac{\delta}{h}y(x_{i-1}), \quad (11)$$

and similarly

$$y(x_{i+1} - \delta) \approx \left(1 - \frac{\delta}{h}\right)y(x_{i+1}) + \frac{\delta}{h}y(x_i). \quad (12)$$

Hence, by making use of (11) and (12) in (10) we obtain

$$\begin{aligned} y(x_{i+1}) - y(x_i) = {} & \frac{h}{2}p(x_{i+1})\left[\left(1 - \frac{\delta}{h}\right)y(x_{i+1}) + \frac{\delta}{h}y(x_i)\right] \\ & + \frac{h}{2}p(x_i)\left[\left(1 - \frac{\delta}{h}\right)y(x_i) + \frac{\delta}{h}y(x_{i-1})\right] \\ & + \frac{h}{2}[q(x_{i+1})y(x_{i+1}) + q(x_i)y(x_i)] + \frac{h}{2}[r(x_{i+1}) + r(x_i)], \\ y(x_{i+1}) - y(x_i) = {} & \frac{h}{2}\left(1 - \frac{\delta}{h}\right)p(x_{i+1})y(x_{i+1}) + \frac{\delta}{2}p(x_{i+1})y(x_i) \\ & + \frac{h}{2}\left(1 - \frac{\delta}{h}\right)p(x_i)y(x_i) + \frac{\delta}{2}p(x_i)y(x_{i-1}) + \frac{h}{2}q(x_{i+1})y(x_{i+1}) \\ & + \frac{h}{2}q(x_i)y(x_i) + \frac{h}{2}[r(x_{i+1}) + r(x_i)]. \end{aligned}$$

This equation leads after simple rearrangement to the final three-term recurrence relationship, namely

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad \text{for } i = 1, 2, \ldots, N-1, \quad (13)$$

where

$$E_i = -\frac{\delta}{2}p_i, \quad (14)$$

$$F_i = 1 + \frac{\delta}{2}p_{i+1} + \frac{h}{2}\left(1 - \frac{\delta}{h}\right)p_i + \frac{h}{2}q_i, \quad (15)$$

$$G_i = 1 - \frac{h}{2}\left(1 - \frac{\delta}{h}\right)p_{i+1} - \frac{h}{2}q_{i+1}, \quad (16)$$

$$H_i = \frac{h}{2}[r_{i+1} + r_i], \quad (17)$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Eq. (13) gives a system of $(N-1)$ equations with $(N+1)$ unknowns $y_0$ to $y_N$. The two given boundary conditions (2) together with these $(N-1)$ equations are then sufficient to solve for the unknowns $y_0$ to $y_N$. The solution of the tridiagonal system (13) can easily be obtained by using an efficient algorithm called 'Thomas Algorithm' also called 'Discrete Invariant Imbedding' [1]. In this algorithm we set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i, \tag{18}$$

where $W_i$ and $T_i$ corresponding to $W(x_i)$ and $T(x_i)$ are to be determined. From (18) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \tag{19}$$

Substituting (19) in (13), we get

$$E_i(W_{i-1}y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_I,$$

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}. \tag{20}$$

By comparing (18) and (20), we get

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}}, \tag{21}$$

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}. \tag{22}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$; we need to know the initial conditions for $W_0$ and $T_0$. This can be done by considering (2)

$$y_0 = \alpha = W_0 y_1 + T_0. \tag{23}$$

If we choose $W_0 = 0$, then $T_0 = \alpha$. With these initial values, we compute sequentially $W_i$ and $T_i$ for $i = 1, 2, \ldots, N-1$; from (21) and (22) in the forward process and then obtain $y_i$ in the backward process from (18) using (2).

Repeat the numerical scheme for different choices of $\delta$ (deviating argument, satisfying the condition $0 < \delta \ll 1$), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$|y(x)^{m+1} - y(x)^m| \leqslant \sigma, \quad 0 \leqslant x \leqslant 1, \tag{24}$$

where $y(x)^m$ is the solution for the $m$th iterate of $\delta$, and $\sigma$ is the prescribed tolerance bound.

### 2.1. Linear problems

To demonstrate the applicability of the numerical integration method, we have applied it to four linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in the literature and because approximate solutions are available for comparison.

**Example 1.** Consider the following homogeneous SPP from Kevorkian and Cole [6, p. 33, Eqs. (2.3.26) and (2.3.27)] with $\alpha = 0$:

$$\varepsilon y''(x) + y'(x) = 0, \quad 0 \leqslant x \leqslant 1$$
$$\text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}.$$

The computational results are presented in Table 1(a) and (b) for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

Table 1
Computational results for Example 1

| $x$ | $y(x)$ | | | Exact solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | 0.9876486 | 0.9899944 | 0.9917358 | 1.0000000 |
| 0.04 | 0.9998419 | 0.9998944 | 0.9999319 | 1.0000000 |
| 0.06 | 0.9999925 | 0.9999934 | 0.9999995 | 1.0000000 |
| 0.08 | 0.9999945 | 0.9999945 | 1.0000000 | 1.0000000 |
| 0.10 | 0.9999946 | 0.9999946 | 1.0000000 | 1.0000000 |
| 0.20 | 0.9999952 | 0.9999952 | 1.0000000 | 1.0000000 |
| 0.40 | 0.9999964 | 0.9999964 | 1.0000000 | 1.0000000 |
| 0.60 | 0.9999976 | 0.9999976 | 1.0000000 | 1.0000000 |
| 0.80 | 0.9999988 | 0.9999988 | 1.0000000 | 1.0000000 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | 0.9998016 | 0.9998477 | 0.9998792 | 1.0000000 |
| 0.04 | 0.9999999 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.06 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.08 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.10 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.20 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.40 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.60 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.80 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Example 2.** Consider the following homogeneous SPP from Bender and Orszag [2, p. 480, problem 9.17] with $\alpha = 0$:

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \qquad 0 \leqslant x \leqslant 1$$
$$\text{with} \quad y(0) = 1 \quad \text{and} \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}}{(e^{m_2} - e^{m_1})},$$

where

$$m_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon},$$

Table 2
Computational results for Example 2

| $x$ | $y(x)$ | | | Exact solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.02 | 0.3834782 | 0.3819607 | 0.3808348 | 0.3756784 |
| 0.04 | 0.3834413 | 0.3833556 | 0.3832939 | 0.3832599 |
| 0.06 | 0.3910828 | 0.3910290 | 0.3909866 | 0.3909945 |
| 0.08 | 0.3989721 | 0.3989189 | 0.3988770 | 0.3988851 |
| 0.10 | 0.4070218 | 0.4069687 | 0.4069269 | 0.4069350 |
| 0.20 | 0.4497731 | 0.4497210 | 0.4496799 | 0.4496879 |
| 0.40 | 0.5492185 | 0.5491707 | 0.5491330 | 0.5491404 |
| 0.60 | 0.6706514 | 0.6706123 | 0.6705817 | 0.6705877 |
| 0.80 | 0.8189330 | 0.8189092 | 0.8188905 | 0.8188942 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.02 | 0.3754841 | 0.3754509 | 0.3754246 | 0.3753479 |
| 0.04 | 0.3829417 | 0.3829373 | 0.3829308 | 0.3829296 |
| 0.06 | 0.3906766 | 0.3906722 | 0.3906657 | 0.3906645 |
| 0.08 | 0.3985677 | 0.3985633 | 0.3985569 | 0.3985557 |
| 0.10 | 0.4066183 | 0.4066139 | 0.4066074 | 0.4066062 |
| 0.20 | 0.4493767 | 0.4493724 | 0.4493661 | 0.4493649 |
| 0.40 | 0.5488553 | 0.5488514 | 0.5488456 | 0.5488445 |
| 0.60 | 0.6703555 | 0.6703524 | 0.6703477 | 0.6703469 |
| 0.80 | 0.8187524 | 0.8187507 | 0.8187476 | 0.8187471 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

and

$$m_2 = \frac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}.$$

The computational results are presented in Table 2(a) and (b) for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

**Example 3.** Now consider the following non-homogeneous SPP:

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \qquad 0 \leqslant x \leqslant 1$$

$$\text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}.$$

The computational results are presented in Table 3(a) and (b) for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

Table 3
Computational results for Example 3

| $x$ | $y(x)$ | | | Exact solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | −0.9648339 | −0.9674433 | −0.9693918 | −0.9776400 |
| 0.04 | −0.9558468 | −0.9561659 | −0.9564115 | −0.9564800 |
| 0.06 | −0.9340470 | −0.9343091 | −0.9345190 | −0.9345200 |
| 0.08 | −0.9112990 | −0.9115546 | −0.9117595 | −0.9117600 |
| 0.10 | −0.8877491 | −0.8879992 | −0.8881995 | −0.8882000 |
| 0.20 | −0.7579996 | −0.7582219 | −0.7583995 | −0.7584000 |
| 0.40 | −0.4385003 | −0.4386670 | −0.4387995 | −0.4388000 |
| 0.60 | −0.0390007 | −0.0391118 | −0.0391996 | −0.0391999 |
| 0.80 | 0.4404994 | 0.4404438 | 0.4404002 | 0.4404000 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | −0.9791212 | −0.9792020 | −0.9792610 | −0.9794040 |
| 0.04 | −0.9581252 | −0.9581594 | −0.9581861 | −0.9582080 |
| 0.06 | −0.9361309 | −0.9361644 | −0.9361906 | −0.9362120 |
| 0.08 | −0.9133366 | −0.9133694 | −0.9133950 | −0.9134160 |
| 0.10 | −0.8897424 | −0.8897744 | −0.8897995 | −0.8898200 |
| 0.20 | −0.7597710 | −0.7597994 | −0.7598217 | −0.7598400 |
| 0.40 | −0.4398281 | −0.4398495 | −0.4398662 | −0.4398800 |
| 0.60 | −0.0398854 | −0.0398996 | −0.0399107 | −0.0399199 |
| 0.80 | 0.4400573 | 0.4400503 | 0.4400447 | 0.4400400 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Example 4.** Finally, we consider the following SPP with variable coefficients from Kevorkian and Cole [6, p. 33, Eqs. (2.3.26) and (2.3.27)] with $\alpha = -1/2$;

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0, \qquad 0 \leqslant x \leqslant 1$$
$$\text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1.$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [7, p. 148, Eq. (4.2.32)]) as our 'exact' solution,

$$y(x) = \frac{1}{2-x} - \frac{1}{2}\exp\left(-\left(x - \frac{x^2}{4}\right)\Big/\varepsilon\right).$$

The computational results are presented in Table 4(a) and (b) for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

Table 4
Computational results for Example 4

| $x$ | $y(x)$ | | | Nayfeh's solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | 0.4997202 | 0.5008049 | 0.5016018 | 0.5050505 |
| 0.04 | 0.5110424 | 0.5109667 | 0.5109015 | 0.5102041 |
| 0.06 | 0.5163831 | 0.5162804 | 0.5161986 | 0.5154639 |
| 0.08 | 0.5217560 | 0.5216525 | 0.5215703 | 0.5208333 |
| 0.10 | 0.5272408 | 0.5271369 | 0.5270546 | 0.5263158 |
| 0.20 | 0.5564888 | 0.5563842 | 0.5563012 | 0.5555556 |
| 0.40 | 0.6259239 | 0.6258209 | 0.6257385 | 0.6250000 |
| 0.60 | 0.7151331 | 0.7150390 | 0.7149630 | 0.7142857 |
| 0.80 | 0.8339487 | 0.8338804 | 0.8338253 | 0.8333333 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | 0.5050477 | 0.5050570 | 0.5050629 | 0.5050505 |
| 0.04 | 0.5103018 | 0.5102878 | 0.5102776 | 0.5102041 |
| 0.06 | 0.5155619 | 0.5155479 | 0.5155377 | 0.5154639 |
| 0.08 | 0.5209316 | 0.5209177 | 0.5209073 | 0.5208333 |
| 0.10 | 0.5264143 | 0.5264003 | 0.5263901 | 0.5263158 |
| 0.20 | 0.5556549 | 0.5556409 | 0.5556305 | 0.5555556 |
| 0.40 | 0.6250984 | 0.6250846 | 0.6250743 | 0.6250000 |
| 0.60 | 0.7143759 | 0.7143634 | 0.7143539 | 0.7142857 |
| 0.80 | 0.8333989 | 0.8333899 | 0.8333829 | 0.8333333 |
| 1.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

### 2.2. Non-linear problems

We have used quasilinearisation process to linearise the non-linear singular perturbation problems and then applied our method on three classical problems.

**Example 5.** Consider the following example from Bender and Orszag [2, p. 463, Eq. (9.7.1)];

$$\varepsilon y'' + 2y' + e^{y} = 0, \qquad 0 \leqslant x \leqslant 1$$
$$\text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 0.$$

We have chosen to use Bender and Orszag's uniformly valid approximation [2, p. 463, Eq. (9.7.6)] for comparison.

Table 5
Computational results for the Example 5

| $x$ | $y(x)$ | | | Bender & Orszag's solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | 0.6713438 | 0.6717810 | 0.6720961 | 0.6733446 |
| 0.04 | 0.6543222 | 0.6542806 | 0.6542470 | 0.6539265 |
| 0.06 | 0.6352631 | 0.6352206 | 0.6351870 | 0.6348783 |
| 0.08 | 0.6165527 | 0.6165122 | 0.6164801 | 0.6161861 |
| 0.10 | 0.5981860 | 0.5981475 | 0.5981169 | 0.5978370 |
| 0.20 | 0.5110986 | 0.5110684 | 0.5110446 | 0.5108256 |
| 0.40 | 0.3568378 | 0.3568198 | 0.3568057 | 0.3566749 |
| 0.60 | 0.2232326 | 0.2232228 | 0.2232150 | 0.2231435 |
| 0.80 | 0.1053979 | 0.1053937 | 0.1053905 | 0.1053605 |
| 1.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.02 | 0.6733644 | 0.6733668 | 0.6733681 | 0.6733446 |
| 0.04 | 0.6539788 | 0.6539733 | 0.6539692 | 0.6539265 |
| 0.06 | 0.6349280 | 0.6349229 | 0.6349189 | 0.6348783 |
| 0.08 | 0.6162335 | 0.6162285 | 0.6162248 | 0.6161861 |
| 0.10 | 0.5978821 | 0.5978773 | 0.5978737 | 0.5978370 |
| 0.20 | 0.5108607 | 0.5108569 | 0.5108542 | 0.5108256 |
| 0.40 | 0.3566956 | 0.3566935 | 0.3566918 | 0.3566749 |
| 0.60 | 0.2231549 | 0.2231536 | 0.2231526 | 0.2231435 |
| 0.80 | 0.1053652 | 0.1053647 | 0.1053643 | 0.1053605 |
| 1.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

$$y(x) = \log\frac{2}{1+x} - (\exp(-2x/\varepsilon))\log 2.$$

For this example, we have boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. Bender and Orszag [2]).

The computational results are presented in Table 5(a) and (b), for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

**Example 6.** Now, consider the following example from Kevorkian and Cole [6, p. 56, Eq. (2.5.1)]:

$$\varepsilon y'' + yy' - y = 0, \qquad 0 \leqslant x \leqslant 1$$
$$\text{with} \quad y(0) = -1 \quad \text{and} \quad y(1) = 3.9995.$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation ([6, pp. 57 and 58, Eqs. (2.5.5), (2.5.11) and (2.5.14)]) for comparison.

Table 6
Computational results for Example 6

| $x$ | $y(x)$ | | | Kevorkian & Cole's solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| 0.02 | 3.0131600 | 3.0144450 | 3.0153750 | 3.0195000 |
| 0.04 | 3.0394860 | 3.0394910 | 3.0394920 | 3.0395000 |
| 0.06 | 3.0594960 | 3.0594970 | 3.0594970 | 3.0595000 |
| 0.08 | 3.0794960 | 3.0794980 | 3.0794970 | 3.0795000 |
| 0.10 | 3.0994960 | 3.0994980 | 3.0994970 | 3.0995000 |
| 0.20 | 3.1994960 | 3.1994970 | 3.1994970 | 3.1995000 |
| 0.40 | 3.3994970 | 3.3994990 | 3.3994970 | 3.3995000 |
| 0.60 | 3.5994980 | 3.5995000 | 3.5994990 | 3.5995000 |
| 0.80 | 3.7994990 | 3.7995000 | 3.7995000 | 3.7995000 |
| 1.00 | 3.9995000 | 3.9995000 | 3.9995000 | 3.9995000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| 0.02 | 3.0194070 | 3.0194290 | 3.0194440 | 3.0195000 |
| 0.04 | 3.0394960 | 3.0394980 | 3.0394980 | 3.0395000 |
| 0.06 | 3.0594960 | 3.0594980 | 3.0594980 | 3.0595000 |
| 0.08 | 3.0794960 | 3.0794980 | 3.0794980 | 3.0795000 |
| 0.10 | 3.0994960 | 3.0994980 | 3.0994990 | 3.0995000 |
| 0.20 | 3.1994960 | 3.1994970 | 3.1994990 | 3.1995000 |
| 0.40 | 3.3994980 | 3.3994980 | 3.3995010 | 3.3995000 |
| 0.60 | 3.5994990 | 3.5994980 | 3.5995000 | 3.5995000 |
| 0.80 | 3.7994990 | 3.7994980 | 3.7995000 | 3.7995000 |
| 1.00 | 3.9995000 | 3.9995000 | 3.9995000 | 3.9995000 |

$$y(x) = x + c_1 \tanh \frac{c_1}{2}\left(\frac{x}{\varepsilon} + c_2\right),$$

where

$$c_1 = 2.9995 \quad \text{and} \quad c_2 = \frac{1}{c_1}\log\left(\frac{c_1 - 1}{c_1 + 1}\right).$$

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. Kevorkian and Cole [6]).

The computational results are presented in Table 6(a) and (b), for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

**Example 7.** Finally, consider the following example from O'Malley [10, p. 9, Eq. (1.10), Case 2]:

Table 7
Computational results for Example 7

| $x$ | $y(x)$ | | | O'Malley's solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| −1.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.98 | −0.9876427 | −0.9899885 | −0.9917358 | −1.0000000 |
| −0.96 | −0.9998360 | −0.9998885 | −0.9999319 | −1.0000000 |
| −0.94 | −0.9999866 | −0.9999875 | −0.9999995 | −1.0000000 |
| −0.92 | −0.9999886 | −0.9999886 | −1.0000000 | −1.0000000 |
| −0.90 | −0.9999887 | −0.9999887 | −1.0000000 | −1.0000000 |
| −0.80 | −0.9999893 | −0.9999893 | −1.0000000 | −1.0000000 |
| −0.40 | −0.9999917 | −0.9999917 | −1.0000000 | −1.0000000 |
| 0.00 | −0.9999940 | −0.9999940 | −1.0000000 | −1.0000000 |
| 0.40 | −0.9999964 | −0.9999964 | −1.0000000 | −1.0000000 |
| 0.80 | −0.9999988 | −0.9999988 | −1.0000000 | −1.0000000 |
| 1.00 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| −1.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.98 | −0.9998016 | −0.9998477 | −0.9998792 | −1.0000000 |
| −0.96 | −0.9999999 | −1.0000000 | −1.0000000 | −1.0000000 |
| −0.94 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| −0.92 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| −0.90 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| −0.80 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| −0.40 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| 0.00 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| 0.40 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| 0.80 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |
| 1.00 | −1.0000000 | −1.0000000 | −1.0000000 | −1.0000000 |

$$\varepsilon y'' - yy' = 0, \qquad -1 \leqslant x \leqslant 1,$$
$$\text{with} \quad y(-1) = 0 \quad \text{and} \quad y(1) = -1.$$

We have chosen to use O'Malley's approximate solution [10, pp. 9 and 10, Eqs. (1.13) and (1.14)] for comparison.

$$y(x) = -\frac{[1 - \exp(-(x+1)/\varepsilon)]}{[1 + \exp(-(x+1)/\varepsilon)]}.$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at the left end of the interval, that is, at $x = -1$ (cf. O'Malley [10]).

The computational results are presented in Table 7(a) and (b), for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

## 3. Right-end boundary layer problems

We now describe the numerical integration method for solving problems with the boundary layer at the right-end of the underlying interval. To be specific we consider the following singular perturbation problem:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \qquad 0 \leqslant x \leqslant 1, \tag{25}$$

$$\text{with} \quad y(0) = \alpha \quad \text{and} \quad y(1) = \beta, \tag{26}$$

where $\varepsilon$ is a small positive parameter ($0 < \varepsilon \ll 1$); $\alpha$, $\beta$ are given constants; $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0, 1]$.

We now assume that $a(x) \leqslant M < 0$ throughout the interval $[0, 1]$, where $M$ is some negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 1$.

The evaluation of the right-end boundary layer for (25) and (26) is similar to that of the left-end boundary layer but there are some differences worth noting. By using Taylor series expansion in the neighbourhood of the point $x$, we have

$$y(x+\delta) \approx y(x) + \delta y'(x) + \frac{\delta^2}{2} y''(x), \tag{27}$$

and consequently, Eq. (25) is replaced by the following first-order differential equation with a small deviating argument:

$$2\varepsilon y(x+\delta) - 2\varepsilon y(x) - 2\varepsilon\delta y'(x) + \delta^2 a(x)y'(x) + \delta^2 b(x)y(x) = \delta^2 f(x). \tag{28}$$

Transition from Eq. (25) to Eq. (28) is admitted, because of the condition that $\delta$ is small ($0 < \delta \ll 1$).

We rewrite Eq. (28) in the following convenient form:

$$y'(x) = p(x)y(x+\delta) + q(x)y(x) + r(x) \quad \text{for } 0 \leqslant x \leqslant 1 - \delta, \tag{29}$$

where

$$p(x) = \frac{-2\varepsilon}{\delta^2 a(x) - 2\varepsilon\delta}, \tag{30}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{\delta^2 a(x) - 2\varepsilon\delta}, \tag{31}$$

$$r(x) = \frac{\delta^2 f(x)}{\delta^2 a(x) - 2\varepsilon\delta}. \tag{32}$$

We will now describe the numerical scheme for solving Eq. (29). As usual, we divide the interval $[0,1]$ into $N$ equal parts with mesh $h$, i.e., $h = 1/N$ and $x_i = ih$ for $i = 0, 1, \ldots, N$.

Integrating Eq. (29) in $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, N-1$; we get

$$y(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} [p(x)y(x+\delta) + q(x)y(x) + r(x)]\,\mathrm{d}x.$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$\begin{aligned} y(x_i) - y(x_{i-1}) = {} & \frac{h}{2}[p(x_{i-1})y(x_{i-1}+\delta) + p(x_i)y(x_i+\delta)] \\ & + \frac{h}{2}[q(x_{i-1})y(x_{i-1}) + q(x_i)y(x_i)] + \frac{h}{2}[r(x_{i-1}) + r(x_i)]. \end{aligned} \tag{33}$$

By means of Taylor series expansion, we have

$$y(x+\delta) \approx y(x) + \delta y'(x),$$

and, then by approximating $y'(x)$ by interpolation formula, we get

$$y(x_i+\delta) \approx y(x_i) + \frac{\delta}{h}[y(x_{i+1}) - y(x_i)],$$

$$y(x_i+\delta) \approx \left(1 - \frac{\delta}{h}\right)y(x_i) + \frac{\delta}{h}y(x_{i+1}), \tag{34}$$

and similarly we have

$$y(x_{i-1}+\delta) \approx \left(1 - \frac{\delta}{h}\right)y(x_{i-1}) + \frac{\delta}{h}y(x_i). \tag{35}$$

Hence, by making use of (34) and (35) in (33) we get

$$
\begin{aligned}
y(x_i) - y(x_{i-1}) = {} & \frac{h}{2} p(x_{i-1}) \left[ \left(1 - \frac{\delta}{h}\right) y(x_{i-1}) + \frac{\delta}{h} y(x_i) \right] \\
& + \frac{h}{2} p(x_i) \left[ \left(1 - \frac{\delta}{h}\right) y(x_i) + \frac{\delta}{h} y(x_{i+1}) \right] \\
& + \frac{h}{2} [q(x_{i-1}) y(x_{i-1}) + q(x_i) y(x_i)] + \frac{h}{2} [r(x_{i-1}) + r(x_i)],
\end{aligned}
$$

$$
\begin{aligned}
y(x_i) - y(x_{i-1}) = {} & \frac{h}{2} \left(1 - \frac{\delta}{h}\right) p(x_{i-1}) y(x_{i-1}) + \frac{\delta}{2} p(x_{i-1}) y(x_i) \\
& + \frac{h}{2} \left(1 - \frac{\delta}{h}\right) p(x_i) y(x_i) + \frac{\delta}{2} p(x_i) y(x_{i+1}) + \frac{h}{2} q(x_{i-1}) y(x_{i-1}) \\
& + \frac{h}{2} q(x_i) y(x_i) + \frac{h}{2} [r(x_{i-1}) + r(x_i)].
\end{aligned}
$$

Finally, this leads after simple rearrangement to the following three-term recurrence relationship:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad \text{for } i = 1, 2, \ldots, N-1, \tag{36}$$

where

$$E_i = -1 - \frac{h}{2} \left(1 - \frac{\delta}{h}\right) p_{i-1} - \frac{h}{2} q_{i-1}, \tag{37}$$

$$F_i = -1 + \frac{\delta}{2} p_{i-1} + \frac{h}{2} \left(1 - \frac{\delta}{h}\right) p_i + \frac{h}{2} q_i, \tag{38}$$

$$G_i = -\frac{\delta}{2} p_i, \tag{39}$$

$$H_i = \frac{h}{2} [r_{i-1} + r_i], \tag{40}$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Eq. (36) gives a system of $(N-1)$ equations with $(N+1)$ unknowns $y_0$ to $y_N$. The two given boundary conditions (26) together with these $(N-1)$ equations are then sufficient to solve for the unknowns $y_0$ to $y_N$. The solution of the tridiagonal system (36) can easily be obtained by using an efficient algorithm called 'Thomas Algorithm' described in the previous section. Repeat the numerical scheme for different choices of $\delta$ (deviating argument, satisfying the condition $0 < \delta \ll 1$), until the solution profiles do not differ materially from iteration to iteration.

**Example 8.** To demonstrate the applicability of the numerical integration method, we will discuss one singular perturbation problem with right-end boundary layer.

Table 8
Computational results for Example 8

| $x$ | $y(x)$ | | | Exact solution |
|---|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ | |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | | |
| 0.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.20 | 0.9999989 | 0.9999997 | 1.0000000 | 1.0000000 |
| 0.40 | 0.9999975 | 0.9999997 | 1.0000000 | 1.0000000 |
| 0.60 | 0.9999962 | 0.9999997 | 1.0000000 | 1.0000000 |
| 0.80 | 0.9999948 | 0.9999997 | 1.0000000 | 1.0000000 |
| 0.90 | 0.9999942 | 0.9999997 | 1.0000000 | 1.0000000 |
| 0.92 | 0.9999940 | 0.9999997 | 1.0000000 | 1.0000000 |
| 0.94 | 0.9999920 | 0.9999987 | 0.9999995 | 1.0000000 |
| 0.96 | 0.9998413 | 0.9998997 | 0.9999318 | 1.0000000 |
| 0.98 | 0.9876480 | 0.9899997 | 0.9917356 | 1.0000000 |
| 1.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ | |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | | |
| 0.00 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.20 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.40 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.60 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.80 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.90 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.92 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.94 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.96 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.98 | 0.9998017 | 0.9998476 | 0.9998792 | 1.0000000 |
| 1.00 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

$$\varepsilon y''(x) - y'(x) = 0; \qquad 0 \leqslant x \leqslant 1$$
$$\text{with} \quad y(0) = 1 \quad \text{and} \quad y(1) = 0.$$

For this example we have $a(x) = -1$, $b(x) = 0$ and $f(x) = 0$. Further we have a boundary layer of width $O(\varepsilon)$ at $x = 1$.

The exact solution is given by

$$y(x) = \frac{1 - \exp((x-1)/\varepsilon)}{1 - \exp(-1/\varepsilon)}.$$

The computational results are presented in Table 8(a) and (b), for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

## 4. Internal layer problems

We will now discuss the singular perturbation problem with an internal layer of the underlying interval. In this case $a(x)$ changes sign in the domain of interest. Without loss of generality, we can take $a(0) = 0$, and the interval to be $[-1, 1]$. With the help of one model example we demonstrate the applicability of the numerical integration method for solving singular perturbation problems with an internal layer.

**Example 9.** Consider the following singular perturbation problem:

$$\varepsilon y''(x) + xy'(x) - y(x) = 0, \qquad -1 \leqslant x \leqslant 1 \tag{41}$$

$$\text{with} \quad y(-1) = 1 \quad \text{and} \quad y(1) = 2. \tag{42}$$

For this example we have $a(x) = x$, $b(x) = -1$ and $f(x) = 0$. Further we have an internal layer of width $O(\sqrt{\varepsilon})$ at $x = 0$. (For details, see O'Malley [10, pp. 168–172, Eq. (8.1), case (i)], and Kevorkian and Cole [6, pp. 41–43, Eqs. (2.3.76) and (2.3.77)]).

We see that the function

$$\begin{aligned} a(x) &= x < 0 \quad \text{for} -1 \leqslant x < 0, \\ a(x) &= x = 0 \quad \text{for } x = 0, \\ a(x) &= x > 0 \quad \text{for } 0 < x \leqslant 1. \end{aligned}$$

Hence, by making use of transitions suggested for left-end and right-end boundary layers, we replace Eq. (41) by the following first-order differential equations with a small deviating argument:

$$y'(x) = p(x)y(x + \delta) + q(x)y(x) + r(x) \quad \text{for} -1 \leqslant x \leqslant -\delta, \tag{43}$$

where $p(x)$, $q(x)$ and $r(x)$ are given by

$$p(x) = \frac{-2\varepsilon}{\delta^2 a(x) - 2\varepsilon\delta}, \qquad q(x) = \frac{2\varepsilon - \delta^2 b(x)}{\delta^2 a(x) - 2\varepsilon\delta}$$

$$\text{and} \quad r(x) = \frac{\delta^2 f(x)}{\delta^2 a(x) - 2\varepsilon\delta},$$

and

$$y'(x) = p(x)y(x - \delta) + q(x)y(x) + r(x) \quad \text{for } \delta \leqslant x \leqslant 1, \tag{44}$$

where $p(x)$, $q(x)$ and $r(x)$ are given by

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)}, \qquad q(x) = \frac{2\varepsilon - \delta^2 b(x)}{2\varepsilon\delta + \delta^2 a(x)}$$

$$\text{and} \quad r(x) = \frac{\delta^2 f(x)}{2\varepsilon\delta + \delta^2 a(x)}.$$

We now divide the interval $[-1, 1]$ into $N$ equal parts with mesh size $h$, i.e., $h = 2/N$ and $x_i = -1 + ih$ for $i = 0, 1, \ldots, N$.

Let us denote $N/2 = L$. Then, integrating using the trapezoidal formula Eq. (43) in $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, L-1$; and Eq. (44) in $[x_i, x_{i+1}]$ for $i = L+1, L+2, \ldots, N-1$; we get a system of $(N-2)$ equations with $(N+1)$ unknowns. From the given boundary conditions (42) we get two equations

$$y_0 = y(-1) = 1,$$

$$y_N = y(1) = 2.$$

Table 9
Computational results for Example 9

| $x$ | $y(x)$ | | |
|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | |
| −1.00 | 1.0000000 | 1.0000000 | 1.0000000 |
| −0.50 | 0.5025128 | 0.5025125 | 0.5025129 |
| −0.10 | 0.1007176 | 0.1006431 | 0.1005968 |
| −0.08 | 0.0812367 | 0.0810065 | 0.0808482 |
| −0.06 | 0.0631051 | 0.0625267 | 0.0620928 |
| −0.04 | 0.0483043 | 0.0471570 | 0.0462338 |
| −0.02 | 0.0401910 | 0.0384762 | 0.0370292 |
| 0.00 | 0.0427526 | 0.0410037 | 0.0395100 |
| 0.02 | 0.0600897 | 0.0583748 | 0.0569278 |
| 0.04 | 0.0881019 | 0.0869545 | 0.0860313 |
| 0.06 | 0.1228018 | 0.1222235 | 0.1217896 |
| 0.08 | 0.1608329 | 0.1606027 | 0.1604443 |
| 0.10 | 0.2002133 | 0.2001388 | 0.2000924 |
| 0.50 | 0.9999993 | 0.9999990 | 0.9999991 |
| 1.00 | 2.0000000 | 2.0000000 | 2.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | |
| −1.00 | 1.0000000 | 1.0000000 | 1.0000000 |
| −0.50 | 0.5025129 | 0.5025131 | 0.5025126 |
| −0.10 | 0.1005030 | 0.1005031 | 0.1005030 |
| −0.08 | 0.0804026 | 0.0804026 | 0.0804026 |
| −0.06 | 0.0603046 | 0.0603036 | 0.0603030 |
| −0.04 | 0.0402802 | 0.0402574 | 0.0402424 |
| −0.02 | 0.0215259 | 0.0213159 | 0.0211514 |
| 0.00 | 0.0131929 | 0.0129429 | 0.0127334 |
| 0.02 | 0.0414236 | 0.0412137 | 0.0410492 |
| 0.04 | 0.0800772 | 0.0800545 | 0.0800394 |
| 0.06 | 0.1200011 | 0.1200002 | 0.1199995 |
| 0.08 | 0.1599986 | 0.1599988 | 0.1599986 |
| 0.10 | 0.1999985 | 0.1999998 | 0.1999986 |
| 0.50 | 0.9999989 | 0.9999995 | 0.9999982 |
| 1.00 | 2.0000000 | 2.0000000 | 2.0000000 |

We need one more equation to solve for the unknowns $(y_0, y_1, \ldots, y_N)$. For this, we consider the original equation at $x = x_L = 0$. Since $a(x) = 0$ at $x = x_L = 0$, we get the following:

$$\varepsilon y''(x_L) + b(x_L)y(x_L) = f(x_L). \tag{45}$$

By making use of the second-order central finite difference approximation for the second-order derivative in Eq. (45) at $x_L$ we get the following equation:

$$[\varepsilon]y_{L-1} - [2\varepsilon - h^2 b_L]y_L + [\varepsilon]y_{L+1} = h^2 f_L. \tag{46}$$

With this Eq. (46) we now have $(N+1)$ equations to solve for the $(N+1)$ unknowns $(y_0, y_1, \ldots, y_N)$.

The matrix problem associated is a tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using an efficient and stable algorithm called Thomas Algorithm. Repeat the numerical scheme for different choices of $\delta$ (deviating argument, satisfying the condition $0 < \delta \ll 1$), until the solution profiles do not differ materially from iteration to iteration.

The computational results are presented in Table 9(a) and (b), for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

## 5. Problems with two boundary layers

The suggestions given for internal layer problems apply mutatis mutandis to problems with two boundary layers. To illustrate this, we will again consider the case where $a(x)$ changes sign in the domain of interest. Without loss of generality, we can take $a(0) = 0$, and the interval to be $[-1, 1]$. Again with the help of one model example we demonstrate the applicability of the numerical integration method for solving singular perturbation problems with two boundary layers.

**Example 10.** Consider the following singular perturbation problem:

$$\varepsilon y''(x) - xy'(x) - y(x) = 0, \qquad -1 \leqslant x \leqslant 1 \tag{47}$$

$$\text{with} \quad y(-1) = 1 \quad \text{and} \quad y(1) = 2. \tag{48}$$

For this example we have $a(x) = -x$, $b(x) = -1$ and $f(x) = 0$. Further we have two boundary layers one at $x = -1$ and one at $x = 1$. (For details, see O'Malley [10, pp. 168–173, Eq. (8.1), case (ii)]).

We see that the function

$$a(x) = -x > 0 \quad \text{for} -1 \leqslant x < 0,$$
$$a(x) = -x = 0 \quad \text{for } x = 0,$$
$$a(x) = -x < 0 \quad \text{for } 0 < x \leqslant 1.$$

Hence, by making use of transitions suggested for left-end and right-end boundary layers, we replace Eq. (47) by the following first-order differential equations with a small deviating argument:

$$y'(x) = p(x)y(x-\delta) + q(x)y(x) + r(x) \quad \text{for} -1+\delta \leqslant x \leqslant 0, \tag{49}$$

where $p(x)$, $q(x)$ and $r(x)$ are given by

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)}, \qquad q(x) = \frac{2\varepsilon - \delta^2 b(x)}{2\varepsilon\delta + \delta^2 a(x)}$$
$$\text{and} \quad r(x) = \frac{\delta^2 f(x)}{2\varepsilon\delta + \delta^2 a(x)},$$

and

$$y'(x) = p(x)y(x+\delta) + q(x)y(x) + r(x) \quad \text{for } 0 \leqslant x \leqslant 1-\delta, \tag{50}$$

where $p(x)$, $q(x)$ and $r(x)$ are given by

$$p(x) = \frac{-2\varepsilon}{\delta^2 a(x) - 2\varepsilon\delta}, \qquad q(x) = \frac{2\varepsilon - \delta^2 b(x)}{\delta^2 a(x) - 2\varepsilon\delta}$$
$$\text{and} \quad r(x) = \frac{\delta^2 f(x)}{\delta^2 a(x) - 2\varepsilon\delta}.$$

As usual, we divide the interval $[-1, 1]$ into $N$ equal parts with mesh size $h$, i.e., $h = 2/N$ and $x_i = -1 + ih$ for $i = 0, 1, \ldots, N$.

Let us denote $N/2 = L$. Then, integrating using the trapezoidal formula Eq. (49) in $[x_i, x_{i+1}]$ for $i = 1, 2, \ldots, L-1$; and Eq. (50) in $[x_{i-1}, x_i]$ for $i = L+1, L+2, \ldots, N-1$; we get a system of $(N-2)$ equations with $(N+1)$ unknowns.

From the given boundary conditions (48) we get two equations

$$y_0 = y(-1) = 1,$$
$$y_N = y(1) = 2.$$

We need one more equation to solve for the unknowns $(y_0, y_1, \ldots, y_N)$. As in the previous section, we again consider the original equation at $x = x_L = 0$. Since $a(x) = 0$ at $x = x_L = 0$, we get the following:

$$\varepsilon y''(x_L) + b(x_L)y(x_L) = f(x_L). \tag{51}$$

By making use of the second-order central finite difference approximation for the second-order derivative in Eq. (51) at $x_L$ we get the following equation:

$$[\varepsilon]y_{L-1} - [2\varepsilon - h^2 b_L]y_L + [\varepsilon]y_{L+1} = h^2 f_L. \tag{52}$$

Hence, with this Eq. (52) we now have $(N+1)$ equations to solve for the $(N+1)$ unknowns $(y_0, y_1, \ldots, y_N)$.

The matrix problem associated is a tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using an efficient and stable algorithm called Thomas algorithm.

Repeat the numerical scheme for different choices of $\delta$ (deviating argument, satisfying the condition $0 < \delta \ll 1$), until the solution profiles do not differ materially from iteration to iteration.

Table 10
Computational results for Example 10

| $x$ | $y(x)$ | | |
|---|---|---|---|
| | $\delta = 0.008$ | $\delta = 0.009$ | $\delta = 0.01$ |
| (a) $\varepsilon = 10^{-3}$ *and* $h = 0.01$ | | | |
| −1.00 | 1.0000000 | 1.0000000 | 1.0000000 |
| −0.98 | 0.0125913 | 0.0101998 | 0.0084301 |
| −0.96 | 0.0001644 | 0.0001079 | 0.0000738 |
| −0.94 | 0.0000022 | 0.0000012 | 0.0000007 |
| −0.92 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.90 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.70 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.30 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.30 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.70 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.90 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.92 | 0.0000001 | 0.0000000 | 0.0000000 |
| 0.94 | 0.0000045 | 0.0000024 | 0.0000013 |
| 0.96 | 0.0003288 | 0.0002159 | 0.0001475 |
| 0.98 | 0.0251827 | 0.0203996 | 0.0168602 |
| 1.00 | 2.0000000 | 2.0000000 | 2.0000000 |
| | $\delta = 0.007$ | $\delta = 0.008$ | $\delta = 0.009$ |
| (b) $\varepsilon = 10^{-4}$ *and* $h = 0.01$ | | | |
| −1.00 | 1.0000000 | 1.0000000 | 1.0000000 |
| −0.98 | 0.0002024 | 0.0001555 | 0.0001232 |
| −0.96 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.94 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.92 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.90 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.70 | 0.0000000 | 0.0000000 | 0.0000000 |
| −0.30 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.30 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.70 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.90 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.92 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.94 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.96 | 0.0000001 | 0.0000001 | 0.0000000 |
| 0.98 | 0.0004048 | 0.0003110 | 0.0002464 |
| 1.00 | 2.0000000 | 2.0000000 | 2.0000000 |

The computational results are presented in Table 10(a) and (b), for $\varepsilon = 10^{-3}$, $10^{-4}$, respectively.

## 6. Discussion and conclusions

As mentioned, the numerical integration method is iterative on the deviating argument $\delta$. The process is to be repeated for different choices of $\delta$ (deviating argument), until the solution profiles do not differ materially from iteration to iteration. The choice of $\delta$ is not unique but can assume any number of values satisfying the condition, $0 < \delta \ll 1$. To reduce the amount of computation, we fix the mesh size $h$ and vary the deviating argument $\delta$. Finally, we pick up the smallest value of $\delta$ which produces the required accuracy. We have implemented this method on total 10 problems (four linear problems with left-end boundary layer, three non-linear problems with left-end boundary layer, one problem with a right-end boundary layer, one problem with an internal layer and one problem with two boundary layers) by taking different values for $\varepsilon$. The computational results are presented in Tables 1–10. We have given here only a few values although the solutions are computed at all the points with mesh size $h$. It can be observed from the tables that the present method approximates the exact solution very well. This shows the efficiency and accuracy of the present method.

We have shown that the numerical integration method is capable of solving general singularly perturbed two-point boundary value problems. This method provides an alternative and supplementary technique to the conventional ways of solving singular perturbation problems. It is a practical method, easily adaptable on a computer to solve singular perturbation problems with a modest amount of problem preparation.

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