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Self-similar solutions of a generalized Burgers equation with nonlinear damping

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Abstract

The nonlinear ordinary differential equation resulting from the self-similar reduction of a generalized Burgers equation with nonlinear damping is studied in some detail. Assuming certain asymptotic conditions at plus infinity or minus infinity, we find a wide variety of solutions—(positive) single hump, monotonic (bounded or unbounded) or solutions with a finite zero. The existence or non-existence of positive bounded solutions with exponential decay to zero at infinity for specific parameter ranges is proved. The analysis relies mainly on the shooting argument.

Keywords: Generalized Burgers equation; Self-similar solution; Connection problem; Shooting argument

1. Introduction

In the present paper we study the self-similar solutions of the generalized Burgers equation, namely

$$u_t + u^\beta u_x + \lambda u^\alpha = \frac{\delta}{2} u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$, $\lambda \in \mathbb{R}$ and $\delta > 0$ (small) are constants. Eq. (1.1) is a generalized Burgers equation with λu^α as the damping term. It reduces to Burgers equation when $\beta = 1$ and $\lambda = 0$.

Lardner and Arya [11] studied a special case of (1.1), namely

$$u_t = uu_x - \lambda u + \frac{\delta}{2} u_{xx}, \tag{1.2}$$

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where $\lambda > 0$ and $\delta > 0$ (small) are constants. The transformation $x \to -x$ changes (1.2) to (1.1) with $\beta = 1$ and $\alpha = 1$. This equation appears in the description of a continuous medium for which the constitutive relation for the stress contains a large linear term proportional to the strain, a small term which is quadratic in strain, and a small dissipative term proportional to the strain rate. The inviscid form of (1.1) arises in several applications, which include nonlinear acoustic propagation [15], the Gunn effect in semiconductors [13], rotating thin liquid films [14], chloride concentration in the kidney [12] and flow of petroleum in underground reservoirs [6,8]. This has also been considered by Bukiet et al. [4] in the inviscid limit.

Sachdev et al. [21] reduced (1.1) to the ODE

$$f'' + 2\eta f' + \frac{4}{\alpha - 1} f - 2^{3/2}\delta^{-1/2} f^{(\alpha-1)/2} f' - 4\lambda f^{\alpha} = 0 \tag{1.3}$$

by the similarity transformation

$$u = t^{1/(1-\alpha)} f(\eta), \qquad \eta = \frac{x}{(2\delta t)^{1/2}}, \tag{1.4}$$

requiring $\beta = (\alpha - 1)/2$. By a simple scaling, (1.3) can be changed to

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g^{(\alpha-1)/2} g' - 4\lambda g^{\alpha} = 0. \tag{1.5}$$

Eq. (1.5) is an important special case which after a transformation belongs to a class of nonlinear ordinary differential equations, called Euler–Painlevé equations first introduced by Sachdev and his collaborators in a series of papers [19–22]. These equations are much more general than the equation studied first by Euler and Painlevé (see Kamke [10, p. 574]), which is exactly linearizable; hence the equations of the former class are referred to as Euler–Painlevé transcendents. It was also pointed out by Sachdev [17] that this class covers a large number of special cases treated by Kamke [10] (see also the more recent work of Sachdev [18]).

We study here the self-similar solutions of (1.1) via (1.5). The study of (1.5) with asymptotic conditions at infinity brings out a rich structure of solutions of Euler–Painlevé transcendents, as evidenced by Theorems 1–5.

Sachdev et al. [21] studied numerically the boundary value problem

$$f'' + 2\eta f' + \frac{4}{\alpha - 1} f - 2^{3/2}\delta^{-1/2} f^{(\alpha-1)/2} f' - 4\lambda f^{\alpha} = 0, \quad -\infty < \eta < \infty, \tag{1.6}$$

$$f \sim A\exp(-\eta^2)H_{\gamma}(\eta) \sim A\exp(-\eta^2)(2\eta)^{(2\gamma)} \quad \text{as} \quad \eta \to \infty, \tag{1.7}$$

$$f \to 0 \quad \text{as } \eta \to -\infty \tag{1.8}$$

and

$$|f| < \infty \quad \text{on } (-\infty, \infty), \tag{1.9}$$

where $\gamma = (3 - \alpha)/(\alpha - 1)$ and H_{γ} is the Hermite function. They obtained precise numerical description of the solutions for $1 < \alpha < 3$, $\lambda > 0$ and $\alpha > 1, \lambda < 0$.

In a related study (see [25]), we have investigated the initial value problem for (1.5), with initial conditions

$$g(0) = \gamma, \quad g'(0) = 0,$$

where $\gamma > 0$ is a positive constant. Depending on the parameters α, λ, and γ, the existence of different types of positive solutions is proved.

In this paper we study the following connection problem for (1.5):

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g|g|^{(\alpha-3)/2} g' - 4\lambda g|g|^{\alpha-1} = 0, \tag{1.10}$$

$$g(\eta) \sim A \exp(-\eta^2)\eta^{(3-\alpha)/(\alpha-1)} \quad \text{as } \eta \to \infty, \tag{1.11}$$

$$g \to 0 \quad \text{as } \eta \to -\infty, \tag{1.12}$$

$$g > 0, \quad |g| < \infty \quad \text{on } (-\infty, \infty), \tag{1.13}$$

where $\alpha > 1$, $A > 0$ and λ is real. The amplitude parameter A is varied for given (α, λ) to see how nonlinearity changes the solution. In order to make the terms containing g well-defined when it becomes negative, we write $g^{(\alpha-1)/2}$ as $g|g|^{(\alpha-3)/2}$ and g^α as $g|g|^{\alpha-1}$.

We have proved Theorems 1–5 for problems (1.10)–(1.11).

Theorem 1. *Assume that* $\alpha > 3$ *and* $\lambda = 0$. *Then for any* $A > 0$, *a unique positive solution to problem* (1.10)–(1.13) *exists and decays algebraically to zero as* $\eta \to -\infty$.

Theorem 2. *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *Then there exist positive solutions* g *of* (1.10) *and* (1.11) *exhibiting each of the following behaviours*:

(i) $g > 0$ *exists on* $(-\infty, \infty)$ *and* $g(\eta) \to 0$ *algebraically as* $\eta \to -\infty$.
(ii) $g > 0$ *exists on* $(-\infty, \infty)$ *and* $g(\eta) \to \gamma_0$, *the constant solution of* (1.10), *as* $\eta \to -\infty$.
(iii) $g > 0$ *exists on* $[a_0, \infty)$ *such that* $g(a_0) = \gamma_0$ *for some* $a_0 \in \mathbb{R}$.

Theorem 3. *Assume that* $\alpha > 3$ *and* $\lambda < 0$. *Then there exists a solution of* (1.10) *and* (1.11), *which is positive on* $(-\infty, \infty)$ *and decays to zero algebraically as* $\eta \to -\infty$.

For the following cases, we show that there does not exist a positive solution on $(-\infty, \infty)$.

Theorem 4. *Assume that* $1 < \alpha < 3$ *and* $\lambda \leqslant 0$ *or* $\alpha = 3, \lambda < 0$. *Then any solution* g *of* (1.10) *and* (1.11) *has a finite zero for all* A *and hence no global positive solution exists*.

Here we prove also the existence of some more solutions of (1.10) which decay exponentially at $\eta = -\infty$ and algebraically at $\eta = +\infty$ for the parametric range $\alpha > 3, \lambda = 0$, following closely the work of Soewono and Debnath [23]. We thus pose the following connection problem:

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g|g|^{(\alpha-3)/2} g' = 0, \tag{1.14}$$

$$g(\eta) \sim A \exp(-\eta^2)|\eta|^{(3-\alpha)/(\alpha-1)} \quad \text{as } \eta \to -\infty, \tag{1.15}$$

$$g(\eta) \to 0 \quad \text{as } \eta \to \infty \tag{1.16}$$

and

$$g(\eta) > 0 \quad \text{for all } \eta \in (-\infty, \infty), \tag{1.17}$$

where $A > 0$ is an arbitrary constant.

Theorem 5. *When* $\alpha > 3$, *there exist solutions* g *of* (1.14)–(1.17) *on* $(-\infty, \infty)$ *decaying algebraically to zero as* $\eta \to +\infty$.

The above theorems are motivated by the extensive numerical study of (1.10), see [24].

The plan of the paper is as follows. Section 2 gives the preliminaries for linear and nonlinear equations. Section 3 sets forth the proofs of the Theorems 1–5. Section 4 contains the conclusions.

2. Some preliminary results

Here we first give the general solution, the asymptotic behaviour and some qualitative properties of the solution of the linearized form of (1.10), namely

$$g''(\eta) + 2\eta g'(\eta) + \frac{4}{\alpha - 1} g(\eta) = 0, \quad \eta \in \mathbb{R}. \tag{2.1}$$

The two linearly independent solutions of (2.1) are

$$g_1(\eta) = e^{-\eta^2/2} U(a, \sqrt{2}\eta) \tag{2.2}$$

and

$$g_2(\eta) = e^{-\eta^2/2} V(a, \sqrt{2}\eta), \tag{2.3}$$

where

$$a = -\left(\gamma + \frac{1}{2}\right), \quad \gamma = \frac{(3-\alpha)}{(\alpha-1)}. \tag{2.4}$$

Here U and V are the parabolic cylinder functions (see [1]). It follows from the asymptotic behaviours of U and V that the asymptotic behaviour of solutions of the linear equation (2.1) as $\eta \to \infty$ is given by

$$g_1(\eta) = A_1 e^{-\eta^2} \eta^{(3-\alpha)/(\alpha-1)} \left(1 - \frac{(\alpha-3)}{(\alpha-1)^2} \frac{(\alpha-2)}{2} \frac{1}{\eta^2} + o\left(\frac{1}{\eta^2}\right)\right), \tag{2.5}$$

$$g_2(\eta) = B_1 \eta^{-2/(\alpha-1)} \left(1 + \frac{(\alpha+1)}{2(\alpha-1)^2\eta^2} + o(\eta^{-4})\right), \tag{2.6}$$

where

$$A_1 = 2^{(3-\alpha)/2(\alpha-1)}, \quad B_1 = \frac{2^{(\alpha-3)/2(\alpha-1)}}{\sqrt{\pi}}. \tag{2.7}$$

Next we give an exact solution of the nonlinear equation (1.10) when $\alpha = 3$, $\lambda = 0$ and $g'(0) = 2^{1/2} g^2(0)$:

$$g(\eta) = \frac{1}{e^{\eta^2}\left(c - \sqrt{\frac{\pi}{2}}\,\mathrm{erf}(\eta)\right)}, \tag{2.8}$$

where $c = 1/g(0)$. The solution (2.8) was first found by Sachdev et al. [21] and subsequently by Cazenave and Escobedo [5].

Let

$$\gamma_0 = (\lambda(\alpha-1))^{-1/(\alpha-1)}. \tag{2.9}$$

$g \equiv \gamma_0$ is an exact solution of (1.10) for $\lambda > 0$.

We will show below that if the nonlinear equation (1.10) admits positive solutions decaying to zero as $\eta \to +\infty$, then they have the leading order behaviours same as g_1 or g_2 as $\eta \to \infty$.

2.1. Asymptotic behaviour of solutions of the nonlinear equation (1.10) as $\eta \to \pm\infty$

We shall analyse here the types of decay of a positive solution of the full Eq. (1.10) as $\eta \to \pm\infty$.

Lemma 2.1. *If g is a positive solution of (1.10) on $(-\infty, \infty)$ and $g \to 0$ as $\eta \to -\infty$, then $g' \to 0$ as $\eta \to -\infty$.*

Proof. It follows from Eq. (1.10) that at an extreme point of the solution g, we have $g'' = -4g/(\alpha-1) + 4\lambda g|g|^{\alpha-1}$. This implies one of the following:

(i) If $\lambda \leqslant 0$, g can have at most one local maximum.
(ii) If $\lambda > 0$, g can have local maximum only when $g < \gamma_0$.

Since $g \to 0$ as $\eta \to -\infty$, we can find η_0 such that $g < \gamma_0$ in $(-\infty, \eta_0)$. Hence g can have at most one local maximum in this interval. Thus, in both these cases, we find η_1 such that $g' > 0$ in $(-\infty, \eta_1)$. Now g' can either be oscillatory or ultimately monotonic as $\eta \to -\infty$. We shall show that in both the cases $g' \to 0$ as $\eta \to -\infty$. Let g' be oscillatory. Then there exists a sequence of points, $\eta_k \to -\infty$ as $k \to \infty$ such that g' has local extremum. By (1.10),

$$g'(\eta_k) = \frac{4g(\eta_k)/(\alpha-1) - 4\lambda g(\eta_k)|g(\eta_k)|^{\alpha-1}}{-2\eta_k + 2^{3/2} g(\eta_k)|g(\eta_k)|^{(\alpha-3)/2}}. \tag{2.10}$$

Using the assumption that $g \to 0$ as $\eta \to -\infty$, we have $g'(\eta_k) \to 0$ as $k \to \infty$ from (2.10). This implies that $g' \to 0$ as $\eta \to -\infty$. Now let us consider the second case wherein g' is ultimately monotonic. Suppose for contradiction that $g' \to K$ as $\eta \to -\infty$, for some $K > 0$. This implies that $g' > K/2$, for $|\eta|$ sufficiently large. An integration of this inequality from η to η_0 gives

$$g(\eta) < g(\eta_0) - \frac{K}{2}(\eta_0 - \eta) \quad \text{for } \eta < \eta_0 \tag{2.11}$$

implying that $g(\eta) \to -\infty$ as $\eta \to -\infty$. This contradicts our assumption that g is positive on $(-\infty, \infty)$. Therefore, g' tends to zero as $\eta \to -\infty$. Hence the lemma. □

Lemma 2.2 and Theorems 2.3 and 2.4 below can be proved following the work of Brezis et al. [3]. Detailed proof can be found in [24].

Lemma 2.2. *Assume that* $\alpha > 1$ *and* $\lambda \in \mathbb{R}$. *Let* g *be a positive solution of* (1.10) *on* $(-\infty, \infty)$. *Further let* $g \to 0$ *as* $\eta \to -\infty$. *Then* $\lim_{\eta \to -\infty} g'/g$ *exists and is zero or* $+\infty$.

2.2. Exponentially decaying solutions

Theorem 2.3. *If* $g(\eta)$ *is a positive solution of* (1.10) *tending to zero as* $\eta \to -\infty$ *and* $\lim_{\eta \to -\infty} g'/g = \infty$, *then*

$$g(\eta) = A e^{-\eta^2} |\eta|^{(3-\alpha)/(\alpha-1)} \left(1 - \frac{(\alpha-2)(\alpha-3)}{(\alpha-1)^2} \frac{1}{2} \frac{1}{|\eta|^2} + o\left(\frac{1}{|\eta|^2}\right)\right) \quad \text{as } \eta \to -\infty, \tag{2.12}$$

where $A > 0$ *is a constant.*

2.3. Algebraically decaying solutions

Theorem 2.4. *If* $g(\eta)$ *is a positive solution of* (1.10) *tending to zero as* $\eta \to -\infty$ *and* $\lim_{\eta \to -\infty} g'/g = 0$, *then for some* $0 < K < 2$,

$$g(\eta) = A|\eta|^{-2/(\alpha-1)} \left(1 + o(|\eta|^{-K})\right) \quad \text{as } \eta \to -\infty, \tag{2.13}$$

where $A > 0$ *is a constant.*

Combining Lemmas 2.1 and 2.2 and Theorems 2.3 and 2.4, we have the following remark.

Remark 2.5. If g is a positive solution of (1.10) tending to zero as $\eta \to -\infty$, then g decays to zero exponentially or algebraically as $\eta \to -\infty$.

Following closely the analysis for $\eta \to -\infty$ we have the following remark.

Remark 2.6. If g is a positive solution of (1.10) tending to zero as $\eta \to +\infty$, then g decays exponentially or algebraically to zero as $\eta \to \infty$. The asymptotic expressions for the solution as $\eta \to \infty$ are essentially the same as in Theorems 2.3 and 2.4.

3. Solutions of (1.10) with exponential decay at $\eta = +\infty$ or $-\infty$

In this section, we shall prove Theorems 1–5.

Proof of Theorem 1. We first show that for every $A > 0$ there exists a unique solution g for the differential equation (1.10) on some interval $[\eta_A, \infty)$ with exponential decay at $\eta = +\infty$ satisfying (1.11). Next we show that g is positive at $+\infty$ for $A > 0$ and it cannot become zero in its maximal interval of existence for $A > 0$. Then we show that g exists globally on $(-\infty, \infty)$ and has algebraic decay at $\eta = -\infty$. The proof as outlined results from the following three lemmas.

The proof of Lemma 3.1 follows from the use of fixed point iteration as in Hastings and McLeod [9]. Unlike their equation, we have a nonlinear term having g' which is handled here by integration by parts to reduce it to an integral equation amenable to fixed point iteration. □

Lemma 3.1. *Under the assumptions of Theorem* 1, *for every* $k_1 > 0$ *there exists a unique positive solution* g_{k_1} *of* (1.10), *which is asymptotic to* $(k_1 \exp(-\eta^2)\eta^{(3-\alpha)/(\alpha-1)})$ *as* $\eta \to \infty$ *on* $[\eta_0(k_1), \infty)$. *Further* g_{k_1} *and its derivatives are continuous functions of* k_1. (*The choice of* $\eta_0(k_1)$ *would be explained in the proof of this lemma.*)

Proof. We first write (1.10) as an inhomogeneous ODE

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g = 2^{3/2} g|g|^{(\alpha-3)/2} g'. \tag{3.1}$$

By the method of variation of parameters, the general solution of (3.1) can be written as

$$g(\eta) = k_1 g_1(\eta) + k_2 g_2(\eta) + \int_\eta^\infty k(s,\eta) R(s)\,\mathrm{d}s, \tag{3.2}$$

where

$$k(s,t) = \frac{\sqrt{\pi}}{2} \mathrm{e}^{(s^2-t^2)/2} (U(a, t\sqrt{2}) V(a, s\sqrt{2}) - U(a, s\sqrt{2}) V(a, t\sqrt{2})), \quad s \geqslant t \tag{3.3}$$

and

$$R(s) \equiv 2^{3/2} g(s)|g(s)|^{(\alpha-3)/2} g'(s). \tag{3.4}$$

Here g_1 and g_2 are as in (2.2) and (2.3) of Section 2. Since we are interested in the solution of (1.10) with asymptotic behaviour $k_1 \exp(-\eta^2)\eta^{(3-\alpha)/(\alpha-1)}$ as $\eta \to \infty$, we put $k_2 = 0$ and replace k_1 by k_1/A_1 where $A_1 = 2^{(3-\alpha)/2(\alpha-1)}$; the constant A_1 appears in the asymptotic behaviour of the solution of linearized form (2.1) of (1.10) (see Eq. (2.7)). Thus,

$$g(\eta) = \frac{k_1}{A_1} g_1(\eta) + 2^{3/2} \int_\eta^\infty k(s,\eta) g(s)|g(s)|^{(\alpha-3)/2} g'(s)\,\mathrm{d}s. \tag{3.5}$$

Integrating by parts the second term on the right side of (3.5) and using the asymptotic behaviours of U, V (see [1]) and g to drop the boundary terms, we get

$$g(\eta) = \frac{k_1}{A_1} g_1(\eta) - \frac{2^{5/2}}{\alpha+1} \int_\eta^\infty k_s(s,\eta)|g(s)|^{(\alpha+1)/2}\, ds. \tag{3.6}$$

We use the contraction mapping theorem to show that (3.6) has a unique solution. The asymptotic behaviours of U and V as $\eta \to \infty$ imply that

$$|k(s,\eta)| \leqslant M e^{s^2-\eta^2} \eta^{(3-\alpha)/(\alpha-1)} s^{-2/(\alpha-1)}, \tag{3.7}$$

$$|k_s(s,\eta)| \leqslant M e^{s^2-\eta^2} \eta^{(3-\alpha)/(\alpha-1)} s^{(\alpha-3)/(\alpha-1)}, \tag{3.8}$$

for some $M > 0$ fixed and $s \geqslant \eta \geqslant \eta_0(k_1)$, where $\eta_0(k_1)$ is sufficiently large. Let $\eta_0(k_1)$ be chosen so large that for $\eta \geqslant \eta_0(k_1)$ the following conditions are satisfied:

$$\text{(i)} \quad \int_\eta^\infty e^{-(\alpha-1)s^2/2} s^{-(\alpha-3)/2}\, ds \leqslant \sqrt{\frac{\pi}{\alpha-1}}\, \text{erfc}\left(\sqrt{\frac{\alpha-1}{4}}\,\eta\right), \tag{3.9}$$

$$\text{(ii)} \quad \frac{2^{5/2}}{\alpha+1}\sqrt{\frac{\pi}{\alpha-1}}\, Mc(\alpha)(4k_1)^{(\alpha-1)/2}\, \text{erfc}\left(\sqrt{\frac{\alpha-1}{4}}\,\eta\right) \leqslant \theta_1, \tag{3.10}$$

$$\text{(iii)} \quad \sup_{[\eta_0(k_1),\infty)} \left|\frac{k_1}{A_1} g_1(\eta) \exp(\eta^2) \eta^{(\alpha-3)/(\alpha-1)}\right| \leqslant \frac{3k_1}{2}, \tag{3.11}$$

where θ_1 is fixed in $(0, 1/4]$, $c(\alpha) = (\alpha+1)/2$, and $\text{erfc}(\eta) = (2/\sqrt{\pi}) \int_\eta^\infty \exp(-t^2)\, dt$ is the complementary error function. Condition (ii) is satisfied since the complementary error function tends to zero as $\eta \to \infty$. Condition (iii) is satisfied thanks to the asymptotic behaviour of g_1 as $\eta \to \infty$ (see (2.5)).

Define

$$C[\eta_0(k_1),\infty) = \{g \mid g \text{ is continuous on } [\eta_0(k_1),\infty)\},$$

$$X = \left\{ g \in C[\eta_0(k_1),\infty) \mid \sup_{[\eta_0(k_1),\infty)} |g(s) e^{s^2} s^{(\alpha-3)/(\alpha-1)}| \leqslant 2k_1 \right\},$$

$$\|g\|_X = \sup_{[\eta_0(k_1),\infty)} |g(s) e^{s^2} s^{(\alpha-3)/(\alpha-1)}|, \qquad d(g_1, g_2) = \|g_1 - g_2\|_X$$

and an operator T on X by

$$Tg(\eta) = \frac{k_1}{A_1} g_1(\eta) - \frac{2^{5/2}}{\alpha+1} \int_\eta^\infty k_s(s,\eta)|g(s)|^{(\alpha+1)/2}\, ds. \tag{3.12}$$

Clearly d satisfies the properties of a metric. It is easy to see that X is a complete metric space with metric d.

Claim. *T is a contraction from X into X.*

Suppose $g_3, g_4 \in X$. We shall show that

$$\|Tg_3 - Tg_4\|_X \leqslant \theta_1 \|g_3 - g_4\|_X, \quad \theta_1 < 1. \tag{3.13}$$

Now, using the definition of T,

$$\begin{aligned}\|Tg_3 - Tg_4\|_X &= \sup_{[\eta_0(k_1),\infty)} e^{\eta^2}\eta^{(\alpha-3)/(\alpha-1)}|Tg_3 - Tg_4| \\ &\leqslant \sup_{[\eta_0(k_1),\infty)} e^{\eta^2}\eta^{(\alpha-3)/(\alpha-1)} \\ &\quad \times \left\{ \frac{2^{5/2}}{\alpha+1}\int_\eta^\infty |k_s(s,\eta)|\left||g_3(s)|^{(\alpha+1)/2} - |g_4(s)|^{(\alpha+1)/2}\right| \mathrm{d}s \right\}.\end{aligned} \tag{3.14}$$

Note that, by the mean value theorem,

$$\left||g_3|^{(\alpha+1)/2} - |g_4|^{(\alpha+1)/2}\right| \leqslant c(\alpha)\left||g_3| + |g_4|\right|^{(\alpha-1)/2}|g_3 - g_4|, \tag{3.15}$$

where $c(\alpha)=(\alpha+1)/2$. By using (3.15) and (3.8) and the fact that $g_3, g_4 \in X$ in (3.14), we get, after some simplification,

$$\begin{aligned}\|Tg_3 - Tg_4\|_X &\leqslant \sup_{[\eta_0(k_1),\infty)} \left\{ \frac{2^{5/2}}{\alpha+1} Mc(\alpha)(4k_1)^{(\alpha-1)/2} \right. \\ &\quad \left. \times \int_\eta^\infty e^{-(\alpha-1)s^2/2} s^{-(\alpha-3)/2}\,\mathrm{d}s \right\} \|g_3 - g_4\|_X \\ &\leqslant \sup_{[\eta_0(k_1),\infty)} \left\{ Mc(\alpha)(4k_1)^{(\alpha-1)/2} \right. \\ &\quad \left. \times \frac{2^{5/2}}{\alpha+1}\sqrt{\frac{\pi}{\alpha-1}}\,\mathrm{erfc}\left(\sqrt{\frac{\alpha-1}{4}}\eta\right) \right\} \|g_3 - g_4\|_X \\ &\quad \text{(by Eq. (3.9))} \\ &\leqslant \theta_1 \|g_3 - g_4\|_X \quad \text{(by Eq. (3.10))}.\end{aligned}$$

Further for $g \equiv 0$, we have $Tg = k_1 g_1/A_1$ and by (3.11), $\|k_1 g_1/A_1\|_X \leqslant 3k_1/2$. For any $g \in X$, we have by Eqs. (3.11) and (3.13) and remembering that $0 < \theta_1 \leqslant 1/4$

$$\|Tg\|_X \leqslant \|T(0)\|_X + \|Tg - T(0)\|_X \leqslant \frac{3}{2}k_1 + \frac{k_1}{2} = 2k_1. \tag{3.16}$$

Thus $Tg \in X$. Hence the claim is proved.

Since T maps X to X and is a contraction on the complete metric space X, by contraction mapping theorem (see [16]), T has a unique fixed point on X defined on $[\eta_0(k_1), \infty)$. This implies that Eq. (3.6) has a unique solution on $[\eta_0(k_1), \infty)$ satisfying $|g| \leqslant 2k_1 e^{-\eta^2}\eta^{(3-\alpha)/(\alpha-1)}$ for all $\eta \geqslant \eta_0(k_1)$. If we use the bound on g and the asymptotic behaviour of complementary error function as $\eta \to \infty$, in the integral equation (3.6), we can show that g has the required asymptotic behaviour at ∞. Thus, we have shown that, for any $k_1 > 0$, there exists $\eta_0(k_1)$ such that the differential equation (1.10) has a unique solution with the asymptotic behaviour $g \sim k_1 e^{-\eta^2} \eta^{(3-\alpha)/(\alpha-1)}$ as $\eta \to \infty$; it exists for all η on $[\eta_0(k_1), \infty)$.

If $k_1 > 0$, then there exists some η_1 sufficiently large such that $g_{k_1} > 0$ on (η_1, ∞) because of the asymptotic behaviour of g_{k_1}. We claim that this g_{k_1} is positive on $(\eta_0(k_1), \infty)$. If not, let η_2 be such that $g_{k_1}(\eta_2) \geqslant 0$ and $g'_{k_1}(\eta_2)$ on (η_2, ∞); then by (3.1) we have at $\eta = \eta_2$

$$g'_{k_1}(\eta_2) = -\int_{\eta_2}^{\infty} \left(\frac{2(\alpha-3)}{(\alpha-1)} g_{k_1}(s) \right) \mathrm{d}s < 0,$$

a contradiction. Hence g_{k_1} cannot have a zero.

We now show the continuous dependence of g on k_1. Let g_{k_1} be the solution of (1.10)–(1.11) on $[\eta_0(k_1), \infty)$ and $k \in (k_1 - \varepsilon, k_1 + \varepsilon)$ for some $\varepsilon > 0$. Define $\eta_s = \sup_{k \in [k_1 - \varepsilon, k_1 + \varepsilon]} \eta_0(k)$. Since g_{k_2}, g_{k_1} are fixed points of (3.6) corresponding to k_2 and k_1 on (η_s, ∞), proceeding as in the proof of the claim we arrive at the following inequality:

$$(1-\theta_1)\|g_{k_2} - g_{k_1}\|_X \leqslant \frac{|k_2 - k_1|}{A_1} \|g_1\|_X. \tag{3.17}$$

This implies that $g_{k_2} \to g_{k_1}$ uniformly on $[\eta_s, \infty)$ as $k_2 \to k_1$. Now the continuous dependence of g with respect to k on $[\eta_{k_1}, \eta_s]$ can be obtained by standard theorems for initial value problems on compact intervals (see [16]).

Integrating (1.10) from η to ∞, we get

$$g'_{k_1}(\eta) = -2\eta g_{k_1}(\eta) + \frac{2^{5/2}}{\alpha+1} |g_{k_1}(\eta)|^{(\alpha+1)/2} - \int_{\eta}^{\infty} \left(\frac{2(\alpha-3)}{\alpha-1} g_{k_1}(s) \right) \mathrm{d}s,$$
$$\eta \in (\eta_0(k_1), \infty). \tag{3.18}$$

On using continuous dependence of g on k_1 in (3.18), we get continuous dependence of g' on k_1. □

For the sake of simplicity, we shall, in the following, use g instead of g_k when there is no ambiguity.

Lemma 3.2. *The positive solution g on $[\eta_0(k), \infty)$ can be extended to $(-\infty, \infty)$ as a positive solution of* (1.10) *and* (1.11) *under the assumptions of Theorem* 1.

Proof. Suppose on the contrary that $\eta_{\max} > -\infty$, $(\eta_{\max}, \infty)$ is the maximal interval of existence for g. Then as $\eta \to \eta_{\max}$, g or g' or both must be unbounded. It is easy to see from (1.10) that g on $(\eta_{\max}, \infty)$ cannot have a positive local minimum. Hence either g has no maximum on $(\eta_{\max}, \infty)$ or g has exactly one maximum and remains positive and bounded near $\eta_{\max}$. In the former case, g has to cross the curve $f(\eta) = (\eta/2^{1/2})^{2/(\alpha-1)}$ on $(0, \infty)$ at some point $\tilde{\eta}$ in $(0, \infty)$ and $g(\eta) > f(\eta)$ on $(\max\{\eta_{\max}, 0\}, \tilde{\eta})$. Then, defining

$$E(\eta) = \frac{g'^2}{2} + \frac{2}{(\alpha-1)} g^2,$$

the derivative of E, on use of (1.10), is

$$E'(\eta) = g'^2(-2\eta + 2^{3/2} g|g|^{(\alpha-3)/2})$$

and is positive on $(\eta_{\max}, \tilde{\eta})$. Thus E is an increasing function on this interval and

$$E(\eta) \leqslant E(\tilde{\eta}) \quad \forall \eta \in (\eta_{\max}, \tilde{\eta}).$$

This implies that g and g' are bounded on $(\eta_{\max}, \tilde{\eta})$ by $E(\tilde{\eta})$. This is a contradiction. Therefore, this case cannot arise. In the second case, when g is bounded on $(\eta_{\max}, \infty)$, g' is bounded in view of (3.18) and (1.11). Hence $\eta_{\max} = -\infty$, that is, g exists on $(-\infty, \infty)$. $\square$

Lemma 3.3. *Assume that $\alpha > 3$ and $\lambda = 0$. If g is a positive solution of* (1.10) *and* (1.11) *on $(-\infty, \infty)$, then g must decay algebraically to 0 as $\eta \to -\infty$.*

Proof. By (1.10), g can have at most one critical point. Since the solution is positive and bounded, $g(\eta) \to C \geqslant 0$ as $\eta \to -\infty$. We will show that $C = 0$.

For contradiction, we let $C > 0$. By integrating (1.10) from η to $\xi > 0$, we get

$$[g'(s)]_\eta^\xi = -[2sg(s)]_\eta^\xi + \left[\frac{2^{5/2}}{\alpha+1}|g(s)|^{(\alpha+1)/2}\right]_\eta^\xi + 2\int_\eta^\xi \left(\frac{\alpha-3}{\alpha-1}\right) g(s)\,\mathrm{d}s.$$

When we let $\xi \to \infty$, $g(\xi)$, $g'(\xi)$ and $\xi g(\xi) \to 0$ in view of (1.11). Thus

$$g'(\eta) = -2\eta g(\eta) + \frac{2^{5/2}}{\alpha+1}|g(\eta)|^{(\alpha+1)/2} - 2\int_\eta^\infty \left(\frac{\alpha-3}{\alpha-1}\right) g(s)\,\mathrm{d}s. \tag{3.19}$$

If $g(\eta) \to C > 0$ as $\eta \to -\infty$, then from (3.19), we conclude that

$$g'(\eta) \to +\infty \quad \text{as } \eta \to -\infty.$$

This is impossible by the earlier lemma. Therefore $C = 0$.

If g decays to 0 exponentially at $-\infty$, then from (3.19) we get by letting η run to $-\infty$,

$$0 = -2\int_{-\infty}^\infty \left(\frac{\alpha-3}{\alpha-1}\right) g(s)\,\mathrm{d}s \neq 0,$$

a contradiction. By Remark 2.5, we conclude that $g \to 0$ algebraically as $\eta \to -\infty$. $\square$

Proof of Theorem 2. The proof of Lemma 3.1 goes through after rewriting the differential equation (1.10) as

$$g(\eta) = \frac{k_1}{A_1} g_1(\eta) - \frac{2^{5/2}}{\alpha+1}\int_\eta^\infty k_s(s,\eta)|g(s)|^{(\alpha+1)/2}\,\mathrm{d}s$$
$$+ 4\lambda \int_\eta^\infty k(s,\eta) g(s)|g(s)|^{\alpha-1}\,\mathrm{d}s.$$

Thus we get a solution g on $[\eta_0(k_1),\infty)$ satisfying (1.10) and (1.11) with $A=k_1$. Notice that

$$g'(\eta)=-\int_{\eta}^{\infty}\left(\frac{2(\alpha-3)}{(\alpha-1)}g(s)+4\lambda g(s)|g(s)|^{\alpha-1}\right)\,\mathrm{d}s$$

$$-2\eta g(\eta)+\frac{2^{5/2}}{\alpha+1}|g(\eta)|^{(\alpha+1)/2}. \tag{3.20}$$

Treating this equation as in the proof of Lemma 3.1, we can conclude that g cannot have a zero on its maximal interval of existence.

Suppose that g is bounded on its maximal interval of existence $(\eta_{\max},\infty)$. Then, using arguments as in Lemma 3.2, we can extend g to $(-\infty,\infty)$ as a positive solution. It is clear from (1.10) that g cannot have a positive local minimum. Hence the solutions are ultimately monotonic as $\eta\to-\infty$.

Suppose that $g(\eta)\to C$ (finite), as $\eta\to-\infty$. By integrating (1.10) from η to ξ, we get

$$[g'(s)]_{\eta}^{\xi}=-[2sg(s)]_{\eta}^{\xi}+\left[\frac{2^{5/2}}{\alpha+1}|g(s)|^{(\alpha+1)/2}\right]_{\eta}^{\xi}$$

$$+\int_{\eta}^{\xi}\left(2\left(\frac{\alpha-3}{\alpha-1}\right)g(s)+4\lambda g(s)|g(s)|^{\alpha-1}\right)\,\mathrm{d}s.$$

$g(\xi)$, $g'(\xi)$ and $\xi g(\xi)\to 0$ as $\xi\to\infty$ in view of (1.11). Thus

$$g'(\eta)=-2\eta g(\eta)+\frac{2^{5/2}}{\alpha+1}|g(\eta)|^{(\alpha+1)/2}$$

$$-\int_{\eta}^{\infty}\left(2\left(\frac{\alpha-3}{\alpha-1}\right)g(s)+4\lambda g(s)|g(s)|^{\alpha-1}\right)\,\mathrm{d}s. \tag{3.21}$$

So, if η_0 is a fixed negative number and $|\eta_0|$ is large, then for $\eta<\eta_0$, we have

$$g'(\eta)=\frac{2^{5/2}C^{(\alpha+1)/2}}{(\alpha+1)}+4C\left(\frac{-1}{(\alpha-1)}+\lambda C^{\alpha-1}\right)\eta$$

$$-\int_{\eta_0}^{\infty}\left(\frac{2(\alpha-3)}{(\alpha-1)}g(s)+4\lambda g^{\alpha+1}(s)\right)\,\mathrm{d}s+\frac{2^{5/2}}{\alpha+1}(g^{(\alpha+1)/2}-C^{(\alpha+1)/2})$$

$$-\int_{\eta}^{\eta_0}\left[\frac{2(\alpha-3)(g(s)-C)}{(\alpha-1)}+4\lambda(g^{\alpha}(s)-C^{\alpha})\right]\,\mathrm{d}s-2\eta(g(\eta)-C).$$

This implies that

$$g'(\eta)=4C\left(\frac{-1}{\alpha-1}+\lambda C^{\alpha-1}\right)\eta+o(\eta).$$

as $\eta\to-\infty$. If $C\neq 0$ or γ_0, $g'(\eta)\to\infty$ or $-\infty$ as $\eta\to-\infty$. Therefore, $g(\eta)$ cannot remain bounded, which is a contradiction. Thus $g(\eta)\to 0$ or γ_0 as $\eta\to-\infty$, leading to cases (i) or (ii) of Theorem 2. We will show in Lemma 3.4 that the case (i) happens for k small. If g is unbounded, case (iii) happens. We show in Lemma 3.5 that a positive solution on $[\eta_0, \infty)$ with $g(\eta_0) = \gamma_0$ exists. Thus, case (iii) occurs. Using the connectedness of $(0, \infty)$ and Lemmas 3.4 and 3.5, we will finally prove in Lemma 3.6 the existence of positive solutions with the behaviour as in (ii). This completes the proof of Theorem 2. □

Remark. Define

$$E(\eta) = \frac{g'^2}{2} + \frac{2}{(\alpha - 1)} g^2 - \frac{4\lambda}{(\alpha + 1)} |g|^{\alpha+1}.$$

Note that a characterization of different types of solutions of (1.10) may be given in the following ways:

(i) If $g, g' \to 0$, then $E \to 0$ as $\eta \to -\infty$.
(ii) If $g \to \gamma_0$, $g' \to 0$ as $\eta \to -\infty$, then $E \to \frac{2}{\alpha+1}(\lambda(\alpha - 1))^{-2/(\alpha-1)}$ as $\eta \to -\infty$.
(iii) If $g(\eta_1) = (\lambda(\alpha - 1))^{-1/(\alpha-1)}$ for some η_1, then $E(\eta_1) \geqslant \frac{2}{\alpha+1}(\lambda(\alpha - 1))^{-2/(\alpha-1)}$.

This characterization will be used in the proof of the next lemma.

Lemma 3.4. *Under the assumptions of Theorem 2, if $k > 0$ is sufficiently small, then there exists a positive solution g of (1.10) and (1.11) on $(-\infty, \infty)$ and $g(\eta) \to 0$ algebraically as $\eta \to -\infty$.*

Proof. We observe that g has to intersect the curve $f(\eta) = (\eta/\sqrt{2})^{2/(\alpha-1)}$, since g has at most one local maximum. Let f and g intersect at the point $\tilde{\eta}$. We have $g_k = 0$ when $k = 0$. By the continuous dependence of g on k, we can choose sufficiently small k such that both $g(\tilde{\eta})$ and $g'(\tilde{\eta})$ are small and $E(\tilde{\eta}) < \frac{1}{2}E_0$, where $E_0 = [2/(\alpha + 1)] \times (\lambda(\alpha - 1))^{-2/(\alpha-1)}$. We shall show that $E(\eta) < \frac{1}{2}E_0$ $\forall \eta$. By (1.10),

$$E'(\eta) = g'^2(-2\eta + 2^{3/2} g|g|^{(\alpha-3)/2}).$$

This implies that $E'(\eta) < 0$ on $(\tilde{\eta}, \infty)$, $E'(\eta) > 0$ on $(-\infty, \tilde{\eta})$ and $E'(\tilde{\eta}) = 0$. Thus $E(\eta)$ has a global maximum at $\eta = \tilde{\eta}$.

If g were unbounded, there would be a point η_1 such that $g(\eta_1) = \gamma_0$. In that case $E(\eta_1) \geqslant g'^2(\eta_1)/2 + E_0 \geqslant E_0$, which is not possible in view of our choice of k. Hence $g < \gamma_0$ for all η. Further observe that if $g \to \gamma_0$ as $\eta \to -\infty$, then $g' \to 0$ as $\eta \to -\infty$ and hence $E(\eta) \to E_0$ as $\eta \to -\infty$. This implies that for some η_2, $E(\eta_2) > E_0/2$, which is again impossible because of our choice of k. Hence g has to tend to zero as $\eta \to -\infty$.

If $g(\eta)$ decays to 0 exponentially at $-\infty$, then from (3.21), we get by letting η tend to $-\infty$,

$$0 = -\int_{-\infty}^{\infty} \left(2\left(\frac{\alpha - 3}{\alpha - 1}\right) g(s) + 4\lambda g|g|^{\alpha-1} \right) \mathrm{d}s \neq 0,$$

a contradiction. From Remark 2.5, we conclude that $g(\eta) \to 0$ algebraically as $\eta \to -\infty$. □

The proof of the following lemma uses arguments similar to those of Soewono and Debnath [23].

Lemma 3.5. *Under the assumptions of Theorem* 2, *there exists a positive solution of* (1.10)–(1.11) *in* $[a_1,\infty)$ *such that* $g(a_1)=\gamma_0$ *for some* $a_1\in\mathbb{R}$.

Proof. Rewriting (1.10) as

$$g''=-2\eta g'-\frac{4}{(\alpha-1)}g+2^{3/2}g|g|^{(\alpha-3)/2}g'+4\lambda g|g|^{\alpha-1}\equiv F(\eta,g,g'). \tag{3.22}$$

First we show that $\beta(\eta)=a\exp(-b\eta)$ is an upper solution of (3.22) on $[a_1,\infty)$ for some positive numbers a and b. Take $\alpha(\eta)\equiv 0$ as a lower solution. We shall show that F satisfies Nagumo's condition with respect to α and β on any interval $J=[a_1,b_1]$. Then by Theorem 1.7.1 of Bernfeld and Lakshmikantham [2], for any c, $\alpha(a_1)\leqslant c\leqslant\beta(a_1)$, the problem

$$g''=F(\eta,g,g'),\qquad g(a_1)=c, \tag{3.23}$$

has a solution $g\in C^2([a_1,\infty),\mathbb{R})$ such that $\alpha(\eta)\leqslant g(\eta)\leqslant\beta(\eta)$ on $[a_1,\infty)$. This implies that g is a solution of (1.10) and (1.11) decaying exponentially at $\eta=+\infty$ by Remark 2.6, and is positive with $g(a_1)=c$.

(i) We now check that β is an upper solution for a suitable choice of a and b.

$$\begin{aligned}
&-2\eta\beta'-\frac{4}{(\alpha-1)}\beta+2^{3/2}\beta^{(\alpha-1)/2}\beta'+4\lambda\beta^{\alpha}-\beta''\\
&=\exp(-b\eta)\left\{\left(2ab\eta+\frac{4a}{(1-\alpha)}-ab^2\right)-2^{3/2}a^{(\alpha+1)/2}b\exp(-b\eta(\alpha-1)/2)\right.\\
&\quad\left.+4\lambda a^{\alpha}\exp(-b\eta(\alpha-1))\right\}\\
&>a\left\{-\frac{4}{(\alpha-1)}\log\left\{\frac{2^{-5/2}a^{(-\alpha+1)/2}}{\lambda}\left(b+\sqrt{b^2+\frac{16\lambda}{(\alpha-1)}}\right)\right\}-\frac{2}{(\alpha-1)}\right.\\
&\quad\left.-b^2\left(1+\frac{1}{4\lambda}\right)-\frac{b}{4\lambda}\left(\sqrt{b^2+\frac{16\lambda}{(\alpha-1)}}\right)\right\}\exp(-b\eta).
\end{aligned} \tag{3.24}$$

Choose positive numbers a and b such that

$$\begin{aligned}
&-\frac{4}{(\alpha-1)}\log\left\{\frac{2^{-5/2}a^{(-\alpha+1)/2}}{\lambda}\left(b+\sqrt{b^2+\frac{16\lambda}{(\alpha-1)}}\right)\right\}-\frac{2}{(\alpha-1)}\\
&-b^2\left(1+\frac{1}{4\lambda}\right)-\frac{b}{4\lambda}\left(\sqrt{b^2+\frac{16\lambda}{(\alpha-1)}}\right)>0.
\end{aligned}$$

This choice is possible since $\log t \to -\infty$ as $t \to 0$. Thus, in conjunction with (3.24), β is an upper solution.

(ii) F satisfies Nagumo's condition with respect to α, β on $J = [a_1, b_1]$. From (3.22), we have

$$\begin{aligned}|F(\eta, g, g')| &= \left| g'(-2\eta + 2^{3/2} g|g|^{(\alpha-3)/2}) - \frac{4}{(\alpha-1)} g + 4\lambda g|g|^{\alpha-1} \right| \\ &\leqslant |g'||2|\eta| + 2^{3/2}|g|^{(\alpha-1)/2}| + 4\lambda|g|^{\alpha} \\ &\leqslant |g'|c_1 + c_2 \equiv h(|g'|),\end{aligned}$$

where $h(s) = c_1 s + c_2$, and c_1 and c_2 depend on J, $|g|$. Now,

$$\int^s \frac{t}{h(t)}\,\mathrm{d}t = \frac{1}{c_1}\left(s - \frac{c_2}{c_1}\log(c_1 s + c_2)\right).$$

Therefore as $s \to \infty$, $\int^s (t/h(t))\,\mathrm{d}t \to \infty$; this implies that $\int^\infty (s/h(s))\,\mathrm{d}s = \infty > \beta(a_1)$. Thus F satisfies Nagumo's condition with respect to α and β. Since $\beta(\eta) \to \infty$ as $\eta \to -\infty$, choose a_1 such that $\beta(a_1) > \gamma_0$. Hence the lemma follows. □

Lemma 3.6. *Under the assumptions of Theorem* 2, *there exists a positive solution* g *of* (1.10) *and* (1.11) *on* $(-\infty, \infty)$ *and* $g(\eta) \to \gamma_0$ *as* $\eta \to -\infty$, *where* γ_0 *is given by* (2.9).

Proof. Define $S_1 = \{A > 0 \,|\, g_A(\eta) > 0$ is the solution of (1.10) and (1.11) and $g_A(\eta_1) = \gamma_0$ for some $\eta_1\}$, $S_2 = \{A > 0 \,|\, g_A(\eta) > 0$ is the solution of (1.10) and (1.11) and $g'_A(\eta_1) = 0$ for some $\eta_1\}$, and $S_3 = \{A > 0 \,|\, g_A(\eta) > 0$ is the solution of (1.10) and (1.11) and $g_A(\eta) \to \gamma_0$ as $\eta \to -\infty\}$.

We now prove that S_1 and S_2 are non-empty open sets. Lemmas 3.4 and 3.5 imply that S_1 and S_2 are non-empty. Suppose $A_1 \in S_1$. By the definition of S_1, there exists η_2 such that $g_{A_1}(\eta_2) > \gamma_0$. By the continuous dependence of solutions on A, there exists a neighbourhood U of A_1 such that $g_A(\eta_2) > \gamma_0$ for all $A \in U$. This implies $U \subset S_1$. Therefore, S_1 is open. Suppose $A_2 \in S_2$. By the definition of S_2, there exists η_3 such that $g'_{A_2}(\eta_3) = 0$. Suppose η_4 is such that $g_{A_2}(\eta_4) = g_{A_2}(\eta_3)/2$, $\eta_4 < \eta_3$. By the continuous dependence of the solutions on A there exists a neighbourhood U of A_2 such that $|g_{A_2}(\eta) - g_A(\eta)| < \varepsilon$ on $[\eta_4, \infty)$. This implies that

$$g_{A_2}(\eta) - \varepsilon < g_A(\eta) < g_{A_2}(\eta) + \varepsilon \quad \text{for all } \eta \in [\eta_4, \infty). \tag{3.25}$$

For the sake of clarity let $A > A_2$. Then $g_A > g_{A_2}$ on $[\eta_4, \infty)$ (see the remark below). This implies that $g_A(\eta_3) > g_{A_2}(\eta_3)$. Choose $\varepsilon = g_{A_2}(\eta_3)/4$. Then, using (3.25), we have $g_A(\eta_4) < 3g_{A_2}(\eta_3)/4 < g_A(\eta_3)$, implying that g_A is increasing on $[\eta_4, \eta_3]$, i.e., $g'_A(\eta) \geqslant 0$ on $[\eta_4, \eta_3]$. Since $g_A \to 0$ as $\eta \to \infty$, we have $g'_A(\eta^*) < 0$ for sufficiently large η^*. Thus there exists $\eta^{**} \in [\eta_4, \eta^*]$ such that $g'_A(\eta^{**}) = 0$. Therefore $A \in S_2$.

Since S_1 and S_2 are non-empty disjoint open subsets of $(0,\infty)$, there exists $A^* \in (0,\infty)$ such that $A^* \notin S_1 \cup S_2$. Hence $A^* \in S_3$. Hence the proof of the lemma follows. □

Remark. Positive solutions of (1.10) and (1.11) are ordered with respect to A for $\alpha > 3, \lambda \geqslant 0$. Suppose that g_3 and g_4 are two solutions of (1.10) and (1.11) intersecting at $\eta = \eta_0$. Further,

$$g_3(\eta) > g_4(\eta) \quad \text{for } \eta > \eta_0 \quad \text{and} \quad g_3(\eta_0) = g_4(\eta_0). \tag{3.26}$$

This implies that

$$g_3'(\eta_0) \geqslant g_4'(\eta_0). \tag{3.27}$$

By (3.20) and (3.26),

$$g_3'(\eta_0) - g_4'(\eta_0) = -\int_{\eta_0}^{\infty} \left(\frac{2(\alpha-3)}{(\alpha-1)}(g_3 - g_4) + 4\lambda(g_3|g_3|^{\alpha-1} - g_4|g_4|^{\alpha-1}) \right) \mathrm{d}s < 0,$$

a contradiction to (3.27). Hence the remark.

Proof of Theorem 3. As in Lemma 3.1, we get the existence of a solution g on $[\eta_0(k),\infty)$ satisfying (1.10) and (1.11) with $A = k$. In this case too $g > 0$ has at most one local maximum. Hence g intersects the curve $f(\eta) = (\eta/\sqrt{2})^{2/(\alpha-1)}$, say, at η_1. In view of the continuous dependence of g on k, we may choose k small such that $g(\eta_1), g'(\eta_1)$ are small enough so that

$$E(\eta_1) < E_1 := \frac{2}{\alpha-1}\left(\frac{(\alpha-3)}{2(\alpha-1)(-\lambda)}\right)^{2/(\alpha-1)}$$

as in the proof of Lemma 3.4; here E is as in Remark above Lemma 3.4. If, for some η_2,

$$g(\eta_2) \geqslant \left(\frac{(\alpha-3)}{2(\alpha-1)(-\lambda)}\right)^{1/(\alpha-1)},$$

then $E(\eta_2) \geqslant E_1$ which is impossible by our choice. This implies that

$$g(\eta) < \left(\frac{(\alpha-3)}{2(\alpha-1)(-\lambda)}\right)^{1/(\alpha-1)}$$

for all η. Now we claim that g is positive on $(-\infty,\infty)$. Suppose on the contrary that g has the first zero from ∞ at $\eta = \eta_0$. Then, by (3.20), we have

$$g'(\eta_0) = -\int_{\eta_0}^{\infty} \left(\frac{2(\alpha-3)}{(\alpha-1)} g + 4\lambda g|g|^{\alpha-1}\right) \mathrm{d}s < 0,$$

a contradiction to the fact that $g'(\eta_0) \geqslant 0$. Hence $g > 0$ on $(-\infty,\infty)$. Following Lemma 3.3, $g \to 0$ as $\eta \to -\infty$ and $g = O(|\eta|^{-2/(\alpha-1)})$ as $\eta \to -\infty$. □

Proof of Theorem 4. If g is positive for all η, then $g'(\eta) > 0$ for all $\eta < 0$ as follows readily from Eq. (3.20). Again from (3.20),

$$g'(\eta) > \int_0^{\infty} \left[\frac{2(3-\alpha)}{(\alpha-1)} g(s) - 4\lambda g(s)|g(s)|^{\alpha-1} \right] ds \equiv \varepsilon > 0$$

for all $\eta < 0$. This implies that g cannot remain positive for all negative η. Hence the theorem. □

3.1. *Solutions with exponential decay at $\eta = -\infty$*

Remark. For $\alpha = 3$ and $\lambda = 0$, an exact solution with exponential decay as $|\eta| \to \infty$ was found earlier by Sachdev, Nair and Tikekar [21] and is given in Section 2. Now, we consider the case $1 < \alpha < 3$. Cazenave and Escobedo [5] studied the ODE

$$u'' + p|u|^{p-1}u' + \frac{\zeta}{2}u' + \frac{1}{2(p-1)}u = 0, \quad \zeta \in \mathbb{R} \tag{3.28}$$

obtained by the similarity reduction of the PDE

$$w_t - w_{xx} + (|w|^{p-1}w)_x = 0, \quad t > 0, \ x \in \mathbb{R}, \ 1 < p < 2. \tag{3.29}$$

They analysed Eq. (3.28) for $1 < p < 2$. Eq. (3.28) can be reduced to (1.14) by simple scaling. Making use of the results of Cazenave and Escobedo [5], we conclude that there are no bounded positive solutions of (1.14). Therefore, the boundary value problem (1.14)–(1.17) does not have a solution for $1 < \alpha < 3$.

Proof of Theorem 5. The local existence on $(-\infty, \eta_0)$, uniqueness and continuous dependence of solutions of (1.14) and (1.15) on the parameter A can be obtained by following the arguments of Lemma 3.1. If $A = 0$, $g \equiv 0$ is a solution of (1.14) and (1.15). By choosing A sufficiently small, and making use of the continuous dependence of solutions on A as in Soewono and Debnath [23], we get the solutions g_A satisfying $g_A(\eta_0) = g_0 < (\eta_0/2^{1/2})^{2/(\alpha-1)}$, where g_A is defined on $(-\infty, \eta_0)$. Following the proof of Lemma 3.2, g_A is bounded. In the manner of the proofs for Lemmas 3.1 and 3.3, we can show that g_A is positive and g_A decays algebraically to zero as $\eta \to \infty$. □

4. Conclusions

In this paper, we have studied self-similar solutions of GBE (1.1) with asymptotic conditions at $x = +\infty$ or as $x = -\infty$. We have proved the existence of bounded, positive solutions with exponential decay to zero as $x \to +\infty$ or $-\infty$ for different parametric ranges. From our analysis we conclude that bounded, positive solutions with exponential decay to zero as $x \to +\infty$ or $-\infty$ exist only for the following cases: (i) $\alpha > 3, \lambda \geqslant 0$ (ii) $\alpha > 3, \lambda < 0$. However, our numerical study (see [24]) suggests that there also exist global positive solutions for the case $1 < \alpha < 3, \lambda > 0$.

The existence of a self-similar solution with exponential decay as $x \to \pm\infty$ for $1 < \alpha < 3$ and $\lambda = 1$ was proved by Escobedo and Zuazua [7]. Their results complement our work. Sachdev et al. [21] showed numerically that self-similar solutions of the GBE (1.1) form intermediate asymptotics in the parametric range $1 < \alpha < 3, \lambda > 0$.

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