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An initial-value approach for solving singularly perturbed two-point boundary value problems

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Abstract

In this paper an initial-value approach is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. This approach is based on the boundary layer behavior of the solution. The method is distinguished by the following fact: The given singularly perturbed two-point boundary value problem is replaced by three first order initial-value problems. Several linear and non-linear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems; Two-point boundary value problems; Ordinary differential equations; Boundary layer; Initial-value approach

1. Introduction

Singularly perturbed second order two-point boundary value problems arise very frequently in fluid mechanics and other branches of Applied Mathematics. These problems have been received a significant amount of attention in past and recent years. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some

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other parts. So, typically there are thin transition layers where the solutions can jump abruptly, while away from the layers the solution behaves regularly and vary slowly. There are a wide variety of techniques for solving singular perturbation problems (cf. [1,3,5]).

A non-asymptotic method, called boundary value technique, has been introduced by Roberts [7] to solve certain classes of singular perturbation problems. He also discussed the analytical and approximate solutions of the problem: $\varepsilon y'' = yy'$ [8]. Roberts [9] has extended his boundary value technique to solve the problem: $\varepsilon y'' + yy' - y = 0$. The concept of replacing singularly perturbed two-point boundary value problem by an initial-value problem is presented by Kadalbajoo and Reddy [2].

In this paper, an initial-value approach is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. This approach is based on the boundary layer behavior of the solution. The method is distinguished by the following fact: The given singularly perturbed two-point boundary value problem is replaced by three first order initial-value problems. The numerical solution of two initial-value problems goes in opposite direction and the third initial-value problem is independent of these two initial-value problems. Several linear and non-linear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

2. Initial-value approach

For convenience, we call our method as the initial-value approach. To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form

$$\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = h(x), \quad x \in [a, b] \tag{1}$$

with

$$y(a) = \alpha \tag{2a}$$

and

$$y(b) = \beta, \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $f(x)$, $g(x)$ and $h(x)$ are sufficiently continuously differentiable functions in $[a, b]$. Further more, we assume that $f(x) \geqslant M > 0$ throughout the interval $[a, b]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = a$.

Because of the boundary layer behavior of the solution of singular perturbation problems, it is known that the solution of (1) and (2) is given by

$$y(x) = p(x) + q(x)\mathrm{e}^{-r(x)/\varepsilon} \tag{3}$$

with

$$r(x) = \int_a^x f(x)\,\mathrm{d}x,$$

where

$$p(x) = \sum_{n=0}^{\infty} p_n(x)\varepsilon^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n(x)\varepsilon^n$$

(cf. [10, p. 292]), i.e.,

$$y(x) = \sum_{n=0}^{\infty} p_n(x)\varepsilon^n + \left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^n\right)\mathrm{e}^{-r(x)/\varepsilon} \tag{4}$$

with

$$r(x) = \int_a^x f(x)\,\mathrm{d}x. \tag{5}$$

Differentiating (4) with respect to 'x' we get

$$\begin{aligned} y'(x) = {} & \sum_{n=0}^{\infty} p'_n(x)\varepsilon^n + \left(\sum_{n=0}^{\infty} q'_n(x)\varepsilon^n\right)\mathrm{e}^{-r(x)/\varepsilon} \\ & - \left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^n\right)(\mathrm{e}^{-r(x)/\varepsilon})\left(\frac{f(x)}{\varepsilon}\right), \end{aligned} \tag{6}$$

$$\begin{aligned} y''(x) = {} & \sum_{n=0}^{\infty} p''_n(x)\varepsilon^n + \left(\sum_{n=0}^{\infty} q''_n(x)\varepsilon^n\right)\mathrm{e}^{-r(x)/\varepsilon} \\ & - 2\left(\sum_{n=0}^{\infty} q'_n(x)\varepsilon^n\right)(\mathrm{e}^{-r(x)/\varepsilon})\left(\frac{f(x)}{\varepsilon}\right) \\ & + \left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^n\right)(\mathrm{e}^{-r(x)/\varepsilon})\left(\frac{f(x)}{\varepsilon}\right)^2 \\ & - \left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^n\right)(\mathrm{e}^{-r(x)/\varepsilon})\left(\frac{f'(x)}{\varepsilon}\right). \end{aligned} \tag{7}$$

Substituting (4), (6) and (7) in (1) we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_n''(x)\varepsilon^{n+1} + \left(\sum_{n=0}^{\infty} q_n''(x)\varepsilon^{n+1}\right)(\mathrm{e}^{-r(x)/\varepsilon}) - 2\left(\sum_{n=0}^{\infty} q_n'(x)\varepsilon^{n}\right)(\mathrm{e}^{-r(x)/\varepsilon})(f(x)) \\
&\quad + \left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^{n-1}\right)(\mathrm{e}^{-r(x)/\varepsilon})(f(x))^2 - \left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^{n}\right)(\mathrm{e}^{-r(x)/\varepsilon})(f'(x)) \\
&\quad + f(x)\left(\sum_{n=0}^{\infty} p_n'(x)\varepsilon^{n}\right) + f(x)\left(\sum_{n=0}^{\infty} q_n'(x)\varepsilon^{n}\right)\mathrm{e}^{-r(x)/\varepsilon} \\
&\quad - (f(x))^2\left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^{n-1}\right)\mathrm{e}^{-r(x)/\varepsilon} + g(x)\left(\sum_{n=0}^{\infty} p_n(x)\varepsilon^{n}\right) \\
&\quad + g(x)\left(\sum_{n=0}^{\infty} q_n(x)\varepsilon^{n}\right)\mathrm{e}^{-r(x)/\varepsilon} = h(x). \qquad (8)
\end{aligned}$$

By restricting these series to their first terms, we get

$$\begin{aligned}
&-2f(x)\mathrm{e}^{-r(x)/\varepsilon}q_0'(x) - q_0(x)\mathrm{e}^{-r(x)/\varepsilon}f'(x) + f(x)p_0'(x) + f(x)q_0'(x)\mathrm{e}^{-r(x)/\varepsilon} \\
&\quad + g(x)p_0(x) + g(x)q_0(x)\mathrm{e}^{-r(x)/\varepsilon} = h(x),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&f(x)p_0'(x) + g(x)p_0(x) + [-2f(x)q_0'(x) - f'(x)q_0(x) + f(x)q_0'(x) \\
&\quad + g(x)q_0(x)]\mathrm{e}^{-r(x)/\varepsilon} = h(x).
\end{aligned}$$

Therefore we have,

$$f(x)p_0'(x) + g(x)p_0(x) = h(x) \qquad (9)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)q_0(x)] = g(x)q_0(x). \qquad (10)$$

The representation (4) and (5) can be inserted to the boundary conditions (2a) and (2b). Now the boundary conditions becomes

$$p_0(a) + q_0(a) = \alpha \qquad (11)$$

and

$$p_0(b) = \beta, \qquad (12)$$

where we have neglected the exponentially small term $\mathrm{e}^{-r(b)/\varepsilon}$ (which is asymptotically zero) in obtaining the boundary condition (12) at $x = b$. First the differential equation (9) can be solved along with the boundary condition (12) to determine $p_0(x)$. Now $q_0(x)$ is determined by solving Eq. (10) subject to the condition $q_0(a) = \alpha - p_0(a)$ where $p_0(a)$ is determined already.

Now from (5) we have $r(x) = \int_a^x f(x)\,\mathrm{d}x$, i.e., $r'(x) = f(x)$ with $r(a) = 0$.

Therefore the three initial-value problems corresponding to (1) and (2) are given by

$$\text{(IVP. I)} \quad f(x)p_0'(x) + g(x)p_0(x) = h(x) \quad \text{with } p_0(b) = \beta, \tag{13}$$

$$\text{(IVP. II)} \quad \frac{\mathrm{d}}{\mathrm{d}x}[f(x)q_0(x)] = g(x)q_0(x) \quad \text{with } q_0(a) = \alpha - p_0(a) \tag{14}$$

and

$$\text{(IVP. III)} \quad r'(x) = f(x) \quad \text{with } r(a) = 0. \tag{15}$$

Remark. Here it is worth to note that these initial-value problems are independent of perturbation parameter ε.

The integration of the first two initial-value problems goes in opposite direction and the second problem is solved only if the solution of the first one is known at $x = a$. The third initial-value problem is independent of the first two initial-value problems. This can be solved independently. There now exist several efficient methods for solving initial-value problems. In order to solve the initial-value problems in our numerical experimentation, we used classical fourth order Runge–Kutta method. In fact, any standard analytical or numerical method can be used. After finding $p_0(x)$ and $q_0(x)$ and $r(x)$ we obtain the solution of (1) and (2) from (3) as

$$y(x) = p_0(x) + q_0(x)\mathrm{e}^{-r(x)/\varepsilon}.$$

3. Numerical examples

To demonstrate the applicability of the method we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example 3.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [1, p. 480, Problem 9.17 with $\alpha = 0$]

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by

$$y(x) = \frac{[(\mathrm{e}^{m_2} - 1)\mathrm{e}^{m_1 x} + (1 - \mathrm{e}^{m_1})\mathrm{e}^{m_2 x}]}{[\mathrm{e}^{m_2} - \mathrm{e}^{m_1}]},$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon)$.

Table 1
Numerical results of Example 3.1

x	$y(x)$	Exact solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	1.0000000	1.0000000
0.001	0.6003270	0.6007918
0.010	0.3712379	0.3719724
0.020	0.3749439	0.3756784
0.030	0.3787160	0.3794502
0.040	0.3825260	0.3832599
0.050	0.3863742	0.3871079
0.100	0.4062043	0.4069350
0.300	0.4962382	0.4969324
0.500	0.6062278	0.6068334
0.700	0.7405963	0.7410401
0.900	0.9047471	0.9049277
1.000	1.0000000	1.0000000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	1.0000000	1.0000000
0.0001	0.6004139	0.6004604
0.0010	0.3682394	0.3683130
0.0020	0.3685792	0.3686527
0.0030	0.3689480	0.3690215
0.0040	0.3693172	0.3693907
0.0050	0.3696867	0.3697602
0.1000	0.4065330	0.4066062
0.3000	0.4965506	0.4966200
0.5000	0.6065004	0.6065609
0.7000	0.7407961	0.7408404
0.9000	0.9048287	0.9048464
1.0000	1.0000000	1.0000000

The numerical results are given in Table 1(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 3.2. Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity [6, Example 2]

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - \mathrm{e}^{-x/\varepsilon})}{(1 - \mathrm{e}^{-1/\varepsilon})}.$$

The numerical results are given in Table 2(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 2
Numerical results of Example 3.2

x	$y(x)$	Exact solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	0.0000000	0.0000000
0.001	−1.0009970	−0.6298573
0.010	−0.9918800	−0.9878747
0.020	−0.9815600	−0.9776400
0.030	−0.9710400	−0.9671600
0.040	−0.9603199	−0.9564800
0.050	−0.9493999	−0.9456000
0.100	−0.8918000	−0.8882000
0.300	−0.6114000	−0.6086000
0.500	−0.2510000	−0.2490000
0.700	0.1894000	0.1906001
0.900	0.7098000	0.7102001
1.000	1.0000000	1.0000000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	0.0000000	0.0000000
0.0001	−1.0001000	−0.6318942
0.0010	−0.9991989	−0.9987538
0.0020	−0.9981956	−0.9977964
0.0030	−0.9971904	−0.9967916
0.0040	−0.9961832	−0.9957848
0.0050	−0.9951739	−0.9947760
0.1000	−0.8901801	−0.8898200
0.3000	−0.6101403	−0.6098600
0.5000	−0.2501001	−0.2499000
0.7000	0.1899399	0.1900600
0.9000	0.7099798	0.7100199
1.0000	1.0000000	1.0000000

Example 3.3. Finally we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [3, p. 33, Eqs. (2.3.26) and (2.3.27) with $\alpha = -1/2$]

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [4, p. 148, Eq. (4.2.32)]) as our 'exact' solution:

$$y(x) = \frac{1}{2 - x} - \frac{1}{2}\mathrm{e}^{-(x - x^2/4)/\varepsilon}.$$

The numerical results are given in Table 3(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 3
Numerical results of Example 3.3

x	$y(x)$	Nayfeh solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	0.0000000	0.0000000
0.001	0.4997493	0.3162644
0.010	0.5020117	0.5024893
0.020	0.5045496	0.5050505
0.030	0.5071134	0.5076142
0.040	0.5097033	0.5102041
0.050	0.5123200	0.5128205
0.100	0.5258163	0.5263158
0.300	0.5877497	0.5882353
0.500	0.6662214	0.6666667
0.700	0.7688746	0.7692308
0.900	0.9089253	0.9090909
1.000	1.0000000	1.0000000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	0.0000000	0.0000000
0.0001	0.4999750	0.3160807
0.0010	0.5002003	0.5002274
0.0020	0.5004506	0.5005005
0.0030	0.5007012	0.5007511
0.0040	0.5009522	0.5010020
0.0050	0.5012032	0.5012531
0.1000	0.5262662	0.5263158
0.3000	0.5881871	0.5882353
0.5000	0.6666222	0.6666667
0.7000	0.7691954	0.7692308
0.9000	0.9090744	0.9090909
1.0000	1.0000000	1.0000000

4. Non-linear problems

Non-linear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using quasilinearization method. The outer solution (the solution of the given problem by putting $\varepsilon = 0$) is taken to be the initial approximation.

5. Non-linear examples

Again to demonstrate the applicability of the method, we have applied it to three non-linear singular perturbation problems with left-end boundary layer.

Example 5.1. Consider the following singular perturbation problem from Bender and Orszag [1, p. 463, Eq. (9.7.1)]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0, \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 0$.

We have chosen to use Bender and Orszag's uniformly valid approximation [1, p. 463, Eq. (9.7.6)] for comparison,

$$y(x) = \log_e(2/(1+x)) - (\log_e 2)e^{-2x/\varepsilon}.$$

For this example, we have boundary layer of thickness $O(\varepsilon)$ at $x = 0$ [1].

The numerical results are given in Table 4(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 5.2. Now consider the following singular perturbation problem from Kevorkian and Cole [3, p. 56, Eq. (2.5.1)]

Table 4
Numerical results of Example 5.1

x	$y(x)$	Bender solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	0.0000000	0.0000000
0.001	0.6913641	0.5983404
0.010	0.6825219	0.6831968
0.020	0.6726859	0.6733446
0.030	0.6629456	0.6635884
0.040	0.6532992	0.6539265
0.050	0.6437448	0.6443570
0.100	0.5972949	0.5978370
0.300	0.4304523	0.4307829
0.500	0.2874905	0.2876821
0.700	0.1624234	0.1625189
0.900	0.0512663	0.0512933
1.000	0.0000000	0.0000000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	0.0000000	0.0000000
0.0001	0.6929690	0.5992399
0.0010	0.6920789	0.6921477
0.0020	0.6910806	0.6911492
0.0030	0.6900831	0.6901517
0.0040	0.6890868	0.6891552
0.0050	0.6880914	0.6881596
0.1000	0.5977829	0.5978370
0.3000	0.4307501	0.4307829
0.5000	0.2876630	0.2876821
0.7000	0.1625094	0.1625189
0.9000	0.0512906	0.0512933
1.0000	0.0000000	0.0000000

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = -1$ and $y(1) = 3.9995$.

We have chosen to use the Kivorkian and Cole's uniformly valid approximation [3, pp. 57–58, Eqs. (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh(c_1(x/\varepsilon + c_2)/2),$$

where $c_1 = 2.9995$ and $c_2 = (1/c_1)\log_e[(c_1 - 1)/(c_1 + 1)]$.

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ [3, pp. 56–66].

The numerical results are given in Table 5(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 5
Numerical results of Example 5.2

x	$y(x)$	Kevorkian solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	−1.0000000	−1.0000000
0.001	2.9508520	2.4569400
0.010	3.0095720	3.0095000
0.020	3.0195710	3.0195000
0.030	3.0295700	3.0295000
0.040	3.0395700	3.0395000
0.050	3.0495690	3.0495000
0.100	3.0995650	3.0995000
0.300	3.2995510	3.2995000
0.500	3.4995360	3.4995000
0.700	3.6995220	3.6995000
0.900	3.8995070	3.8995000
1.000	3.9995000	3.9995000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	−1.0000000	−1.0000000
0.0001	2.9508230	2.4560400
0.0010	3.0015250	3.0005000
0.0020	3.0025240	3.0015000
0.0030	3.0035230	3.0025000
0.0040	3.0045220	3.0035000
0.0050	3.0055210	3.0045000
0.1000	3.1004240	3.0995000
0.3000	3.3002180	3.2995000
0.5000	3.5000130	3.4995000
0.7000	3.6998080	3.6995000
0.9000	3.8996030	3.8995000
1.0000	3.9995000	3.9995000

Example 5.3. Finally we consider the following singular perturbation problem from O'Malley [5, p. 9, Eq. (1.10) case 2]:

$$\varepsilon y''(x) - y(x)y'(x) = 0, \quad x \in [-1, 1]$$

with $y(-1) = 0$ and $y(1) = -1$.

We have chosen to use O'Malley's approximate solution [5, pp. 9–10, Eqs. (1.13) and (1.14)] for comparison,

$$y(x) = -(1 - e^{-(x+1)/\varepsilon})/(1 + e^{-(x+1)/\varepsilon}).$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = -1$ (cf. [5, pp. 9–10, Eqs. (1.10), (1.13), (1.14), case 2] and [8]).

The numerical results are given in Table 6(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 6
Numerical results of Example 5.3

x	$y(x)$	O'Malley solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
−1.000	0.0000000	0.0000000
−0.999	−1.0000000	−0.4621121
−0.980	−1.0000000	−1.0000000
−0.960	−1.0000000	−1.0000000
−0.940	−1.0000000	−1.0000000
−0.920	−1.0000000	−1.0000000
−0.900	−1.0000000	−1.0000000
−0.800	−1.0000000	−1.0000000
−0.400	−1.0000000	−1.0000000
0.000	−1.0000000	−1.0000000
0.400	−1.0000000	−1.0000000
0.800	−1.0000000	−1.0000000
1.000	−1.0000000	−1.0000000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
−1.0000	0.0000000	0.0000000
−0.9999	−1.0000000	−0.4621824
−0.9980	−1.0000000	−1.0000000
−0.9960	−1.0000000	−1.0000000
−0.9940	−1.0000000	−1.0000000
−0.9920	−1.0000000	−1.0000000
−0.9900	−1.0000000	−1.0000000
−0.8000	−1.0000000	−1.0000000
−0.4000	−1.0000000	−1.0000000
0.0000	−1.0000000	−1.0000000
0.4000	−1.0000000	−1.0000000
0.8000	−1.0000000	−1.0000000
1.0000	−1.0000000	−1.0000000

6. Right-end boundary layer problems

Finally, we discuss our method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form

$$\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = h(x), \quad x \in [a, b] \tag{16}$$

with

$$y(a) = \alpha \tag{17a}$$

and

$$y(b) = \beta, \tag{17b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $f(x)$, $g(x)$ and $h(x)$ are sufficiently continuously differentiable functions in $[a, b]$. Further more, we assume that $f(x) \leqslant M < 0$ throughout the interval $[a, b]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = b$.

Because of the boundary layer behavior of the solution of singular perturbation problems, it is known that the solution of (16) and (17) is given

$$y(x) = p(x) + q(x)\mathrm{e}^{-r(x)/\varepsilon} \tag{18}$$

with

$$r(x) = \int_b^x f(x)\,\mathrm{d}x, \tag{19}$$

where $p(x) = \sum_{n=0}^{\infty} p_n(x)\varepsilon^n$ and $q(x) = \sum_{n=0}^{\infty} q_n(x)\varepsilon^n$.

Restricting these series to their first terms and substituting (18) in (16) we get

$$f(x)p_0'(x) + g(x)p_0(x) = h(x) \tag{20}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)q_0(x)] = g(x)q_0(x). \tag{21}$$

The boundary conditions becomes

$$p_0(a) = \alpha \tag{22}$$

and

$$p_0(b) + q_0(b) = \beta. \tag{23}$$

Now from (19) we have $r(x) = \int_b^x f(x)\,\mathrm{d}x$, i.e., $r'(x) = f(x)$ with $r(b) = 0$.

Therefore the three initial-value problems corresponding to (16) and (17) are given by

$$\text{(IVP. I)} \quad f(x)p_0'(x) + g(x)p_0(x) = h(x) \quad \text{with } p_0(a) = \alpha, \tag{24}$$

$$\text{(IVP. II)} \quad \frac{d}{dx}[f(x)q_0(x)] = g(x)q_0(x) \quad \text{with } q_0(b) = \beta - p_0(b) \tag{25}$$

and

$$\text{(IVP. III)} \quad r'(x) = f(x) \quad \text{with } r(b) = 0. \tag{26}$$

We used classical forth order Runge–Kutta method to solve these initial-value problems. After finding $p_0(x)$ and $q_0(x)$ and $r(x)$ we obtain the solution of (16) and (17) from (18) as

$$y(x) = p_0(x) + q_0(x)e^{-r(x)/\varepsilon}.$$

7. Examples with right-end boundary layer

To illustrate the method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval we considered two examples.

Example 7.1. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at $x = 1$. i.e.; at the right end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{(e^{(x-1)/\varepsilon} - 1)}{(e^{-1/\varepsilon} - 1)}.$$

The numerical results are given in Table 7(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 7.2. Now we consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0, \quad x \in [0, 1]$$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$; and $y(1) = 1 + 1/e$.

Clearly this problem has a boundary layer at $x = 1$. The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}.$$

The numerical results are given in Table 8(panels A and B) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 7
Numerical results of Example 7.1

x	$y(x)$	Exact solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	1.0000000	1.0000000
0.200	1.0000000	1.0000000
0.400	1.0000000	1.0000000
0.600	1.0000000	1.0000000
0.800	1.0000000	1.0000000
0.900	1.0000000	1.0000000
0.920	1.0000000	1.0000000
0.940	1.0000000	1.0000000
0.960	1.0000000	1.0000000
0.980	1.0000000	1.0000000
0.999	0.6321205	0.6320939
1.000	0.0000000	0.0000000
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	1.0000000	1.0000000
0.2000	1.0000000	1.0000000
0.4000	1.0000000	1.0000000
0.6000	1.0000000	1.0000000
0.8000	1.0000000	1.0000000
0.9000	1.0000000	1.0000000
0.9200	1.0000000	1.0000000
0.9400	1.0000000	1.0000000
0.9600	1.0000000	1.0000000
0.9800	1.0000000	1.0000000
0.9999	0.6321205	0.6321816
1.0000	0.0000000	0.0000000

8. Discussion and conclusions

We have presented an initial-value approach for solving singularly perturbed two-point boundary value problems. In general, the numerical solution of a boundary value problem will be more difficult matter than the numerical solution of the corresponding initial-value problems. Hence, we prefer always to convert the second order problem into first order problems. The solution of the given singularly perturbed two-point boundary value problem is computed numerically by solving three initial-value problems. It is worth to note that these initial-value problems are independent of perturbation parameter ε. We have implemented the present method on three linear examples, three nonlinear examples, with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. To solve initial-value problems we used the classical fourth order Runge–Kutta method. In fact any standard analytical or numerical method can be used. Numerical results are presented in

Table 8
Numerical results of Example 7.2

x	$y(x)$	Exact solution
Panel A: $\varepsilon = 10^{-3}$, $h = 10^{-3}$		
0.000	1.0000000	1.0000000
0.200	0.8185673	0. 8187308
0.400	0.6700525	0.6703200
0.600	0.5484830	0.5488116
0.800	0.4489703	0.4493290
0.900	0.4062046	0.4065697
0.920	0.3981533	0.3985190
0.940	0.3902616	0.3906278
0.960	0.3825263	0.3828929
0.980	0.3749442	0.3753111
0.999	0.7355264	0.7357859
1.000	1.3678790	1.3678790
Panel B: $\varepsilon = 10^{-4}$, $h = 10^{-4}$		
0.0000	1.0000000	1.0000000
0.2000	0.8187149	0.8187308
0.4000	0.6702932	0.6703200
0.6000	0.5487787	0.5488117
0.8000	0.4492931	0.4493290
0.9000	0.4065330	0.4065697
0.9200	0.3984824	0.3985191
0.9400	0.3905911	0.3906278
0.9600	0.3828562	0.3828929
0.9800	0.3752744	0.3753111
0.9999	0.7357357	0.7356979
1.0000	1.3678790	1.3678790

tables. It can be observed from the tables that the present method approximates the exact solution very well.

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