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The method of inner boundary condition for singular boundary value problems

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Abstract

A method of inner boundary condition is presented for solving two-point singular boundary value problem. The original interval is divided into two parts. An inner boundary condition is obtained by using a series solution. Then, a special finite difference method of order two is employed to solve the problem in the interval $[\delta, 1]$. The method is implemented on several numerical examples and results are compared with exact solutions.

Keywords: Ordinary differential equations; Singular boundary value problem; Series solution; Finite difference method

1. Introduction

Consider a linear second order differential equation

$$v''(t) + P(t)v'(t) + Q(t)v(t) = R(t), \tag{1}$$

with boundary conditions

$$v(0) = a \tag{1a}$$

and

$$v(1) = b. \tag{1b}$$

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The coefficient functions $P(t)$ and $Q(t)$ fail to be analytic at $t = 0$, where a and b are given constants. These problems are called as singular boundary value problems.

In the last few years, considerable effort has been made in developing methods for numerically solving singular boundary value problems. Typically, these problems arise very frequently in chemical and mechanical engineering, physics and many other applications. Finite difference methods have been widely used for scalar equations by Jamet [5], Gustafsson [4], De Hoog and Weiss [2]. Weinmuller [9] has studied analytical properties of these problems. Kadalbajoo and Raman [6] studied the discrete invariant imbedding approach.

In this paper, we present an inner boundary condition for solving two-point singular boundary value problem. The original interval is divided into two parts. An inner boundary condition is obtained by using a series solution. Then, a special finite difference method of order two is employed to solve the problem in the interval $[\delta, 1]$. The method is implemented on several numerical examples and results are compared with exact solutions.

2. Derivation of inner boundary condition

We use series expansion in a small interval near δ and Eq. (1) has a solution of the form

$$v(t) = \sum_{r=0}^{\infty} a_r t^{r+m}, \quad a_0 \neq 0. \tag{2}$$

The indicial roots m and coefficients a_r are obtained by differentiating (2) substituting in (1), and comparing the coefficients of like powers of t on the two sides of the equation. In general, the solution can be written as

$$v(t) = \sum_{j=1}^{n} \beta_j L_j(t) + L_{n+1}(t), \quad n \leqslant 2. \tag{3}$$

For $t \in (0, \delta]$, where $L_1(t)$ and $L_2(t)$ are two linearly independent solutions of Eq. (1), and $L_{n+1}(t)$ is the particular solution to (1). Keller [7] and Coddington and Levinson [1] have discussed the basic theoretical results of series expansion about a singular point. Series solution may be valid for entire interval, but due to its slow convergence, we find the series expansion in the interval $(0, \delta]$ only. To derive the regular problem, we thus have to derive the inner boundary condition at δ. To do this, we have from Eq. (3)

$$L_1(\delta)\beta_1 + L_2(\delta)\beta_2 = v(\delta) - L_{n+1}(\delta), \tag{4}$$

$$L_1'(\delta)\beta_1 + L_2'(\delta)\beta_2 = v'(\delta) - L_{n+1}'(\delta), \tag{5}$$

where the prime denotes the differentiation. From Eqs. (4) and (5), we have

$$\beta_1 = \frac{[v(\delta) - L_{n+1}(\delta)]L_2'(\delta) - [v'(\delta) - L_{n+1}'(\delta)]L_2(\delta)}{L_1(\delta)L_2'(\delta) - L_2(\delta)L_1'(\delta)}, \tag{6}$$

$$\beta_2 = \frac{[v'(\delta) - L_{n+1}'(\delta)]L_1(\delta) - [v(\delta) - L_{n+1}(\delta)]L_1'(\delta)}{L_1(\delta)L_2'(\delta) - L_2(\delta)L_1'(\delta)}. \tag{7}$$

We have from Eqs. (1b) and (3)

$$L_1(0)\beta_1 + L_2(0)\beta_2 = v(0) - L_{n+1}(0). \tag{8}$$

From Eqs. (6)–(8), we have

$$\frac{R(\delta)L_2'(\delta) - R'(\delta)L_2(\delta)}{S(\delta)}L_1(0) + \frac{R'(\delta)L_1(\delta) - R(\delta)L_1'(\delta)}{S(\delta)}L_2(0) = a - L_{n+1}(0), \tag{9}$$

where

$$R(t) = v(t) - L_{n+1}(t), \tag{10}$$

$$S(t) = L_1(t)L_2'(t) - L_2(t)L_1'(t). \tag{11}$$

For our convince Eq. (9) can be written as

$$[L_1(0)L_2'(\delta) - L_2(0)L_1'(\delta)]R(\delta) + [L_1(\delta)L_2(0) - L_2(\delta)L_1(0)]R'(\delta) = S(\delta)[a - L_{n+1}(0)] \tag{12}$$

or

$$pv(\delta) + qv'(\delta) = r, \tag{13}$$

where

$$p = L_1(0)L_2'(\delta) - L_2(0)L_1'(\delta), \tag{14}$$

$$q = L_1(\delta)L_2(0) - L_2(\delta)L_1(0) \tag{15}$$

and

$$r = S(\delta)[a - L_{n+1}(0)] + pL_{n+1}(\delta) + qL_{n+1}'(\delta). \tag{16}$$

Thus the regular boundary value problem over $[\delta, 1]$ is given by Eqs. (1), (13) and (1b).

3. Special second order finite difference method

A finite difference scheme is often a convenient choice of method for the numerical solution of two point boundary value problems [3,8]. We briefly describe the special second order Finite difference scheme for a general two-point boundary value problem given:

$$v''(t) + a(t)v'(t) - b(t)v(t) = f(t), \tag{17}$$

$$\text{for } t_1 \leqslant t \leqslant t_2 \quad \text{with } v(t_1) = L_1 \text{ and } v(t_2) = L_2, \tag{18}$$

where $a(t)$, $b(t)$ and $f(t)$ are assumed to be sufficiently continuously differentiable functions $b(t) \geqslant 0$, $a(t) > 0$ on $[t_1, t_2]$ and L_1, L_2 are given constants. Throughout the discussion the symbols μ, δ denote the usual central difference operators and E, D denote the displacement and differentiation respectively, defined by Fox. As usual, we divide the interval $[t_1, t_2]$ into N equal parts of mesh size h. Consider a typical pivotal point in the mesh, at $t = t_i + rh$. The following expression can written for v, v' and v'':

$$v_r = v(t_i + rh) = E^r v(t_i), \tag{19}$$

$$v'_r = Dv_r, \tag{20}$$

$$v''_r = D^2 v_r. \tag{21}$$

The displacement operator

$$E = e^{hD} \tag{22}$$

can be related to the central difference operators μ, δ by using the following expressions (for details, see [3, pp. 10–11]);

$$hD = \mu\delta - \tfrac{1}{6}\mu\delta^3 + \tfrac{1}{30}\mu\delta^5 + \cdots, \tag{23}$$

$$h^2D^2 = \delta^2 - \tfrac{1}{12}\delta^4 + \tfrac{1}{90}\delta^6 + \cdots \tag{24}$$

Substituting Eqs. (22)–(24) in Eqs. (19)–(21), we get

$$v_r = [1 + r\mu\delta + \tfrac{1}{2}r^2\delta^2 + \tfrac{1}{6}r(r^2-1)\mu\delta^3 + \tfrac{1}{24}r^2(r^2-1)\delta^4 + \cdots]v_i, \tag{25}$$

$$v'_r = \frac{1}{h}\left[\mu\delta + r\delta^2 + \frac{1}{6}(3r^2-1)\mu\delta^3 + \frac{1}{12}r(2r^2-1)\delta^4 + \cdots\right]v_i, \tag{26}$$

$$v''_r = \frac{1}{h^2}\left[\delta^2 + r\mu\delta^3 + \frac{6r^2-1}{12}\delta^4 + \frac{r(2r^2-3)}{12}\mu\delta^5 + \cdots\right]v_i. \tag{27}$$

Now, we substitute Eqs. (25)–(27) in Eq. (17) with $r = 1/2$, to obtain

$$\begin{aligned}(v_{i+1} - 2v_i + v_{i-1}) + ha_{i+1/2}(v_{i+1} - v_i) - \frac{h^2}{8}b_{i+1/2}(3v_{i+1} + 6v_i - v_{i-1})\\ = h^2 f_{i+1/2} + \varphi_{i+1/2}v_i,\end{aligned} \tag{28}$$

where

$$\varphi_{i+1/2} = \left\{ -\frac{1}{2}\mu\delta^3 - \frac{1}{24}\delta^4 + \cdots \right\} + \frac{h}{24}a_{i+1/2}\left\{ \mu\delta^3 + \frac{1}{2}\delta^4 + \cdots \right\} + \frac{h^2}{16}b_{i+1/2}\left\{ -\mu\delta^3 - \frac{1}{8}\delta^4 + \cdots \right\}.$$

A similar procedure can be followed with $r = -1/2$, to obtain

$$(v_{i+1} - 2v_i + v_{i-1}) + ha_{i-1/2}(v_i - v_{i-1}) - \frac{h^2}{8}b_{i-1/2}(3v_{i-1} + 6v_i - v_{i+1}) = h^2 f_{i-1/2} + \varphi_{i-1/2}v_i, \tag{29}$$

$$\varphi_{i-1/2} = \left\{ \frac{1}{2}\mu\delta^3 - \frac{1}{24}\delta^4 + \cdots \right\} + \frac{h}{24}a_{i-1/2}\left\{ \mu\delta^3 - \frac{1}{2}\delta^4 + \cdots \right\} + \frac{h^2}{16}b_{i-1/2}\left\{ \mu\delta^3 - \frac{1}{8}\delta^4 + \cdots \right\}.$$

A set of difference equations similar to that of the classical finite difference scheme can be obtained by adding Eqs. (28) and (29), as

$$E_i v_{i-1} - F_i v_i + G_i v_{i+1} = H_i + \varphi v_i, \tag{30}$$

where

$$E_i = 1 - \frac{h}{2}a_{i-1/2} - \frac{h^2}{16}(3b_{i-1/2} - b_{i+1/2}), \tag{31}$$

$$F_i = 2 + \frac{h}{2}(a_{i+1/2} - a_{i-1/2}) + \frac{3h^2}{8}(b_{i+1/2} + b_{i-1/2}), \tag{32}$$

$$G_i = 1 + \frac{h}{2}a_{i+1/2} + \frac{h^2}{16}(b_{i-1/2} - 3b_{i+1/2}), \tag{33}$$

$$H_i = \frac{h^2}{2}(f_{i+1/2} + f_{i-1/2}), \tag{34}$$

$$\varphi = \left\{ -\frac{1}{24}\delta^4 + \frac{41}{5760}\delta^6 + \cdots \right\} + \left\{ \frac{h}{24}(\mu a_i)\mu\delta^3 + \frac{h}{96}(\delta a_i)\delta^4 + \cdots \right\} + \left\{ -\frac{h^2}{32}(\delta b_i)\mu\delta^3 - \frac{h^2}{128}(\mu b_i)\delta^4 + \cdots \right\}. \tag{35}$$

Eqs. (30)–(34) are the basis of the present scheme. They can be observed from (35) that the magnitude of each term in the truncation error is reduced by at least a factor of two from the classical finite difference scheme. It is of interest to note that a feature of the present scheme which is different from the classical finite difference scheme is that to evaluate the coefficients (31)–(34), values of $a(t)$, $b(t)$, and $f(t)$ are required at the grid locations $(t_i + h/2)$ and

$(t_i - h/2)$. If the truncation error term in (30) is neglected. The matrix problem associated with (30) is a tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using Thomas algorithm.

Repeat the numerical scheme for different choices of δ (terminal point) until the solution profiles do not differ materially from iteration. For computational point of view, we use absolute error criteria, namely

$$|v^{m+1}(t) - v^m(t)| \leqslant \sigma, \quad \delta \leqslant t \leqslant 1,$$

where $v^m(t)$ is the solution for the mth iterate of δ and σ is the prescribed tolerance bound.

4. Numerical experiments

To demonstrate the applicability of this numerical method, we have applied to three examples. These examples have been chosen because either analytical or approximate solutions are available for comparison.

Example 1. Consider the problem

$$v'' - \frac{0.5}{t}v' = -1, \quad 0 < t < 1 \text{ with } v(0) = 5,\ v(1) = 5.$$

The exact solution is given by $v(t) = 5 - t^2 + t^{3/2}$.

The computational results are presented in Table 1.

Example 2. Now consider the problem

$$(t^\alpha v')' = \beta t^{(\alpha+\beta-2)}[(\alpha + \beta - 1) + \beta t^\beta]v, \quad 0 < t < 1,$$

with boundary conditions $v(0) = 1$, $v(1) = e$.

Table 1
Numerical results for Example 1

t	$v(t)$ exact	$\delta = 0.1$		$\delta = 0.2$		$\delta = 0.5$	
		$h = 1/80$	$h = 1/160$	$h = 1/80$	$h = 1/160$	$h = 1/80$	$h = 1/160$
0.1	5.021623	5.021606	5.021629				
0.2	5.049443	5.049427	5.049469	5.049436	5.049445		
0.3	5.074317	5.074309	5.074330	5.074321	5.074304		
0.4	5.092982	5.092986	5.092996	5.092995	5.092976		
0.5	5.103553	5.103554	5.103570	5.103561	5.103556	5.103521	5.103548
0.6	5.104758	5.104753	5.104773	5.104757	5.104760	5.104723	5.104751
0.7	5.095662	5.095659	5.095666	5.095660	5.095654	5.095634	5.095648
0.8	5.075541	5.075541	5.075547	5.075541	5.075537	5.075522	5.075534
0.9	5.043815	5.043816	5.043809	5.043817	5.043805	5.043809	5.043805
1.0	5.000000	5.000000	5.000000	5.000000	5.000000	5.000000	5.000000

Table 2
Numerical results for Example 2

t	$v(t)$ exact	$\delta = 0.1$		$\delta = 0.2$		$\delta = 0.5$	
		$h = 1/80$	$h = 1/160$	$h = 1/80$	$h = 1/160$	$h = 1/80$	$h = 1/160$
0.1	1.000100	1.001025	1.001150				
0.2	1.001601	1.002264	1.002397	1.001537	1.001606		
0.3	1.008133	1.008598	1.008736	1.008045	1.008134		
0.4	1.025931	1.026236	1.026376	1.025823	1.025926		
0.5	1.064494	1.064668	1.064814	1.064369	1.064485	1.064658	1.064701
0.6	1.138373	1.138435	1.138589	1.138228	1.138359	1.138430	1.138512
0.7	1.271376	1.271349	1.271502	1.271216	1.271352	1.271348	1.271453
0.8	1.506216	1.506127	1.506274	1.506050	1.506186	1.506128	1.506247
0.9	1.927262	1.927157	1.927275	1.927123	1.927237	1.927158	1.927266
1.0	2.718283	2.718282	2.718282	2.718282	2.718282	2.718282	2.718282

The exact solution is given by $v(t) = \exp(t^\beta)$. The computational results are presented in Table 2 for $\alpha = 0.5$, $\beta = 4$.

Example 3. We consider the problem

$$(t^{1/2}v')' - t^{1/2}v = -\tfrac{1}{2}(3 + t^{-1/2})\mathrm{e}^{-t}, \quad 0 < t < 1,$$

with boundary conditions $v(0) = 1$, $v(1) = 2\mathrm{e}^{-1}$.

The exact solution is given by $v(t) = (t^{1/2} + 1)\mathrm{e}^{-t}$. The computational results are presented in Table 3.

Table 3
Numerical results for Example 3

t	$v(t)$ exact	$\delta = 0.1$		$\delta = 0.2$		$\delta = 0.4$	
		$h = 1/80$	$h = 1/160$	$h = 1/80$	$h = 1/160$	$h = 1/80$	$h = 1/160$
0.1	1.190972	1.191462	1.191037				
0.2	1.184878	1.185297	1.184948	1.185046	1.184929		
0.3	1.146581	1.146922	1.146643	1.146723	1.146624		
0.4	1.094268	1.094541	1.094316	1.094383	1.094297	1.094311	1.094259
0.5	1.035413	1.035628	1.035445	1.035505	1.035427	1.035448	1.035398
0.6	0.973919	0.974084	0.973939	0.973990	0.973922	0.973946	0.973901
0.7	0.912058	0.912176	0.912067	0.912109	0.912054	0.912077	0.912040
0.8	0.851221	0.851296	0.851223	0.851253	0.851215	0.851233	0.851206
0.9	0.792275	0.792312	0.792276	0.792291	0.792271	0.792281	0.792268
1.0	0.735759	0.735759	0.735759	0.735759	0.735759	0.735759	0.735759

5. Discussion and conclusion

We have described and to demonstrated the applicability of the special second order finite difference method, by solving some model examples. It is observed from the results that the present method is simple, accurate, stable and easy to implement on computer. The numerical results for examples for different mesh sizes and three different values of δ are presented in Tables 1–3. It can be observed from these tables that the computed solutions compare well with the exact solutions.

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