An engineer's approach to a general algorithm for finding the sum of powers of natural numbers

R. Venkatachalam[1] and Umesh Chandra Sharma†[2]
[1] *Regional Engineering College, Warangal 506004, India*
[2] *Faculty of Science, Garyounis University, P.O. Box 9480, Benghazi, Libya*
E-mail: chalamrv@yahoo.com

Abstract Convenient formulae for finding the sums of *k*th power of the first *n* natural numbers may be useful in some engineering applications. The formulae for $k = 1, 2$, and 3, are commonly found in the literature. In this paper, an attempt is made to develop a general algorithm for finding the sum for any positive integer value of *k*. The development of the algorithm is entirely an engineering approach, based purely on a simple geometric interpretation and does not involve any deep mathematics. This algorithm may be used to derive the formulae for different values of *k*. Some of the possible engineering applications of these formulae are also discussed.

Keywords powers of natural numbers; sum; general algorithm; geometric interpretations

1 Introduction

The sums of the first *n* natural numbers, the sums of the squares and cubes of the first *n* natural numbers, are being used to illustrate the method of limiting process to find the area under specific curves [1]. The sums of the powers of the first *n* natural numbers have applications in the theory of probability. For instance, if it is desired to find the expected value of the *k*th power of the random variable in a finite sample space with equally likely outcomes, a need arises for a formula which helps to find the value of the sum of the *k*th power of the first *n* natural numbers. This type of problem may be encountered in the Game Theory [2].

In the area of mechanical engineering, situations may arise where the formulae for finding the sum of the *k*th power of natural numbers, are required. Some striking examples are discussed later in this article in section 3.

The formulae up to the third power are commonly found in the literature. However, a general formula is not yet available. Usually, these formulae are derived by using identities [3] or by using the method of mathematical induction [1, 4]. Durell [5] has demonstrated the use of the method of differences and derived the formulae up to the third power. It was also mentioned about the increase of difficulty with higher powers [6]. In this paper, an attempt is made to arrive at a general algorithm to obtain the sum of the *k*th power of the first *n* natural numbers. The method is purely based on a simple geometric interpretation and does not involve

† Formerly, Assistant Professor of Mathematics, Birla Institute of Technology & Science, Pilani – 333031, India.

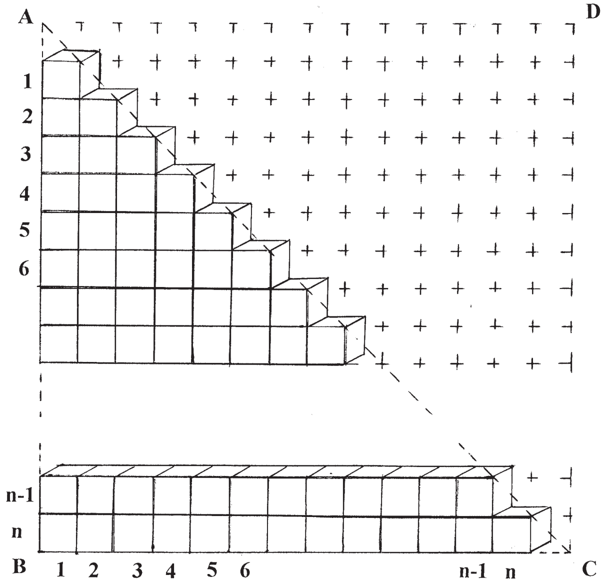

Fig. 1 *Arrangement of blocks for finding the sum of* n *natural numbers.*

any deep mathematics. This algorithm can be used to derive formula for any specific value of k.

The main objective of the present paper is to inculcate important skills such as, creativity, physical visualization, reasoning and interpretation, among the budding engineers.

2 Development of the algorithm

In this work, a block in the form of a cube is considered as a building block of the method. Throughout, the approach involves only the counting of the number of blocks. The method is described starting with a simple problem of finding the sum of the first n natural numbers. Later on, this method is extended to find the sum of any power of the first n natural numbers.

2.1 Finding the sum of the first n natural numbers

Fig. 1 shows an arrangement of the building blocks. In the topmost row there is only one block to represent the first natural number, 1. In the next row, there are two blocks to represent the next natural number, 2. This is continued and in the bottom most row, there are n blocks to represent the nth natural number. The overall arrangement of the blocks resembles a right-angled triangle. The sum of the first n natural numbers is clearly the total number of blocks in this arrangement.

A line AC may now be imagined which touches the free corner of the extreme right block in each row. With AC as diagonal, a square $ABCD$ may now be con-

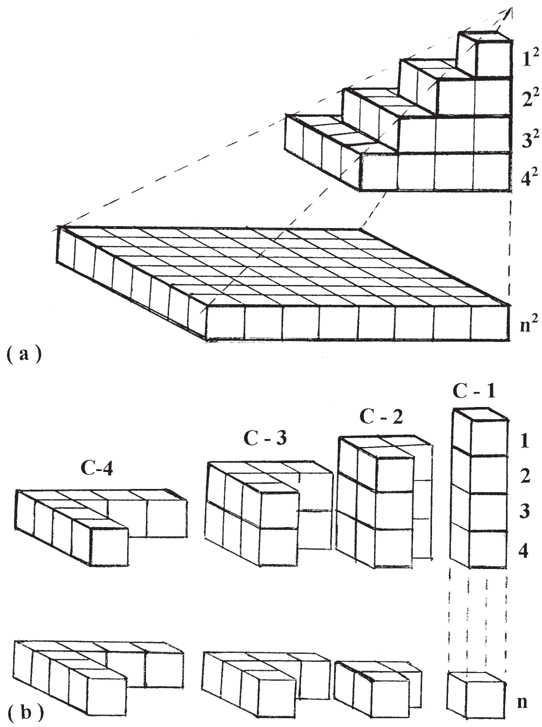

Fig. 2 *Finding the sum of squares of* n *natural numbers. (a) Arrangement of blocks; (b) splitting up into* n *columns.*

structed. This square provides space for one more row and one more column. The number of blocks that can be accommodated in the square *ABCD* is $(n + 1)^2$. From this, $(n + 1)$ diagonal blocks are removed first. Then half of the remaining blocks may be realized as the original blocks arranged. With this interpretation, the sum of the first n natural numbers may now be expressed as

$$\sum_{i=1}^{n} i = \frac{\{(n+1)^2 - (n+1)\}}{2} \tag{1}$$

which may be simplified as

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{2}$$

Though this is a very well known result, it will serve as a starting step for the approach adopted in this paper.

2.2 Finding the sum of the squares of the first *n* natural numbers

Fig. 2(a) shows the arrangement of the building blocks in the form of a pyramid.

The topmost layer has only one block and represents 1^2. The next layer of blocks has four building blocks representing 2^2, and so on. The bottom-most layer has n^2 blocks. The total number of blocks accommodated in the pyramid is the sum of the squares of the first n natural numbers.

For the purpose of counting the total number of blocks, the pyramid is now split up into n columns as shown in Fig. 2(b). In the column $C - 1$, there are n layers, each layer with one block. In the column $C - 2$, there are $(n - 1)$ layers, each with three blocks. In the column $C - 3$, there are $(n - 2)$ layers, each with five blocks, and so on. The total number of blocks may now be expressed as,

$$\sum_{i=1}^{n} i^2 = 1(n) + 3(n-1) + 5(n-2) + 7(n-3) + \cdots + (2n-1)1 \tag{3}$$

which may be rewritten as

$$\sum_{i=1}^{n} i^2 = \sum_{j=1}^{n} (2j-1)(n-j+1) \tag{4}$$

Expanding the term inside the summation and using the result in equation (2), equation (4) may be simplified as

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{5}$$

This result is also very well known. However, it serves as the next step in the development of the approach.

2.3 Finding the sum of the cubes of the first n natural numbers

Cubes may be made out of the building blocks and stacked as shown in Fig. 3. This arrangement is a pyramid with curved slant edges. The topmost layer has only one block, representing 1^3. The next two layers together have eight blocks, representing 2^3. The next three layers have, in total, twenty-seven blocks, representing 3^3, and so on. The total number of blocks in this arrangement may easily be realized as the sum of the cubes of the first n natural numbers.

For the purpose of counting, the arrangement of blocks shown in Fig. 3, may be split up into n columns, as shown in Fig. 4. The column $C - 1$ has $(1 + 2 + 3 + \cdots + n)$ layers, each with one block. The column $C - 2$ has $(2 + 3 + 4 + \cdots + n)$ layers, each with three blocks. The column $C - 3$ has $(3 + 4 + 5 + \cdots + n)$ layers each with five blocks, and so on. With this interpretation, the total number of blocks may now be expressed as

$$\sum_{i=1}^{n} i^3 = 1\sum_{i=1}^{n} i + 3\sum_{i=2}^{n} i + 5\sum_{i=3}^{n} i + \cdots + (2n-1)\sum_{i=n}^{n} i \tag{6}$$

and may be rewritten as

$$\sum_{i=1}^{n} i^3 = \sum_{j=1}^{n} i(2j-1)\sum_{i=j}^{n} i \tag{7}$$

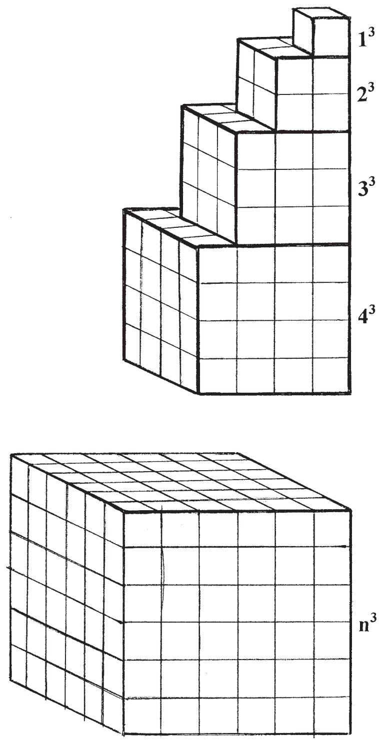

Fig. 3 *Finding the sum of cubes of* n *natural numbers; arrangement of blocks.*

Expanding the terms inside the summation and using equations (2) and (5), equation (7) may be simplified as

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{8}$$

This is also a well known formula and it forms the basis for the next step.

2.4 Finding the sum of the fourth power of the first *n* natural numbers

The sum of the fourth power of the first *n* natural numbers may be represented as

$$\sum_{i=1}^{n} i^4 = \sum_{i=1}^{n} i \cdot i^3 \tag{9}$$

which in the expanded form may be written as

$$\sum_{i=1}^{n} i^4 = 1(1)^3 + 2(2)^3 + 3(3)^3 + \cdots + n(n)^3 \tag{10}$$

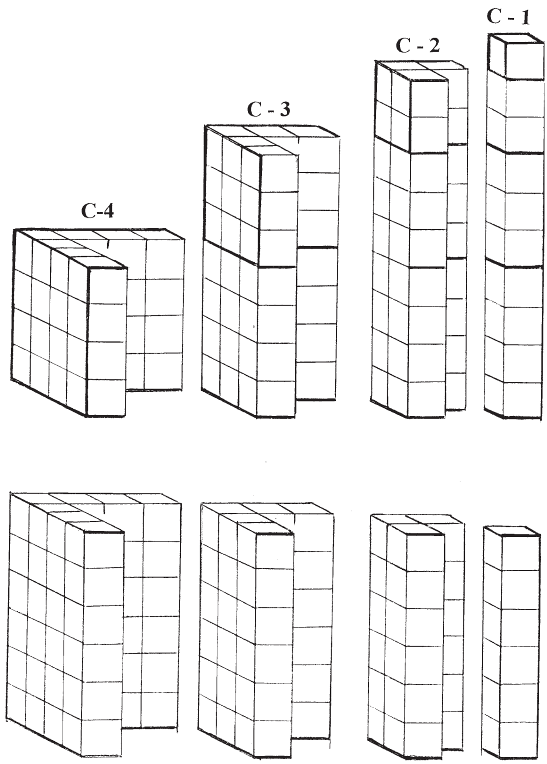

Fig. 4 *Finding the sum of cubes of* n *natural numbers; splitting up the arrangement of blocks into* n *columns.*

Equation (10) suggests an arrangement of the blocks as shown in Fig. 5. The total number of building blocks is nothing but the sum of the fourth power of the n natural numbers. Following the geometric interpretation adopted, that is, splitting up the arrangement into n columns, and counting, it may be shown easily that

$$\sum_{i=1}^{n} i^4 = \sum_{j=1}^{n} (2j-1) \sum_{i=j}^{n} i^2 \tag{11}$$

Using equations (2), (5) and (8) in equation (11), it may be shown that

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30} \tag{12}$$

Obtaining the formula in equation (12) from equation (11) is illustrated in Appendix A. The geometric interpretation developed so far enables us to formulate a general algorithm.

2.5 Finding the sum of the kth power of the first n natural numbers

The sum of the kth power of n natural numbers may be expressed as

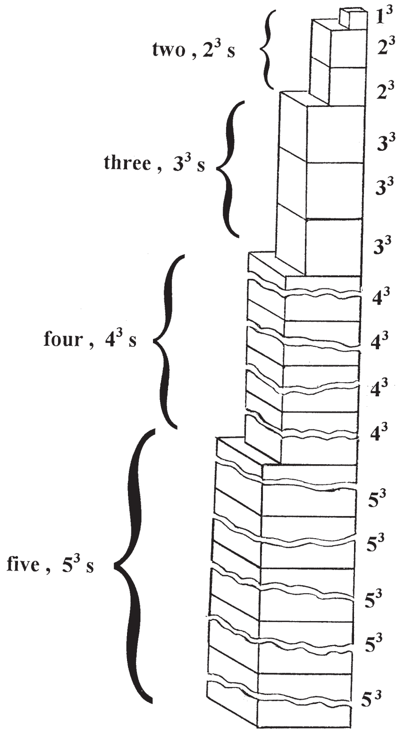

Fig. 5 *Arrangement of blocks to find the sum of 4th power of* n *natural numbers.*

$$\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k-3} i^3 \tag{13}$$

This suggests that i^{k-3} number of cubes of size i are to be arranged as a group, taking $i = 1$ to n, and stacked. The whole assembly must be split up into n columns. The first column contains one block in each layer. The second column contains three blocks in each layer. Third column contains five blocks in each layer, and so on. The sum of the kth power of n natural numbers may now be expressed conveniently as

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{n} (2j-1) \sum_{i=j}^{n} i^{k-2} \tag{14}$$

Equation (14) may be regarded as one general algorithm and can be used to obtain the formula for the sum of any power of the n natural numbers. Using the algorithm in equation (14), formulae are derived for the sum of different powers of natural numbers. Some of them are presented in Appendix B.

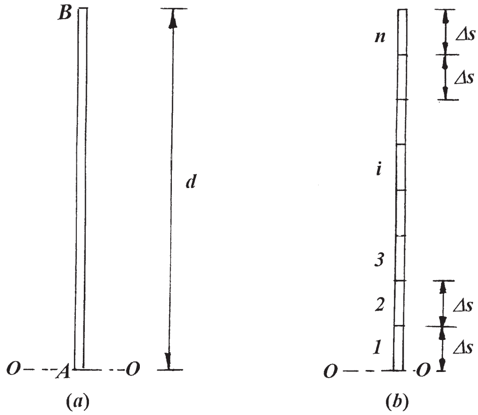

Fig. 6 *Determination of moment of inertia of a line.*

3 Discussion

In earlier attempts [1, 3–6], the formulae were derived up to $k = 3$ and they were completely mathematical approaches. The present approach is based on a simple geometric interpretation only. It is found possible to give a physical interpretation to the problem for any positive integer value of k.

The formulae for a particular value of k is found to be a polynomial of degree $(k + 1)$ without a constant term, and it depends on all the previous formulae, as it is demonstrated in Appendix A.

The formulae presented in Appendix B are derived using the general algorithm expressed in equation (14). These formulae are very convenient and may find applications in the field of engineering. Some applications may be described briefly as follows.

3.1 Determination of volumes

The building blocks when stacked as shown in Figs 2, 3 and 5, are seen to give a different shape of a pyramid in each arrangement. A given object may be imagined to be made-up of different pyramids, each representing a particular summation. Thus one may find the volume of a given irregular object using the formulae presented in Appendix B.

3.2 Determination of moments of inertia

The formula for $k = 2$ may be used to obtain the moments of inertia. As an example, the moment of inertia of a line AB of length d, about an axis $O - O$ passing through the end A (Fig. 6(a)), may be obtained by first splitting the line into n bits each of length Δs, as shown in Fig. 6(b), and then expressing the moment of inertia as

$$I = \sum_{i=1}^{n} \Delta s (i \Delta s)^2 = (\Delta s)^3 \sum_{i=1}^{n} i^2 \qquad (15)$$

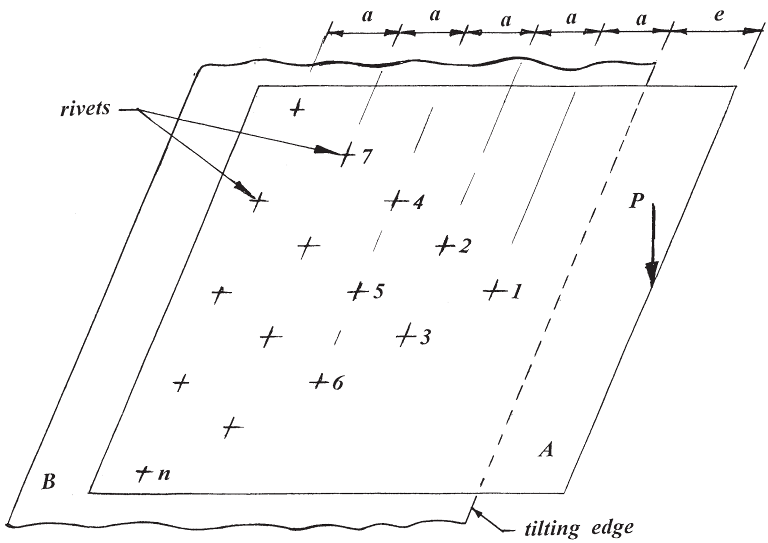

Fig. 7 *Design of a riveted joint.*

Replacing (Δs) by (d/n) and using equation (B-2), in equation (15), the moment of inertia may be expressed as

$$I = (d/n)^3 \left(\frac{1}{6}\right) n(n+1)(2n+1) \tag{16}$$

which for large values of n, becomes

$$I = d^3/n \tag{17}$$

The moments of inertia of lines are useful in the design of fillet welds [7]. This approach may also be employed to obtain the moments of inertia of areas which are useful in the design of beams [8], and also the mass moments of inertia of objects which are useful in the studies of dynamics. The formulae for the moments of inertia for regular shapes are well known. The present discussion only represents a different interpretation. However, this may be found useful for obtaining the moments of inertia of irregular shaped objects.

3.3 Design of riveted joints

Fig. 7 shows a member ***A*** in the shape of a plate, being fastened to a rigid member ***B*** by means of a group of rivets. When a load ***P*** is applied perpendicular to the member ***A*** as shown, all the rivets will be subjected to a direct tensile load. If the number of rivets is more than one, the situation becomes statically indeterminate [9, 10], and the tensile load in each rivet may be shown to be expressed as

$$F_i = Kx_i \quad i = 1, 2, 3, \ldots, n \tag{18}$$

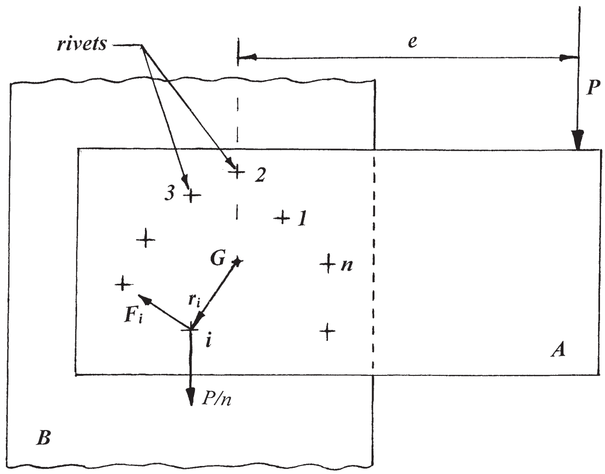

Fig. 8 *Design of an eccentrically loaded riveted joint.*

where,

F_i = tensile load in the ith rivet

x_i = distance of the ith rivet from the tilting edge

and the parameter K is given by,

$$K = \frac{Pe}{\sum_{i=1}^{n} x_i^2} \tag{19}$$

For a rivet arrangement shown in Fig. 7, the expression for K becomes,

$$K = \frac{Pe}{a^2 \sum_i i^3} \tag{20}$$

and the equation (B-3) may be found useful to obtain the value of K.

3.4 Design of riveted joints under eccentric loading

Fig. 8 shows a bracket ***A*** being fastened to a pillar ***B***, by means of a group of rivets. When a load ***P*** is applied on the bracket ***A*** as shown, the rivets are subjected to shear loads. Each rivet undergoes a *direct shear load* which may be assumed to be shared equally by all the rivets, and a *secondary shear load* which may be expressed as [11],

$$F_i = Kr_i \quad i = 1, 2, 3, \ldots, n \tag{21}$$

where,

F_i = secondary shear load in the ith rivet

r_i = distance of the ith rivet from the centroid G of the group of rivets

and the parameter K is given by,

$$K = \frac{Pe}{\sum_{i=1}^{n} r_i^2} \tag{22}$$

The direct shear load (P/n) is parallel to the line of action of $\boldsymbol{P}$ and the secondary shear load $\boldsymbol{F}_i$ is perpendicular to $\boldsymbol{r}_i$. As before, for a particular arrangement of the rivets, the expression for K may be shown to involve a summation of powers of natural numbers.

3.5 Optimization in engineering designs

Many times an engineer wishes to minimize an objective function which may represent the cost or volume or weight of a component or a system to be designed. The objective function, in general, is a function of several design variables, x_i, $i = 1, 2, 3, \ldots, n$, each representing a particular dimension or a parameter. The task of the engineer is to find that set of values of x_i, for which the objective function takes the minimum value. That is, the minimization is to be done in an n dimensional Euclidean space. The objective function $f(\boldsymbol{x})$, in general, may be so complicated that it may be very difficult to solve simultaneously the set of the necessary conditions, $\partial f/\partial x_i = 0$, $i = 1, 2, 3, \ldots, n$. Hence, the design engineer relies on numerical methods. There are several methods available and almost all of them have one common feature. That is, to minimize the function, first a *starting point* $\boldsymbol{x}^{(1)}$ is chosen and then moved by a length λ called the *step length*, along a direction $\boldsymbol{S}$ which is called the *search direction*. The optimum step length λ^* for which the objective function takes the minimum value is then determined. The new point $\boldsymbol{x}^{(2)}$ is then arrived at according to

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} + \lambda^* \boldsymbol{S} \tag{23}$$

The point $\boldsymbol{x}^{(2)}$ now becomes the starting point for the next iteration and the process is repeated till an insignificant minimization of $f(\boldsymbol{x})$ from one iteration to the next iteration is noticed.

All the methods available adopt this procedure. But, they differ only in the way the search direction is generated [12]; for instance, in *univariate* method, the search is done parallel to each coordinate direction x_i, one at a time; in the *steepest descent* method, the search is done along the negative of the gradient of the function at the starting point $\boldsymbol{x}^{(1)}$, and so on.

There is one method called the *random walk* method [13] in which, random numbers between 0 and 1 are generated in the computer, and are assigned to the various components of the search direction vector ***S***. In all these methods, in order to reduce the numerical difficulties, the search direction ***S*** is normalized by dividing each of its components by its magnitude which is the square root of the sum of the squares of various components.

Instead of assigning random numbers to various components of ***S***, one may also think of an alternative scheme, which is to consider a string of n natural numbers (not necessarily the first n) and assign them to various components randomly. In that case, when ***S*** is to be normalized, the magnitude of ***S*** involves the sum of squares of natural numbers and equation (B-2) will be found useful. If one wishes to assign to various components of ***S***, squares of natural numbers instead of only the natural numbers, the magnitude of ***S*** will involve the sum of the fourth power of natural numbers and equation (B-4) will be found useful. Thus, the formulae listed in Appendix B may find their use in such applications.

4 Concluding remarks

The significant conclusions that may be drawn on the basis of the present work may be summarized as follows.

1 A general algorithm for finding the sum of kth power of the first n natural numbers is developed.
2 The development of the algorithm is based purely on a geometric interpretation.
3 The method is developed starting with a simple problem of finding the sum of the first n natural numbers.
4 The algorithm developed appears to be very simple and convenient to derive formulae for the sum of various powers of n natural numbers.
5 Some of the formulae derived, are listed in Appendix B. These are very handy and convenient to use.
6 The derivation of a formula for the sum of kth power of n natural numbers requires the formulae for the sum the n natural numbers with powers less than k. This shows that these formulae are interdependent. As an example for illustration, the derivation of the formula for $k = 4$, is presented in Appendix A.
7 From the formulae listed in Appendix B, it is observed that $n(n + 1)$ factor is one thing common in all. Further, the formulae for odd values of k have one more such factor and the formulae for even values of k have a factor $(2n + 1)$.
8 In general, the formula for the sum of the kth powers is a polynomial in n of degree $(k + 1)$, with roots $n = 0$ and $n = -1$, owing to the fact that $n(n + 1)$ is a factor always. The polynomial is without a constant term, which follows from the fact that $n = 0$ is a root.
9 Some applications of the formulae presented in Appendix B, in the field of engineering are also discussed. However, it is hoped that these formulae may find very important applications in future.

Appendix A

The derivation of the formula for the sum of fourth power of n natural numbers is illustrated here. The algorithm in equation (14) for $k = 4$, may be written as

$$\sum_{i=1}^{n} i^4 = \sum_{j=1}^{n} (2j-1) \sum_{i=j}^{n} i^2 \tag{A-1}$$

$$\sum_{i=j}^{n} i^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{j-1} i^2 \tag{A-2}$$

Using equation (5) in equation (A-2),

$$\sum_{i=j}^{n} i^2 = \frac{\left(C - j + 3j^2 - 2j^3\right)}{6} \tag{A-3}$$

where

$$C = n(n+1)(2n+1) \tag{A-4}$$

Using equation (A-3) in equation (A-1), and expanding the terms, it may be rewritten as

$$\sum_{i=1}^{n} i^4 = \left(\frac{1}{6}\right) \sum_{j=1}^{n} \left(-4j^4 + 8j^3 - 5j^2 + (2C+1)j - C\right) \tag{A-5}$$

or

$$\sum_{i=1}^{n} i^4 = \left(\frac{4}{5}\right) \sum_{j=1}^{n} j^3 - \left(\frac{1}{2}\right) \sum_{j=1}^{n} j^2 + \left(\frac{2C+1}{10}\right) \sum_{j=1}^{n} j - \left(\frac{C}{10}\right) n \tag{A-6}$$

The dependence on the summations of previous powers, may be observed in equation (A-6). Using the formulae in equations (2), (5) and (8), in equation (A-6), the sum of the fourth power of n natural numbers may be obtained as

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)}{30} \tag{A-7}$$

Appendix B

The formulae for the sum of various powers of the first n natural numbers may be listed as

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{B-1}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{B-2}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{B-3}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30} \tag{B-4}$$

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12} \tag{B-5}$$

$$\sum_{i=1}^{n} i^6 = \frac{n(n+1)(2n+1)\left(3n^4+6n^3-3n+1\right)}{42} \tag{B-6}$$

$$\sum_{i=1}^{n} i^7 = \frac{n^2(n+1)^2\left(3n^4+6n^3-n^2-4n+2\right)}{24} \tag{B-7}$$

$$\sum_{i=1}^{n} i^8 = \frac{n(n+1)(2n+1)\left(5n^6+15n^5+5n^4-15n^3-n^2+9n-3\right)}{90} \tag{B-8}$$

Except the formulae in equations (B-1) and (B-2), all are derived using equation (14).

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