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A numerical method for solving two-point boundary value problems over infinite intervals

A.S.V. Ravi Kanth, Y.N. Reddy *

Department of Mathematics and Humanities, Regional Engineering College, Warangal 506 004, India

Abstract

In this paper we present a numerical method for the solution of a two-point boundary value problem posed on an infinite interval involving a second order linear differential equation. By reducing the infinite interval to a finite interval that is large and imposing approximate asymptotic boundary condition at the far end, the resulting boundary value problem is treated by using fourth order finite difference method. The stability of the method is analyzed and the theory is illustrated by solving test examples.

Keywords: Ordinary differential equations; Two-point boundary value problem; Infinite interval; Fourth order finite difference method

1. Introduction

During the last few years much progress has been made in the numerical treatment of boundary value problems over infinite intervals. Typically, these problems arise very frequently in fluid dynamics, aerodynamics, quantum mechanics, electronics, and other domains of science. A few notable examples are the Von Karman swirling flows [7,8], combined forced and free convection over a horizontal plate [10] and eigenvalue problem for the Schrodinger equation [6]. In many cases, the domain of the governing equations of these problems is

* Corresponding author.
E-mail address: ynreddy@recw.ernet.in (Y.N. Reddy).

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infinite or semi-infinite so that the special treatment is required for these so called infinite interval problems. The analytical solutions for these problems are not readily attainable and thus the problem is brought to the problem of finding efficient computational algorithms for obtaining numerical solution.

Before computing the solution, we plummet the infinite interval to a finite but large one, so that a finite point represent infinity. This is standard approach of solving such problems that are posed on infinite intervals. This done in different approaches. One approach is to replace the boundary conditions at infinity by the same conditions at a finite value N (the truncated boundary). In many cases this simple approach is sufficient and efficient and it provides good results only for very large value of N. Another approach, De Hoog and Weiss [3] proposed an analytical transformation of the independent variable that reduces the original problem to a boundary value problem over a finite interval. Usually, that produces a singularity of the second kind at the origin and must be solved by suitable difference methods. Finally, Hoog and Wiess [4], Lentini and Keller [6] and Markowich [8] approach to performing a preliminary asymptotic analysis to find the appropriate boundary conditions to be imposed at a truncated boundary. Since the imposed conditions are related to the asymptotic behavior of the solution, for the same value of N this approach usually yields a more accurate solution than the previous approach.

In this paper we present a numerical method for the solution of a two-point boundary value problem posed on an infinite interval involving a second order linear differential equation. By reducing the infinite interval to a finite interval that is large and imposing approximate asymptotic boundary condition at the far end, the resulting boundary value problem is treated by using fourth order finite difference method. The stability of the method is analyzed and the theory is illustrated by solving test examples.

2. Asymptotic boundary condition

Consider the linear two-point boundary value problem of the form

$$\mathrm{Ly}(x) = y''(x) + P(x)y'(x) - Q(x)y(x) = R(x) \tag{1}$$

with

$$y(a) = b \tag{2}$$

$$y(\infty) = c \quad \text{or} \quad \lim_{x\to\infty} y(x) = c \tag{3}$$

where $P(x)$, $Q(x)$ and $R(x)$ are continuous functions and $Q(x) > 0$. In order to find the appropriate asymptotic boundary conditions for Eq. (1), rewrite (1) as a first order system in the form let $y(x) = u(x)$, $y'(x) = u'(x) = v(x)$, we have

$$u'(x) = v(x) \tag{4}$$

$$v'(x) + P(x)v(x) - Q(x)u(x) = R(x) \tag{5}$$

and correspondingly (2) and (3) become

$$u(a) = b \tag{6}$$

$$\operatorname*{Lim}_{x\to\infty} u(x) = u_\infty = c \tag{7}$$

Letting $U = (u, v)^{\mathrm{t}}$ (t denotes transpose), we can write the first order system (4) and (5) in the matrix vector form

$$u'(x) = A(x)U + b(x) = F(x, u) \tag{8}$$

where

$$A(x) = \begin{bmatrix} 0 & 1 \\ Q(x) & -P(x) \end{bmatrix} \quad \text{and} \quad b(x) = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$$

A general theory for linear and nonlinear systems of the form (8) on semi-infinite interval has been developed by Lentini and Keller [6]. We assume that

(i) $\mathrm{Lim}_{x\to\infty} A(x) = A$, constant matrix,
(ii) $\mathrm{Lim}_{x\to\infty} \frac{\mathrm{d}A(x)}{\mathrm{d}x} = 0$,
(iii) $A(x)$ is piecewise continuously differentiable on (a, ∞) and
(iv) u_∞ is required to be the root of $\mathrm{Lim}_{x\to\infty} F(x, u) = 0$.

We also assume that A is in the canonical form such that $A = EJE^{-1} \neq 0$ (a zero matrix) and J has the block diagonal form $J = \mathrm{diag}(J^+, J^0, J^-)$, where J^+ contains eigenvalues of A with positive real part, J^0 the eigenvalues of A with zero real part and J^- the eigenvalues of A with a negative real part. The main idea is to find all bounded solutions and to eliminate the contribution from the unbounded solution of Eq. (8). The behavior at infinity of the solution of the system (8) is essentially given by the eigenvalues of the matrix.

$$A_\infty = \operatorname*{Lim}_{x\to\infty} A(x) = \begin{bmatrix} 0 & 1 \\ K & L \end{bmatrix}$$

where $K = \mathrm{Lim}_{x\to\infty} Q(x)$, $L = \mathrm{Lim}_{x\to\infty} - P(x)$.

Suppose the matrix A_∞ has the eigenvalues λ_1 and λ_2, then depending upon $\mathrm{Re}\,\lambda_1$, $\mathrm{Re}\,\lambda_2 \geqslant 0$, we find the linearly independent solution which decay exponentially at infinity and the linearly independent solutions which are unbounded as $x \to \infty$. Since we need only one condition at the far end we expect only one eigenvalue with a positive real part and say, this eigenvalue is λ_1. we introduce the projection matrix P_n of the form $P_n = [1, 0]$. (If the eigenvalue λ_2 is with a positive real part, we have $P_n = [0, 1]$.) Let E be a matrix of eigenvectors of A_∞,

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

By calculating E^{-1} for which $E^{-1}A_\infty E = \mathrm{diag}(\lambda_1, \lambda_2)$ we can write the asymptotic boundary condition as

$$\lim_{x\to\infty} P_n E^{-1} F(x, u) = 0 \tag{9}$$

Eq. (9) yields the condition at $x = N$ where N is chosen by taking different values of X for which the computed solution approximates the actual solution.

3. Fourth order finite difference scheme

In order to solve the finite interval problem obtained above, we describe the fourth order finite difference scheme for a more general two-point boundary value problem as follows. For the sake of brevity, we assume that the asymptotic boundary condition (9) is of the form

$$\alpha y(x_\infty) + \beta y'(x_\infty) = 0 \tag{10}$$

For $x_\infty = N$, N large but finite, where α, β are known constants such that $\alpha\beta \geqslant 0$ and $|\alpha| + |\beta| \neq 0$. This guarantees (for details see Keller [2]) the unique solution of the two-point boundary value given by $Ly(x) = y''(x) + P(x)y'(x) - Q(x)y(x) = R(x)$ with boundary conditions

$$y(a) = c \quad \text{and} \quad \alpha y(x_\infty) + \beta y'(x_\infty) = 0$$

A finite difference scheme is often a convenient choice of method for the numerical solution of two-point boundary value problems. Throughout the discussion the symbols μ, δ denote the usual central difference operators, defined by Fox [1]. As usual, we consider the mesh with grid points $a = x_0 < x_1 < \cdots, x_n = N$ with mesh size $h = x_i - x_{i-1}$. We use central difference formulae to obtain the finite difference representations of (1) at a typical mesh point x_i, $i = 0, 1, 2, \ldots, N$ as:

$$\frac{1}{h^2}\{\delta^2 - \tfrac{1}{12}\delta^4\}y_i + \frac{1}{h}P_i\{\mu\delta - \tfrac{1}{6}\mu\delta^3\}y_i - Q_i y_i = R_i + \varphi_i y_i \tag{11}$$

where

$$\varphi_i = -\frac{1}{h^2}\left[\tfrac{1}{96}\delta^6 + \cdots\right] - \frac{1}{h}P_i\left[\tfrac{1}{30}\mu\delta^5 + \cdots\right] \tag{12}$$

and φ_i is $O(h^4)$. Clearly, the left hand side of (11) is not tridiagonal, because it involves the differences $\mu\delta^3 y_i$ and $\delta^4 y_i$. To make it tridiagonal, we obtain the tridiagonal estimates of $\mu\delta^3 y_i$ and $\delta^4 y_i$ as follows.

Differentiating (1) once with respect to x, then using central difference formulae produces a tridiagonal $O(h^5)$ approximation for $\mu\delta^3 y_i$ as follows:

$$\mu\delta^3 y_i = -hP_i\delta^2 y_i - h^2(P_i' - Q_i)\mu\delta y_i + h^3 Q_i' y_i + h^3 R_i + \Psi_i y_i \tag{13}$$

where

$$\Psi_i = \tfrac{1}{4}\mu\delta^5 + \cdots + hP_i\{\tfrac{1}{12}\delta^4 + \cdots\} + h^2(P_i' - Q_i)\{\tfrac{1}{6}\mu\delta^3 + \cdots\} \tag{14}$$

Similarly, differentiating (1) twice, then using (1) and central difference formulae, produces a $O(h^6)$ approximation for $\delta^4 y_i$, as follows

$$\begin{aligned}\delta^4 y_i &= h^2(P_i^2 - P_i' + Q_i)\delta^2 y_i + h^3(2P_iP_i' - P_i'' - P_iQ_i + 2Q_i')\mu\delta y_i \\ &\quad - h^4(P_i'Q_i + P_iQ_i' - Q_i'')y_i + h^4(R_i'' - R_i'P_i - R_iP_i') + \pi_i y_i \end{aligned}\tag{15}$$

where

$$\begin{aligned}\pi_i &= \tfrac{1}{6}\delta^6 - \cdots - h^2(P_i^2 - P_i' + Q_i)\{\tfrac{1}{12}\delta^4 + \cdots\} \\ &\quad - h^3(2P_iP_i' - P_i'' - P_iQ_i + 2Q_i')\{\tfrac{1}{6}\mu\delta^3 + \cdots\}\end{aligned}\tag{16}$$

Now, we substitute (13) and (15) in (11) to obtain an equation of the form

$$E_i y_{i-1} F_i y_i + G_i y_{i+1} = H_i + K_i y_i \tag{17}$$

where

$$E_i = 1 - \frac{h}{2}P_i + \frac{h^2}{12}(P_i^2 + P_i' - Q_i) - \frac{h^3}{24}(P_i'' - P_iQ_i - 2Q_i')$$

$$F_i = 2 + \frac{h^2}{6}(5Q_i + P_i' + P_i^2) + \frac{h^4}{24}(Q_i'' + P_iQ_i' - P_i'Q_i)$$

$$G_i = 1 + \frac{h}{2}P_i + \frac{h^2}{12}(P_i^2 + P_i' - Q_i) + \frac{h^3}{24}(P_i'' - P_iQ_i - 2Q_i')$$

$$H_i = h^2 R_i + \frac{h^4}{12}(R_i'' + R_i'P_i - R_iP_i')$$

and the error term is given by

$$\begin{aligned}k_i &= \tfrac{1}{360}\delta^6 + \cdots + \tfrac{1}{120}hP_i\mu\delta^5 + \cdots + \tfrac{1}{144}h^2(P_i^2 + P_i' - Q_i) \\ &\quad \delta^4 + \cdots + \tfrac{1}{72}h^3(P_i'' - P_iQ_i - 2Q_i')\mu\delta^3 + \cdots\end{aligned}$$

Note that the error term associated with (17) is $O(h^6)$ and that the over all accuracy of the method is $O(h^4)$. If the error term in (17) is neglected, the matrix problem associated with (17) is a tridiagonal algebraic system

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, n \tag{18}$$

The boundary conditions (2) and (10) can be written in the form

$$y_0 = b \tag{19}$$

$$2\alpha h y_n + \beta y_{n+1} - \beta y_{n-1} = 0 \tag{20}$$

where the value y_{n-1} at the fictitious point $x = (n+1)h$ needs to be eliminated between Eqs. (18) and (20).

Let a difference relation of the form

$$y_{i+1} = W_i y_i + T_i, \quad i = 0, 1, 2, \ldots, n-1 \tag{21}$$

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined from (21). By substituting (21) in (18), we get

$$y_i = \frac{E_i}{F_i - G_i W_i} y_{i-1} + \frac{H_i - G_i T_i}{G_i W_i - F_i} \tag{22}$$

By comparing the Eqs. (22) with (21), we get the recurrence relations

$$W_{i-1} = \frac{E_i}{F_i - G_i W_i} \tag{23}$$

$$T_{i-1} = \frac{H_i - G_i T_i}{G_i W_i - F_i} \tag{24}$$

To solve these recurrence relations for W_i and T_i $(i = n-2, n-3, \ldots, 0)$ we need to know the values of W_i and T_i at $i = n-1$. To do this, we have from Eq. (10)

$$2h\alpha y_n + \beta y_{n+1} - \beta y_{n-1} = 0 \tag{25}$$

From Eq. (18) at $x = x_n$, we have

$$E_n y_{n-1} - F_n y_n + G_n y_{n+1} = H_n \tag{26}$$

By using (25) and (26), we get

$$W_{n-1} = \frac{E_n + G_n}{F_n + \frac{2h\alpha}{\beta} G_n} \tag{27}$$

$$T_{n-1} = -\frac{H_n}{\frac{2h\alpha}{\beta} G_n + F_n} \tag{28}$$

starting with these initial values of W_{n-1} and T_{n-1}, the values of W_i and T_i $(i = n-2, n-3, \ldots, 0)$ are obtained by using (23) and (24). Using these values of W_i's and T_i's are knowing the values of y_0 (initial condition) solutions y can be obtained by using (21).

4. Stability

We will now show that the method is computationally stable. By stability, we mean the effect of an error made in one stage of calculation is not propagated into errors at latter stages of computation. That is, local errors are not magnified by further computation.

Let us now examine the recurrence relation given by (23) suppose a small error e_i has been introduced in the calculation of W_i, then we have

$$\widetilde{W}_i = W_i + e_i \tag{29}$$

and we are actually solving

$$\widetilde{W}_{i-1} = \frac{E_i}{F_i - G_i \widetilde{W}_i} \tag{30}$$

From (27) and (30), we have

$$\begin{aligned} e_{i-1} &= \frac{E_i}{F_i - G_i(W_i + e_i)} - \frac{E_i}{F_i - G_i W_i} = \frac{E_i G_i e_i}{[F_i - G_i(W_i + e_i)][F_i - E_i W_i]} \\ e_{i-1} &= \left[W_{i-1}^2 \frac{G_i}{E_i} \right] e_i \end{aligned} \tag{31}$$

Under the assumption that initially the error is small. Let us assume that $G_i > 0$ and $E_i > 0$ for all i. From the definitions of E_i, F_i and G_i, we also have $F_i > 0$ and $F_i > E_i + G_i$ for all i.

We now make use of the assumptions on E_i and G_i to show $0 < W_i < 1$ for $i = n-1, n-2, \ldots, 0$. From (27)

$$W_{n-1} = \frac{E_n + G_n}{F_n + \frac{2h\alpha}{\beta} G_n}$$

Under the above mentioned conditions, it is then easy to verify that $0 < W_{n-1} < 1$. Also,

$$\begin{aligned} W_{n-2} &= \frac{E_{n-1}}{F_{n-1} - G_{n-1} W_{n-1}} \\ &< \frac{E_{n-1}}{F_{n-1} - G_{n-1}} \qquad \text{since } W_{n-1} < 1 \\ &< 1 \end{aligned}$$

and this $0 < W_{n-2} < 1$. Successively it follows that $0 < W_i < 1$, $i = n-3, \ldots, 0$. Then it follows that Eq. (28) that $e_{i-1} < e_i$, provided $G_i \geqslant E_i$ and thus the recurrence relation (27) is stable. Similar arguments will show that the recurrence relation (28) is also stable.

5. Computational results

In this section, we have implemented the present method on two examples. The numerical results are compared with exact results. The applicability of results show that the present method approximate the solution very well.

Example 1. Consider the boundary value problem

$$\mathrm{LY}(x) = -y'' - 2y' + 2y = \mathrm{e}^{-2x} \tag{32}$$

with

$$y(0) = 1.0 \tag{33}$$

$$y(\infty) = 0.0 \tag{34}$$

This problem has earlier been considered by Robertson [9], Kadalbajoo and Raman [5], and its exact solution is given by

$$y(x) = \tfrac{1}{2}\mathrm{e}^{-(1+\sqrt{3})x} + \tfrac{1}{2}\mathrm{e}^{-2x}$$

The asymptotic boundary condition for this example can be written as

$$\frac{1}{\sqrt{3}}y(x_\infty) + \frac{\sqrt{3}(-1+\sqrt{3})}{6}y'(x_\infty) = 0 \tag{35}$$

The boundary value problem given by (32), (33) and (35) has been solved using fourth order finite difference method and the numerical results are presented in Tables 1–3.

Table 1
Computational results for Example 1 ($N = 8$)

X	Exact	FFDM		
		$h = 1/16$	$h = 1/32$	$h = 1/64$
0.0	0.100000E+01	0.100000E+01	0.100000E+01	0.100000E+01
1.0	0.100210E+00	0.100211E+00	0.100215E+00	0.100220E+00
2.0	0.112759E−01	0.112760E−01	0.112766E−01	0.112776E−01
3.0	0.137723E−02	0.137725E−02	0.137733E−02	0.137748E−02
4.0	0.176704E−03	0.176706E−03	0.176717E−03	0.176737E−03
5.0	0.232839E−04	0.232830E−04	0.232844E−04	0.232882E−04
6.0	0.311011E−05	0.310742E−05	0.310760E−05	0.311051E−05
7.0	0.418238E−06	0.412565E−06	0.412558E−06	0.417943E−06
8.0	0.564286E−07	0.446244E−07	0.445621E−07	0.556647E−07

Table 2
Computational results for Example 1 ($N = 9$)

X	Exact	FFDM		
		$h = 1/16$	$h = 1/32$	$h = 1/64$
0.0	0.100000E+01	0.100000E+01	0.100000E+01	0.100000E+01
1.0	0.100210E+00	0.100211E+00	0.100215E+00	0.100220E+00
2.0	0.112759E−01	0.112760E−01	0.112766E−01	0.112776E−01
3.0	0.137723E−02	0.137725E−02	0.137733E−02	0.137748E−02
4.0	0.176704E−03	0.176706E−03	0.176717E−03	0.176737E−03
5.0	0.232839E−04	0.232842E−04	0.232857E−04	0.232882E−04
6.0	0.311011E−05	0.310998E−05	0.311018E−05	0.311051E−05
7.0	0.418238E−06	0.417874E−06	0.417898E−06	0.417943E−06
8.0	0.564286E−07	0.556608E−07	0.556598E−07	0.556647E−07
9.0	0.762547E−08	0.602791E−08	0.601950E−08	0.601786E−08

Table 3
Computational results for Example 1 ($N = 10$)

X	Exact	FFDM		
		$h = 1/16$	$h = 1/32$	$h = 1/64$
0.0	0.100000E+01	0.100000E+01	0.100000E+01	0.100000E+01
1.0	0.100210E+00	0.100211E+00	0.100215E+00	0.100220E+00
2.0	0.112759E−01	0.112760E−01	0.112766E−01	0.112776E−01
3.0	0.137723E−02	0.137725E−02	0.137733E−02	0.137748E−02
4.0	0.176704E−03	0.176706E−03	0.176717E−03	0.176737E−03
5.0	0.232839E−04	0.232843E−04	0.232857E−04	0.232883E−04
6.0	0.311011E−05	0.311015E−05	0.311034E−05	0.311068E−05
7.0	0.418238E−06	0.418220E−06	0.418246E−06	0.418291E−06
8.0	0.564286E−07	0.563793E−07	0.563826E−07	0.563886E−07
9.0	0.762547E−08	0.752156E−08	0.752142E−08	0.752208E−08
10.0	0.103126E−08	0.815049E−09	0.813913E−09	0.813692E−09

Example 2. A second example

$$\mathrm{LY}(x) = -y'' + \left(1 + \frac{1}{x}\right)y = \frac{1}{x^2} \tag{36}$$

with

$$y(1) = 1.0 \tag{37}$$

$$y(\infty) = 0.0 \tag{38}$$

This problem has earlier been considered by Fox [1] and later by Robertson [9]. The asymptotic boundary condition for this example can be written as

$$\tfrac{1}{2}y(x_\infty) + \tfrac{1}{2}y'(x_\infty) = 0 \tag{39}$$

Table 4
Computational results for Example 2 ($N = 15$)

X	FFDM		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1.0	0.000000E+00	0.000000E+00	0.000000E+00
2.0	0.109816E+00	0.109817E+00	0.109817E+00
3.0	0.852347E−01	0.852352E−01	0.852347E−01
5.0	0.390871E−01	0.390874E−01	0.390872E−01
7.0	0.200437E−01	0.200437E−01	0.200439E−01
9.0	0.119536E−01	0.119536E−01	0.119537E−01
11.0	0.792264E−02	0.792261E−02	0.792265E−02
13.0	0.545571E−02	0.545546E−02	0.545541E−02
15.0	0.245363E−02	0.245168E−02	0.245120E−02

Table 5
Computational results for Example 2 ($N = 20$)

X	FFDM		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1.0	0.000000E+00	0.000000E+00	0.000000E+00
3.0	0.852347E−01	0.852352E−01	0.852347E−01
5.0	0.390872E−01	0.390874E−01	0.390872E−01
7.0	0.200441E−01	0.200442E−01	0.200443E−01
9.0	0.119571E−01	0.119571E−01	0.119572E−01
11.0	0.795100E−02	0.795099E−02	0.795104E−02
13.0	0.568391E−02	0.568389E−02	0.568391E−02
16.0	0.374146E−02	0.374144E−02	0.374145E−02
20.0	0.134216E−02	0.134103E−02	0.134075E−02

Table 6
Computational results for Example 2 ($N = 25$)

X	FFDM		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1.0	0.000000E+00	0.000000E+00	0.000000E+00
3.0	0.852347E−01	0.852352E−01	0.852347E−01
5.0	0.390872E−01	0.390874E−01	0.390872E−01
7.0	0.200441E−01	0.200442E−01	0.200443E−01
9.0	0.119571E−01	0.119571E−01	0.119572E−01
11.0	0.795110E−02	0.795109E−02	0.795114E−02
13.0	0.568469E−02	0.568467E−02	0.568469E−02
16.0	0.375894E−02	0.375893E−02	0.375893E−02
17.0	0.333260E−02	0.333259E−02	0.333259E−02
25.0	0.845859E−03	0.845133E−03	0.844953E−03

The boundary value problem given by (36), (37) and (39) has been solved by the method described earlier and the computational results are presented in Tables 4–6.

6. Discussion and conclusion

A fourth order finite difference method is used for the approximate solution of the two-point boundary value problems over infinite intervals. The method has been analyzed for stability. It is a practical method and can easily be implemented on a computer to solve such problems. Test examples, tackled earlier by Fox [1] and Robertson [9] has been solved to demonstrate the efficiency of the proposed method. For these examples, the asymptotic boundary condition at $x_\infty = N$ was first derived and the value $x_\infty = N$ was the varied until no significant change in the solution was noticed. The computational results for the Example 1 are presented in Tables 1–3 at different mesh sizes and different values of $x_\infty = N$. The values of N taken for computation are 8, 9 and 10 for different values of h. The computational results are presented in tables. From these tables it can be observed that the approximate results are very close to the exact solutions. It can be seen that the computed solutions for $N = 10$ show six places of decimal accuracy and thus $N = 10$ can be taken to represent the point at infinity for the problem. The computational results for Example 2 for different values of h are given in Tables 4–6. It is evident from this tables that the solutions decay relatively slowly which agrees with the observation made by Fox [1]. The computed solutions also compare very well with that obtained by Robertson [9] and at $x_\infty = N = 9$.

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