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Method of reduction of order for solving singularly perturbed two-point boundary value problems

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Abstract

In this paper, a method of reduction of order is proposed for solving singularly perturbed two-point boundary value problems with a boundary layer at one end point. It is distinguished by the following fact: the original singularly perturbed boundary value problem is replaced by a pair of initial value problems. Classical fourth order Runge–Kutta method is used to solve these initial value problems. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory.

Keywords: Ordinary differential equations; Singular perturbations; Boundary value problems; Initial value methods; Boundary layer

1. Introduction

Singular perturbation problems occur very frequently in fluid mechanics and other branches of applied mathematics. There are a wide variety of methods for the solution of the singular perturbation problems. Many of these methods consists of: (a) dividing the problem into an inner region (boundary layer) problem and an outer region problem; (b) expressing the inner and outer

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solutions as asymptotic expansions; (c) equating various terms in the inner and outer expressions to determine the constants in these expressions; and (d) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution. Typically, the inner region problems are obtained from the original problem by rescaling the independent variable. These methods and their variations have been used successfully on a variety of linear and nonlinear singular perturbation problems. However, there can be difficulties in applying these methods, such as the matching of the coefficients of the inner and outer expansions. Success may depend on finding the proper scaling or the proper transformation to express the dependent and independent variables. For a detailed discussion, one may refer to O'Malley [7], Kevorkian and Cole [5], Bender and Orszag [1], Nayfeh [6] and research papers by Kadalbajoo and Reddy [3,4], Hu et al. [2] and Reinhardt [8].

In view of the wealth of literature on singular perturbation problems, we raise the question of whether there are other ways to attack singular perturbation problems, namely, ways that are very easy to use and ready for computer implementation. In response to this need for a fresh approach to singular perturbation problems, we propose and illustrate in this paper, the method of reduction of order for solving singularly perturbed two-point boundary value problems with a boundary layer at one end point of the underlying interval. It is distinguished by the following fact: the original singularly perturbed boundary value problem is replaced by a pair of initial value problems. This replacement is significant from the computational point of view. Classical fourth order Runge–Kutta method is used to solve these initial value problems. Numerical experience with several examples is described.

2. Method of reduction of order

For convenience, we call our method as the method of reduction of order. To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [p,q] \tag{1}$$

with

$$y(p) = \alpha \quad \text{and} \quad y(q) = \beta, \tag{2}$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$ and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[p,q]$. Furthermore, we assume that $a(x) \geqslant M > 0$ throughout the interval $[p,q]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = p$.

The method of reduction of order consists the following steps:

Step 1. Obtain the reduced problem by setting $\varepsilon = 0$ in Eq. (1) and solve it for the solution with the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of (1) and (2), i.e.;

$$a(x)y_0'(x) + b(x)y_0(x) = f(x) \tag{3}$$

with

$$y_0(q) = \beta. \tag{4}$$

Step 2. Setup the two first-order equations equivalent to Eq. (1) as follows:

$$z'(x) + [b(x) - a'(x)]y(x) = f(x) \tag{5}$$

and

$$\varepsilon y'(x) + a(x)y(x) = z(x). \tag{6}$$

Step 3. Setup the initial conditions as follows:

Using $y_0(x)$, the solution of the reduced problem, in Eq. (6) we have

$$z(q) = \varepsilon y_0'(q) + a(q)y_0(q). \tag{7}$$

This will be the initial condition for Eq. (5) and $y(p) = \alpha$ will be the initial condition for Eq. (6).

Step 4. Get the pair of initial value problems as follows:

Replacing $y(x)$ by $y_0(x)$ in (5), we get

$$z'(x) + [b(x) - a'(x)]y_0(x) = f(x). \tag{8}$$

Now the differential equation (8) with condition (7) constitute an initial value problem and the differential equation (6) with the condition $y(p) = \alpha$ constitute another initial value problem.

Therefore the pair of initial value problems corresponding to (1) and (2) are given by

$$\text{(i) } z'(x) + [b(x) - a'(x)]y_0(x) = f(x) \quad \text{with } z(q) = \varepsilon y_0'(q) + a(q)y_0(q), \tag{9}$$

$$\text{(ii) } \varepsilon y'(x) + a(x)y(x) = z(x) \quad \text{with } y(p) = \alpha. \tag{10}$$

Thus in a manner of speaking, we have replaced the original boundary value problem (1) and (2) by a pair of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. We solve these initial value problems (9) and (10) to obtain the solution over the interval $[p, q]$. There now exist a number of efficient methods for the solution of initial value problems. In order to solve the initial value problems in our numerical experimentation, we make use of classical fourth order Runge–Kutta method. In fact, any standard analytical or numerical method can be used.

3. Numerical examples

To demonstrate the applicability of the method of reduction of order, we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in the literature and because approximate solutions are available for comparison.

Example 1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [1, p. 480; problem (9.17)] with $\alpha = 0$;

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad x \in [0, 1] \tag{11}$$

with

$$y(0) = 1 \quad \text{and} \quad y(1) = 1. \tag{12}$$

The exact solution is given by

$$y(x) = [(\mathrm{e}^{m_2} - 1)\mathrm{e}^{m_1 x} + (1 - \mathrm{e}^{m_1})\mathrm{e}^{m_2 x}]/[\mathrm{e}^{m_2} - \mathrm{e}^{m_1}], \tag{13}$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon)$.

From step 1, the reduced problem is $y_0'(x) - y_0(x) = 0$; $y_0(1) = 1$. The solution of this problem is $y_0(x) = \mathrm{e}^{x-1}$.

From step 2, the two first-order equations equivalent to Eq. (11) are

$$z'(x) - y(x) = 0 \tag{14}$$

and

$$\varepsilon y'(x) + y(x) = z(x). \tag{15}$$

From step 3, we have

$$z(1) = \varepsilon y_0'(1) + a(1)y_0(1), \quad \text{i.e.,} \quad z(1) = \varepsilon + 1. \tag{16}$$

Replacing $y(x)$ by $y_0(x)$ in (14) we get

$$z'(x) - y_0(x) = 0. \tag{17}$$

Hence the pair of initial value problems corresponding to (11) and (12) are

(i) $z'(x) = \mathrm{e}^{x-1}$ with $z(1) = \varepsilon + 1$,
(ii) $\varepsilon y'(x) + y(x) = z(x)$ with $y(0) = 1$.

The numerical results are given in Tables 1 and 2 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 2. Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt [8, Example 2].

Table 1
Numerical results of Example 1 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.02	0.3758743	0.3756784
0.04	0.3834493	0.3832599
0.06	0.3911773	0.3909945
0.08	0.3990614	0.3988851
0.10	0.4071048	0.4069350
0.20	0.4498258	0.4496879
0.40	0.5492191	0.5491404
0.60	0.6706182	0.6705877
0.80	0.8188952	0.8188942
1.00	1.0000010	1.0000000

Table 2
Numerical results of Example 1 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.02	0.3753700	0.3753479
0.04	0.3829486	0.3829296
0.06	0.3906853	0.3906645
0.08	0.3985749	0.3985557
0.10	0.4066239	0.4066062
0.20	0.4493805	0.4493649
0.40	0.5488544	0.5488445
0.60	0.6703461	0.6703468
0.80	0.8187459	0.8187471
1.00	0.9999999	1.0000000

$$\varepsilon y''(x) + y'(x) = 1 + 2x; \quad x \in [0, 1] \tag{18}$$

with

$$y(0) = 0 \quad \text{and} \quad y(1) = 1. \tag{19}$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)(1 - \mathrm{e}^{-x/\varepsilon})/(1 - \mathrm{e}^{-1/\varepsilon}). \tag{20}$$

From step 1, the reduced problem is $y_0'(x) = 1 + 2x$; $y_0(1) = 1$. The solution of this problem is

$$y_0(x) = x^2 + x - 1. \tag{21}$$

From step 2, the two first-order equations equivalent to Eq. (18) are

$$z'(x) = 1 + 2x \tag{22}$$

and

$$\varepsilon y'(x) + y(x) = z(x). \tag{23}$$

Now from step 3, we have $z(1) = \varepsilon y_0'(1) + a(1)y_0(1)$, i.e.; $z(1) = 3\varepsilon + 1$. Hence the pair of initial value problems related to (18) and (19) are

(i) $z'(x) = 1 + 2x$ with $z(1) = 3\varepsilon + 1$,

(ii) $\varepsilon y'(x) + y(x) = z(x)$ with $y(0) = 0$.

The numerical results are given in Tables 3 and 4 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 3. Finally we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [5, p. 33; Eqs. (2.3.26) and (2.3.27)] with $\alpha = -1/2$;

Table 3
Numerical results of Example 2 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.02	−0.9778339	−0.9776400
0.04	−0.9566700	−0.9564800
0.06	−0.9347062	−0.9345200
0.08	−0.9119419	−0.9117600
0.10	−0.8883780	−0.8882000
0.20	−0.7585579	−0.7584000
0.40	−0.4389180	−0.4388000
0.60	−0.0392781	−0.0391999
0.80	0.4403619	0.4404000
1.00	1.0000020	1.0000000

Table 4
Numerical results of Example 2 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.02	−0.9794231	−0.9794040
0.04	−0.9582251	−0.9582080
0.06	−0.9362285	−0.9362120
0.08	−0.9134333	−0.9134160
0.10	−0.8898363	−0.8898200
0.20	−0.7598536	−0.7598400
0.40	−0.4398913	−0.4398800
0.60	−0.0399276	−0.0399201
0.80	0.4400363	0.4400399
1.00	0.9999999	1.0000000

$$\varepsilon y''(x) + \left(1 - \tfrac{1}{2}x\right)y'(x) - \tfrac{1}{2}y(x) = 0; \quad x \in [0, 1] \tag{24}$$

with

$$y(0) = 0 \quad \text{and} \quad y(1) = 1. \tag{25}$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [6, p. 148; Eq. (4.2.32)]) as our 'exact' solution;

$$y(x) = 1/(2 - x) - (1/2)e^{-(x - x^2/4)/\varepsilon}. \tag{26}$$

From step 1, the reduced problem is

$$(1 - \tfrac{1}{2}x)y_0'(x) - \tfrac{1}{2}y_0(x) = 0; \quad y_0(1) = 1.$$

The solution of this problem is $y_0(x) = 1/(2 - x)$.

From step 2, the two first-order equations equivalent to Eq. (24) are

$$z'(x) = 0 \tag{27}$$

and

$$\varepsilon y'(x) + (1 - \tfrac{1}{2}x)y(x) = z(x). \tag{28}$$

Now from step 3, we have $z(1) = \varepsilon y_0'(1) + a(1)y_0(1)$, i.e.; $z(1) = \varepsilon + 1/2$.

Hence the pair of initial value problems related to (24) and (25) are

$$\text{(i) } z'(x) = 0 \quad \text{with } z(1) = \varepsilon + 1/2, \tag{29}$$

$$\text{(ii) } \varepsilon y'(x) + (1 - \tfrac{1}{2}x)y(x) = z(x) \quad \text{with } y(0) = 0. \tag{30}$$

The numerical results are given in Tables 5 and 6 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 5
Numerical results of Example 3 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.02	0.5058028	0.5050505
0.04	0.5109587	0.5102041
0.06	0.5162208	0.5154639
0.08	0.5215923	0.5208333
0.10	0.5270767	0.5263158
0.20	0.5563236	0.5555556
0.40	0.6257619	0.6250000
0.60	0.7149862	0.7142857
0.80	0.8338451	0.8333333
1.00	1.0000080	1.0000000

Table 6
Numerical results of Example 3 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.02	0.5051258	0.5050505
0.04	0.5102796	0.5102041
0.06	0.5155396	0.5154639
0.08	0.5209092	0.5208333
0.10	0.5263919	0.5263158
0.20	0.5556324	0.5555555
0.40	0.6250762	0.6250000
0.60	0.7143557	0.7142857
0.80	0.8333843	0.8333333
1.00	1.0000000	1.0000000

4. Non-linear problems

We now extend this method of reduction of order for a class of non-linear singularly perturbed two-point boundary value problems with left-end boundary layer of the underlying interval. For this we consider a class of non-linear singularly perturbed two-point boundary value problems of the form:

$$\varepsilon y''(x) + [a(y(x))]' + b(x, y(x)) = f(x), \quad x \in [p, q] \tag{31}$$

with

$$y(p) = \alpha \quad \text{and} \quad y(q) = \beta, \tag{32}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(y(x))$, $b(x, y)$ and $f(x)$ are sufficiently continuously differentiable functions in $[p, q]$. Furthermore, we assume that (31) and (32) has a solution which displays a boundary layer of width $O(\varepsilon)$ at $x = p$ for small values of ε.

Step 1. Obtain the reduced problem by setting $\varepsilon = 0$ in Eq. (31) and solve it for the solution with the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of (31) and (32), i.e.;

$$[a(y_0(x))]' + b(x, y_0(x)) = f(x) \tag{33}$$

with

$$y_0(q) = \beta. \tag{34}$$

Step 2. Setup the two first-order equations equivalent to Eq. (31) as follows:

$$z'(x) + b(x, y(x)) = f(x) \tag{35}$$

and

$$\varepsilon y'(x) + a(y(x)) = z(x). \tag{36}$$

Step 3. Setup the initial conditions as follows:
Using $y_0(x)$, the solution of the reduced problem, in Eq. (36) we have

$$z(q) = \varepsilon y_0'(q) + a(y_0(q)). \tag{37}$$

This will be the initial condition for Eq. (35) and $y(p) = \alpha$ will be the initial condition for Eq. (36).

Step 4. Get the pair of initial value problems as follows: Replacing $y(x)$ by $y_0(x)$ in (35), we get

$$z'(x) + b(x, y_0(x)) = f(x). \tag{38}$$

Now the differential equation (38) with condition (37) constitute an initial value problem and the differential equation (36) with the condition $y(p) = \alpha$ constitute another initial value problem.

Therefore the pair of initial value problems corresponding to Eqs. (31) and (32) are given by

$$\text{(i) } z'(x) + b(x, y_0(x)) = f(x) \quad \text{with } z(q) = \varepsilon y_0'(q) + a(y_0(q)), \tag{39}$$

$$\text{(ii) } \varepsilon y'(x) + a(y(x)) = z(x) \quad \text{with } y(p) = \alpha. \tag{40}$$

Thus in a manner of speaking, we have replaced the original boundary value problem (31) and (32) by a pair of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. We solve these initial value problems (39) and (40) to obtain the solution over the interval $[p, q]$.

5. Non-linear examples

Again to demonstrate the applicability of the method of reduction of order, we have applied it to three non-linear singular perturbation problems with left-end boundary layer.

Example 4. Consider the following singular perturbation problem from Bender and Orszag [1, p. 463; Eqs. (9.7.1)];

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \quad x \in [0, 1] \tag{41}$$

with

$$y(0) = 0 \quad \text{and} \quad y(1) = 0. \tag{42}$$

We have chosen to use Bender and Orszag's uniformly valid approximation [1, p. 463; Eq. (9.7.6)] for comparison,

$$y(x) = \log_e(2/(1+x)) - (\log_e 2)e^{-2x/\varepsilon}. \tag{43}$$

For this example, we have boundary layer of thickness $O(\varepsilon)$ at $x = 0$ (cf. [1]).

From step 1, the reduced problem is

$$2y_0'(x) + e^{y_0(x)} = 0; \quad y_0(1) = 0. \tag{44}$$

The solution of this problem is

$$y_0(x) = \log_e(2/(x+1)). \tag{45}$$

From step 2, the two first-order equations equivalent to Eq. (41) are

$$z'(x) + e^{y(x)} = 0 \tag{46}$$

and

$$\varepsilon y'(x) + 2y(x) = z(x). \tag{47}$$

From step 3, we have

$$z(1) = \varepsilon y_0'(1) + a(y_0(1)), \quad \text{i.e.,} \ z(1) = -\varepsilon/2. \tag{48}$$

Replacing $y(x)$ by $y_0(x)$ in (46) we get $z'(x) + e^{y_0(x)} = 0$.

Hence the pair of initial value problems related to (41) and (42) are

$$\text{(i)}\ z'(x) = -2/(x+1) \quad \text{with } z(1) = -\varepsilon/2, \tag{49}$$

$$\text{(ii)}\ \varepsilon y'(x) + 2y(x) = z(x) \quad \text{with } y(0) = 0. \tag{50}$$

The numerical results are given in Tables 7 and 8 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 7
Numerical results of Example 4 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.02	0.6735372	0.6733446
0.04	0.6541116	0.6539265
0.06	0.6350562	0.6348783
0.08	0.6163570	0.6161861
0.10	0.5980009	0.5978370
0.20	0.5109591	0.5108256
0.40	0.3567606	0.3566749
0.60	0.2231937	0.2231435
0.80	0.1053829	0.1053605
1.00	0.0000000	0.0000000

Table 8
Numerical results of Example 4 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.02	0.6733618	0.6733446
0.04	0.6539454	0.6539265
0.06	0.6348970	0.6348783
0.08	0.6162032	0.6161861
0.10	0.5978512	0.5978370
0.20	0.5108411	0.5108256
0.40	0.3566844	0.3566750
0.60	0.2231481	0.2231436
0.80	0.1053628	0.1053605
1.00	0.0000000	0.0000000

Example 5. Now consider the following singular perturbation problem from Kevorkian and Cole [5, p. 56; Eqs. (2.5.1)];

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0; \quad x \in [0, 1] \tag{51}$$

with $y(0) = -1$ and $y(1) = 3.9995$.

We have chosen to use Kevorkian and Cole's uniformly valid approximation [5, pp. 57 and 58; Eqs. (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh(c_1(x/\varepsilon + c_2)/2), \tag{52}$$

where $c_1 = 2.9995$ and $c_2 = (1/c_1)\log_e[(c_1 - 1)/(c_1 + 1)]$.

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. [2]).

First we rewrite the given equation as

$$\varepsilon y''(x) + [y(x)^2/2]' - y(x) = 0.$$

From step 1, the reduced problem is $[y_0(x)^2/2]' - y_0(x) = 0$; $y_0(1) = 3.9995$; whose solution is

$$y_0(x) = x + 2.9995. \tag{53}$$

From step 2, the two first-order equations equivalent to Eq. (50) are

$$z'(x) - y(x) = 0 \tag{54}$$

and

$$\varepsilon y'(x) + \frac{y(x)^2}{2} = z(x). \tag{55}$$

Now from step 3, we have $z(1) = \varepsilon y_0'(1) + a(y_0(1))$, i.e.; $z(1) = \varepsilon + (3.9995)^2/2$.

Replacing $y(x)$ by $y_0(x)$ in (54) we have $z'(x) - y_0(x) = 0$.
Hence the pair of initial value problems related to (50) and (51) are

$$\text{(i)}\ z'(x) = x + 2.9995 \quad \text{with } z(1) = \varepsilon + (3.9995)^2/2, \tag{56}$$

$$\text{(ii)}\ \varepsilon y'(x) + \frac{y(x)^2}{2} = z(x) \quad \text{with } y(0) = -1. \tag{57}$$

The numerical results are given in Tables 9 and 10 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 6. Finally we consider the following singular perturbation problem from O'Malley [7, p. 9; Eqs. (1.10) case 2];

$$\varepsilon y''(x) - y(x)y'(x) = 0; \quad x \in [-1, 1] \tag{58}$$

Table 9
Numerical results of Example 5 with $\varepsilon = 10^{-3}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	−1.0000000	−1.0000000
0.02	3.0194650	3.0195000
0.04	3.0394760	3.0395000
0.06	3.0594940	3.0595000
0.08	3.0794360	3.0795000
0.10	3.0995030	3.0995000
0.20	3.1995000	3.1995000
0.40	3.3994760	3.3995000
0.60	3.5994310	3.5995000
0.80	3.7994730	3.7995000
1.00	3.9994990	3.9995000

Table 10
Numerical results of Example 5 with $\varepsilon = 10^{-4}$, $h = 10^{-5}$

x	$y(x)$	Exact solution
0.00	−1.0000000	−1.0000000
0.02	3.0209980	3.0195000
0.04	3.0398800	3.0395000
0.06	3.0586450	3.0595000
0.08	3.0772960	3.0795000
0.10	3.0958400	3.0995000
0.20	3.2018430	3.1995000
0.40	3.4039620	3.3995000
0.60	3.5978090	3.5995000
0.80	3.8039610	3.7995000
1.00	3.9995010	3.9995000

with

$$y(-1) = 0 \quad \text{and} \quad y(1) = -1. \tag{59}$$

We have chosen to use O'Malley's approximate solution [7, pp. 9 and 10; Eqs. (1.13) and (1.14)] for comparison,

$$y(x) = -(1 - \mathrm{e}^{-(x+1)/\varepsilon})/(1 + \mathrm{e}^{-(x+1)/\varepsilon}).$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at the left end of the interval. That is at $x = -1$ (cf. [7]).

First we rewrite the given equation as

$$\varepsilon y''(x) - [y(x)^2/2]' = 0. \tag{60}$$

From step 1, the reduced problem is $[y_0(x)^2/2]' = 0$; $y_0(1) = -1$; whose solution is $y_0(x) = -1$.

From step 2, the two first-order equations equivalent to Eq. (60) are

$$z'(x) = 0$$

and

$$\varepsilon y'(x) - \frac{y(x)^2}{2} = z(x).$$

Now from step 3, we have $z(1) = \varepsilon y_0'(1) + a(y_0(1))$, i.e.; $z(1) = -1/2$.

Hence the pair of initial value problems related to (58) and (59) are

$$\text{(i) } z'(x) = 0 \quad \text{with } z(1) = -1/2, \tag{61a}$$

$$\text{(ii) } \varepsilon y'(x) - \frac{y(x)^2}{2} = z(x) \quad \text{with } y(-1) = 0. \tag{61b}$$

The numerical results are given in Tables 11 and 12 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 11
Numerical results of Example 6 with $\varepsilon = 10^{-3}$, $h = 0.002$

x	$y(x)$	Exact solution
−1.00	0.0000000	0.0000000
−0.96	−1.0000000	−1.0000000
−0.92	−1.0000000	−1.0000000
−0.88	−1.0000000	−1.0000000
−0.84	−1.0000000	−1.0000000
−0.80	−1.0000000	−1.0000000
−0.60	−1.0000000	−1.0000000
−0.20	−1.0000000	−1.0000000
0.20	−1.0000000	−1.0000000
0.60	−1.0000000	−1.0000000
1.00	−1.0000000	−1.0000000

Table 12
Numerical results of Example 6 with $\varepsilon = 10^{-4}$, $h = 0.0002$

x	$y(x)$	Exact solution
−1.00	0.0000000	0.0000000
−0.96	−1.0000000	−1.0000000
−0.92	−1.0000000	−1.0000000
−0.88	−1.0000000	−1.0000000
−0.84	−1.0000000	−1.0000000
−0.80	−1.0000000	−1.0000000
−0.60	−1.0000000	−1.0000000
−0.20	−1.0000000	−1.0000000
0.20	−1.0000000	−1.0000000
0.60	−1.0000000	−1.0000000
1.00	−1.0000000	−1.0000000

6. Right-end boundary layer problems

Finally, we extend this method of reduction of order for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [p, q] \tag{62}$$

with

$$y(p) = \alpha \quad \text{and} \quad y(q) = \beta, \tag{63}$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$ and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[p, q]$. Furthermore, we assume that $a(x) \leqslant M < 0$ throughout the interval $[p, q]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = q$.

Step 1. Obtain the reduced problem by setting $\varepsilon = 0$ in Eq. (62) and solve it for the solution with the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of (62) and (63), i.e.;

$$a(x)y_0'(x) + b(x)y_0(x) = f(x) \tag{64}$$

with

$$y_0(p) = \alpha. \tag{65}$$

Step 2. Setup the two first-order equations equivalent to the Eq. (62) as follows:

$$z'(x) + [b(x) - a'(x)]y(x) = f(x) \tag{66}$$

and

$$\varepsilon y'(x) + a(x)y(x) = z(x). \tag{67}$$

Step 3. Set up the initial conditions as follows:
Using $y_0(x)$, the solution of the reduced problem, in Eq. (67) we have

$$z(p) = \varepsilon y_0'(p) + a(p)y_0(p). \tag{68}$$

This is the initial condition for Eq. (66) and $y(q) = \beta$ will be the initial condition for Eq. (67).

Step 4. Get the pair of initial value problems as follows: Replacing $y(x)$ by $y_0(x)$ in (66), we get

$$z'(x) + [b(x) - a'(x)]y_0(x) = f(x). \tag{69}$$

Now the differential equation (69) with condition (68) constitute an initial value problem and the differential equation (67) with the condition $y(q) = \beta$ constitute another initial value problem.

Therefore the pair of initial value problems corresponding to Eqs. (62) and (63) are given by

$$\text{(i) } z'(x) + [b(x) - a'(x)]y_0(x) = f(x) \quad \text{with } z(p) = \varepsilon y_0'(p) + a(p)y_0(p), \tag{70}$$

$$\text{(ii) } \varepsilon y'(x) + a(x)y(x) = z(x) \quad \text{with } y(q) = \beta. \tag{71}$$

Thus in a manner of speaking, we have replaced the original boundary value problem (62) and (63) by a pair of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. We solve these initial value problems (70) and (71) to obtain the solution over the interval $[p, q]$.

7. Examples with right-end boundary layer

To illustrate the method of reduction of order for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval we considered two examples.

Example 7. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; \quad x \in [0, 1] \tag{72}$$

with

$$y(0) = 1 \quad \text{and} \quad y(1) = 0. \tag{73}$$

Clearly, this problem has a boundary layer at $x = 1$. i.e.; at the right-end of the underlying interval.

The exact solution is given by

$$y(x) = (e^{(x-1)/\varepsilon} - 1)/(e^{-1/\varepsilon} - 1). \quad (74)$$

From step 1, the reduced problem is $y_0'(x) = 0$; $y_0(0) = 1$; whose solution is $y_0(x) = 1$.

From step 2, the two first-order equations equivalent to Eq. (72) are

$$z'(x) = 0$$

and $\varepsilon y'(x) - y(x) = z(x)$.

From step 3, we have

$$z(0) = \varepsilon y_0'(0) + a(0)y_0(0) \quad \text{i.e.; } z(0) = -1 \quad (75)$$

Hence the pair of initial value problems related to (72) and (73) are

$$\text{(i) } z'(x) = 0 \quad \text{with } z(0) = -1, \quad (76)$$

$$\text{(ii) } \varepsilon y'(x) - y(x) = z(x) \quad \text{with } y(1) = 0. \quad (77)$$

The numerical results are given in Tables 13 and 14 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 8. We now consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0; \quad x \in [0,1] \quad (78)$$

with

$$y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon); \quad \text{and} \quad y(1) = 1 + 1/e. \quad (79)$$

Clearly this problem has a boundary layer at $x = 1$. The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}. \quad (80)$$

Table 13
Numerical results of Example 7 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.20	1.0000000	1.0000000
0.40	1.0000000	1.0000000
0.60	1.0000000	1.0000000
0.80	1.0000000	1.0000000
0.90	1.0000000	1.0000000
0.92	1.0000000	1.0000000
0.94	1.0000000	1.0000000
0.96	1.0000010	1.0000000
0.98	1.0007730	1.0000000
1.00	0.0000000	0.0000000

Table 14
Numerical results of Example 7 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.20	1.0000000	1.0000000
0.40	1.0000000	1.0000000
0.60	1.0000000	1.0000000
0.80	1.0000000	1.0000000
0.90	1.0000000	1.0000000
0.92	1.0000000	1.0000000
0.94	1.0000000	1.0000000
0.96	1.0000000	1.0000000
0.98	1.0000000	1.0000000
1.00	0.0000000	0.0000000

From step 1, the reduced problem is; $y_0'(x) + y_0(x) = 0$; $y_0(0) = 1$. The solution of this problem is

$$y_0(x) = \mathrm{e}^{-x}. \tag{81}$$

From step 2, the two first-order equations equivalent to Eq. (78) are

$$z'(x) - (1 + \varepsilon)y(x) = 0 \tag{82}$$

and

$$\varepsilon y'(x) - y(x) = z(x). \tag{83}$$

Now from step 3, we have $z(0) = \varepsilon y_0'(0) + a(0)y_0(0)$, i.e.; $z(0) = -\varepsilon - 1$.
Replacing $y(x)$ by $y_0(x)$ in (82) we have $z'(x) - (1 + \varepsilon)y_0(x) = 0$.
Hence the pair of initial value problems related to (78) and (79) are

Table 15
Numerical results of Example 8 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.20	0.8187319	0.8187308
0.40	0.6703215	0.6703200
0.60	0.5488131	0.5488116
0.80	0.4493306	0.4493290
0.90	0.4065710	0.4065697
0.92	0.3985204	0.3985190
0.94	0.3906293	0.3906278
0.96	0.3828960	0.3828929
0.98	0.3769802	0.3753111
1.00	1.3678790	1.3678790

Table 16
Numerical results of Example 8 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.20	0.8187292	0.8187308
0.40	0.6703162	0.6703200
0.60	0.5488135	0.5488117
0.80	0.4493309	0.4493290
0.90	0.4065703	0.4065697
0.92	0.3985207	0.3985191
0.94	0.3906303	0.3906278
0.96	0.3828928	0.3828929
0.98	0.3753136	0.3753111
1.00	1.3678790	1.3678790

$$\text{(i) } z'(x) = (1+\varepsilon)\mathrm{e}^{-x} \quad \text{with } z(0) = -\varepsilon - 1, \tag{84}$$

$$\text{(ii) } \varepsilon y'(x) - y(x) = z(x) \quad \text{with } y(1) = 1 + 1/\mathrm{e}. \tag{85}$$

The numerical results are given in Tables 15 and 16 for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

8. Discussion and conclusions

We have presented and illustrated the method of reduction of order for solving singularly perturbed two-point boundary value problems. The solution of the given singularly perturbed boundary value problem is computed numerically by solving a pair of initial value problems, which are deduced from the original problem. This method is very easy to implement on any computer with minimum problem preparation. We have implemented the present method on three linear examples, three non-linear examples with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. To solve the initial value problems we used the classical fourth order Runge–Kutta method. In fact any standard analytical or Numerical method can be used. Computational results are presented in tables. Here we have given results for only few values, although the solutions are computed at all points with mesh size h. The approximate solution is compared with exact solution. It can be observed from the results that the present method agrees with exact solution very well, which shows the efficiency of the method.

References

[1] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.

[2] X.C. Hu, T.A. Manteuffel, S. Mccormick, T.F. Russell, Accurate discretization for singular perturbations: the one-dimensional case, SIAM J. Numer. Anal. 32 (1) (1995) 83–109.
[3] M.K. Kadalbajoo, Y.N. Reddy, Initial value technique for a class of non-linear singular perturbation problems, J. Optim. Theory Appl. 53 (1987) 395–406.
[4] M.K. Kadalbajoo, Y.N. Reddy, Asymptotic and numerical analysis of singular perturbations; a survey, Appl. Math. Comput. 30 (1) (1989), part 1.
[5] J. Kevorkian, J.D. Cole, Perturbation Methods in Applied Mathematics, Springer, New York, 1981.
[6] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1973.
[7] R.E. O'Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.
[8] H.J. Reinhardt, Singular perturbations of difference methods for linear ordinary differential equations, Appl. Anal. 10 (1980) 53–70.