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International Journal of Engineering Science 42 (2004) 1035–1059

International Journal of Engineering Science

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Oscillatory flow of a micropolar fluid generated by the rotary oscillations of two concentric spheres

T.K.V. Iyengar *, V. Geetha Vani

Department of Mathematics and Humanities, National Institute of Technology, Warangal 506 004, Andhra Pradesh, India

Received 11 December 2002; received in revised form 3 September 2003; accepted 18 October 2003
Available online 27 April 2004

(Communicated by A.C. ERINGEN)

Abstract

In this paper we examine the flow of an incompressible micropolar fluid between two concentric spheres, generated by their rotary oscillations about a common diameter. The spheres are assumed to be oscillating with the same amplitude but with different angular speeds. The speeds of oscillation are assumed to be small so that the nonlinear terms in the equations of motion can be neglected under the usual Stokesian assumption. The analytical expressions for velocity and microrotation components are determined in terms of modified Bessel functions of first and second kind. The couples experienced by the inner and outer spheres are calculated and are expressed in terms of two real parameters K and K' whose variation is studied numerically. The variations of K and K' with respect to micropolarity parameter and frequency parameter are displayed graphically.

1. Introduction

The study of micropolar fluids initiated by Eringen [1] is a general microcontinuum approach in which microrotational effects are present and surface and body couples are permitted. The theory involves two basic and independent kinematical vector fields viz., the velocity vector representing the translational velocity of the fluid particles and the microrotation vector representing the angular velocity of the particles. Rigid particles contained in a small volume element can rotate about the centroid of the volume element and the microrotation vector describes this rotation in an average sence. This is a local rotation of the particles and is an addition to the usual rigid body

* Corresponding author.
E-mail address: iyengar@nitw.ernet.in (T.K.V. Iyengar).

0020-7225/$ - see front matter © 2004 Published by Elsevier Ltd.
doi:10.1016/j.ijengsci.2003.10.006

motion of the entire volume element. This theory of micropolar fluids is a special case of the theory of simple microfluids [2] introduced by Eringen himself.

Ever since the theory of micropolar fluids appeared, several workers attempted diverse fluid flow problems concerning these fluids. One interesting class of problems that attracted the attention of researchers is constituted by the axisymmetric flow problems in micropolar fluids. Lakshmana Rao et al. [3,4], Lakshmana Rao and Bhujanga Rao [5], have dealt with axisymmetric micropolar fluids in sphere geometry. Lakshmana Rao and Iyengar discussed the micropolar fluid flow problems concerning spheroids [6–9]. Iyengar and Srinivasacharya have discussed the flow problems involving near sphere [10,11]. Ramkissoon and Majumdar derived an elegant formula for the drag on an axially symmetric body in the Stokes flow of a micropolar fluid [12]. While Ramkissoon derived a formula for the couple on an axially symmetric body slowly rotating about its axis of symmetry [13], recently Srinivasa charya and Iyengar derived a formula for the drag on an axisymmetric body performing rectilinear oscillations along its axis of symmetry in an incompressible micropolar fluid medium [14]. These authors have also discussed the rotary oscillations of an approximate sphere in an incompressible micropolar fluid [15]. In all these problems, the thrust of investigation has been to obtain the drag or couple (as the case may be) on the body under consideration.

In this paper we consider an incompressible micropolar fluid present in the region between two concentric spheres ($r = a$, $r = b$, $a < b$). Let the inner sphere and outer sphere perform rotary oscillations with the same frequency, but with different angular speeds Ω_1 and Ω_2 about a common diameter. We assume Ω_1 and Ω_2 to be small so that the nonlinear terms and gyroinertial terms can be neglected from the equations governing the flow under the usual Stokesian approximation. The expressions for the velocity component and the two microrotation components are obtained under the hyperstick conditions on the boundary. The stress and couple stress components are calculated on the inner and outer sphere and these are expressed in terms of two parameters K and K'. The numerical variation of these parameters is studied for diverse values of the micropolarity, gyrorotation and frequency parameters through figures.

2. Basic equations and formulation of the problem

The field equations governing an incompressible micropolar fluid flow are [1],

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \bar{q}) = 0 \tag{1}$$

$$\rho \frac{\mathrm{d}\bar{q}}{\mathrm{d}t} = \rho \bar{f} - \operatorname{grad} p + k \operatorname{curl} \bar{\upsilon} - (\mu + k) \operatorname{curl} \operatorname{curl} \bar{q} + (\lambda_1 + 2\mu + k) \operatorname{grad} (\operatorname{div} \bar{q}) \tag{2}$$

$$\rho j \frac{\mathrm{d}\bar{\upsilon}}{\mathrm{d}t} = \rho \bar{l} - 2k\bar{\upsilon} + k \operatorname{curl} \bar{q} - \gamma \operatorname{curl} \operatorname{curl} \bar{\upsilon} + (\alpha + \beta + \gamma) \operatorname{grad} (\operatorname{div} \bar{\upsilon}) \tag{3}$$

in which $\bar{q}$, $\bar{\upsilon}$ are velocity and microrotation vectors, $\bar{f}$, $\bar{l}$ are body force per unit mass, body couple per unit mass respectively and p is the fluid pressure at any point. ρ and j are the density of the fluid and gyration parameters respectively and are assumed to be constants. The material constants (λ_1, μ, k) are viscosity coefficients and (α, β, γ) are gyroviscosity coefficients. These constants confirm to the inequalities,

$$\begin{aligned} &k \geqslant 0, \quad 2\mu + k \geqslant 0, \quad 3\lambda_1 + 2\mu + k \geqslant 0 \\ &\gamma \geqslant 0, \quad |\beta| \leqslant \gamma, \quad 3\alpha + \beta + \gamma \geqslant 0 \end{aligned} \tag{4}$$

The stress tensor t_{ij} and the couple stress tensor m_{ij} are given by

$$t_{ij} = (-p + \lambda_1 + \text{div } \bar{q})\delta_{ij} + (2\mu + k)e_{ij} + k\varepsilon_{ijm}(w_m - \upsilon_m) \tag{5}$$

$$m_{ij} = \alpha(\text{div } \bar{\upsilon})\delta_{ij} + \beta\upsilon_{i,j} + \gamma\upsilon_{j,i} \tag{6}$$

in which the symbols p, δ_{ij}, e_{ij}, w_m, υ_m and $\upsilon_{j,i}$ have their usual meanings.

Consider the concentric spheres with radii 'a' and 'b' ($a < b$). Let (r, θ, ϕ) be a spherical polar coordinate system with the common centre of the spheres as the origin. Let the region $a < r < b$ be filled by an incompressible micropolar fluid and the spheres $r = a$ and $r = b$ perform rotary oscillations with small angular speeds Ω_1 and Ω_2 about the common diameter ($\theta = 0$) with the same amplitude of oscillation σ. The fluid flow generated is axially symmetric and the field variables will be independent of ϕ. We choose the field vectors $\bar{q}$, $\bar{\upsilon}$ in the form

$$\bar{q} = V(r, \theta)\overline{e_\phi}\, e^{i\sigma t} \tag{7}$$

$$\bar{\upsilon} = (\mathscr{A}(r, \theta)\overline{e_r} + B(r, \theta)\overline{e_\theta})\, e^{i\sigma t} \tag{8}$$

Assuming that the flow is Stokesian, we neglect the nonlinear terms in the equations of motion and hence the field equations (1)–(3) simplify to

$$\text{div } \bar{q} = 0 \tag{9}$$

$$\rho\frac{\partial\bar{q}}{\partial t} = -\text{grad } p + k \text{ curl } \bar{\upsilon} - (\mu + k) \text{ curl curl } \bar{q} \tag{10}$$

$$\rho j\frac{\partial\bar{\upsilon}}{\partial t} = -2k\bar{\upsilon} + k \text{ curl } \bar{q} - \gamma \text{ curl curl } \bar{\upsilon} + (\alpha + \beta + \gamma)\text{grad div } \bar{\upsilon} \tag{11}$$

Defining the functions $f(r, \theta)$ and $g(r, \theta)$ through

$$\text{div } \bar{\upsilon} = f(r, \theta)e^{i\sigma t} \tag{12}$$

$$\text{curl } \bar{\upsilon} = g(r, \theta)e^{i\sigma t}\overline{e_\phi} \tag{13}$$

Eqs. (7)–(13) lead to

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \tag{14}$$

$$i\rho\sigma V(r,\theta) = kg + \frac{(\mu+k)}{r\sin\theta}E^2(r\sin\theta V) \quad (15)$$

$$(2k + i\rho j\sigma)(\mathscr{A}(r,\theta)\overline{e_r}\mathscr{B}(r,\theta)\overline{e_\theta}) = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[r\sin\theta(kV-\gamma g)\overline{e_r}] - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}[r\sin\theta(kV-\gamma g)\overline{e_\theta}] + (\alpha+\beta+\gamma)\left\{\frac{\partial f}{\partial r}\overline{e_r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\overline{e_\theta}\right\} \quad (16)$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad (17)$$

Further we note that

$$(\nabla^2 - p^2)f = 0 \quad (18)$$

where

$$p^2 = \frac{a^2(2k + i\rho j\sigma)}{\alpha+\beta+\gamma} \quad (19)$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} \quad (20)$$

From (16) we get

$$\left[i\rho j\sigma + \frac{k(2\mu+k)}{\mu+k}\right]g = \frac{-i\rho\sigma}{(\mu+k)}kV + \frac{\gamma}{r\sin\theta}E^2(r\sin\theta V) \quad (21)$$

Eliminating 'g' from (15) and (21) we get

$$\gamma(\mu+k)E^4(r\sin\theta V) - [k(2\mu+k) + i\rho\sigma(\gamma + j(\mu+k))]E^2(r\sin\theta V) + i\rho\sigma(2k + i\rho j\sigma)(r\sin\theta V) = 0 \quad (22)$$

As in [4], this can be rewritten in the form

$$(E^2-\alpha^2)(E^2-\beta^2)(r\sin\theta V) = 0 \quad (23)$$

where

$$\alpha^2 + \beta^2 = \frac{[k(2\mu+k) + i\rho\sigma(\gamma + j(\mu+k))]}{\gamma(\mu+k)}a^2$$

$$\alpha^2\beta^2 = \frac{\mathrm{i}\rho\sigma(2k+\mathrm{i}\rho j\sigma)}{\gamma(\mu+k)}a^4 \tag{24}$$

Eqs. (18) and (23) enable us to determine f and V respectively. We can determine the microrotation components $\mathscr{A}$, $\mathscr{B}$ through the equations

$$(2k+\mathrm{i}\rho j\sigma)\mathscr{A} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\gamma(\mu+k)}{k}E^2(r\sin\theta V) + \left(k - \frac{\mathrm{i}\rho\sigma\gamma}{k}\right)(r\sin\theta V)\right] + (\alpha+\beta+\gamma)\frac{\partial f}{\partial r} \tag{25}$$

$$(2k+\mathrm{i}\rho j\sigma)\mathscr{B} = \frac{-1}{r\sin\theta}\frac{\partial}{\partial r}\left[\frac{\gamma(\mu+k)}{k}E^2(r\sin\theta V) + \left(k - \frac{\mathrm{i}\rho\sigma\gamma}{k}\right)(r\sin\theta V)\right] + \frac{(\alpha+\beta+\gamma)}{r}\frac{\partial f}{\partial\theta} \tag{26}$$

which are together equivalent to (16). The arbitrary constants that arise in the solutions obtained can be determined through the hyperstick boundary conditions, which mean that

$$\begin{aligned} V(r,\theta) &= \Omega_1 r\sin\theta \text{ on } r=a \\ &= \Omega_2 r\sin\theta \text{ on } r=b \end{aligned} \tag{27}$$

and

$$(\bar{v})_{\text{Boundary}} = \frac{1}{2}\text{curl }(\bar{q}_{\text{Boundary}}) \tag{28}$$

$$\begin{aligned} A(r,\theta) &= \Omega_1\cos\theta \text{ on } r=a \\ &= \Omega_2\cos\theta \text{ on } r=b \end{aligned} \tag{29}$$

$$\begin{aligned} B(r,\theta) &= -\Omega_1\sin\theta \text{ on } r=a \\ &= -\Omega_2\sin\theta \text{ on } r=b \end{aligned} \tag{30}$$

3. Solution of the problem

Using the method of separation of variables, the solution of (18) is seen to be

$$f(r,\theta) = \left[Ar^{-1/2}I_{3/2}(pr) + Br^{-1/2}K_{3/2}(pr)\right]\cos\theta \tag{31}$$

The solution of Eq. (23) can be obtained by superposing the solutions of

$$\begin{aligned} (E^2-\alpha^2)(r\sin\theta V) &= 0 \\ (E^2-\beta^2)(r\sin\theta V) &= 0 \end{aligned} \tag{32}$$

and hence using again the method of separation of variables, we see that

$$V(r,\theta) = r^{-1/2}\left[CI_{3/2}(\alpha r) + DK_{3/2}(\alpha r) + EI_{3/2}(\beta r) + FK_{3/2}(\beta r)\right]\sin\theta \tag{33}$$

The constants A, B, C, D, E, F that appear in (31) and (33) are arbitrary constants to be determined using the boundary conditions (27), (29), (30). Using the expressions of 'f' and 'V' of (31) and (33) in (25) and (26), we get

$$\begin{aligned}(2k + \mathrm{i}\rho j\sigma)\mathscr{A} = \big\{(\alpha+\beta+\gamma)\big\{A\big[-2r^{-3/2}I_{3/2}(pr) + pr^{-1/2}I_{1/2}(pr)\big] + B\big[-2r^{-3/2}K_{3/2}(pr) \\ - pr^{-1/2}K_{1/2}(pr)\big]\big\} + 2\big\{A_\alpha\big[Cr^{-3/2}I_{3/2}(\alpha r) + Dr^{-3/2}K_{3/2}(\alpha r)\big] \\ + A_\beta\big[Er^{-3/2}I_{3/2}(\beta r) + Fr^{-3/2}K_{3/2}(\beta r)\big]\big\}\big\}\cos\theta\end{aligned} \tag{34}$$

$$\begin{aligned}(2k + \mathrm{i}\rho j\sigma)\mathscr{B} = \Big\{(\alpha+\beta+\gamma)\big[Ar^{-3/2}I_{3/2}(pr) + Br^{-3/2}K_{3/2}(pr)\big] + A_\alpha\Big\{C\big[r^{-3/2}I_{3/2}(\alpha r) \\ - r^{-1/2}\alpha I_{1/2}(\alpha r)\big] + D\Big[r^{-3/2}K_{3/2}(\alpha r) + r^{-1/2}\alpha K_{1/2}(\alpha r)\Big]\Big\} + A_\beta\Big\{E\big[r^{-3/2}I_{3/2}(\beta r) \\ - r^{-1/2}\beta I_{1/2}(\beta r)\big] + F\big[r^{-3/2}K_{3/2}(\beta r) + r^{-1/2}\beta K_{1/2}(\beta r)\big]\Big\}\Big\}\sin\theta\end{aligned} \tag{35}$$

where

$$\begin{aligned}A_\alpha &= \frac{\gamma(\mu+k)\alpha^2 + k^2 - \mathrm{i}\rho\sigma\gamma}{a^2k(2k + \mathrm{i}\rho j\sigma)} \\ A_\beta &= \frac{\gamma(\mu+k)\beta^2 + k^2 - \mathrm{i}\rho\sigma\gamma}{a^2k(2k + \mathrm{i}\rho j\sigma)}\end{aligned} \tag{36}$$

4. Determination of arbitrary constants

Before proceeding to determine the arbitrary constants, let us introduce the nondimensionalization scheme, $r = a\tilde{r}$, $V = a\Omega_1\widetilde{V}$, $\mathscr{A} = \Omega_1\widetilde{\mathscr{A}}$, $\mathscr{B} = \Omega_1\widetilde{\mathscr{B}}$ and later drop the tildes. The boundary conditions (27), (29), (30) take the nondimensional formv

$$V(r,\theta) = r\sin\theta \text{ on } r = 1 \tag{37}$$

$$V(r,\theta) = \Omega r\sin\theta \text{ on } r = \eta \tag{38}$$

$$A(r,\theta) = \cos\theta \text{ on } r = 1 \tag{39}$$

$$A(r,\theta) = \Omega\cos\theta \text{ on } r = \eta \tag{40}$$

$$B(r,\theta) = -\sin\theta \text{ on } r = 1 \tag{41}$$

$$B(r,\theta) = -\Omega\sin\theta \text{ on } r = \eta \tag{42}$$

where $\eta = \frac{b}{a}$ and $\Omega = \frac{\Omega_2}{\Omega_1}$.

The expressions for $V(r,\theta)$, $\mathscr{A}(r,\theta)$ and $\mathscr{B}(r,\theta)$ are given by

$$V(r,\theta) = r^{-1/2}\left[CI_{3/2}(\alpha r) + DK_{3/2}(\alpha r) + EI_{3/2}(\beta r) + FK_{3/2}(\beta r)\right]\sin\theta \tag{43}$$

$$\mathscr{A} = \Big\{ -\frac{1}{p^2}\{A[-2r^{-3/2}I_{3/2}(pr) + pr^{-1/2}I_{1/2}(pr)] + B[-2r^{-3/2}K_{3/2}(pr) - pr^{-1/2}K_{1/2}(pr)]\} + 2\{A_\alpha[Cr^{-3/2}I_{3/2}(\alpha r) + Dr^{-3/2}K_{3/2}(\alpha r)] + A_\beta[Er^{-3/2}I_{3/2}(\beta r) + Fr^{-3/2}K_{3/2}(\beta r)]\}\Big\}\cos\theta \tag{44}$$

$$\mathscr{B} = \Big\{ -\frac{1}{p^2}\left(Ar^{-3/2}I_{3/2}(pr) + Br^{-3/2}K_{3/2}(pr)\right) + A_\alpha\{C(r^{-3/2}I_{3/2}(\alpha r) - r^{-1/2}\alpha I_{1/2}(\alpha r)) + D(r^{-3/2}K_{3/2}(\alpha r) + r^{-1/2}\alpha K_{1/2}(\alpha r))\} + A_\beta\{E(r^{-3/2}I_{3/2}(\beta r) - r^{-1/2}\beta I_{1/2}(\beta r)) + F(r^{-3/2}K_{3/2}(\beta r) + r^{-1/2}\beta K_{1/2}(\beta r))\}\Big\}\sin\theta \tag{45}$$

Hence the conditions (37)–(42) on (43)–(45) give rise to the following 6×6 system of linear equations for the determination of the arbitrary constants A, B, C, D, E, F.

$$MX = N \tag{46}$$

where the elements of the matrix M are given below:

$$\begin{aligned}
&M_{11} = M_{12} = M_{21} = M_{22} = 0\\
&M_{23} = \eta^{-3/2}I_{3/2}(\alpha\eta), \quad M_{24} = \eta^{-3/2}K_{3/2}(\alpha\eta)\\
&M_{25} = \eta^{-3/2}I_{3/2}(\beta\eta), \quad M_{26} = \eta^{-3/2}K_{3/2}(\beta\eta)\\
&(M_{13}, M_{14}, M_{15}, M_{16}) = (M_{23}, M_{24}, M_{25}, M_{26}) \text{ with } \eta = 1.\\
&M_{41} = -\frac{1}{p^2}[-2\eta^{-3/2}I_{3/2}(p\eta) + p\eta^{-1/2}I_{1/2}(p\eta)]\\
&M_{42} = -\frac{1}{p^2}[-2\eta^{-3/2}K_{3/2}(p\eta) - p\eta^{-1/2}K_{1/2}(p\eta)]\\
&M_{43} = 2A_\alpha\eta^{-3/2}I_{3/2}(\alpha\eta), \quad M_{44} = 2A_\alpha\eta^{-3/2}K_{3/2}(\alpha\eta)\\
&M_{45} = 2A_\beta\eta^{-3/2}I_{3/2}(\beta\eta), \quad M_{46} = 2A_\beta\eta^{-3/2}K_{3/2}(\beta\eta)\\
&(M_{31}, M_{32}, M_{33}, M_{34}, M_{35}, M_{36}) = (M_{41}, M_{42}, M_{43}, M_{44}, M_{45}, M_{46}) \text{ with } \eta = 1.\\
&M_{61} = -\frac{1}{p^2}[\eta^{-3/2}I_{3/2}(p\eta)], \quad M_{62} = -\frac{1}{p^2}[\eta^{-3/2}K_{3/2}(p\eta)]\\
&M_{63} = A_\alpha[\eta^{-3/2}I_{3/2}(\alpha\eta) - \alpha\eta^{-1/2}I_{1/2}(\alpha\eta)]\\
&M_{64} = A_\alpha[\eta^{-3/2}K_{3/2}(\alpha\eta) + \alpha\eta^{-1/2}K_{1/2}(\alpha\eta)]\\
&M_{65} = A_\beta[\eta^{-3/2}I_{3/2}(\beta\eta) - \beta\eta^{-1/2}I_{1/2}(\beta\eta)]\\
&M_{66} = A_\beta[\eta^{-3/2}K_{3/2}(\beta\eta) - \beta\eta^{-1/2}K_{1/2}(\beta\eta)]\\
&(M_{51}, M_{52}, M_{53}, M_{54}, M_{55}, M_{56}) = (M_{61}, M_{62}, M_{63}, M_{64}, M_{65}, M_{66}) \text{ with } \eta = 1.
\end{aligned} \tag{47}$$

Further

$$X = [A, B, C, D, E, F]^{\mathrm{T}}$$
$$N = [1, \Omega, 1, \Omega, -1, -\Omega]^{\mathrm{T}}.$$

The constants A, B, C, D, E, F can be readily determined. The expressions for these are not explicitly written as they can be obtained in a straight forward manner, of course, through calculation.

5. Stress and couple stress components

The components of the stress tensor t_{ij} are given by

$$\begin{aligned}
t_{rr} &= t_{\theta\theta} = t_{\phi\phi} = -p \\
t_{r\theta} &= t_{\theta r} = 0 \\
t_{r\phi} &= \mu\frac{\partial V}{\partial r} - (\mu + k)\frac{V}{r} + k\mathscr{B} \\
t_{\phi r} &= (\mu + k)\frac{\partial V}{\partial r} - \mu\frac{V}{r} - k\mathscr{B} \\
t_{\theta\phi} &= \frac{\mu}{r}\frac{\partial V}{\partial \theta} - (\mu + k)\frac{V\cot\theta}{r} - k\mathscr{A} \\
t_{\phi\theta} &= \frac{(\mu + k)}{r}\frac{\partial V}{\partial \theta} - \mu\frac{V\cot\theta}{r} + k\mathscr{A}
\end{aligned} \tag{48}$$

and the components of couple stress tensor are seen to be

$$\begin{aligned}
m_{rr} &= \alpha f + (\beta + \gamma)\frac{\partial \mathscr{A}}{\partial r} \\
m_{\theta\theta} &= \alpha f + (\beta + \gamma)\left(\frac{1}{r}\frac{\partial \mathscr{B}}{\partial \theta} + \frac{\mathscr{A}}{r}\right) \\
m_{\phi\phi} &= \alpha f + (\beta + \gamma)\left(\frac{\mathscr{A}}{r} + \frac{\cot\theta}{r}\mathscr{B}\right) \\
m_{r\theta} &= \beta\left(\frac{1}{r}\frac{\partial \mathscr{A}}{\partial \theta} - \frac{\mathscr{B}}{r}\right) + \gamma\frac{\partial \mathscr{B}}{\partial r} \\
m_{\theta r} &= \beta\frac{\partial \mathscr{B}}{\partial r} + \gamma\left(\frac{1}{r}\frac{\partial \mathscr{A}}{\partial \theta} - \frac{\mathscr{B}}{r}\right) \\
m_{r\phi} &= m_{\phi r} = m_{\theta\phi} = m_{\phi\theta} = 0
\end{aligned} \tag{49}$$

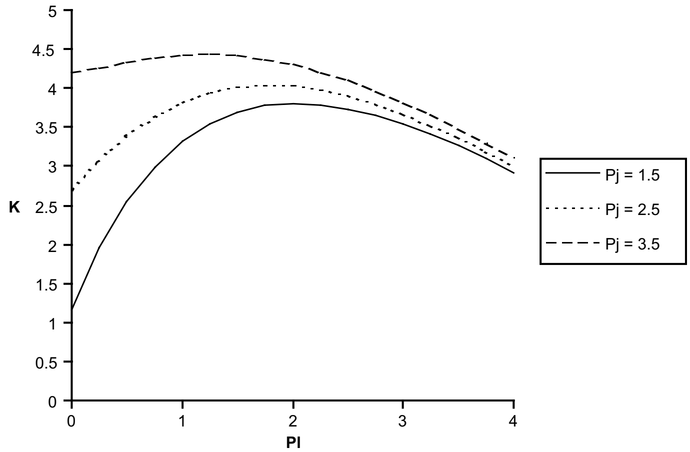

Fig. 1. Variation of K w.r.t. Pl (inner sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.3.

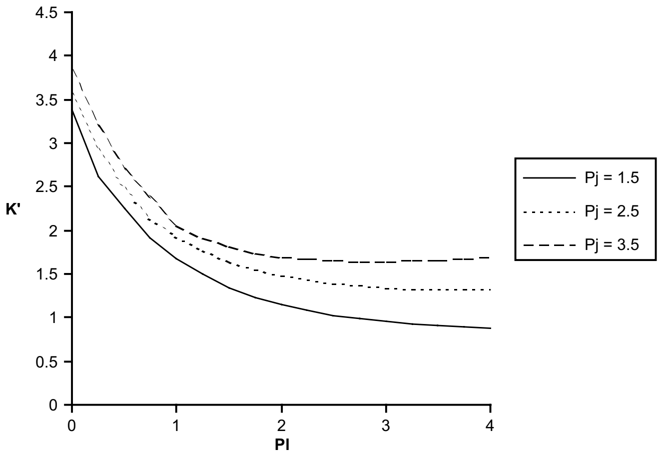

Fig. 2. Variation of K' w.r.t. Pl (inner sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.3.

These can be put in nondimensional form using

$$t_{r\phi} = \mu\Omega_1\tilde{t}_{r\phi},\ \ t_{\theta\phi} = \mu\Omega_1\tilde{t}_{\theta\phi} \text{ etc.}$$

$$m_{rr} = \Omega_1\frac{\gamma}{a}\tilde{m}_{rr},\ \ m_{r\theta} = \Omega_1\frac{\gamma}{a}\tilde{m}_{r\theta} \text{ etc}$$

where $\tilde{t}_{r\phi}, \tilde{t}_{\theta\phi}, \ldots, \tilde{m}_{rr}, \tilde{m}_{r\theta}, \ldots$ are nondimensional quantities (We shall drop the tildes later).

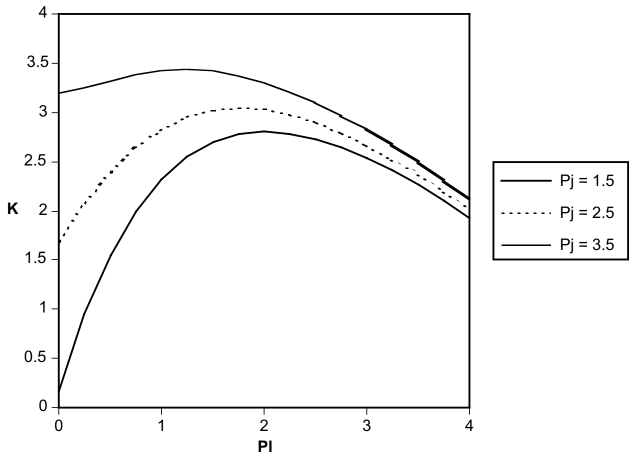

Fig. 3. Variation of K w.r.t. Pl (inner sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.5.

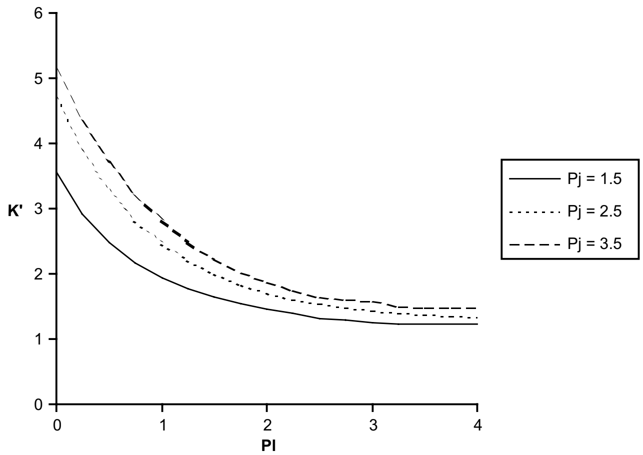

Fig. 4. Variation of K' w.r.t. Pl (inner sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.5.

6. Calculation of the couple acting on the spheres

The couple acting on the inner sphere/outer sphere has contributions both from the surface stress tensor and couple stress tensor.

The contribution of the surface stress tensor to the couple is given by

$$N_S = \int \bar{r} \times (\bar{n} : t) \cdot \overline{K} \, ds \tag{50}$$

where

$$\bar{r} = a\overline{e_r}, \; (\bar{n} : t) = t_{rr}\bar{e}_r + t_{r\theta}\overline{e_\theta} + t_{r\phi}\overline{e_\phi}$$

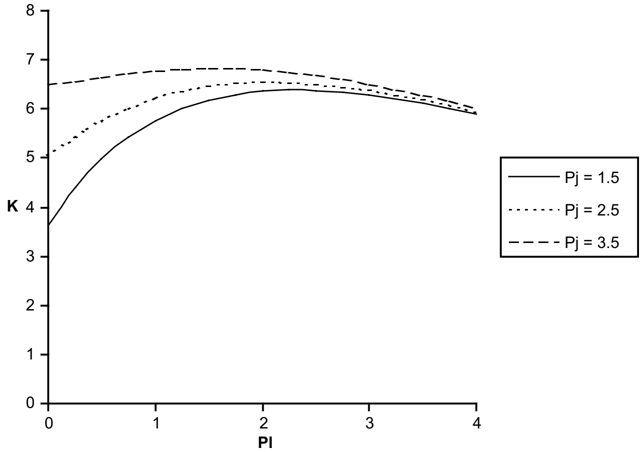

Fig. 5. Variation of K w.r.t. Pl (inner sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.5.

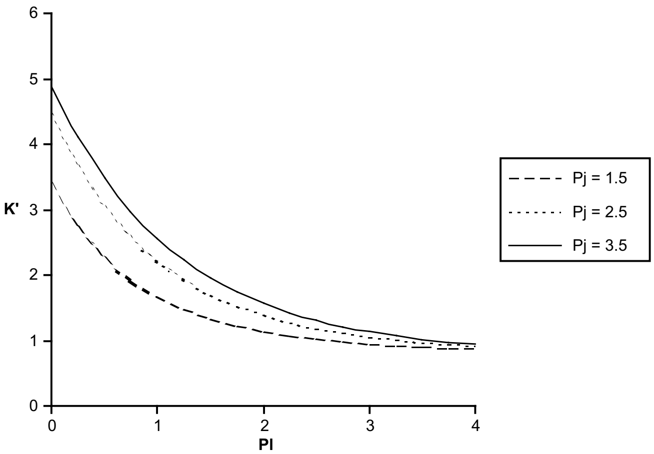

Fig. 6. Variation of K' w.r.t. Pl (inner sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.5.

and $\overline{K}$ is the unit vector in the direction of the axis of rotation and the integral is taken over the surface of the boundary.

We find that

$$N_S = 2\pi a^3 \int_0^{\pi} (t_{r\phi}) \sin^2\theta \, d\theta \tag{51}$$

in dimensional form where the integrand is calculated on $r = a$ or $r = b$ as the case may be. The contribution of the couple stress tensor to the couple is given by

$$N_C = \int (\bar{n} : m) \cdot \overline{K} \, ds \tag{52}$$

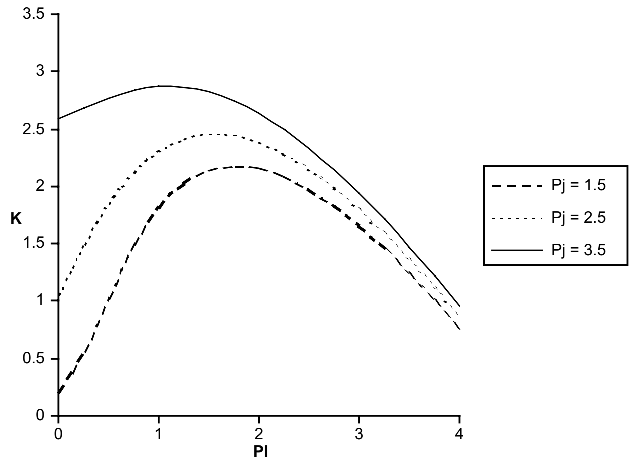

Fig. 7. Variation of K w.r.t. Pl (inner sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.5.

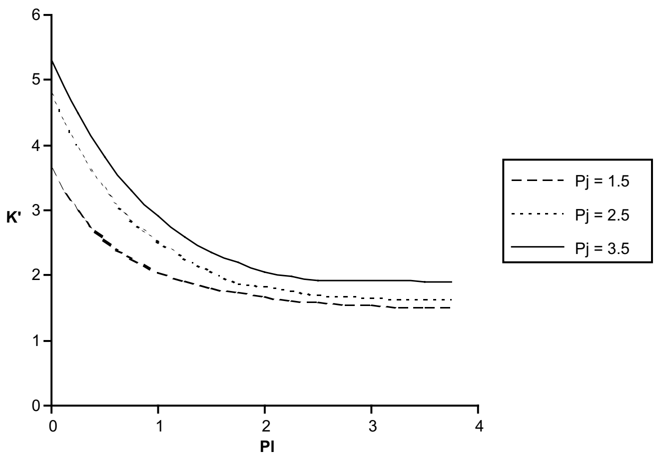

Fig. 8. Variation of K' w.r.t. Pl (inner sphere) for Pa = 3.5 Pt = 0.8, Pg = 0.5.

where

$$(\bar{n} : m) = m_{rr}\overline{e_r} + m_{r\theta}\overline{e_\theta} + m_{r\phi}\overline{e_\phi}$$

and this is seen to be

$$2\pi a^2 \int_0^\pi (m_{rr}\cos\theta - m_{r\theta}\sin\theta)\sin\theta \, d\theta$$

where the integrand is calculated on $r = a$ or $r = b$ as the case may be.

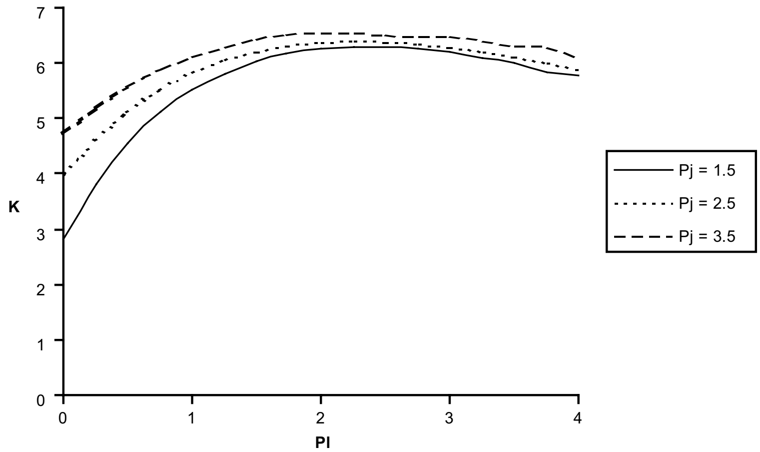

Fig. 9. Variation of K w.r.t. Pl (inner sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.3.

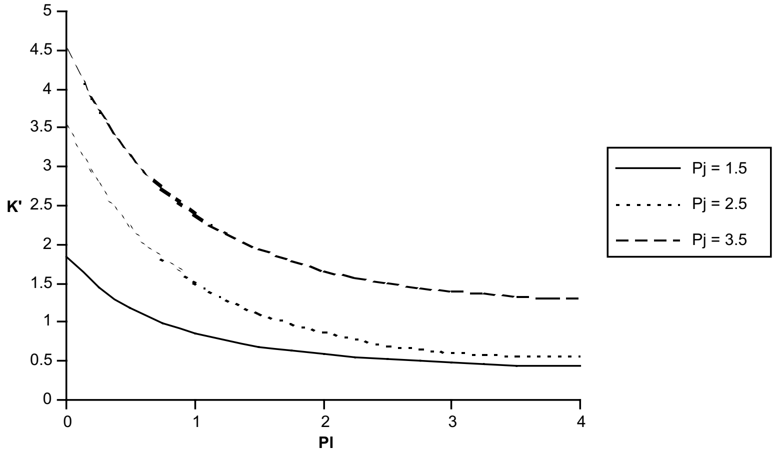

Fig. 10. Variation of K' w.r.t. Pl (inner sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.3.

It is interesting to see that, in nondimensional form

$$m_{rr} = \frac{(\alpha + \beta + \gamma)}{\gamma} f(r, \theta) \tag{53}$$

on $r = 1$ and $r = \eta$.

Further

$$m_{r\theta} = \left(\frac{\partial B}{\partial r}\right) \tag{54}$$

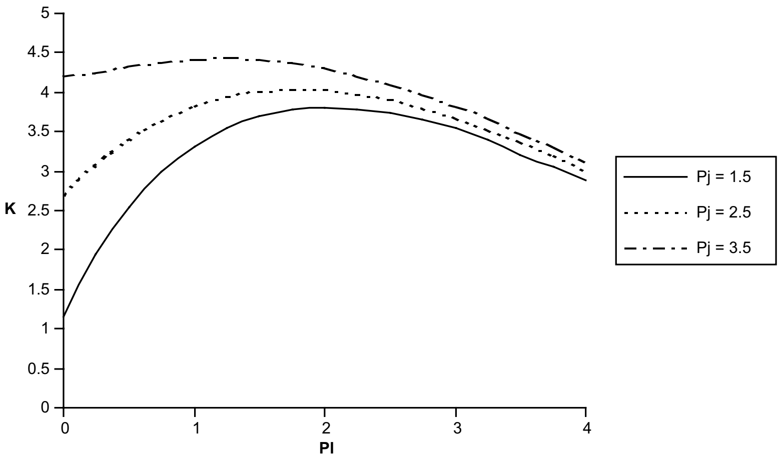

Fig. 11. Variation of K w.r.t. Pl (inner sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.3.

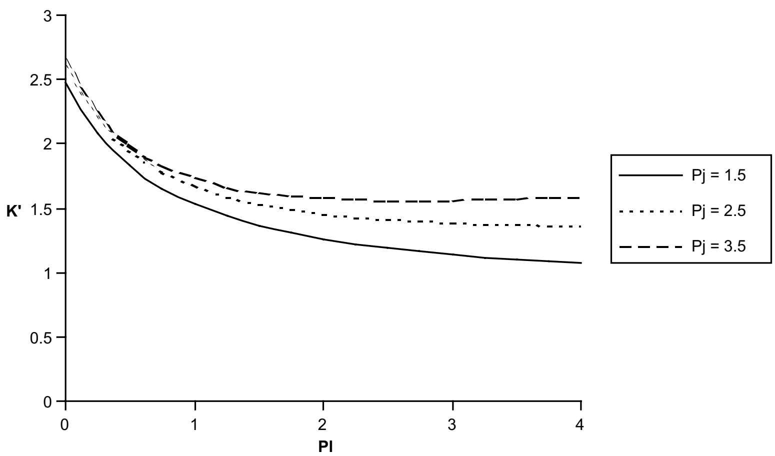

Fig. 12. Variation of K' w.r.t. Pl (inner sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.3.

and on $r = 1$ and on $r = \eta$, it takes the delightfully simple form

$$m_{r\theta} = g(r, \theta). \tag{55}$$

6.1. Couple acting on the inner sphere

The nondimensional N_S on the inner sphere $r = 1$ is seen to be

$$N_S = -\frac{4}{3}\left\{\left[\frac{3\mu + 2k}{\mu}\right] - \alpha\left[CI_{1/2}(\alpha) - DK_{1/2}(\alpha)\right] - \beta\left[EI_{1/2}(\beta) - FK_{1/2}(\beta)\right]\right\}e^{i\sigma t} \tag{56}$$

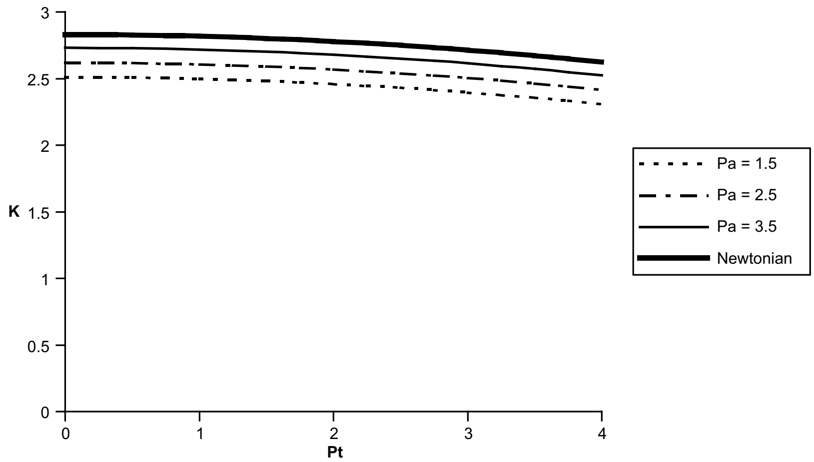

Fig. 13. Variation of K w.r.t. Pt (inner sphere) for Pl = 0.2, Pg = 0.3, Pj = 1.5.

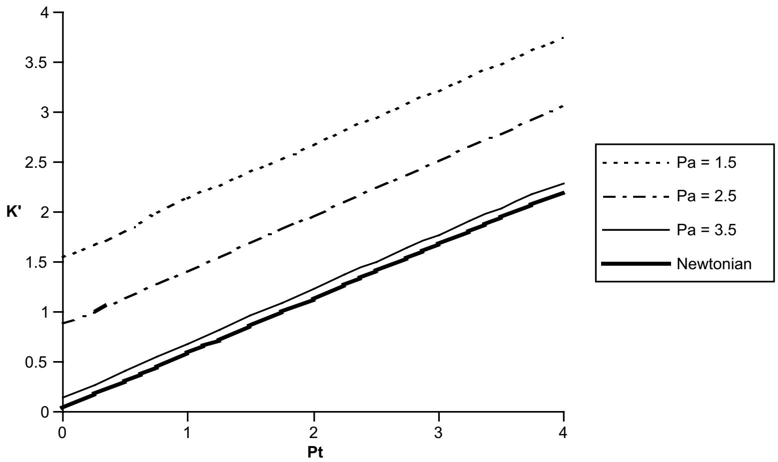

Fig. 14. Variation of K' w.r.t. Pt (inner sphere) for Pl = 0.2, Pg = 0.3, Pj = 1.5.

The nondimensional N_C on the inner sphere $r = 1$ is seen to be

$$N_C = \frac{2}{3}\left(\frac{\alpha+\beta+\gamma}{\gamma}\right)\left[AI_{3/2}(p) + BK_{3/2}(p)\right] - \frac{4}{3}\left(\frac{i\rho\sigma a^2}{k}\right) + \frac{4}{3}\left(\frac{\mu+k}{k}\right)\left\{\alpha^2\left[CI_{3/2}(\alpha) + DK_{3/2}(\alpha)\right] + \beta^2\left[EI_{3/2}(\beta) + FK_{3/2}(\beta)\right]\right\}e^{i\sigma t} \tag{57}$$

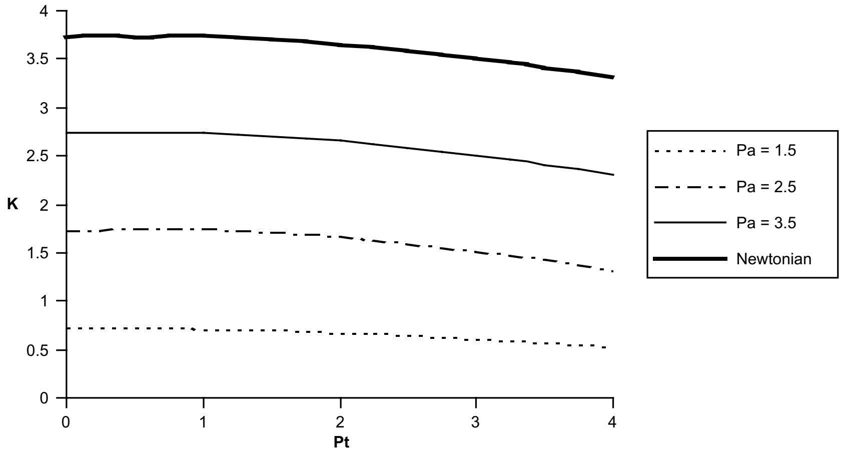

Fig. 15. Variation of K w.r.t. Pt (inner sphere) for Pl = 0.2, Pg = 0.3, Pj = 2.5.

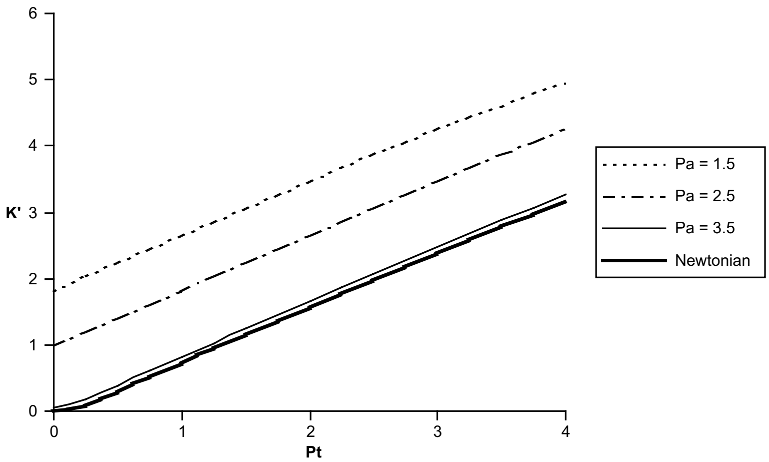

Fig. 16. Variation of K' w.r.t. Pt (inner sphere) for Pl = 0.2, Pg = 0.3, Pj = 2.5.

Hence the total couple on the inner sphere is seen to be

$$\begin{aligned} N &= N_S + N_C \\ &= -\frac{4}{3}\left\{\left[\frac{3\mu+2k}{\mu}\right] - \alpha\left[CI_{1/2}(\alpha) - DK_{1/2}(\alpha)\right] - \beta\left[EI_{1/2}(\beta) - FK_{1/2}(\beta)\right]\right\} \\ &\quad + \frac{2}{3}\left(\frac{\alpha+\beta+\gamma}{\gamma}\right)\left[AI_{3/2}(p) + BK_{3/2}(p)\right] - \frac{4}{3}\left(\frac{\mathrm{i}\rho\sigma a^2}{k}\right) + \frac{4}{3}\left(\frac{\mu+k}{k}\right)\left\{\alpha^2\left[CI_{3/2}(\alpha)\right.\right. \\ &\quad \left.\left. + DK_{3/2}(\alpha)\right] + \beta^2\left[EI_{3/2}(\beta) + FK_{3/2}(\beta)\right]\right\}\mathrm{e}^{\mathrm{i}\sigma t} \end{aligned} \tag{58}$$

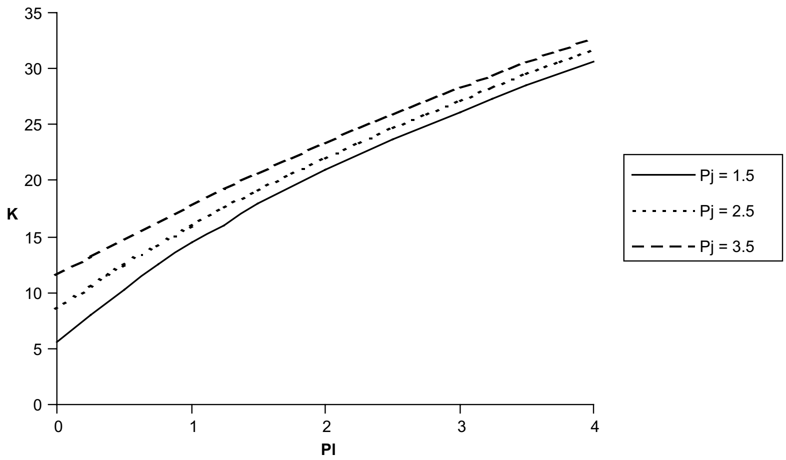

Fig. 1*. Variation of K w.r.t. Pl (outer sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.3.

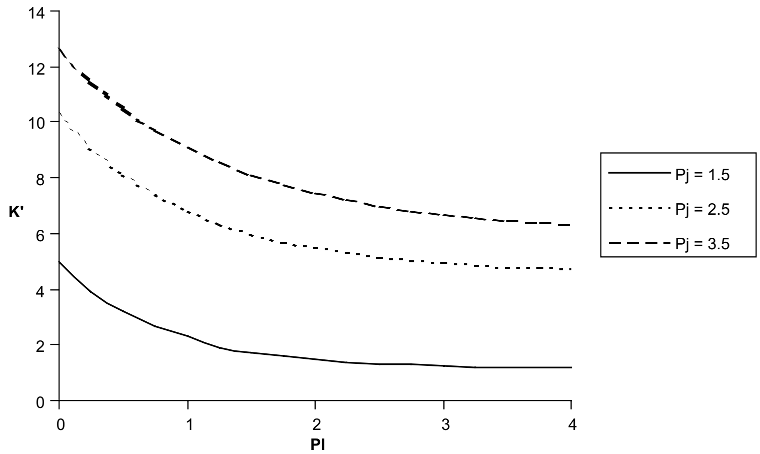

Fig. 2*. Variation of K' w.r.t. Pl (outer sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.3.

6.2. Couple acting on the outer sphere

The nondimensional N_S on the outer sphere $r = \eta$ is seen to be

$$N_S = \left\{ -\frac{4}{3}\left[\frac{3\mu + 2k}{\mu}\right]\Omega + \frac{4}{3}\eta^{-1/2}\left[\alpha\left[CI_{1/2}(\alpha) - DK_{1/2}(\alpha)\right] - \beta\left[EI_{1/2}(\beta) - FK_{1/2}(\beta)\right]\right]\right\}e^{i\sigma t} \tag{59}$$

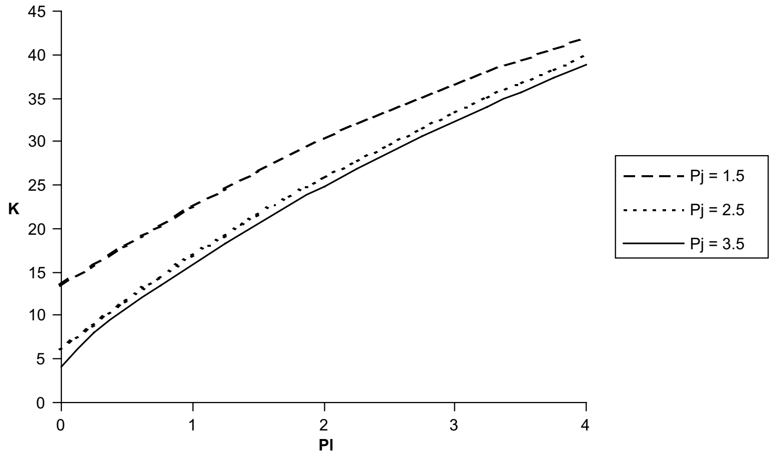

Fig. 3*. Variation of K w.r.t. Pl (outer sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.5.

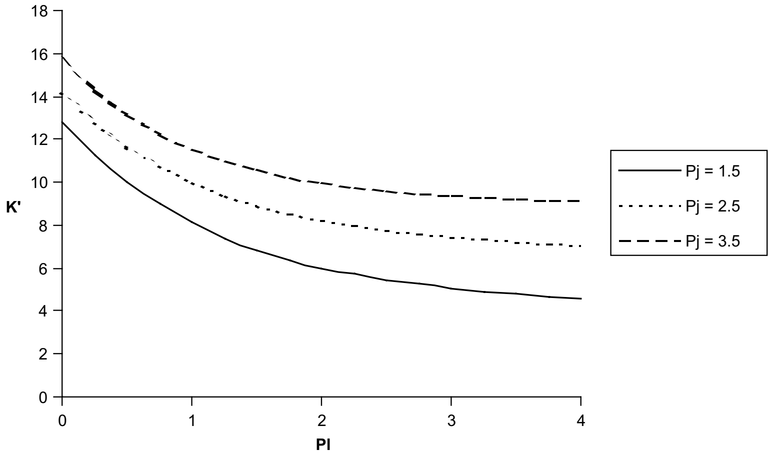

Fig. 4*. Variation of K' w.r.t. Pl (outer sphere) for Pa = 2.5 Pt = 0.8 Pg = 0.5.

The nondimensional N_C on the outer sphere $r = \eta$ is seen to be

$$N_C = -\frac{4}{3}\left(\frac{i\rho\sigma a^2 - (\mu+k)\alpha^2}{k}\right)\Omega + \frac{4}{3}\left(\frac{\alpha+\beta+\gamma}{\gamma}\right)\left[A\eta^{-1/2}I_{3/2}(p\eta) + B\eta^{-1/2}K_{3/2}(p\eta)\right] + \frac{4}{3}\left(\frac{\mu+k}{k}\right)\left\{\left(\beta^2-\alpha^2\right)\left[E\eta^{-1/2}I_{3/2}(\beta\eta) + FK_{3/2}(\beta\eta)\right]\right\}e^{i\sigma t} \tag{60}$$

Hence the total couple on the outer sphere is seen to be

$$N = N_S + N_C \tag{61}$$

and this can be simplified as in the earlier case. As $\eta \to \infty$ and $\Omega = 0$, the above expression reduces to the expression for the couple on a single sphere performing rotary oscillations as seen in [4] by Lakshmana Rao and Bhujanga Rao with the present hyperstick boundary conditions.

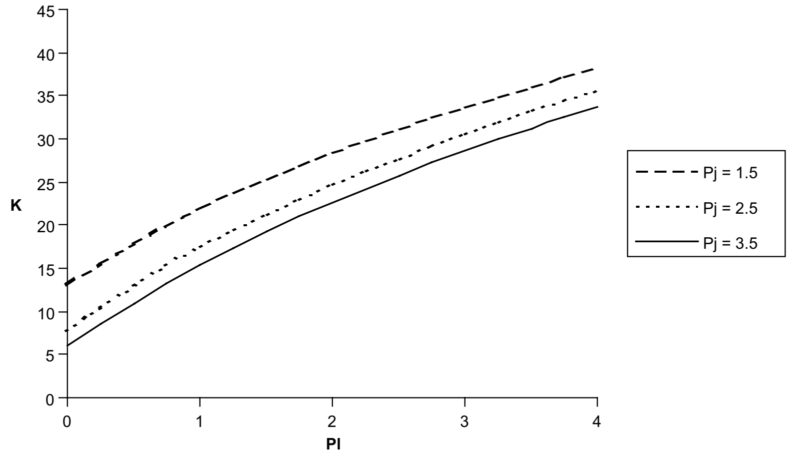

Fig. 5*. Variation of K w.r.t. Pl (outer sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.5.

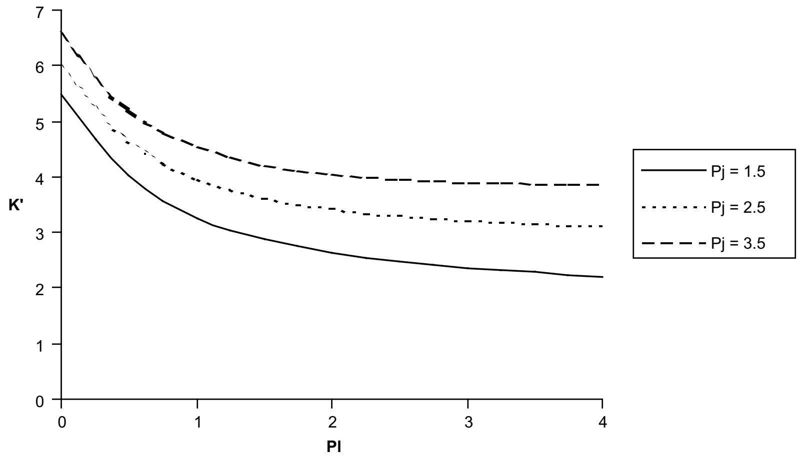

Fig. 6*. Variation of K' w.r.t. Pl (outer sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.5.

6.3. Newtonian fluid

If the fluid is Newtonian, the velocity component V can be determined from the equation

$$\mathrm{i}\rho\sigma V(r,\theta) = \frac{\mu}{r\sin\theta}E^2(r\sin\theta V) \tag{62}$$

which is Eq. corresponding to (15) with $k = 0$. Eq. (16) does not obviously arise in this case. The above Eq. (62) can be written as

$$\left(E^2 - \frac{\mathrm{i}\rho\sigma}{\mu}\right)(r\sin\theta V) = 0 \tag{63}$$

This can be obtained from (22) by taking $\gamma = 0$ and allowing k tend to zero.

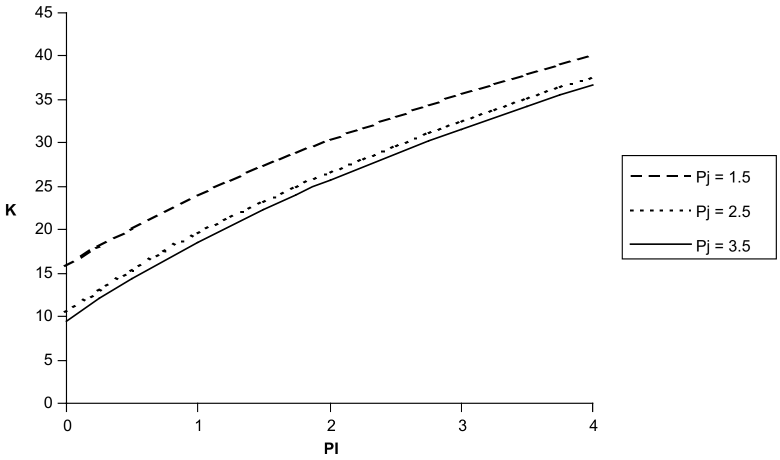

Fig. 7*. Variation of K w.r.t. Pl (outer sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.5.

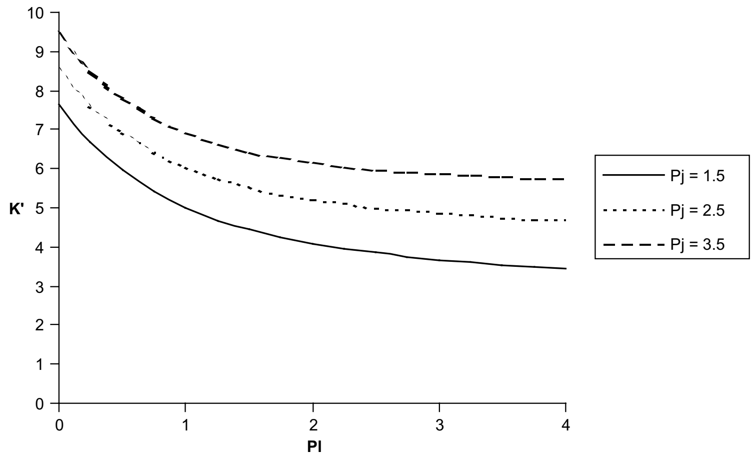

Fig. 8*. Variation of K' w.r.t. Pl (outer sphere) for Pa = 3.5, Pt = 0.8, Pg = 0.5.

Defining

$$V = a\Omega_1 \widetilde{V}, \quad r = a\tilde{r} \tag{64}$$

and dropping tildes later we have the nondimensional equation

$$(E^2 - q^2)(r \sin\theta V) = 0 \tag{65}$$

where V is to satisfy the boundary conditions (37) and (38). Solving (65) we get

$$V(r,\theta) = r^{-1/2}\left[CI_{3/2}(qr) + DK_{3/2}(qr)\right]\sin\theta \tag{66}$$

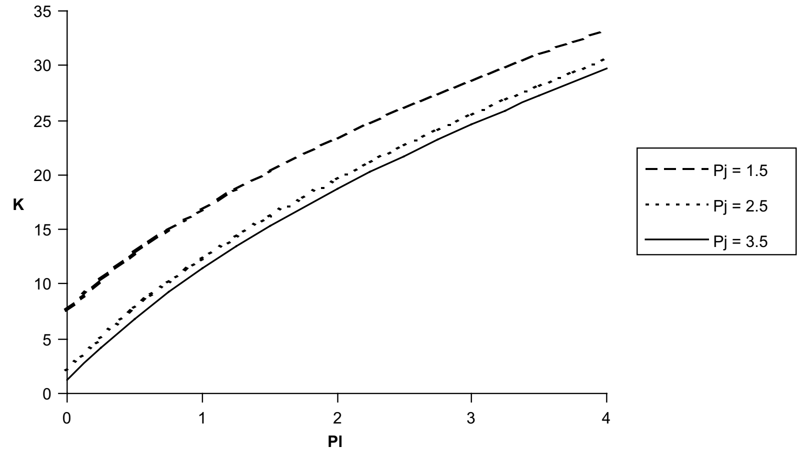

Fig. 9*. Variation of K w.r.t. Pl (outer sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.5.

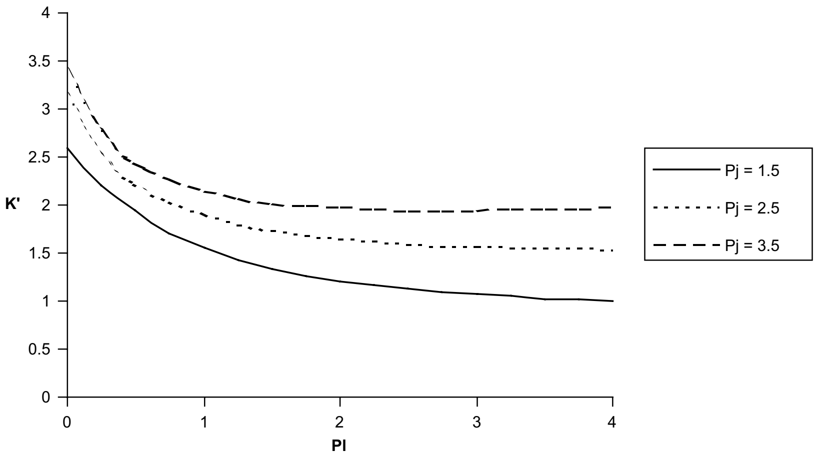

Fig. 10*. Variation of K' w.r.t. Pl (outer sphere) for Pa = 1.5, Pt = 0.8, Pg = 0.3.

where

$$C = \frac{\left[\eta^{-3/2}K_{3/2}(q\eta) - \Omega K_{3/2}(q)\right]}{\text{Den}} \tag{67}$$

$$D = \frac{\left[\Omega I_{3/2}(q) - \eta^{-3/2}I_{3/2}(q\eta)\right]}{\text{Den}} \tag{68}$$

$$Den = \eta^{-3/2}\left[I_{3/2}(q)K_{3/2}(q\eta) - K_{3/2}(q)I_{3/2}(q\eta)\right] \tag{69}$$

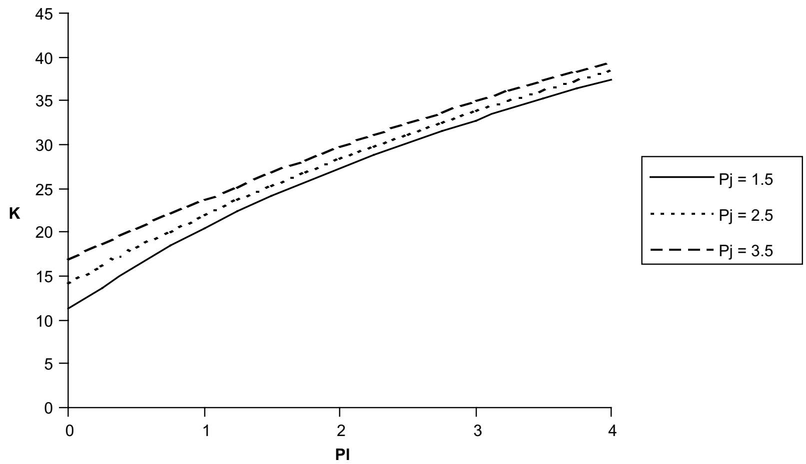

Fig. 11*. Variation of K w.r.t. Pl (outer sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.3.

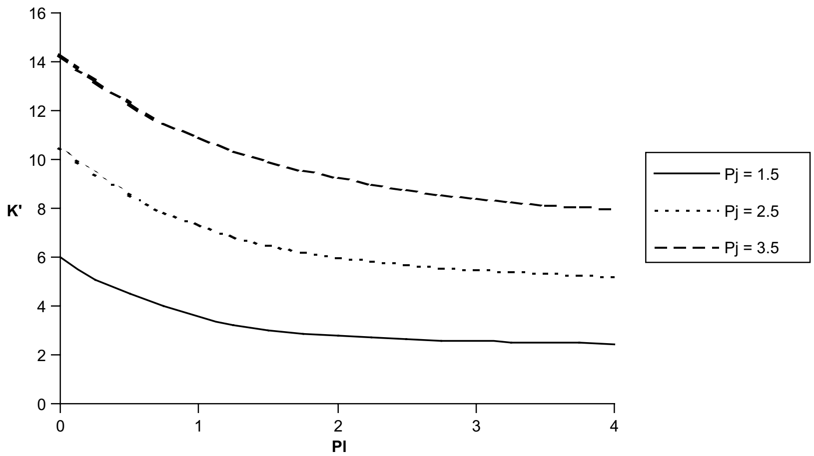

Fig. 12*. Variation of K' w.r.t. Pl (outer sphere) for Pa = 2.5, Pt = 0.8, Pg = 0.3.

and

$$q^2 = \frac{\mathrm{i}\rho\sigma a^2}{\mu} \tag{70}$$

Proceeding as before, the couple on the inner sphere is seen to be

$$N_{\text{inner}} = -4 + \frac{4}{3} q \left[C I_{1/2}(q) - D K_{1/2}(q) \right] \mathrm{e}^{\mathrm{i}\sigma t} \tag{71}$$

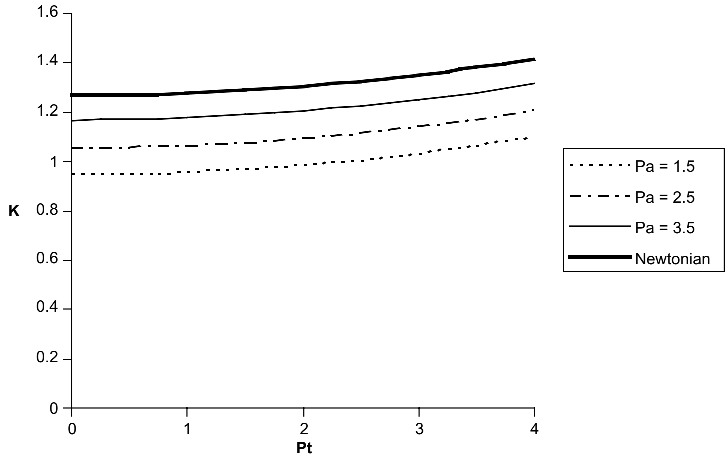

Fig. 13*. Variation of K w.r.t. Pt (outer sphere) for Pl = 0.2, Pg = 0.3, Pj = 1.5.

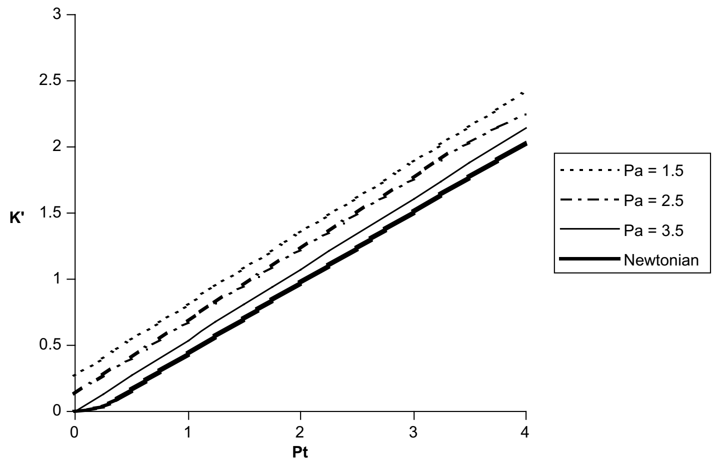

Fig. 14*. Variation of K' w.r.t. Pt (outer sphere) for Pl = 0.2, Pg = 0.3, Pj = 1.5.

and the couple on the outer sphere is seen to be

$$N_{\text{outer}} = -4\Omega + \frac{4}{3}\eta^{-1/2}q\left[CI_{1/2}(q\eta) - DK_{1/2}(q\eta)\right]e^{i\sigma t} \tag{72}$$

6.4. Numerical work

The expressions for the couple on the inner and outer spheres (given in (58) and (61) for the micropolar case and (70) and (71) for the Newtonian case) are all expressed in the form

$$(-K + iK')e^{i\sigma t} \tag{73}$$

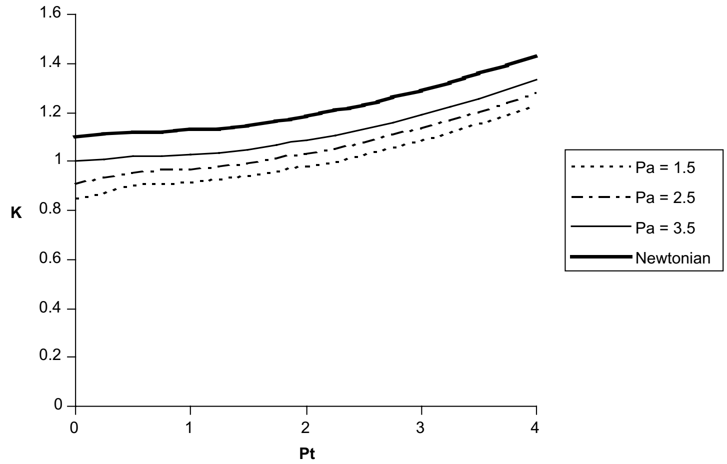

Fig. 15*. Variation of K w.r.t. Pt (outer sphere) for Pl = 0.2, Pg = 0.3, Pj = 2.5.

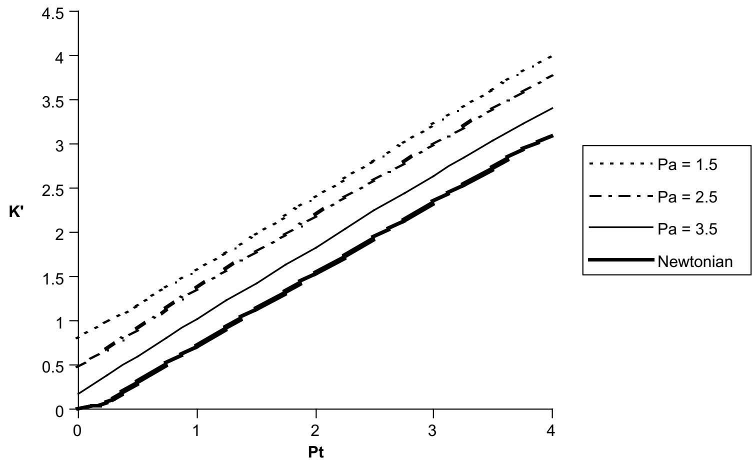

Fig. 16*. Variation of K' w.r.t. Pt (outer sphere) for Pl = 0.2, Pg = 0.3, Pj = 2.5.

The couple parameters K and K' are evaluated for diverse values of

$$f_p = \frac{\rho\sigma a^2}{\mu + k}, \quad m_p = \frac{k}{\mu + k}$$
$$\mathrm{Pj} = \frac{j(\mu + k)}{\gamma}, \quad \lambda^2 = \frac{k(2\mu + k)a^2}{\gamma(\mu + k)}, \quad \mathrm{Pg} = \frac{\gamma}{(\alpha + \beta + \gamma)} \tag{74}$$

Pt, Pl, Pg, Pj, Pa in Figs. 1–16 and (1*)–(16*) respectively stand for f_p, λ^2, Pg, Pj and $(1/m_p)$. The values of the parameters considered are indicated on the figures. Figs. 1, 3, 5, 7, 9, 11 indicate the variation of K on the inner sphere with respect to Pl and for a fixed set of values of Pa, Pt, Pg with

Pj varying. Figs. 2, 4, 6, 8, 10, 12 depict the variation of K' on the inner sphere. Figs. 13–16 indicate the variation of K and K' on the inner sphere with respect to the frequency parameter Pt and a fixed set of values of Pl, Pg and Pj with Pa varying.Figs. (1*)–(16*) concern the couple parameters K and K' on the outer sphere.

For a fixed Pa, Pg, Pt and Pj as λ increases, the parameter K increases initially but decreases subsequently. Irrespective of other parameters, for large λ the difference in K's seems to be decreasing. The couple parameter K' decreases as λ increases for fixed values of other parameters. This trend is seen even with respect to the couple parameters K and K' on the outer sphere. As the gyration parameter Pj increases the parameters K and K' are both increasing. As k tends to zero, i.e. as Pa $\to \infty$, the fluid tends to be Newtonian. The graphs 13–16 indicate that, as k tends to zero, i.e. as the fluid tends to become Newtonian, the parameter K increases while the parameter K' steadily decreases.

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