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Flow past an axisymmetric body embedded in a saturated porous medium

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Abstract The problem of viscous fluid past an axisymmetric body embedded in a fluid saturated porous medium is studied using the Brinkman's extension. A general formula for the drag on the body is derived in the form of a limit of an expression involving the stream function characterizing the flow. The flow past an axisymmetric approximate sphere is also considered. The stream function in this case is obtained in terms of Bessel functions and Gegenbauer's functions. The drag acting on the body is evaluated by using the formula derived. Its variation is studied with respect to geometric and permeability parameters. The special cases of flow past a sphere and a spheroid are obtained from the present analysis. ***To cite this article: D. Srinivasa Charya, J.V. Ramana Murthy, C. R. Mecanique 330 (2002) 417–423.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

fluid mechanics / axisymmetric body / porous medium / drag / approximate sphere

Écoulement derrière un corps à symétrie axiale plongé dans un milieu poreux saturé

Résumé Nous étudions le problème d'écoulement derrière un corps axisymétrique plongé dans un milieu poreux saturé en utilisant l'extension de Brinkman. Nous donnons une formule générale pour la force de traînée exercée sur ce corps, sous forme de limite d'une expression contenant la fonction de courant caractéristique de l'écoulement. L'écoulement derrière une sphère axisymétrique approchée est également considéré. Dans ce cas précis, la fonction de courant s'exprime à l'aide des fonctions de Bessel et de Gegenbauer. Cette formule permet d'évaluer la traînée exercée sur le corps. Nous étudions également la variation de cette force en fonction des paramètres géométriques et de la perméabilité. La présente analyse permet d'obtenir les résultats concrets dans les cas d'une sphère et d'un corps sphéroidal. ***Pour citer cet article : D. Srinivasa Charya, J.V. Ramana Murthy, C. R. Mecanique 330 (2002) 417–423.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

mécanique des fluides / corps à symétrie axiale / milieux porex / force de traînée / sphère approchée

1. Introduction

The study of flow of fluids through porous medium is very important in ground water recharge, acquifiers and oil technology. Several authors have considered the flow past a body embedded in a porous medium using Darcy's model. However, the Darcy law appears to be inadequate for the flows with high porosity

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and large shear rates and for flows near the surface of the bounded porous medium. To model such flows Brinkman [1] has suggested a modification to Darcy's law. The validity of this equation was theoretically justified by Tam [2] and Lundgren [3].

The study of uniform flow past an axisymmetric body has been considered both analytically and experimentally by many researchers for Newtonian as well as different types of non-Newtonian fluids. The Stokes flow past a class of axisymmetric bodies with uniform stream at infinity parallel to the axis of symmetry was studied by Payne and Pell [4] and a general formula for the drag experienced by the body was obtained in terms of the stream function. Padmavathi et al. [5] studied the general non-axisymmetric Stokes flow past a porous sphere of Brinkman's model and derived a formula for the drag and couple. Bhupen Barman [6] studied the flow of viscous liquid past a sphere embedded in Brinkman's porous medium. Ganapathy [7] considered the flow past a sphere and studied the heat transfer due to forced convection.

In the present paper, we consider the flow of viscous fluid past an impermeable axisymmteric body embedded in a saturated porous medium with Brinkman's model. The fluid assumes a uniform velocity far away from the body. We derive a general formula for the drag on the body and obtain the expression for the drag on an approximate sphere. We recover from this the expression for the drag on a sphere and obtain that on a spheroid as special cases.

2. Formulation of the problem

Consider an axisymmetric body submerged in an infinite expanse of fluid saturated porous medium of uniform porosity, assuming that the fluid is at rest at infinity. Let (n, s, ϕ) be intrinsic coordinate system with scale factors $h_1 = 1$, $h_2 = 1$ and $h_3 = 1/\varpi$ (ϖ is the distance from the point of consideration to axis of symmetry) and $\vec{n}$, $\vec{s}$, $\vec{i}_\phi$ be corresponding unit base vectors. The flow generated is axially symmetric and all the flow functions are independent of ϕ.

The equations of motion of the fluid in steady state in the porous medium based on Brinkman's model are

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

$$\frac{\mu}{k}\vec{q} = \nabla \cdot \Pi \tag{2}$$

where

$$\Pi = -PI + \mu\left(\nabla\vec{q} + (\nabla\vec{q})^{\mathrm{T}}\right) \tag{3}$$

is the stress tensor at any point of the fluid, μ is the viscosity of the fluid, k is the permeability of the porous medium, P is the pressure and $\vec{q}$ is the velocity vector of the flow.

Introducing the stream function ψ, the above equation can be written as

$$\frac{\mu}{k}\nabla \times \left[\vec{i}_\phi \frac{1}{\varpi}\psi\right] = \nabla \cdot \Pi \tag{4}$$

Eliminating pressure from the above equation, we get

$$E^2\left(E^2 - \alpha^2\right)\psi = 0 \tag{5}$$

where

$$E^2 = \varpi\left[\frac{\partial}{\partial n}\left(\frac{1}{\varpi}\frac{\partial}{\partial n}\right) + \frac{\partial}{\partial s}\left(\frac{1}{\varpi}\frac{\partial}{\partial s}\right)\right] \tag{6}$$

is the Stokes stream function operator and $\alpha^2 = 1/k$. Thus, our goal is to find the solution of (5) subject to the following boundary conditions:

(i) Far away from the body, there is practically no flow and hence the stream function ψ tends to zero as $r \to \infty$;

(ii) At the boundary of the body, the velocity $\vec{q}$ of the fluid element on the body is equal to the velocity $\vec{q}_B$ of the body. Since the body is stationary, i.e. $\vec{q}_B = 0$ and hence $\vec{q} = 0$ on the boundary.

The general solution of (5) in terms of spherical harmonics is

$$\psi = \psi^P + \psi^D \tag{7}$$

with

$$\psi^P = \sum_{n=0}^{\infty} A_n r^{-n+1} \vartheta_n(\zeta) \tag{8}$$

$$\psi^D = \sum_{n=0}^{\infty} B_n R_n^{(1)}(r) \vartheta_n(\zeta) \tag{9}$$

where

$$R_n^{(1)}(r) = r^n \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{n-1} \frac{1}{r}\,\mathrm{e}^{-\alpha r} \tag{10}$$

and $\vartheta_n(\zeta)$ is Gegenbauer function of the first kind with $\zeta = \cos\theta$.

3. Drag on an axisymmetric body

We shall integrate Eq. (4) over the volume V bounded by the axisymmetric body B and a large concentric sphere S of radius R. We convert the volume integrals to surface integrals using the Gauss divergence theorem and we retain only the z-component of the vector equation:

$$-\iint_S \vec{i}_z \cdot \Pi_n \,\mathrm{d}A - \iint_B \vec{i}_z \cdot \Pi_n \,\mathrm{d}A + \mu\alpha^2 \iint_S \vec{i}_z \cdot \left[\vec{n} \times i_\phi \frac{\psi}{\varpi}\right] \mathrm{d}A$$
$$+ \mu\alpha^2 \iint_B \vec{i}_z \cdot \left[\vec{n} \times i_\phi \frac{\psi}{\varpi}\right] \mathrm{d}A = 0 \tag{11}$$

where i_z is the unit vector along z-axis, $\vec{n}$ is the local outward normal, $\Pi_n = \Pi \cdot \vec{n}$ and $\mathrm{d}A$ is the surface element. In the above, the second integral is simply the force exerted by the body on the fluid. In view of the condition $\vec{q} = 0$ on the boundary, the fourth integral vanishes. The first integral can be evaluated individually as below:

From the constitutive equations of viscous fluid, Π_n is

$$\Pi_n = -p\vec{n} - 2\mu\nabla\left[\frac{1}{\varpi}\frac{\partial\psi}{\partial s}\right] + \frac{\mu}{\varpi}E^2\psi \tag{12}$$

Substituting Π_n in the first integral of (11), it can be put in the form

$$\iint_S \vec{i}_z \cdot \Pi_n \,\mathrm{d}A = \pi\mu \int_{\mathcal{C}} \left[\varpi \frac{\partial}{\partial n}\left(E^2 - \alpha^2\right)\psi - 2\frac{\partial\varpi}{\partial s}E^2\psi\right] \mathrm{d}s \tag{13}$$

where $\mathrm{d}A = 2\pi\varpi\,\mathrm{d}s$ and $\mathcal{C}$ is the cross section of the large sphere S in the meridian plane. If there are no sources, then A_0 is zero. $R_n^{(1)}(r)$ is exponentially small at large r, and hence we neglect it in (7), if R is large enough. Then (13) reduce to

$$\iint_S \vec{i}_z \cdot \Pi_n \,\mathrm{d}A = \frac{2}{3}\mu\pi\alpha^2 A_2 \tag{14}$$

Similarly the third integral in Eq. (11) is evaluated using the procedure that is similar to that just outlined for the first integral, and its value is determined to be $(4/3)\pi A_2$. Using Eqs. (7)–(10), we may express A_2 in the form

$$A_2 = 2\lim_{r\to\infty} \frac{r^3\psi}{\varpi^2} \tag{15}$$

Hence the force on the body is given by

$$F = 4\pi\mu\alpha^2 \lim_{r\to\infty} \frac{r^3\psi}{\varpi^2} \tag{16}$$

This is analogous to the result of Payne and Pell [4].

If the fluid is not rest at infinity the above formula is not applicable. If, however ψ_∞ denotes the stream function corresponding to the fluid motion at infinity then the stream function $\psi - \psi_\infty$ gives a state of rest at infinity. Hence the drag is given by

$$F = 4\pi\mu\alpha^2 \lim_{r\to\infty} \frac{r^3(\psi - \psi_\infty)}{\varpi^2} \tag{17}$$

4. Drag on an approximate sphere

Let (r, θ, ϕ) denote a spherical polar coordinate system with $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ as the corresponding unit base vectors and $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$ as the scale factors. Consider the body $r = a[1 + f(\theta)]$ where $f(\theta)$ is a function of θ which can be expressed as $f(\zeta) = \sum \beta_m \vartheta_m(\zeta)$ where $\vartheta_m(\zeta) = \{P_{m-2}(\zeta) - P_m(\zeta)\}/(2m-1)$, $\zeta = \cos\theta$, in which $P_m(\zeta)$ is Legendre function of the first kind. For small β_m we refer to this body as an approximate sphere.

Consider a steady incompressible viscous fluid flow past an approximate sphere embedded in a saturated porous medium of uniform porosity with a uniform velocity U far away from the body along the axis of symmetry $\theta = 0$. Since the flow generated is in the meridian plane and is axially symmetric, all the quantities are independent of ϕ.

The solution of (5) in spherical polar coordinate system, using the regularity condition at infinity (i.e. $\psi \to \frac{1}{2}Ur^2\sin^2\theta$ as $r \to \infty$) is

$$\psi = \left[Ur^2 + B_2r^{-1} + C_2\sqrt{r}K_{3/2}(\alpha r)\right]\vartheta_2(\zeta) + \sum_{n=3}^{\infty}\left[B_nr^{-n+1} + C_n\sqrt{r}K_{n-1/2}(\alpha r)\right]\vartheta_n(\zeta) \tag{18}$$

where $K_{3/2}(\alpha r)$ and $K_{n-1/2}(\alpha r)$ are modified Bessel functions.

Let us introduce the following non-dimensionalization scheme before proceeding to the implementation of the boundary conditions to determine the arbitrary constants in the expression of ψ:

$$r = a\tilde{r}, \qquad \psi = Ua^2\tilde{\psi}, \qquad B_n = Ua^{n+1}\widetilde{B}_n, \qquad C_n = Ua^{3/2}\widetilde{C}_n \tag{19}$$

Introducing these in (18) and then dropping the tildes the expression for ψ in nondimensional form is seen to be

$$\psi = \left[r^2 + B_2r^{-1} + C_2\sqrt{r}K_{3/2}(a\alpha r)\right]\vartheta_2(\zeta) + \sum_{n=3}^{\infty}\left[B_nr^{-n+1} + C_n\sqrt{r}K_{n-1/2}(a\alpha r)\right]\vartheta_n(\zeta) \tag{20}$$

The boundary conditions at the surface of the approximate sphere become

$$\psi = 0, \qquad \psi_r = 0 \tag{21}$$

We first propose to develop the solutions corresponding to the boundary $r = [1 + \beta_m\vartheta_m(\zeta)]$. Assuming that the coefficient β_m is sufficiently small, so that squares and higher powers of β_m may be neglected, we replace r^k by $1 + k\beta_m\vartheta_m(\zeta)$ where k is positive or negative

If the body is the sphere, then the expression for ψ is given by

$$\psi = \left[r^2 + B_2r^{-1} + C_2\sqrt{r}K_{3/2}(a\alpha r)\right]\vartheta_2(\zeta) \tag{22}$$

Comparing (20) with the above expression, we note that the terms involving B_n and C_n for $n > 2$ in (20) are extra terms here which are not present in ψ for the case of sphere. The body in the present problem is an approximate sphere and the motion is expected not to be far different from that which occurs when the body is a sphere. All the coefficients B_n and C_n for $n > 2$ will be of order β_m. Therefore in these terms involving

B_n and C_n for $n > 2$, we disregard the departure from a spherical form and set $r = 1$ (r is non-dimensional) while implementing the boundary conditions as in [8,9]. Hence the boundary conditions (21) imply

$$\left[1 + B_2 + C_2 K_{3/2}(a\alpha)\right]\vartheta_2(\zeta) + [2 - B_2]\beta_m\vartheta_m(\zeta)\vartheta_2(\zeta) + \sum\left[B_n + C_n K_{n-1/2}(a\alpha)\right]\vartheta_n(\zeta) = 0 \quad (23)$$

$$\left[2 - B_2 - C_2\left\{K_{3/2}(a\alpha) + a\alpha K_{1/2}(a\alpha)\right\}\right]\vartheta_2(\zeta) + \left[2 + 2B_2 + C_2 K_{3/2}(a\alpha)\right]\beta_m\vartheta_m(\zeta)\vartheta_2(\zeta) + \sum\left[(1-n)B_n - C_n\left\{(n-1)K_{n-1/2}(a\alpha) + a\alpha K_{n-3/2}(a\alpha)\right\}\right]\vartheta_n(\zeta) = 0 \quad (24)$$

Equating the leading coefficients to zero in (23) and (24), we get

$$B_2 = -1 - \frac{3K_{3/2}(\alpha)}{a\alpha K_{1/2}(a\alpha)}, \qquad C_2 = \frac{3}{a\alpha K_{3/2}(a\alpha)} \quad (25)$$

Using the values B_2 and C_2 in (23) and (24) and making use of the following identity

$$\vartheta_m(\zeta)\vartheta_2(\zeta) = \frac{-(m-2)(m-3)}{2(2m-1)(2m-3)}\vartheta_{m-2}(\zeta) + \frac{m(m-1)}{(2m+1)(2m-1)}\vartheta_m(\zeta) - \frac{(m+1)(m+2)}{2(2m-1)(2m-3)}\vartheta_{m+2}(\zeta) \quad (26)$$

we get

$$B_n = C_n = 0 \quad \text{for } n \neq m-2,\ m,\ m+2 \quad (27)$$

and for $n = m-2$, m, $m+2$ we get the following system of equations

$$B_n + C_n K_{n-1/2}(a\alpha) = b_n\varepsilon_1 \quad (28)$$

$$(1-n)B_n - C_n\left\{(n-1)K_{n-1/2}(a\alpha) + a\alpha K_{n-3/2}(a\alpha)\right\} = b_n\varepsilon_2 \quad (29)$$

where

$$b_{m-2} = \frac{-(m-2)(m-3)}{2(2m-1)(2m-3)}, \qquad b_m = \frac{m(m-1)}{(2m+1)(2m-3)}, \qquad b_{m+2} = \frac{-(m+1)(m+2)}{2(2m-1)(2m+1)} \quad (30)$$

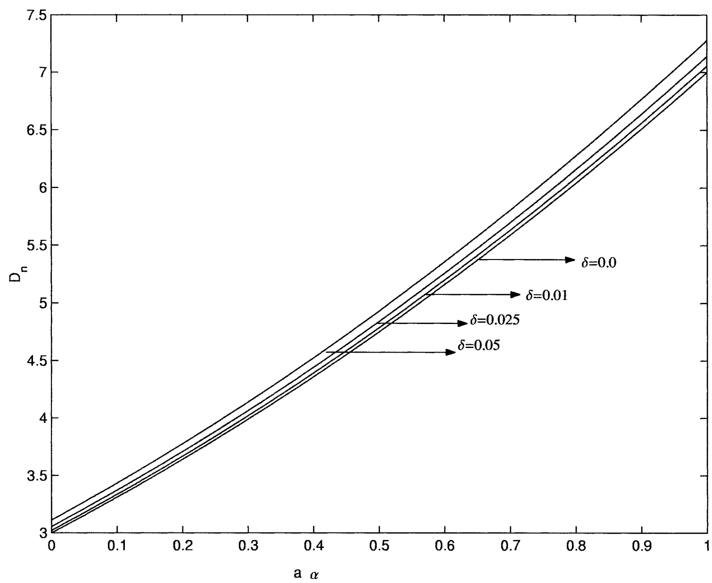

Figure 1. Variation of drag coefficient (D_N) with $a\alpha$ (approximate sphere with $\beta_2 = \beta_4 = \delta$).

and

$$\varepsilon_1 = -3 - \frac{3K_{3/2}(\alpha)}{a\alpha K_{1/2}(a\alpha)}, \qquad \varepsilon_2 = \frac{3K_{3/2}(\alpha)}{a\alpha K_{1/2}(a\alpha)} \tag{31}$$

Solving these equations we get the expressions for B_n and C_n.

In case the equation of approximate sphere is $r = a[1 + \sum \beta_m \vartheta_m(\zeta)]$, we employ the same technique as above and determine the corresponding arbitrary constants in the expansion for ψ and superimpose the expressions thus obtained.

The drag (D) experienced by an approximate sphere, using the formula (17), is given by

$$D = -2\pi\mu U a\left[\left\{3 + 3a\alpha + (a\alpha)^2\right\} + \frac{3}{5}\left\{2 + 2a\alpha + (a\alpha)^2\right\}\left(2\beta_2 - \frac{1}{7}\beta_4\right)\right] \tag{32}$$

It is interesting to note that though the boundary surface is $r = a[1 + \sum_{m=2}^{\infty} \beta_m \vartheta_m(\zeta)]$, the coefficients β_2 and β_4 only contribute to the drag. This implies that the drag on the approximate sphere is relatively insensitive to the details of the surface geometry. This is in tune with the observations made by Iyengar and Srinivasacharya [8,9] in case of micropolar fluids.

The variation of the drag coefficient $D_N = -D/(2\pi\mu U a)$ with respect to the permeability constant is shown in Fig. 1 for various values of $\beta_2 = \beta_4 = \delta$. It is observed that as the permeability constant $a\alpha$ increases the drag coefficient on an approximate sphere increases. Also D_N increases as δ increases.

5. Special cases

5.1. Sphere

If $\beta_m = 0$ for $m \geqslant 2$, we get the case of the sphere. In this case the drag simplifies to

$$D = -2\pi\mu U a\left\{3 + 3(a\alpha) + (a\alpha)^2\right\} \tag{33}$$

It can be observed that this drag is less than the drag on an approximate sphere and also the drag given in Eq. (32) is more than that of the drag on the sphere in a viscous fluid. As $k \to \infty$, i.e. as $(a\alpha) \to 0$, the drag in (33) gives the Stokesian drag $-6\pi\mu U a$.

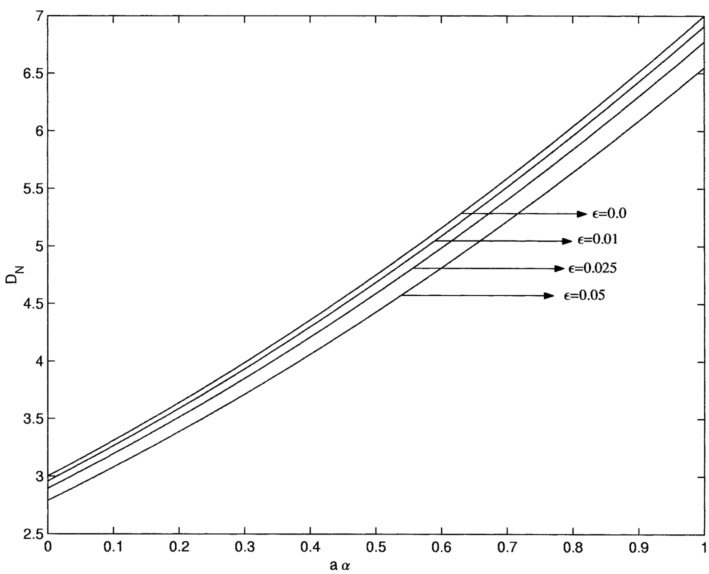

Figure 2. Variation of drag coefficient (D_N) with $a\alpha$ (oblate spheroid).

5.2. Oblate spheroid

Consider the oblate spheroid given by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{a^2(1-\varepsilon)^2} = 1 \tag{34}$$

whose equatorial radius is 'a' in which ε is so small that ε^2 and higher powers may be neglected. Following Happel and Brenner [10] its polar equation can be put in the form $r = c[1 + 2\varepsilon\vartheta_2(\zeta)]$ where $c = a(1-\varepsilon)$ ([10], p. 144).

Using (20), the expression for $\psi(r, \theta)$ can be determined. Using the formula (17), the drag is seen to be

$$D = -2\pi\mu U a\left[\left\{3 + 3(a\alpha) + (a\alpha)^2\right\} - \frac{3\varepsilon}{5}\left\{7 + 7(a\alpha) + (a\alpha)^2\right\}\right] \tag{35}$$

The variation of the drag coefficient $D_N = -D/(2\pi\mu U a)$ on the spheroid with permeablility parameter is shown in Fig. 2. In this case also there is an increase in the drag coefficient as the permeability parameter increases.The drag on the spheroid is less than that of the approximate sphere and sphere.

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