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Crank–Nicolson finite difference method based on a midpoint upwind scheme on a non-uniform mesh for time-dependent singularly perturbed convection–diffusion equations

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A numerical approach is proposed to examine the singularly perturbed time-dependent convection–diffusion equation in one space dimension on a rectangular domain. The solution of the considered problem exhibits a boundary layer on the right side of the domain. We semi-discretize the continuous problem by means of the Crank–Nicolson finite difference method in the temporal direction. The semi-discretization yields a set of ordinary differential equations and the resulting set of ordinary differential equations is discretized by using a midpoint upwind finite difference scheme on a non-uniform mesh of Shishkin type. The resulting finite difference method is shown to be almost second-order accurate in a coarse mesh and almost first-order accurate in a fine mesh in the spatial direction. The accuracy achieved in the temporal direction is almost second order. An extensive amount of analysis has been carried out in order to prove the uniform convergence of the method. Finally we have found that the resulting method is uniformly convergent with respect to the singular perturbation parameter, i.e. ε-uniform. Some numerical experiments have been carried out to validate the proposed theoretical results.

Keywords: Crank–Nicolson finite difference Scheme; Midpoint upwind; Shishkin mesh; Singular perturbation; Singularly perturbed convection–diffusion equation

AMS Subject Classification: 65M06, 65M12, 65M15

1. Introduction

We consider a time-dependent singularly perturbed convection–diffusion problem with variable coefficients and a small parameter $\varepsilon \ll 1$ multiplied by the highest order spatial derivative; this small parameter is known as the singular perturbation parameter. These problems arise in various fields of engineering and science, for example, in the mathematical modelling of steady and unsteady viscous flow problems with high Reynolds numbers [1], the convective heat transport problems with high Peclet numbers [2], the linearized Burgers equation or Navier–Stokes

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equations at high Reynolds numbers [3], simulation of oil extraction from underground reservoirs [4], and the drift diffusion equation of semiconductor device modelling [5]. In general the solutions of the time-dependent convection–diffusion equations possess a boundary layer on the right side of the rectangular domain, when the singular perturbation parameter ε is small, i.e. $\varepsilon \ll 1$ [6]. Due to the presence of the singular perturbation parameter ε, wild oscillations occur in the computed solutions using classical finite difference schemes, unless the mesh discretization used is very fine [7]. To tackle such situations we need to derive a method using a class of special piecewise uniform meshes introduced in [8], which are constructed *a priori* as a function of the parameter ε, the coefficient of convection term and the number of points N used in the spatial mesh.

The derivation of the ε-uniform convergence method based on a fitted mesh for ordinary differential equations has been given in [8, 9] which also contain numerical experiments for such meshes. Time-dependent problems have been discussed in [10, 11], which contain results based on a finite difference scheme used to solve the parabolic problem without convection term. Clavero *et al.* [12] have considered the time-dependent singularly perturbed convection–diffusion problem and gave a numerical scheme comprising an Euler implicit and standard upwind finite difference operator on the fitted mesh.

In recent years, many numerical techniques have been developed to solve time-dependent problems for convection–diffusion equations with variable coefficients. In [13], a finite element technique was used to construct mass lumped and non-lumped difference schemes of order one in both variables. In [14], a family of difference schemes exponentially fitted in spatial variables was defined. Assuming a CFL condition, the authors proved that the methods are uniformly convergent of order one. Syam [15] has given higher order predictor methods for numerical tracing of implicitly defined curves. He used mainly Newton and Hermite interpolation polynomials and approximated line integrals by a Gauss–Legendre polynomial. In [16], the Crank–Nicolson finite difference scheme is used to solve Burgers' equations and accuracy of order two is obtained in both variables.

The midpoint upwind scheme was introduced by Abrahamsson, Keller and Kriess [17], who examined it on an equidistant grid that is second-order accurate away from the boundary layer. In the present paper, we consider the problem given in [12]. We construct a numerical method based on the Crank–Nicolson finite difference and with a midpoint upwind finite difference operator on a piecewise uniform mesh, which is second-order accurate in time and almost second-order accurate in space in a coarse mesh and of almost first order in a fine mesh. In particular, we analyse the accuracy of the proposed method on a piecewise uniform mesh by reducing it to a system of ordinary differential equations. We prove that the numerical solution generated by the proposed method converges uniformly to the solution of the continuous problem with respect to the singular perturbation parameter.

A description of the contents of the paper is as follows. In section 2, we describe the continuous problem, its reduced problem and the classical bounds on the solution. In section 3, we describe the discretization in the temporal direction by means of the Crank–Nicolson finite difference method and the error in the temporal direction has been shown to be of second order and free from the parameter ε. In section 4, asymptotic analysis of the solution of the semi-discretized problem is given and in order to prove the ε-uniform convergence, sharper bounds on the derivatives are obtained by means of a decomposition of the solution into smooth and singular components. In section 5, the formulation of the numerical method comprising a discrete operator on the Shishkin mesh is given. In section 6, it is shown that the discrete operator satisfies the discrete maximum principle. The numerical solution is decomposed into smooth and singular components and the error estimates for the smooth and singular solutions have been obtained separately. The ε-uniform convergence of the numerical solution (generated by the proposed method) to the solution of the continuous problem is shown.

Finally section 7 describes the numerical experiments to corroborate the results predicted by the theory.

Throughout this paper the constant C (sometimes subscripted) will be a positive generic constant, independent of the mesh parameters, i.e. Δx, Δt and the singular perturbation parameter, ε, and the norm $\|\cdot\|$ (sometimes subscripted) used is the pointwise maximum norm.

2. Continuous problem

We consider the following singularly perturbed parabolic problem

$$u_t - \varepsilon u_{xx} + a(x)u_x + b(x)u = f(x,t), \quad \text{in} \quad \Omega \times (0,T], \tag{1}$$

where $\Omega = (0,1)$ with initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 1,$$

and boundary conditions

$$u(0,t) = 0 = u(1,t), \quad 0 \le t \le T,$$

where ε is the singular perturbation parameter, and $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions with

$$a(x) \ge \alpha > 0, \quad \text{on} \quad \bar{\Omega}, \tag{2}$$

$$b(x) \ge \beta > 0, \quad \text{on} \quad \bar{\Omega}. \tag{3}$$

We impose the compatibility conditions

$$u_0(0) = 0 \quad \text{and} \quad u_0(1) = 0,$$

so that the data match at the two corners $(0,0)$ and $(1,0)$. These conditions guarantee that there exists a constant C such that for all $(x,t) \in \bar{\Omega} \times [0,T]$

$$|u(x,t) - u_0(x)| \le Ct \tag{4}$$

$$|u(x,t)| \le C(1-x). \tag{5}$$

The reduced problem (i.e. the set $\varepsilon = 0$ in equation 1) is given by

$$u_t^0 + a(x)u_x^0 + b(x)u^0 = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{6}$$

$$u^0(x,0) = u_0^0(x), \quad 0 \le x \le 1,$$

$$u^0(0,t) = 0, \quad 0 \le t \le T.$$

This is a first-order hyperbolic equation with initial data specified along two sides $t = 0$ and $x = 0$ of the domain $\bar{\Omega}$. For small values of ε the solution $u(x,t)$ of equation (1) will be very close to $u^0(x,t)$. In order to obtain error bounds on the solution of the difference scheme it is assumed that the solution of the reduced problem (6) is sufficiently smooth.

For the bounds on the derivatives of the solution $u(x,t)$ of equation (1), we may assume without loss of generality that the initial condition is zero [13, 18].

LEMMA 2.1

$$|u(x,t)| \le C, \quad (x,t) \in \bar{\Omega} \times [0,T].$$

Proof By equation (4) we have

$$|u(x,t)| \le Ct, \quad (x,t) \in \bar{\Omega} \times [0,T], \tag{7}$$

since $t \in (0, T]$ therefore,

$$|u(x,t)| \le C, \quad (x,t) \in \bar{\Omega} \times [0,T]. \tag{8}$$

■

3. Temporal discretization

We discretize the continuous problem (1) in the temporal direction by means of the Crank–Nicolson method. In this case, we get a system of ordinary differential equations with boundary conditions. Discretization by the proposed method yields the following system of differential equations,

$$u_0 = u_0(x), \quad 0 \le x \le 1, \tag{9a}$$

$$\frac{u_{j+1} - u_j}{\Delta t} - \varepsilon \frac{(u_{j+1})_{xx} + (u_j)_{xx}}{2} + a(x) \frac{(u_{j+1})_x + (u_j)_x}{2} + b(x) \frac{u_{j+1} + u_j}{2}$$
$$= \frac{f(x, t_{j+1}) + f(x, t_j)}{2}. \tag{9b}$$

with boundary conditions,

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0, \quad t \ge 0, \tag{9c}$$

where u_{j+1} is the solution of equation (9) at the $(j+1)$th time level. Here $u_j = u(x, t_j)$, and Δt is the time step; the subscript denotes the jth time level, i.e. $t_j = n\Delta t$.,

Rewrite equation (9) as

$$u_0 = u_0(x), \quad 0 \le x \le 1, \tag{10a}$$

$$-\frac{\varepsilon}{2}(u_{j+1})_{xx} + \frac{a(x)}{2}(u_{j+1})_x + d(x) u_{j+1}$$
$$= \frac{f(x, t_{j+1}) + f(x, t_j)}{2} + \frac{\varepsilon}{2}(u_j)_{xx} - \frac{a(x)}{2}(u_j)_x$$
$$- c(x) u_j, \quad 0 < x < 1, \quad t > 0, \tag{10b}$$

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0, \quad t \ge 0, \tag{10c}$$

where $d(x) = (1/\Delta t + (b(x)/2))$ and $c(x) = (b(x)/2 - (1/\Delta t))$. Here $d(x) > 0$, since $b(x) \ge 0$.

We write the above equation (10) in operator form, with initial condition

$$u_0 = u_0(x), \qquad 0 \le x \le 1, \tag{11a}$$

$$L^c(u_{j+1}(x)) = g(x, t_{j+1}), \quad 0 \le x \le 1, \tag{11b}$$

and with boundary conditions,

$$u_{j+1}(0) = 0, \quad u_{j+1}(1) = 0, \quad t \geq 0, \tag{11c}$$

where

$$L^c(u_{j+1}) \equiv -\frac{\varepsilon}{2}(u_{j+1})_{xx} + \frac{a(x)}{2}(u_{j+1})_x + d(x)u_{j+1}$$

and

$$g(x, t_{j+1}) = \frac{f(x, t_{j+1}) + f(x, t_j)}{2} + \frac{\varepsilon}{2}(u_j)_{xx} - \frac{a(x)}{2}(u_j)_x - c(x)u_j \Delta t.$$

LEMMA 3.1 (Maximum principle) *Let* $\psi_{j+1}(x) \in C^2(\bar{\Omega})$. *If* $\psi_{j+1}(0) \geq 0$, $\psi_{j+1}(1) \geq 0$ *and* $L^c\psi_{j+1}(x) \geq 0$ *for all* $(x) \in \Omega$, *then* $\psi_{j+1}(x) \geq 0$ *for all* $x \in \bar{\Omega}$.

Proof Assume that there exists $x^* \in \bar{\Omega}$ such that

$$\psi_{j+1}(x^*) = \min_{x \in \bar{\Omega}} \psi_{j+1}(x) < 0.$$

It is clear that the point $(x^*) \notin \{0, N\}$ which implies that $x^* \in \Omega$.

Using the differential operator on ψ gives

$$L^c\psi = -\frac{\varepsilon}{2}(\psi_{j+1})_{xx} + \frac{a(x)}{2}(\psi_{j+1})_x + d(x)\psi_{j+1} \tag{12}$$

and at the point x^* the value of the above operator becomes

$$L^c\psi_{j+1}(x^*) = -\frac{\varepsilon}{2}(\psi_{j+1})_{xx}(x^*) + \frac{a(x^*)}{2}(\psi_{j+1})_x(x^*) + d(x^*)\psi_{j+1}(x^*). \tag{13}$$

Since we have,

$$(\psi_{j+1})_{xx}(x^*) \geq 0 \quad \text{and} \quad (\psi_{j+1})_x(x^*) = 0,$$

using the above estimates in equation (13), we have

$$L^c\psi_{j+1}(x^*) < 0,$$

which is a contradiction as

$$L^c\psi_{j+1}(x) \geq 0 \quad \text{for all} \quad x \in \Omega.$$

Therefore we can conclude that the minimum of $\psi_{j+1}(x)$ is non-negative. ■

The local truncation error of the time semi-discretization method, i.e. equation (10), is given by $e_{j+1} \equiv u_{j+1} - \hat{u}_{j+1}$, where $\hat{u}_{j+1}$ is the computed solution of the boundary value problem

$$-\frac{\varepsilon}{2}(\hat{u}_{j+1})_{xx} + \frac{a(x)}{2}(\hat{u}_{j+1})_x + d(x)\hat{u}_{j+1} = g(x, t_{j+1}), \quad 0 < x < 1, \quad t > 0, \tag{14a}$$

$$\hat{u}_{j+1}(0) = 0, \quad \hat{u}_{j+1}(1) = 0, \quad t \geq 0. \tag{14b}$$

The local error estimate of each time step contributes to the global error in the temporal discretization which is defined, at t_j, as $E_j \equiv u(x, t_j) - \hat{u}_j(x)$.

LEMMA 3.2 (Local error estimate) *Suppose that*

$$\left|\frac{\partial^i}{\partial t^i}u(x,t)\right| \le C, \quad (x,t) \in \bar{\Omega} \times [0,T], \quad 0 \le i \le 2.$$

The local error estimate in the temporal direction is given by

$$\|e_{j+1}\|_\infty \le C_1(\Delta t)^3. \tag{15}$$

Proof Using Taylor's theorem we have

$$u(x,t_{j+1}) = u(x,t_{j+1/2}) + \frac{\Delta t}{2}u_t(x,t_{j+1/2}) + \frac{(\Delta t)^2}{4.2!}u_{tt} + O((\Delta t)^3), \tag{16}$$

$$u(x,t_j) = u(x,t_{j+1/2}) - \frac{\Delta t}{2}u_t(x,t_{j+1/2})\frac{(\Delta t)^2}{4.2!}u_{tt} + O((\Delta t)^3), \tag{17}$$

By using equations (16) and (17), we have

$$\begin{aligned}\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} &= u_t\left(x, t_j + \frac{\Delta t}{2}\right) + O((\Delta t)^2),\\ &= \varepsilon\left(u\left(x, t_j + \frac{\Delta t}{2}\right)\right)_{xx} - \frac{a(x)}{2}\left(u\left(x, t_j + \frac{\Delta t}{2}\right)\right)_x\\ &\quad - b(x)u\left(x, t_j + \frac{\Delta t}{2}\right)\\ &\quad f\left(x, t_j + \frac{\Delta t}{2}\right) + O((\Delta t)^2),\end{aligned}$$

where

$$\begin{aligned}f\left(x, t_j + \frac{\Delta t}{2}\right) &= \frac{f(x,t_{j+1}) + f(x,t_j)}{2} + O((\Delta t)^2),\\ u\left(x, t_j + \frac{\Delta t}{2}\right) &= \frac{u(x,t_{j+1}) + u(x,t_j)}{2} + O((\Delta t)^2).\end{aligned}$$

Now it can be seen that the local error is the solution of

$$\begin{aligned}L^c e_{j+1} &= O((\Delta t)^3),\\ e_{j+1}(0) &= 0 = e_{j+1}(1).\end{aligned}$$

Then, the maximum principle on the operator gives the required result. ■

THEOREM 3.3 (Global error estimate) *The global error estimate at t_j as E_j is given by*

$$\|E_j\|_\infty \le C(\Delta t)^2, \quad \forall j \le \frac{T}{\Delta t}. \tag{18}$$

Proof Using the local error estimate up to the jth time step given by Lemma 3.2, we get the following global error estimate at the jth time step:

$$\begin{aligned}\|E_j\|_\infty &= \left\|\sum_{l=1}^{j+1} e_l\right\|_\infty, \quad j \le \frac{T}{\Delta t}, \qquad (19)\\ &\le \|e_1\|_\infty + \|e_2\|_\infty + \cdots + \|e_j\|_\infty,\end{aligned}$$

$$\begin{aligned}&\le C_1(j.\Delta t).(\Delta t)^2, \quad \text{using equation (15)},\\ &\le C_1 T(\Delta t)^2, \quad \text{since} \quad j\cdot\Delta t \le T,\\ &= C(\Delta t)^2, \quad C = C_1 T\end{aligned}$$

where C is a positive constant independent of ε and Δt. ■

4. Asymptotic behaviour of the solution of semi-discrete problems

In order to define the local error bounds to the finite difference scheme in the next section, we need error bounds for the exact solutions of the previous semi-discrete problems.

THEOREM 4.1 *The exact solution of equation* (9) *satisfies*

$$\left|\frac{d^i u_{j+1}}{dx^i}\right| \le C\left(1+\varepsilon^{-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right), \quad 0\le i\le 4. \tag{20}$$

Proof The maximum principle for L^c together with the smoothness requirements imposed on f and on u gives $\|u_{j+1}\| \le C$. The proof for the bounds of its derivatives are given in [19]. ■

4.1 *Decomposition of the solution*

Since the above bounds on the derivatives of the solution are not sharp enough for the proof of ε-uniform convergence, we need to derive stronger bounds. The stronger bounds are now obtained based on a method originally given by Shishkin. This can be achieved by the following decomposition of the solution into a smooth component and a singular component. Let

$$\underbrace{u_{j+1}(x)}_{\text{Solution}} = \underbrace{v_{j+1}(x)}_{\text{Smooth component}} + \underbrace{w_{j+1}(x),}_{\text{Singular component}} \qquad 0\le x\le 1,$$

where the smooth component satisfies the non-homogeneous problem

$$\begin{aligned}&L^c v_{j+1}(x,t) = g(x,t_{j+1}) \quad 0\le x\le 1,\\ &v_{j+1}(0) = u_{j+1}(0),\end{aligned} \tag{21}$$

and the singular component satisfies the homogeneous problem

$$\begin{aligned}&L^c w_{j+1}(x) = 0 \quad 0\le x\le 1,\\ &w_{j+1}(0) = 0,\\ &w_{j+1}(1) = u_{j+1}(1) - v_{j+1}(1).\end{aligned} \tag{22}$$

We take further decomposition in the smooth component v_{j+1}

$$v_{j+1}(x) = ((v_{j+1})_0 + \varepsilon(v_{j+1})_1 + \varepsilon^2(v_{j+1})_2)(x), \quad 0\le x\le 1,$$

where $(v_{j+1})_0$ is the solution of the reduced problem and v_1 and v_2 are the solutions of the following equations (23) and (24), respectively

$$a(x)((v_{j+1})_1)_x + d(x)(v_{j+1})_1 = ((v_{j+1})_0)_{xx}, \quad 0 \le x \le 1 \tag{23}$$
$$(v_{j+1})_1(0) = 0,$$
$$L^c(v_{j+1})_2(x) = ((v_{j+1})_1)_{xx}, \quad 0 \le x \le 1 \tag{24}$$
$$(v_{j+1})_2(0) = 0 = (v_{j+1})_2(1).$$

Clearly we have

$$\begin{aligned} L^c v_{j+1}(x) &= g(x, t_{j+1}) \quad 0 \le x \le 1, \\ v_{j+1}(0) &= u_{j+1}(0), \\ v_{j+1}(1) &= ((v_{j+1})_0 + \varepsilon(v_{j+1})_1)(1). \end{aligned}$$

THEOREM 4.2 *Let v_{j+1} be the solution of equation* (21). *Then v_{j+1} and its derivatives satisfy the bounds*

$$\|v_{j+1}^{(k)}\| \le C(1 + \varepsilon^{2-k}), \quad k = 0, 1, 2. \tag{25}$$

In general, the bounds on the derivatives of v_{j+1} satisfy

$$\|v_{j+1}^{(k)}\| \le C, \quad k = 0, 1, 2, 3. \tag{26}$$

Proof Since $(v_{j+1})_0$ and $(v_{j+1})_1$ are the solutions of equations (6) and (23), respectively, which are independent of ε, therefore for all non-negative integers k such that $0 \le k \le 3$,

$$\|(v_{j+1}^{(k)})_0\| \le C, \tag{27}$$

and

$$\|(v_{j+1}^{(k)})_1\| \le C. \tag{28}$$

As $(v_{j+1})_2$ is the solution of equation (24), by using Theorem 4.1 for all non-negative integers k such that $0 \le k \le 3$ we have,

$$\|(v_{j+1}^{(k)})_2\| \le C(1 + \varepsilon^{-k}\mathrm{e}^{-\alpha(1-x)/\varepsilon}). \tag{29}$$

Now combining the above three estimates for all non-negative integers k such that $0 \le k \le 3$, gives

$$\begin{aligned} \left\|v_{j+1}^{(k)}\right\| &\le \left\|\left(v_{j+1}^{(k)}\right)_0\right\| + \varepsilon\left\|\left(v_{j+1}^{(k)}\right)_1\right\| + \varepsilon^2\left\|\left(v_{j+1}^{(k)}\right)_2\right\| \\ &\le C + \varepsilon C + \varepsilon^2 C(1 + \varepsilon^{-k}\mathrm{e}^{-\alpha(1-x)/\varepsilon}) \\ &\le C(1 + \varepsilon^{2-k}\mathrm{e}^{-\alpha(1-x)/\varepsilon}) \\ &\le C(1 + \varepsilon^{2-k}), \quad \text{since} \quad \mathrm{e}^{-\alpha(1-x)/\varepsilon} \le 1, \\ &\le C, \text{ for } k = 0, 1, 2. \end{aligned}$$

The same argument works for $k = 3$ [20]. ■

A sharper bound on the singular component w_{j+1}, is given by:

THEOREM 4.3 *Let $w_{j+1}(x)$ be the solution of equation (22). The bounds of w_{j+1} and its derivatives satisfy the following estimate*

$$|w_{j+1}(x)| \leq C\varepsilon^{-k}\mathrm{e}^{-\alpha(1-x)/\varepsilon}, \quad k = 0, 1, 2, 3 \quad 0 \leq x \leq 1. \tag{30}$$

Proof First we prove the result for $k = 0$. Consider the barrier functions

$$\psi^{\pm}_{j+1}(x) = C\mathrm{e}^{-\alpha(1-x)/\varepsilon} \pm w_{j+1}(x), \quad 0 \leq x \leq 1.$$

The values of $\psi^{\pm}_{j+1}(x)$ at the boundaries are

$$\begin{aligned}
\psi^{\pm}_{j+1}(0) &= C\mathrm{e}^{-\alpha/\varepsilon} \pm w_{j+1}(0),\\
&= C\mathrm{e}^{-\alpha/\varepsilon},\\
&\geq 0,\\
\psi^{\pm}_{j+1}(1) &= w_{j+1}(1),\\
&\geq 0, \quad \text{by choosing } C \text{ sufficiently large.}
\end{aligned}$$

Thus by the above estimates we have

$$\begin{aligned}
\psi^{\pm}_{j+1}(x) &\geq 0, \quad 0 \leq x \leq 1\\
L^c\psi^{\pm}_{j+1}(x) &= \left(-\frac{\varepsilon}{2}\left(\psi^{\pm}_{j+1}\right)_{xx} + \frac{a(x)}{2}\left(\psi^{\pm}_{j+1}\right)_{x} + d(x)\psi^{\pm}_{j+1}\right)(x), \quad \forall x \in [0,1]\\
&\geq C\mathrm{e}^{-\alpha(1-x)/\varepsilon}\left[-\frac{\alpha^2}{2\varepsilon} + \frac{a(x)\alpha}{2\varepsilon} + d(x)\right],\\
&\geq C\mathrm{e}^{-\alpha(1-x)/\varepsilon},\\
&\geq 0, \quad \forall x \in [0,1].
\end{aligned}$$

Now by using maximum principle (Lemma 3.1) on the operator L^c, we get

$$\psi^{\pm}_{j+1}(x) \geq 0, \quad 0 \leq x \leq 1, \quad \text{i.e.}$$

$$|w_{j+1}(x)| \leq C\mathrm{e}^{-\alpha(1-x)/\varepsilon}, \quad 0 \leq x \leq 1.$$

For $k = 1, 2, 3$, the proof follows in a similar way [9]. ■

5. Discretization in the spatial direction

5.1 *Shishkin mesh*

Shishkin meshes are piecewise-uniform meshes which condense approximately in the boundary layer regions as $\varepsilon \to 0$. This is accomplished by the use of the transition parameter τ, which depends naturally on ε, and crucially on N.

Thus for a given N and ε, the interval [0, 1] is divided into parts, $[0, 1-\tau]$, $[1-\tau, 1]$ where the transition point τ is given by

$$\tau \equiv \min\left\{\frac{1}{2}, m\varepsilon\ln N\right\},$$

where m is a constant which we choose such that $m \geq 1/\alpha$. It is clear that when $\tau = (1/2)$ the mesh is uniform otherwise the mesh condenses near the right boundary. The value of the constant C depends on the scheme being used.

Define the fitted piecewise-uniform mesh (Shishkin mesh) that discretizes the interval [0, 1] with N piecewise uniform subintervals as

$$h_i = x_i - x_{i-1} = \begin{cases} H = \dfrac{2(1-\tau)}{N} & \text{if } 0 \le i \le \dfrac{N}{2}; \\ h = \dfrac{2\tau}{N} & \text{if } \dfrac{N}{2} < i \le N, \end{cases}$$

and the piecewise-uniform mesh $\bar{\Omega}^N$ with the spatial nodal values x_i for $i = 0, 1, \ldots, N$ is given as

$$\bar{\Omega}_\tau^N = \left\{ x_i : x_i = \begin{cases} \dfrac{2(1-\tau)}{N} i & \text{for } 0 \le i \le \dfrac{N}{2}; \\ (1-\tau) + \dfrac{2\tau}{N}\left(i - \dfrac{N}{2}\right) & \text{for } \dfrac{N}{2} < i \le N \end{cases} \right. .$$

5.2 *Difference scheme*

Now we define the finite difference approximation of problems (9) using modified upwind scheme, i.e. the midpoint upwind scheme on a piecewise uniform mesh of Shishkin type $\bar{\Omega}^N = \{x_i\}_{i=0}^{i=N}$.

$$U_0^N(x_i) = u_0(x_i), \quad x_i \in \Omega^N. \tag{31a}$$

$$\frac{U_{j+1}^N(x_i) - U_j^N(x_i)}{\Delta t} - \frac{\varepsilon}{2}\delta^2\left(U_{j+1}^N(x_i) + U_j^N(x_i)\right) + \frac{a(x_{i-1/2})}{2}\left(D^- U_{j+1}^N(x_i) + D^- U_j^N(x_i)\right)$$
$$+ \frac{b(x_{i-1/2})}{2}\left(U_{j+1}^N(x_i) + U_j^N(x_i)\right) = \frac{f(x_{i-1/2}, t_{j+1}) + f(x_{i-1/2}, t_j)}{2}, \tag{31b}$$

with boundary conditions

$$U_{j+1}^N(0) = 0, \quad U_{j+1}^N(1) = 0 \tag{31c}$$

where $U_{j+1}^N(x_i)$ is the approximate solution of $u_{j+1}(x)$ at the point x_i, $i = 0, 1, \ldots, N$.

Rewrite the above equations in the following form

$$U_0^N(x_i) = u_0(x_i), \quad x_i \in \Omega^N. \tag{32a}$$

$$\begin{aligned} &- \frac{\varepsilon}{2}\delta^2 U_{j+1}^N(x_i) + \frac{a(x_{i-1/2})}{2} D^- U_{j+1}^N(x_i) + \frac{d(x_{i-1/2})}{2} U_{j+1}^N(x_i) \\ &= (f(x_{i-1/2}, t_{j+1}) + f(x_{i-1/2}, t_j))/2 + \frac{\varepsilon}{2}\delta^2 U_j^N(x_i) \\ &- \frac{a(x_{i-1/2})}{2} D^- U_j^N(x_i) - \frac{c(x_{i-1/2})}{2} U_j^N(x_i), \end{aligned} \tag{32b}$$

with boundary conditions

$$U_{j+1}^N(0) = 0, \quad U_{j+1}^N(1) = 0. \tag{32c}$$

Write the above equation (32) in operator form

$$U_0^N(x_i) = u_0(x_i), \quad x_i \in \Omega^N. \tag{33a}$$

$$L^{c,N} U_{j+1}^N(x_i) = g_{j+1}(x_{i-1/2}) \quad x_i \in \Omega^N, \tag{33b}$$

$$U_{j+1}^N(0) = 0, \quad U_{j+1}^N(1) = 0. \tag{33c}$$

where $g_{j+1}(x_{i-1/2}) = (f(x_{i-1/2}, t_{j+1}) + f(x_{i-1/2}, t_j))/2 + (\varepsilon/2)\delta^2 U_j^N(x_i) - (a(x_{i-1/2})/2)$ $D^- U_j^N(x_i)$ $-(c(x_{i-1/2})/2)U_j^N(x_i)$, and the operator $L^{c,N}$ is given by

$$L^{c,N} \equiv -\frac{\varepsilon}{2}\delta^2 + \frac{a(x_{i-1/2})}{2}D^- + \frac{d(x_{i-1/2})}{2}I, \tag{34}$$

$d(x_{j-1/2}) = (b(x_{i-1/2})/2 + (1/\Delta t))$, $c(x_{j-1/2}) = (b(x_{i-1/2})/2 - (1/\Delta t))$, $a(x_{i-1/2}) = (a(x_{i-1}) + a(x_i)/2)$, $b(x_{i-1/2}) = (b(x_{i-1}) + b(x_i)/2)$ and $f_{j+1}(x_{i-1/2}) = (f_{j+1}(x_{i-1}+) f_{j+1}(x_i)/2)$.

The first- and second-order differences are defined by

$$D^+ Z_{i,j} = \frac{Z_{i+1,j} - Z_{i,j}}{h_{i+1}}, \quad D^- Z_{i,j} = \frac{Z_{i,j} - Z_{i-1,j}}{h_i},$$
$$\delta^2 Z_{i,j} = \frac{(D^+ - D^-)Z_{i,j}}{\bar{h}_i}, \quad \bar{h}_i = \frac{2}{h_i + h_{i+1}}.$$

6. Stability and convergence analysis

LEMMA 6.1 (Discrete maximum principle) *Assume that* $\Delta t(4\varepsilon/h_i h_{i+1} + (a_{i-1} + a_i)/h_i + (b_{i-1} + b_i)) \le 4$. *Then the discrete operator* $L^{c,N}$ *satisfies a discrete maximum principle, i.e. if* $\{\phi_i\}$ *and* $\{\psi_i\}$ *are mesh functions satisfying* $\phi_0 \le \psi_0$ *and* $L^{c,N}\phi_i \le L^{c,N}\psi_i$ *for* $i = 1, 2, \ldots, N-1$, *then* $\phi_i \le \psi_i$ *for all* i.

Proof The matrix associated with operator $L^{c,N}$ is of type $(N+1) \times (N+1)$ and satisfies the properties of being M matrix under the given conditions in Lemma 6.1. ■

LEMMA 6.2 *Let* $Z_{j+1}(x_i) = 1 + x_i$ *for* $0 \le i \le N$. *Then there exists a positive constant* C *such that* $L^{c,N} Z_{j+1}(x_i) \ge C$ *for* $1 \le i \le N-1$.

Proof Applying the discrete operator $L^{c,N}$ on the mesh function $Z_{j+1}(x_i)$, we have

$$\begin{aligned}
L^{c,N} Z_{j+1}(x_i) &= -\frac{\varepsilon}{2}\delta^2 Z_{j+1}(x_i) + \frac{a(x_{i-1/2})}{2} D^- Z_{j+1}(x_i) \\
&\quad + d(x_{i-1/2}) Z_{j+1}(x_i), \quad 0 \le i \le N, \\
&= -\frac{\varepsilon}{2}\delta^2(1 + x_i) + \frac{a(x_{i-1/2})}{2} D^-(1 + x_i) \\
&\quad + d(x_{i-1/2})(1 + x_i), \quad 0 \le i \le N, \\
&= 0 + \frac{a(x_{i-1/2})}{+} d(x_{i-1/2})(1 + x_i), \\
&\ge \alpha + 2d(x_{i-1/2}), \\
&\ge C, \quad \text{since} \quad d(x_{i-1/2}) \quad \text{is bounded}
\end{aligned}$$

where C is a positive constant. ■

LEMMA 6.3 *For* $i = 0, \ldots, N$, *and for fixed* j, *define the mesh function*

$$S_{j+1}(x_i) = \prod_{k=1}^{i} \left(1 + \frac{\alpha h_k}{\varepsilon}\right)$$

with the usual convention that if $i = 0$, *then* $S_0 = 1$. *Then for* $i = 1, \dots, N-1$, *we have*

$$L^{c,N} S_{j+1}(x_i) \le \frac{C}{\max\{\varepsilon, h_i\}} S_{j+1}(x_i),$$

for some constant C.

Proof Now $(S_{j+1}(x_i) - S_{j+1}(x_{i-1}))/h_i = \alpha S_{j+1}(x_{i-1})/\varepsilon$, so

$$\begin{aligned} L^{c,N} S_{j+1}(x_i) &= -\frac{\varepsilon}{h_i + h_{i+1}} \frac{\alpha(S_{j+1}(x_i) - S_{j+1}(x_{i-1}))}{\varepsilon} + \left(\frac{a_{i-1/2}}{2}\right) \frac{\alpha S_j(x_{i-1})}{\varepsilon} \\ &\quad + d_{i-1/2} S_{j+1}(x_i), \\ &= \frac{\alpha}{2\varepsilon} S_{j+1}(x_{i-1}) \left(a_{i-1/2} - \frac{2\alpha h_i}{h_i + h_{i+1}}\right) + d_{i-1/2} S_{j+1}(x_i), \\ &= \frac{\alpha}{2(\varepsilon + \alpha h_i)} S_{j+1}(x_i) \left(a_{i-1/2} - \frac{2\alpha h_i}{h_i + h_{i+1}} + d_{i-1/2} \frac{\varepsilon + \alpha h_i}{\alpha}\right), \end{aligned}$$

from which the result follows. ∎

LEMMA 6.4 *Let* $r_{j+1}(x)$ *be any smooth function defined on* $[0, 1]$. *For* $i = 1, \dots, N-1$, *define the truncation error of* $L^{c,N}$ *to* $r_{j+1}(x)$ *at* $x_{i-1/2}$ *to be* $L^{c,N}(r_{j+1}(x_i) - (L^c r_{j+1})(x_{i-1/2})$. *Then there exists a constant* C *such that*

$$|L^{c,N}(r_{j+1}(x_i)) - (L^c r_{j+1})(x_{i-1/2})| \le C\varepsilon \int_{x_{i-1}}^{x_{i+1}} |r'''_{j+1}(\xi)|\, d\xi + Ch_i \int_{x_{i-1}}^{x_i} |r'''_{j+1}(\xi)|\, d\xi. \tag{35}$$

Proof The proof is similar to the proof of Lemma 3.3 in [21]. ∎

LEMMA 6.5 *For each* i, *we have*

$$e^{-\alpha(1-x_i)/\varepsilon} \le \prod_{k=i+1}^{N} \left(1 + \frac{\alpha h_k}{\varepsilon}\right).$$

Proof For each k, we have

$$e^{-\alpha h_k/\varepsilon} = (e^{\alpha h_k})^{-1} \le \left(1 + \frac{\alpha h_k}{\varepsilon}\right).$$

On multiplying these above inequalities for $k = i+1, \dots, N$, we get the desired result. ∎

LEMMA 6.6 *For the Shishkin mesh defined above, there exists a constant* C *such that*

$$\prod_{k=i+1}^{N} \left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1} \le C N^{-4(1-i/N)} \quad \frac{N}{2} \le i \le N.$$

Proof Suppose $N/2 \le i \le N$. By Lemma 4.1(b) in Kellogg and Tsan [21],

$$\begin{aligned}\prod_{k=i+1}^{N}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1} &\le \mathrm{e}^{-\alpha(1-x_i)}/(\varepsilon+\alpha h)\\ &= \mathrm{e}^{-4(N-i)(N^{-1}\ln N)/(1+4N^{-1}\ln N))}\\ &= N^{-4(N-i)N^{-1}/(1+4N^{-1}\ln N))}\\ &= N^{-4(1-i/N)}N^{16(1-i/N)(N^{-1}\ln N)/(1+4N^{-1}\ln N)}.\end{aligned}$$

Since $N^{16(1-i/N)(N^{-1}\ln N)/(1+4N^{-1}\ln N)}$ is bounded for $N \ge 2$, therefore we have the desired result. ■

6.1 *Decomposition of numerical solution*

In this section we decompose the numerical solution U^N_{j+1} into smooth and singular components in a similar way to the continuous case,

$$\underbrace{U^N_{j+1}(x_i)}_{\text{Numerical Solution}} = \underbrace{V^N_{j+1}(x_i)}_{\text{Smooth Component}} + \underbrace{W^N_{j+1}(x_i)}_{\text{Singular Component}},$$

where the smooth component $V^N(x_i)$ satisfies the non-homogeneous equation

$$\begin{aligned} L^{c,N}V^N_{j+1}(x_i) &= g(x_{i-1/2}, t_{j+1}) \\ V^N_{j+1}(0) &= v_{j+1}(0),\\ V^N_{j+1}(N) &= v_{j+1}(N),\end{aligned} \tag{36}$$

and the singular component $W^N_{j+1}(x_i)$ satisfies the homogeneous equation

$$\begin{aligned} L^{c,N}W^N_{j+1}(x_i) &= 0 \\ W^N_{j+1}(0) &= w_{j+1}(0),\\ W^N_{j+1}(N) &= w_{j+1}(N).\end{aligned} \tag{37}$$

Now the error in the numerical solution can also be decomposed as

$$(U^N_{j+1} - u_{j+1})(x_i) = (V^N_{j+1} - v)(x_i) + (W^N_{j+1} - w_{j+1})(x_i).$$

We estimate the error in the smooth and singular solution separately.

THEOREM 6.7 (Error in the smooth component) *The error in the smooth component V^N_{j+1} satisfies the following estimate at the $(j+1)$th time level*:

$$|V^N_{j+1}(x_i) - v_{j+1}(x_i)| \le CN^{-1}(\varepsilon + N^{-1}), \quad x_i \in \bar{\Omega}^N. \tag{38}$$

Proof For the error in the smooth part

$$|L^{c,N}(v_{j+1}(x_i) - V_{j+1}^N(x_i))| = |L^{c,N}v_{j+1}(x_i) - (L^c v)_{j+1}(x_{i-1/2})|,$$
$$\leq C(\varepsilon + h_i)(h_i + h_{i+1}) \text{ by using Lemma 6.4.}$$

Hence using the barrier function $\psi_{j+1}(x_i) = CH(\varepsilon + H)(1 + x_i)$, where $H = \max_i\{h_i\}$ and applying Lemma 6.1 and Lemma 6.2, we have

$$|v_{j+1}(x_i) - V_{i,j+1}^N| \leq CH(\varepsilon + H).$$
$$= CN^{-1}(\varepsilon + N^{-1}). \quad \blacksquare$$

THEOREM 6.8 (Error in the singular component) *The error in the singular component satisfies the following bound*

$$|w_{j+1}(x_i) - W_{j+1}^N(x_i)| \leq \begin{cases} CN^{-2}, & \text{for } 0 \leq i < \dfrac{N}{2}, \\ CN^{-1}(\varepsilon + N^{-4(1-i/N)\ln N}) & \text{for } \dfrac{N}{2} \leq i \leq N. \end{cases} \quad (39)$$

Proof Consider the barrier function $\psi_{j+1}(x_i)$

$$\psi_{j+1}(x_i) = C\left[\prod_{k=1}^{N}\left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1}\right] S_{j+1}(x_i).$$

Now using the discrete maximum principle, Lemma 6.1, we have

$$|W_{j+1}^N(x_i)| \leq \psi_{j+1}(x_i) = C\left[\prod_{k=1}^{N}\left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1}\right]. \quad (40)$$

The estimate $|w_{j+1}(x_i) - W_{j+1}^N(x_i)|$ can be written as:

$$|w_{j+1}(x_i) - W_{j+1}^N(x_i)| \leq |w_{j+1}(x_i)| + |W_{j+1}^N(x_i)|$$
$$\leq Ce^{-\alpha(1-x_i)/\varepsilon} + C\prod_{k=1}^{N}\left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1}$$
$$\text{by Theorem 4.3, equation (40)}$$
$$\leq C\prod_{k=i+1}^{N}\left(1 + \frac{\alpha h_k}{\varepsilon}\right)^{-1} \text{ by Lemma 6.5.}$$

In particular for $0 < i \leq N/2$, this inequality becomes

$$|w_{j+1}(x_i) - W_{j+1}^N(x_i)| \leq CN^{-2} \text{ using Lemma 6.6.} \quad (41)$$

Taking $i = N/2$ in equation (41), we have

$$|w_{j+1}(x_{N/2}) - W_{j+1}^N(x_{N/2})| \leq CN^{-2}.$$

Furthermore,

$$|w_{j+1}(x_N) - W_{j+1}^N(x_N)| = 0.$$

Some modifications in Lemma 6.4 yield

$$|L^{c,N}(r_{j+1}(x_i)) - (L^c r_{j+1})(x_{i-1/2})| \le C \int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon |r'''_{j+1}(\xi)| + |r''_{j+1}(\xi)|\right] d\xi, \quad \frac{N}{2} < i < N, \tag{42}$$

for $N/2 < i < N$,

$$\begin{aligned} |L^{c,N}(w_{j+1}(x_i) - W^N_{j+1}(x_i))| &\le C\varepsilon^{-1}[\mathrm{e}^{-\alpha(1-x_{i+1})/\varepsilon} - \mathrm{e}^{\alpha(1-x_{i-1})/\varepsilon}] \quad \text{using equation (42)},\\ &= C\varepsilon^{-1}\mathrm{e}^{-\alpha(1-x_i)/\varepsilon}(\mathrm{e}^{\alpha h/\varepsilon} - \mathrm{e}^{-\alpha h/\varepsilon}),\\ &= C\varepsilon^{-1}\mathrm{e}^{-\alpha(1-x_i)/\varepsilon}\sinh(\alpha h/\varepsilon),\\ &\le (C\varepsilon^{-1}N^{-1}\ln N)\mathrm{e}^{-\alpha(1-x_{i+1})/\varepsilon}, \quad \sinh t \le Ct, \quad 0 \le t \le 1,\\ &\le (C\varepsilon^{-1}N^{-1}\ln N)\prod_{k=i+1}^{N}\left(1 + \frac{\alpha h}{\varepsilon}\right)^{-1}, \quad \text{by Lemma 6.6.} \end{aligned}$$

Construct the barrier function ψ_{j+1}

$$\psi_{j+1}(x_i) = C\left\{N^{-2} + (N^{-1}\ln N)\left[\prod_{k=1}^{N}(1 + \frac{\alpha h_k}{\varepsilon})^{-1}\right] S_{j+1}(x_i)\right\},$$

for $N/2 \le i \le N$ and sufficiently large C. Then by Lemma 6.1, we have

$$|w_{j+1}(x_i) - W^N_{j+1}(x_i)| \le \psi_{j+1}(x_i).$$

Using Lemma 6.6 for $i = N/2, \dots, N$, we have

$$|w_{j+1}(x_i) - W^N_{j+1}(x_i)| \le C\max\{N^{-2}, N^{-5+4i/N\ln N}\}. \tag{43}$$

Combining equations (41) and (43) yields the required result. ■

THEOREM 6.9 *The error in the solution satisfies the following estimate*

$$|u_{j+1}(x_i) - U^N_{i,j+1}| \le \begin{cases} CN^{-1}(\varepsilon + N^{-1}) & \text{for } 0 \le i \le \dfrac{N}{2},\\ CN^{-1}(\varepsilon + N^{-4(1-i/N)\ln N}) & \text{for } \dfrac{N}{2} \le i \le N. \end{cases} \tag{44}$$

Proof Combining the estimate for the error given by Theorem 6.7 in the smooth component and by Theorem 6.8 in the singular component gives the result. ■

THEOREM 6.10 *The solution of problems* (1), (9) *and* (34) *given by* $u(x_i, t_{j+1})$, $u_{j+1}(x_i)$ *and* $U^N_{j+1}(x_i)$ *respectively, satisfies the following error estimate*

$$\|u(x_i, t_{j+1}) - U^N_{j+1}(x_i)\| \le \begin{cases} (\Delta t)^2 + CN^{-1}(\varepsilon + N^{-1}) & \text{for } 0 \le i \le \dfrac{N}{2},\\ (\Delta t)^2 + CN^{-1}(\varepsilon + N^{-4(1-i/N)\ln N}) & \text{for } \dfrac{N}{2} \le i \le N. \end{cases} \tag{45}$$

Proof

$$
\begin{aligned}
\|u(x_i,t_{j+1})-U^N_{j+1}(x_i)\| &= \|u(x_i,t_{j+1})-u_{j+1}(x_i)+u_{j+1}(x_i)-U^N_{j+1}(x_i)\|,\\
&\leq \|u(x_i,t_{j+1})-u_{j+1}(x_i)\|+\|u_{j+1}(x_i)-U^N_{j+1}(x_i)\|,\\
&\leq \begin{cases} (\Delta t)^2 + CN^{-1}(\varepsilon+N^{-1}), & \text{for } \; 0\leq i\leq \dfrac{N}{2},\\[2mm] (\Delta t)^2 + CN^{-1}(\varepsilon+N^{-4(1-i/N)\ln N}), & \text{for } \; \dfrac{N}{2}\leq i\leq N,\end{cases}
\end{aligned}
$$

by Theorems 3.3 and 6.9. ∎

7. Numerical results

In this section we present the numerical results which validate the theoretical results. Nevertheless, it is seen that the numerical behaviour of the proposed method using a fitted piecewise uniform mesh is ε-uniform. The problem is solved using the proposed method comprising Crank–Nicolson time discretization and midpoint upwind finite difference operators on a piecewise uniform mesh, i.e. a Shishkin mesh with N points. The Shishkin mesh used in these computations is of the form described in section 5 and so is condensed on the right side boundary $x = 1$. In all the examples we begin with $N = 8$ with time step $\Delta t = 0.1$ and we multiply N by two.

We will show computationally that the numerical solutions given by the proposed method converge uniformly with respect to ε.

Example 1 In this example, we take $a(x) = 2 - x^2, b(x) = x, f = 10t^2 e^{-t} x(1 - x), u_0 = 0$ and $T = 2$.

Table 1. Maximum pointwise errors by using the proposed method on Shishkin mesh for Example 1 for various values of ε and N.

	$\varepsilon = 10^{-6}$	
N	Maximal error for $0 \leq i \leq N/2$	Maximal error for $N/2 < i$
8	1.072573E−002	6.507748E−002
16	3.902427E−003	3.216956E−002
32	1.371070E−003	2.080475E−002
64	5.608386E−004	1.402070E−002
128	2.577317E−004	9.734363E−003
256	1.252182E−004	6.376307E−003
	$\varepsilon = 10^{-12}$	
N	Maximal error for $0 \leq i \leq N/2$	Maximal error for $N/2 < i$
8	1.072598E−002	6.507757E−002
16	3.902411E−003	3.216808E−002
32	1.370903E−003	2.080332E−002
64	5.605606E−004	1.401936E−002
128	2.573487E−004	9.733703E−003
256	1.247449E−004	6.375862E−003

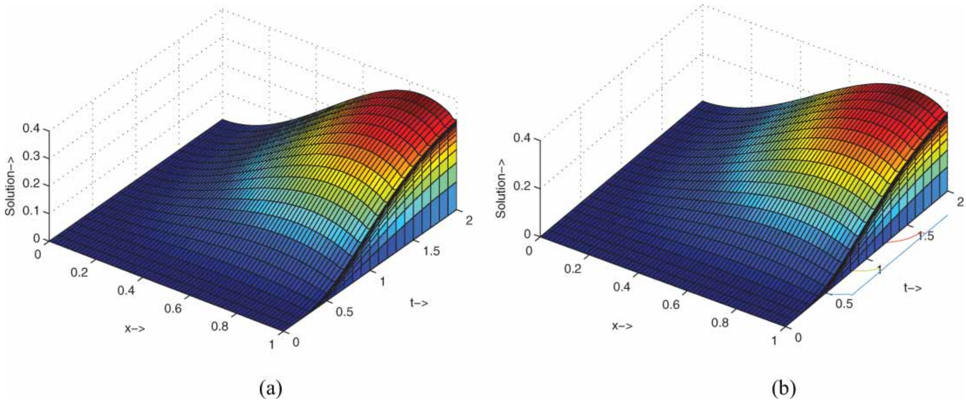

Figure 1. Numerical solutions for Example 1 for $N = 128$, $\Delta t = 0.1$ (a) for $\varepsilon = 10^{-6}$ and (b) for $\varepsilon = 10^{-12}$.

The exact solution for this example is not known and we estimate the maximum pointwise errors E_ε^N by

$$E_\varepsilon^N = \max_{\Omega^N} |U^N(x_i, t_j) - U^{2N}(x_i, t_j)|,$$

where $U^{2N}(x_i, t_j)$ is the computed solution corresponding to $2N$ points. The computed errors for this example are given in table 1 by using the proposed method on a fitted piecewise uniform mesh, i.e. a Shishkin mesh for different values of ε and N. Computed results are plotted in figure 1.

Example 2 We take $a(x) = 2 - x^2$, $b(x) = x^2 + 1 + \cos(\pi x)$, $f = 10t^2 e^{-t} x(1 - x)$, $u_0 = 0$ and $T = 1$.

Table 2. Maximum pointwise errors using the proposed method on a Shishkin mesh for Example 2 for various values of ε and N.

	$\varepsilon = 10^{-6}$	
N	Maximal error for $0 \le i \le N/2$	Maximal error for $N/2 < i$
8	1.255190E−002	2.733708E−002
16	6.146872E−003	1.420798E−002
32	2.807309E−003	9.267409E−003
64	1.333453E−003	5.863876E−003
128	6.487930E−004	3.497379E−003
256	3.198603E−004	1.969163E−003
	$\varepsilon = 10^{-12}$	
N	Maximal error for $0 \le i \le N/2$	Maximal error for $N/2 < i$
8	1.255208E−002	2.733715E−002
16	6.146877E−003	1.420822E−002
32	2.807334E−003	9.267692E−003
64	1.333468E−003	5.864240E−003
128	6.488029E−004	3.497815E−003
256	3.198657E−004	1.969650E−003

As the exact solution of this problem is not known, again we estimate the maximum pointwise errors by

$$E_\varepsilon^N = \max_{\Omega^N} |U^N(x_i, t_j) - U^{2N}(x_i, t_j)|.$$

Computed errors for this example are given in table 2 by using the proposed method on a fitted piecewise uniform mesh, i.e. a Shishkin mesh for different values of ε and N. Numerical results generated by the proposed method are shown in figure 2.

Example 3 In this example we take $a(x) = 1 + x^2 + x, b(x) = 1 + x^2, f(x) = \sin(\pi x(1 - x)), u_0 = 0$ and $T = 1$.

For this example we have computed the maximum pointwise errors by

$$E_\varepsilon^N = \max_{\Omega^N} |U^N(x_i, t_j) - U^{2N}(x_i, t_j)|$$

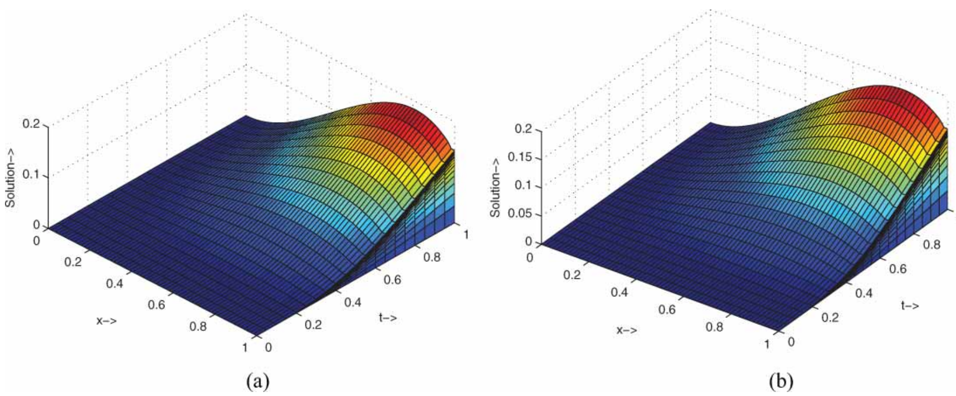

Figure 2. Numerical solutions for Example 2 for $N = 128$, $\Delta t = 0.1$ (a) for $\varepsilon = 10^{-6}$ and (b) for $\varepsilon = 10^{-12}$.

Table 3. Maximum pointwise errors using the proposed method on a Shishkin mesh for Example 3 for various values of ε and N.

	$\varepsilon = 10^{-6}$	
N	Maximal error for $0 \le i \le N/2$	Maximal error for $N/2 < i$
8	9.329105E−002	2.520934E−001
16	4.880965E−002	1.181521E−001
32	1.815794E−002	7.129879E−002
64	8.818542E−003	4.428023E−002
128	5.229191E−003	2.721459E−002
256	2.892490E−003	1.636164E−002
	$\varepsilon = 10^{-12}$	
N	Maximal error for $0 \le i \le N/2$	Maximal error for $N/2 < i$
8	9.329105E−002	2.520920E−001
16	4.880947E−002	1.181506E−001
32	1.815785E−002	7.129768E−002
64	8.818712E−003	4.427945E−002
128	5.229338E−003	2.721409E−002
256	2.892585E−003	1.636135E−002

for $\Delta t = 0.2$ and different values of N. Computed maximum pointwise errors are displayed in table 3 for various values of ε and N. It is clear from the tabulated values that our proposed method works well in the coarse mesh and validates the theoretical results.

8. Conclusions

In this paper we propose a numerical scheme for solving a singularly perturbed parabolic initial-boundary value problem with boundary layer on the right side of the domain. The method comprises Crank–Nicolson discretization in time and midpoint-upwind discretization on a non-uniform mesh, i.e. a Shishkin mesh in the spatial direction and a uniform mesh in the temporal direction. The method is almost second-order accurate in the coarse mesh and almost first order in the fine mesh in the spatial direction and second order in the time direction. An extensive amount of analysis has been carried out to obtain the parameter uniform error estimates.

In support of the predicted theory some test examples are solved using the proposed method. To illustrate the performance of the proposed method, the maximum pointwise errors are given in tables 1–3 in coarse and fine mesh separately. Numerical solution profiles are given in figures 1 and 2, which show the physical behaviour of the computed solution obtained by the proposed scheme. The errors in tables 1–3 show that the proposed method converges uniformly with respect to the perturbation parameter and the convergence behaviour matched the theoretical result.

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