2010 5th International Conference on Industrial and Information Systems, ICIIS 2010, Jul 29 - Aug 01, 2010, India

Designing the Inverse of an LPTV System Using $\mathcal{H}_\infty$ Optimization

Siddhartha Sankar Bora
Department of Electronics and Communication Engineering
National Institute of Technology
Warangal, India 506004
Email: siddharth.nitw@gmail.com

Ravindra Dhuli and Brejesh Lall
Department of Electrical Engineering
Indian Institute of Technology
Delhi, India, 110016
Email: ravindra_dhuli@yahoo.co.in
brejesh@ee.iitd.ac.in

***Abstract*—This paper investigates the problem of designing the inverse of a linear periodic time varying (LPTV) system. Input and output switching representations of LPTV systems are useful in reducing the problem to that of designing the synthesis filters of the uniform filter bank (UFB) for perfect reconstruction. We use the fact that the cascade connection of the LPTV system and its inverse has to result in a finite delay system. Our design aims at minimizing the worst case error norm. This problem is solved by minimizing the $\mathcal{H}_\infty$ norm of the corresponding error system. For the choice of FIR synthesis systems, the problem is modeled as a standard Linear Matrix Inequalities(LMI) problem.**

I. INTRODUCTION

Linear periodically time varying systems (LPTV) are an important class of multirate systems having applications in the areas including but not limited to blind channel estimation [1], speech scrambling [2]. The task of finding the inverse of an LPTV is an important requirement in many applications like scrambled signal recovery [2] and equalization based on modulation induced cyclo-stationarity [1]. So, the problem of LPTV inversion has drawn a lot of research attention [3], [4], [5], [6].

The output of the LPTV contains aliasing terms and LPTV inversion is equivalent to alias cancelation [3]. The switching representations of the LPTV gives better insight into the system behavior. The alias terms in the output are quantified using these representations. To design the inverse of an LPTV we consider the cascade of two LTPV systems. First LPTV in the cascade is represented by the output switching representation and the second LPTV is represented with input switching representation. Using the standard multirate building blocks, we observe that the cascade is equivalent to an UFB [7]. The analysis stage is directly related to the first LPTV whereas the synthesis stage is directly related to the second LPTV. Now, if the two LPTV systems are to be the inverse of each other then the corresponding UFB should give perfect reconstruction. This implies that if we design the synthesis stage for perfect reconstruction, the problem of inverting the LTPV is solved. But an attempt to reduce the UFB directly to an ideal delay is not possible for an arbitrary analysis stage. Hence we adopt a different approach, where in we minimize the worst case error norm. The problem is converted to the standard $\mathcal{H}_\infty$ optimization in the control literature [8]. For the choice of FIR synthesis filters the problem is solvable using the LMI tool box in MATLAB.

This paper is organized as follows: Section II contains a brief review of the LPTV systems and their input and output switching representations. In section III, the problem of designing the inverse of the LPTV is reduced to that of a synthesis filter bank design. In section IV, the problem is modeled into an $\mathcal{H}_\infty$ optimization problem. In section V, an example to highlight the results is included. Section VI contains the conclusions.

A. Notations

Capital bold letters are reserved for matrices. The z-transform of $x(n)$ is denoted by $X(z)$. The modulo-M operation is denoted by $<.>_M$. The state space representation of $\mathbf{H}(z)$ is denoted as $\mathbf{H}(z) = \{A_\mathbf{H}, B_\mathbf{H}, C_\mathbf{H}, D_\mathbf{H}\}$

II. REVIEW OF LPTV SYSTEMS

In this section we review the basics of LPTV systems. The switching representations and their corresponding multirate equivalents are also discussed [9], [10].

A. Linear Periodically Time Varying System

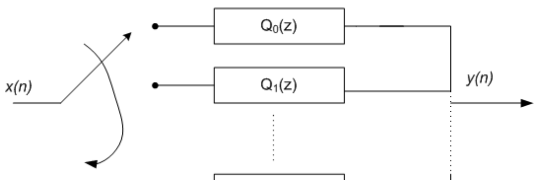
A discrete time system is called a linear periodically time varying (LPTV) system of order M, if a shift of M samples in input produces a corresponding shift of M samples in the output. For any discrete time linear system, the input-output relationship is given by the generalized convolution sum [11]

$$y(n) = \sum_{k=-\infty}^{+\infty} h(n,k)x(k) \quad \forall n,k \in \mathbf{Z}, \tag{1}$$

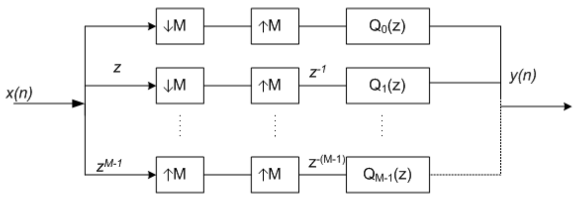
where $x(n)$ and $y(n)$ are the input and output sequences respectively. The kernel of the system is given by $h(n,k)$, defined as the response of the system to an impulse at an instant $n = k$. For an LPTV system of order M, the kernel $h(n,k)$ satisfies the condition[7]

$$h(n+M, k+M) = h(n,k) \tag{2}$$

The kernel is unique for a given linear system.

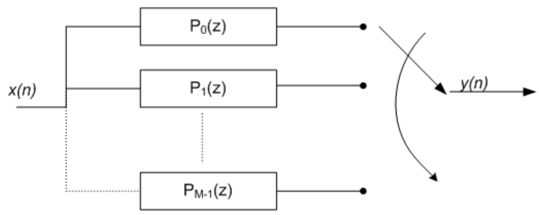


(a)

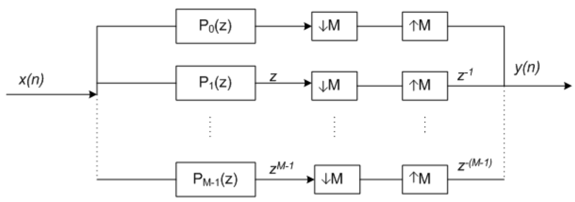


(b)

Fig. 1. (a) Input switching representation of an LPTV. (b) Input switching representation of an LPTV using multirate building blocks



(a)



(b)

Fig. 2. (a) Output switching representation of an LPTV. (b) Output switching representation of an LPTV using multirate building blocks

B. Input switching representation of an LPTV

It is understood that the system behavior changes periodically. To understand it clearly, we present the input switching representation. We define $q(m,k) = h(m+k,k)$ which is periodic in k for an LPTV system i.e.,

$$q(m,k+M) = q(m,k) \forall m,k \in \mathbf{Z} \tag{3}$$

So, the input-output relationship of the LPTV system can be re-written as,

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{+\infty} q(n-k,k)x(k) \\ &= \sum_{i=0}^{M-1} \sum_{r=-\infty}^{+\infty} q(n-rM-i, rM+i)x(rM+i) \end{aligned} \tag{4}$$

We define,

$$\begin{aligned} x_i(n) &= x(n); < n >_M = i \\ &= 0; \text{otherwise} \end{aligned} \tag{5}$$

A periodic switching operation on $x(n)$ result in the sequences $x_i(n)$. So, the LPTV system can be represented as a periodic switch at the input followed by a set of LTI systems as shown in Fig. 1(a). The periodic input switching can be achieved by using standard multirate building blocks as shown in Fig. 1(b). The multirate building blocks $z^i, \downarrow M, \uparrow M$ and z^{-i} are used in order to realize the ith channel [12]. Here, the decimator followed by expander replaces the samples other than multiples of M with zeros. The advance and delay operators are chosen in such a way that (5) is satisfied in the ith branch.

Now we proceed to the output switching representations.

C. Output switching representation of an LPTV

To obtain the output switching representation, we define $p(n,m) = h(n, n-m)$, which is periodic in n.

$$p(n+M,m) = p(n,m) \quad \forall n,m \in \mathbf{Z} \tag{6}$$

Thus,(6) gives

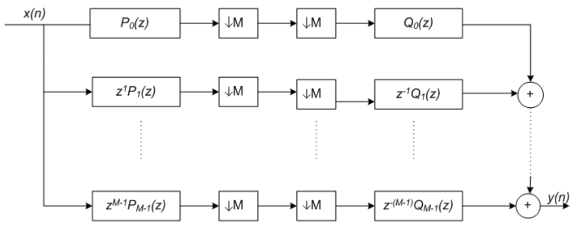
$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{+\infty} p(n,n-k)x(k) \\ &= \sum_{k=-\infty}^{+\infty} p(i,n-k)x(k) \end{aligned} \tag{7}$$

The LPTV system can be represented by periodically switching among the outputs of a set of LTI systems, $p_i(n) = p(i,n)$ as shown in Fig. 2(a). This output switching can be achieved by using multirate building blocks as shown in Fig. 2(b). As in the input representations, the multirate building blocks $z^i, \downarrow M, \uparrow M$ and z^{-i} are used in order to realize the ith channel.

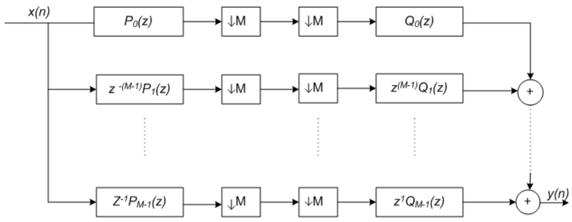
Using these switching representations we now show how the cascade of two LPTV systems is represented by an UFB.

III. Cascade connection of two LPTV systems as an UFB

Let us consider an LPTV of order M, with an output representation $P_i(z), i = 0,1,...,M-1$ as shown in Fig. 2(a). Consider another LPTV that has an input representation $Q_i(z), i = 0,1,...,M-1$ as shown in Fig. 1(a). The switching operation can be realized using the multirate building blocks as shown in Fig. 2(b) and 1(b) respectively. By careful observation we can note that the cascade $z^i, \downarrow M, \uparrow M$ and z^{-i} is equivalent to the cascade $z^{-(M-i)}, \downarrow M, \uparrow M$ and z^{M-i}.



(a)



(b)

Fig. 3. (a) Cascade connection of two LPTV systems (b) Cascade connection of two LPTV systems, equivalent to Fig. 3(a)

This cascade system kills the samples at sampling instant other than $<n>_M = i$. Thus the cascade of LPTV result in an UFB with the i-th analysis filter $z^i P_i(z)$ and the corresponding synthesis filter $z^{-i}Q_i(z)$ and equivalently $z^{-(M-i)}P_i(z)$ as the analysis filter and $z^{M-i}Q_i(z)$ as the synthesis filter. Thus, the cascade connection of the inverse LPTV systems can be represented as an UFB as shown in Fig. 3(a) and equivalently Fig. 3(b).

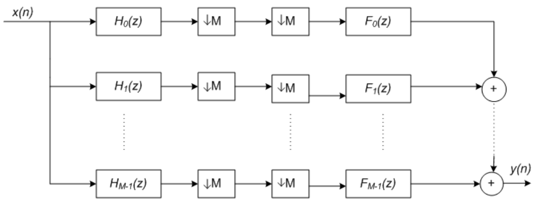
To design the inverse of a given LPTV system we need its output switching representation $P_i(z)$. From this, we find the respective analysis filters $z^{-(M-i)}P_i(z)$. Next we design the synthesis filters for perfect reconstruction. From the synthesis filters we obtain the input switching representation of the required inverse LPTV. We can change the specifications obtained to any other representation using the polyphase decomposition [7]. For any given LPTV system we need the inverse. Hence we try to minimize the worst case error norm. This criterion reduces to that of minimizing $\mathcal{H}_\infty$ norm of the error system between the ideal delay z^{-d} and the UFB.

IV. Modeling the Problem to an $\mathcal{H}_\infty$ Optimization Problem

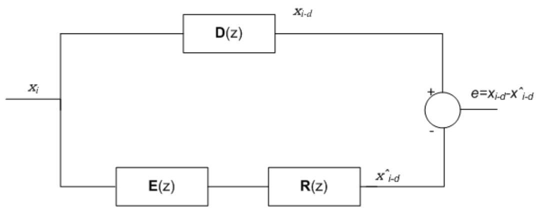
A. State space model of the UFB

Let us consider an M-channel UFB as shown in Fig. 4(a). By applying blocking and polyphase representations, the above filterbank can be reduced to the error system $\mathbf{T}_d(z)$, showed in Fig. 4(b).

Here, $\mathbf{E}(z)$ is the M-input M-output(MIMO) LTI system resulting from M-order type-I polyphase decomposition of the analysis filter bank $H(z)$.



(a)



(b)

Fig. 4. (a) An M-channel UFB (b) Corresponding error system

$$\begin{aligned} H(z) &= \begin{bmatrix} H_0(z) & H_1(z) & .... & H_{(M-1)}(z) \end{bmatrix}^T \\ &= \mathbf{E}(z^M)\begin{bmatrix} 1 & z^{-1} & .... & z^{-(M-1)} \end{bmatrix}^T \end{aligned} \tag{8}$$

where,

$$\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) & .. & E_{0(M-1)}(z) \\ E_{10}(z) & E_{11}(z) & .. & E_{1(M-1)}(z) \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ E_{(M-1)0}(z) & .. & .. & E_{(M-1)(M-1)}(z) \end{bmatrix} \tag{9}$$

For $j = 0, 1, 2, ..., (M-1)$, $E_{ij}(z)$s are polyphase components of the ith analysis filter $H_i(z)$.

Similarly, $\mathbf{R}(z)$ is the MIMO LTI system resulting from M-order type-II polyphase decomposition of the synthesis filter bank $F(z)$.

$$\begin{aligned} F(z) &= \begin{bmatrix} F_0(z) & F_1(z) & F_2(z) & ... & F_{M-1}(z) \end{bmatrix} \\ &= \begin{bmatrix} z^{-(M-1)} & z^{-(M-2)} & .... & z^{-1} & 1 \end{bmatrix} \mathbf{R}(z^M) \end{aligned} \tag{10}$$

where,

$$\mathbf{R}(z) = \begin{bmatrix} R_{00}(z) & R_{10}(z) & .. & R_{(M-1)0}(z) \\ R_{01}(z) & R_{11}(z) & .. & R_{(M-1)1}(z) \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ R_{0(M-1)}(z) & .. & .. & R_{(M-1)(M-1)}(z) \end{bmatrix} \tag{11}$$

For $j = 0, 1, 2, ..., (M-1)$, $R_{ij}(z)$s are the polyphase components of the ith synthesis filter $F_i(z)$.

$\mathbf{D}(z)$ is the delay system with an ideal delay d and is represented as,

$$\mathbf{D}(z) = z^{-1}\begin{bmatrix} 0_{(M-r)\times r} & I_{(M-r)} \\ z^{-1}I_r & 0_{r\times(M-r)} \end{bmatrix}_{M\times M} \tag{12}$$

where, $d = lM + r + M - 1; l, r \in \mathbf{Z}$. $0_{m\times n}$ stands for zero matrix of size $m \times n$ and $I_{m\times m}$ stands for identity matrix of size m.

Say, the state space representations of $\boldsymbol{E}(z), \boldsymbol{R}(z)$ and $\boldsymbol{D}(z)$ are given as,

$$\begin{aligned} \mathbf{E}(z) &= \{A_{\mathbf{E}}, B_{\mathbf{E}}, C_{\mathbf{E}}, D_{\mathbf{E}}\} \\ \mathbf{R}(z) &= \{A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}, D_{\mathbf{R}}\} \\ \mathbf{D}(z) &= \{A_{\mathbf{D}}, B_{\mathbf{D}}, C_{\mathbf{D}}, D_{\mathbf{D}}\} \end{aligned} \tag{13}$$

Then, the state space representation of the error system $\mathbf{T}_d(z) = \mathbf{D}(z) - \mathbf{R}(z)\mathbf{E}(z)$ is given by [13],

$$\mathbf{T}_d(z) = \{A_{\mathbf{T}}, B_{\mathbf{T}}, C_{\mathbf{T}}, D_{\mathbf{T}}\} \tag{14}$$

where,

$$\begin{aligned} A_{\mathbf{T}} &= \begin{bmatrix} A_{\mathbf{D}} & 0 & 0 \\ 0 & A_{\mathbf{E}} & 0 \\ 0 & B_{\mathbf{R}}C_{\mathbf{E}} & A_{\mathbf{R}} \end{bmatrix} \\ B_{\mathbf{T}} &= \begin{bmatrix} B_{\mathbf{D}} \\ B_{\mathbf{E}} \\ B_{\mathbf{R}}D_{\mathbf{E}} \end{bmatrix} \\ C_{\mathbf{T}} &= \begin{bmatrix} C_{\mathbf{D}} & -D_{\mathbf{R}}C_{\mathbf{E}} & -C_{\mathbf{R}} \end{bmatrix} \\ D_{\mathbf{T}} &= D_{\mathbf{D}} - D_{\mathbf{R}}D_{\mathbf{E}} \end{aligned}$$

For the allowed delay d, the state space matrices of $\mathbf{D}(z)$ are available. Since the output switching representation of the the LPTV under consideration is known, the state space matrices of $\mathbf{E}(z)$ are also available. We consider synthesis FIR filters are known length. Moreover, the length is considered to be an integral multiple of M. In such a case, of the state space matrices of $\mathbf{R}(z)$, $A_{\mathbf{R}}$ and $B_{\mathbf{R}}$ are readily available without the knowledge of the coefficients of the synthesis FIR filters. These coefficients occur only in the matrices $C_{\mathbf{R}}$ and $D_{\mathbf{R}}$. Thus, when approaching the problem with the criterion of minimized $\mathcal{H}_\infty$ norm, the matrices $C_{\mathbf{R}}$ and $D_{\mathbf{R}}$ are the only variables, the remaining matrices being fixed. So, the problem of designing the synthesis filters can be re-stated as the following optimization problem.

Find matrices $C_{\mathbf{R}}$ and $D_{\mathbf{R}}$ to minimize $|\boldsymbol{T}_d(z)|_\infty$

Once we find out the matrices $C_{\mathbf{R}}$ and $D_{\mathbf{R}}$, we can easily extract the FIR synthesis filter coefficients from them.

B. Modeling into Linear Matrix Inequalities(LMI)

A discrete time system $\mathbf{T}_d(z) = \{A_{\mathbf{T}}, B_{\mathbf{T}}, C_{\mathbf{T}}, D_{\mathbf{T}}\}$ has an l_2 induced norm less than or equal to α if and only if [13],

$$\begin{bmatrix} A'_{\mathbf{T}}PA_{\mathbf{T}} - P & A'_{\mathbf{T}}PB_{\mathbf{T}} & C'_{\mathbf{T}} \\ B'_{\mathbf{T}}PA_{\mathbf{T}} & B'_{\mathbf{T}}PB_{\mathbf{T}} - \alpha^2 I & D'_{\mathbf{T}} \\ C_{\mathbf{T}} & D_{\mathbf{T}} & -I \end{bmatrix} \leq 0 \tag{15}$$

Thus, our problem reduces to the following optimization problem.

minimize α

subject to $\begin{bmatrix} A'_{\mathbf{T}}PA_{\mathbf{T}} - P & A'_{\mathbf{T}}PB_{\mathbf{T}} & C'_{\mathbf{T}} \\ B'_{\mathbf{T}}PA_{\mathbf{T}} & B'_{\mathbf{T}}PB_{\mathbf{T}} - \alpha^2 I & D'_{\mathbf{T}} \\ C_{\mathbf{T}} & D_{\mathbf{T}} & -I \end{bmatrix} \leq 0$

and $P > 0$

This is a standard LMI problem and can be solved by using Robust Control Toolbox functions in MATLAB. For the optimized value of α, the corresponding matrices C_R and D_R can be obtained. The synthesis FIR filters can be extracted from C_R and D_R. After we obtain the synthesis FIR filters, the input switching representation of the inverse LPTV system can be found out by delaying the synthesis FIR filters appropriately. The relationship is given by the following equation.

$$Q_i(z) = z^{-i}F_i(z); i = 0, 1, ..., M-1 \tag{16}$$

In the next section, we present a simulation example to demonstrate the efficiency of our method.

V. Simulation Results

We consider an LPTV of order 4, with the output switching representation given by the FIR filters $P_0(z), P_1(z), P_2(z)$ and $P_3(z)$ of length 15 which correspond to frequency bands $[0, (\pi/4)], [(\pi/4), (\pi/2)], [(\pi/2), (3\pi/4)]$ and $[(3\pi/4), \pi]$ respectively. These FIR filters can be obtained using MATLAB'S $fir1$ function. The frequency response of these FIR filters are given in Fig. 5(a). The analysis filters of the corresponding UFB in Fig. 3(b) are given by,

$$\begin{aligned} H_0(z) &= P_0(z) \\ H_1(z) &= z^{-3}P_1(z) \\ H_2(z) &= z^{-2}P_2(z) \\ H_3(z) &= z^{-1}P_3(z) \end{aligned} \tag{17}$$

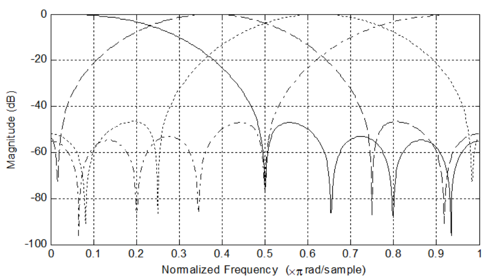
We perform $\mathcal{H}_\infty$ norm optimization for perfect reconstruction on this UFB allowing a delay of 18 samples and taking the length of the synthesis FIR filters $F_0(z), F_1(z), F_2(z)$ and $F_3(z)$ as 24. The signal is reconstructed with negligible error as shown in Fig. 6(a). The error is given by,

$$\frac{||y(n) - x(n-18)||_2}{||x(n)||_2} = 0.065 \tag{18}$$

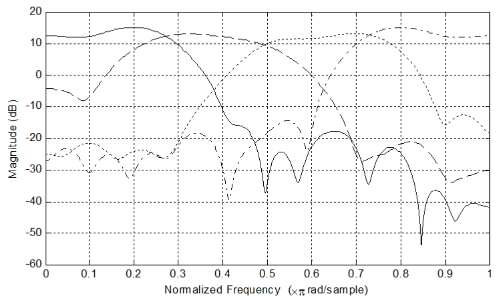
The input switching representation of the inverse LPTV system is obtained by delaying the synthesis FIR filters $F_0(z), F_1(z), F_2(z)$ and $F_3(z)$ that we have just designed.i.e.

$$\begin{aligned} Q_0(z) &= F_0(z) \\ Q_1(z) &= z^{-3}F_1(z) \\ Q_2(z) &= z^{-2}F_2(z) \\ Q_3(z) &= z^{-1}F_3(z) \end{aligned} \tag{19}$$

The frequency responses of $Q_0(z), Q_1(z), Q_2(z)$ and $Q_3(z)$ are shown in Fig. 5(b).

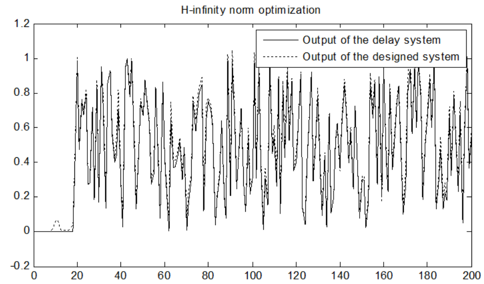


(a) — $P_0(z)$, - - - $P_1(z)$, ... $P_2(z)$, -.- $P_3(z)$

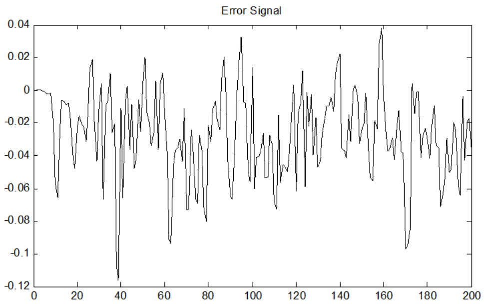


(b) — $Q_0(z)$, - - - $Q_1(z)$, ... $Q_2(z)$, -.- $Q_3(z)$

Fig. 5. (a) Frequency responses of the FIR filters in the output switching representation of the LPTV (b) Frequency responses of the FIR filters in the input switching representation in the designed inverse LPTV



(a)



(b)

Fig. 6. (a) Reconstructed signal (b) Error signal

VI. Conclusion

In this paper we presented a design method for the inversion of a given LPTV system. We allow some finite delay in the processing. Using the input and output switching representations the problem is converted into that of designing a UFB. Our design criterion imposes no conditions on the LPTV for inversion. The main advantage of our method is that we can design inverse for any given LPTV system where as earlier methods [4], [5], [6], [7] are imposing conditions on the LPTV system for which we have to design inverse. We present simulation results for an LPTV of order 4. The MATLAB codes used in this simulation are available on e-mail request.

References

[1] E. Serpedin, G.B. Giannakis, Blind channel identification and equalization with modulation-induced cyclostationarity, *IEEE Transactions on Signal Processing*, vol. 46, no. 7, pp. 1930–1944, 1998

[2] C. W. King, C. A. Lin, A unified approach to scrambling filter design,*IEEE Transactions on Signal Processing*, vol. 43, no. 8, pp. 1753–1765, 1995

[3] M. Vetterli, Invertibility of linear periodic time-varying filters, *IEEE Transactions on Signal Processing*, vol. 50, no. 9, pp. 2221–2229, 2002.

[4] K. Kazlauskas, Inversion of linear periodically time-varying digital filters *IEEE Transactions on Circuits and Systems II:Analog and Digital Signal Processing*, vol. 41, no. 2, pp. 173–175, 1994

[5] J. Y. Wu, C. A. Lin, Optimal approximate inverse of linear periodic filters,*IEEE Transactions on Signal Processing*, vol. 52, no. 9, pp. 2371–2382, 2004

[6] W. Chauvet, B. Lacaze, D. Roviras, A. Duverdier, Characterization of a set of invertible LPTV filters using circulant matrices, *Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing(ICASSP 03)*, vol. 6, pp. 45–48, 2003

[7] A. S. Mehr, T. Chen, Representation of linear periodically time- varying and multirate systems, *IEEE Transactions on Signal Processing*, vol. 50, no. 9, pp. 2221–2229, 2002.

[8] B. A. Francis,A Course in H-infnity Control Theory, Lecture Notes in Control and Information Science, vol. 88, Springer-Verlag, Berlin, 1987.

[9] R. Meyer, C. Burrus, A unified analysis of multirate and periodically time-varying digital filters, *IEEE Transactions on Circuits and Systems*, vol. 22, no. 3, pp. 162168, 1975.

[10] C. Loeffler, C. Burrus, Optimal design of periodically time-varying and multirate digital filters, *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 32, no. 5, pp. 991997, 1984.

[11] A. V. Oppenheim, R. W. Schafer, J. R. Buck, Discrete-Time Signal Processing (2nd Edition), Prentice-Hall, Englewood Cliffs, NJ, 1999.

[12] P. P. Vaidyanathan, Multirate Systems and Filterbanks, Prentice Hall,Englewood Cliffs, NJ, 1948.

[13] A. S. Mehr and T. Chen, Design of Nonuniform Multirate Filter Banks by Semidefinite Programming,*IEEE Transactions on Circuits and SystemsII: Analog and Digital Signal Processing*, vol. 47, no. 11, 2000.