A simple tool for engineers for generating formulae for the sum of powers of natural numbers

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Abstract The formulae for the sum of powers of natural numbers are in the form of polynomials. These formulae are derived using an algorithm which was developed from a simple geometric interpretation. It has been found that these polynomials are interdependent. In this paper, a close study is made and many interesting features are brought out. A method is proposed to generate a new polynomial based on the features of these polynomials. Another simplified algorithm is also presented which is found to be more suitable for the automatic generation of the polynomials. The latter may be realized as a convenient and handy tool for generating the formulae, especially for engineers.

Keywords powers; summation; natural numbers; algorithm; creativity

Introduction

An algorithm for finding the sum of the kth power of the first n natural numbers for $k \geq 3$, has been developed [1]. The development was based on purely a geometric interpretation. Using this algorithm, formulae for different values of k have been derived and presented, and some possible applications of these formulae in the field of engineering have been discussed [1]. The main aim of that paper was to inculcate among young and budding engineers essential engineering aptitudes, such as creativity, and geometric and physical interpretation. The paper showed how a simple geometric interpretation enabled interesting mathematical formulae to be derived without the use of any deep mathematics.

In this paper, the formulae are further discussed and it is shown how the formulae for different values of k may be derived. Then a simplified procedure is presented which is more suitable for research engineers for automatic generation of the formulae.

Observations of the formulae

It has been shown that the sum of the kth power of the first n natural numbers may be expressed as a polynomial of order $(k + 1)$ without a constant term [1]. Hence, it may be expressed as:

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} a_{k,j} n^j \tag{1}$$

where $a_{k,j}$ is the coefficient of n^j of the polynomial.

The formulae presented in reference [1], together with some more formulae, may be expressed in polynomial form and the values of various coefficients may be tabulated as in Table 1. It may be noted that $k = 0$ is an obvious result which is included only to complete the triangular form of Table 1. From the values listed in Table 1, various interesting features may now be observed and described, as follows.

(1) The sum of the coefficients of any such polynomial is always unity. That is,

$$\sum_{j=1}^{k+1} a_{k,j} = 1 \qquad \text{for all values of } k \tag{2a}$$

This is as expected, because all these formulae must satisfy for $n = 1$, for all values of k.

(2) The coefficient of n^{k+1} is the reciprocal of $(k + 1)$. That is,

$$a_{k,k+1} = \frac{1}{k+1} \qquad \text{for all values of } k \tag{2b}$$

(3) The coefficient of n^k is always (1/2). That is,

$$a_{k,k} = \frac{1}{2} \qquad \text{for all values of } k \tag{2c}$$

TABLE 1 *Coefficients of the polynomials representing the sum of the* kth *power of the first* n *natural numbers*

k	n^1	n^2	n^3	n^4	n^5	n^6	n^7	n^8	n^9	n^{10}	n^{11}
0	$\frac{1}{1}$										
1	$\frac{1}{2}$	$\frac{1}{2}$									
2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$								
3	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$							
4	$\frac{-1}{30}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$						
5	0	$\frac{-1}{12}$	0	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{6}$					
6	$\frac{1}{42}$	0	$\frac{-1}{6}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{7}$				
7	0	$\frac{1}{12}$	0	$\frac{-7}{24}$	0	$\frac{7}{12}$	$\frac{1}{2}$	$\frac{1}{8}$			
8	$\frac{-1}{30}$	0	$\frac{2}{9}$	0	$\frac{-7}{15}$	0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{9}$		
9	0	$\frac{-3}{20}$	0	$\frac{1}{2}$	0	$\frac{-7}{10}$	0	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{10}$	
10	$\frac{5}{66}$	0	$\frac{-1}{2}$	0	1	0	−1	0	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{11}$

(4) The product of the coefficient of n^{k+1} of one polynomial and the coefficient of n^k of the next polynomial is always (1/12). This may also be expressed, in view of equation 2b, as,

$$a_{k,k-1} = \frac{k}{12} \qquad \text{for all values of } k \tag{2d}$$

(5) Some coefficients are zeros. They are,

$$a_{k,k-j} = 0 \qquad \text{for } k = 3, 4, 5, 6, \cdots \tag{2e}$$
$$\text{and } j = 2, 4, 6, 8, \cdots$$

(6) Some coefficients are negative. They are,

$$a_{k,k-j} < 0 \qquad \text{for } k = 4, 5, 6, 7, \cdots \tag{2f}$$
$$\text{and } j = 3, 7, 11, \cdots$$

All these observations have been proved elsewhere by the authors [2], and for the sake of conciseness the proofs are not presented here. These are, however, useful in obtaining a formula for a particular value of k. If, for example, one wishes to obtain the formula for $k = 8$, the observations in equations 2a–f may be used as follows.

First, the polynomial for $k = 8$ may be written from equation 1 as,

$$\sum_{i=1}^{n} i^8 = a_{8,1}n + a_{8,2}n^2 + a_{8,3}n^3 + \cdots + a_{8,9}n^9 \tag{3}$$

The polynomial in equation 3 has nine coefficients, ($a_{8,j}$, $j = 1$ to 9). However, *six* of them may be assigned values, straight away, using the observations in equation set 2:

$$\text{from (2b)} \qquad a_{8,9} = \frac{1}{9} \tag{4a}$$

$$\text{from (2c)} \qquad a_{8,8} = \frac{1}{2} \tag{4b}$$

$$\text{from (2d)} \qquad a_{8,7} = \frac{2}{3} \tag{4c}$$

$$\text{from (2e)} \qquad a_{8,6} = a_{8,4} = a_{8,2} = 0 \tag{4d}$$

Equation 3 may now be written as:

$$a_{8,1}n + a_{8,3}n^3 + a_{8,5}n^5 = \sum_{i=1}^{n} i^8 - \frac{2}{3}n^7 - \frac{1}{2}n^8 - \frac{1}{9}n^9 \tag{5}$$

By taking three different values of n, say $n = 1$, 2 and 3, three simultaneous equations may be derived from equation 5:

$$a_{8,1} + a_{8,3} + a_{8,5} = -5/18 \tag{6a}$$

$$a_{8,1} + 4a_{8,3} + 16a_{8,5} = -119/18 \tag{6b}$$

$$a_{8,1} + 9a_{8,3} + 81a_{8,5} = -215/18 \tag{6c}$$

It may be noted that equation 6a also follows from the observation in equation 2a. Solving equation 6, the unknown coefficients may be obtained:

$$a_{8,1} = -1/30 \qquad a_{8,3} = 2/9 \qquad \text{and} \qquad a_{8,5} = 7/15 \tag{7}$$

It is worth noting here that the number of equations to be solved for polynomials for $k = 2m$ and $k = 2m + 1$, for $m = 2, 3, 4, \ldots$, is $(m - 1)$. The number of equations to be solved increases with the value of k. Further, the process of computation may disturb the rational form of the coefficients, leading to inaccuracies in the formulae.

A simplified algorithm

The method described above, though it appears to be attractive and fast, suffers from the drawbacks mentioned. The authors have developed another algorithm, which overcomes these difficulties. This is based on the mathematical treatment described in [2]. In view of conciseness, the final algorithm only is presented here. The algorithm is,

$$\boldsymbol{A} = (k/k+1)\{\boldsymbol{B} + \boldsymbol{C} - \boldsymbol{DE}\} \tag{8}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$ are matrices described as follows.

$\boldsymbol{A}$ is a $1 \times (k + 1)$ matrix containing the coefficients to be determined for a polynomial of a particular value of k:

$$\boldsymbol{A} = \begin{bmatrix} a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k} & a_{k,k+1} \end{bmatrix} \tag{9a}$$

$\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are all matrices containing, essentially, the coefficients of the polynomial for $(k - 1)$, but arranged in different ways:

$$\boldsymbol{B} = \begin{bmatrix} a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k} & 0 \end{bmatrix} \tag{9b}$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,k-1} & a_{k-1,k} \end{bmatrix} \tag{9c}$$

$$\boldsymbol{D} = \begin{bmatrix} a_{k-1,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-1,k-1} \end{bmatrix} \tag{9d}$$

Matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are of size $1 \times (k + 1)$ and matrix $\boldsymbol{D}$ *is* of *size* $1 \times (k - 1)$.

Matrix $\boldsymbol{E}$ is of order $(k - 1)$ by $(k + 1)$ and contains the coefficients of all the previous polynomials and a column of zeros:

$$\boldsymbol{E} = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & \cdots & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & a_{k-1,4} & a_{k-1,5} & \cdots & a_{k-1,k} & 0 \end{bmatrix} \tag{9e}$$

Zero elements are included in matrices $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ in order to maintain their compatibility in matrix equation 8.

By substituting appropriate matrices into equation 8, the coefficients of the required polynomial can be generated.

An example

For the purpose of illustration, the polynomial for $k = 5$ is obtained using the algorithm described in equation 8. For this purpose, consider the part of Table 1 up to $k = 4$. Various matrices may now be constructed:

$$\boldsymbol{A} = \begin{bmatrix} a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \end{bmatrix} \tag{10a}$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{-1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 \end{bmatrix} \tag{10b}$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & \frac{-1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{bmatrix} \tag{10c}$$

$$\boldsymbol{D} = \begin{bmatrix} \frac{-1}{30} & 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \tag{10d}$$

$$\boldsymbol{E} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{-1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 \end{bmatrix} \tag{10e}$$

By substituting matrices $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ into equation set 9, matrix $\boldsymbol{A}$ may be obtained very easily as:

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{-1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \tag{11}$$

which contains the coefficients of the polynomial for $k = 5$. The dependence of the polynomial for $k = 5$ on the polynomials for $k < 5$ may be noticed.

References

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