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Effect of couple stresses on the flow in a constricted annulus

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Abstract The flow of an incompressible couple stress fluid in an annulus with local constriction at the outer wall is considered. This configuration is intended as a simple model for studying blood flow in a stenosed artery when a catheter is inserted into it. The effects couple stress fluid parameters α and σ, height of the constriction (ε), and ratio of radii (k) on the impedance and wall shear stresses are studied graphically. Graphical results show that the resistance to the flow as well as the wall shear stress increases as the ratio of the radii increases and decreases as the couple stress fluid parameters increases.

Keywords Blood flow · Catheterized artery · Constriction · Annulus · Couple stress fluid

1 Introduction

The flows of the fluid in pipes of different shapes are important in many biological and biomedical systems, the human cardiovascular system and in several technological devices. The abnormal and unnatural growth in the lumen of an artery is called stenosis. Localized atherosclerotic constrictions in arteries (arterial stenosis) are found predominantly in the internal carotid artery, which supplies blood to the brain, the coronary artery, which supplies blood to the cardiac muscles, and the femoral artery, which supplies blood to the lower limbs. Catheterization refers to a procedure in which a long, thin, flexible plastic tube (catheter) is inserted into artery. Catheter procedures can both diagnose and treat heart and blood vessel conditions. Angiography, which is used for diagnosis, is the most common type of heart catheter procedure. The insertion of a catheter in an artery will form an annular region between the walls of the catheter and artery. This will alter the flow field, modifying the pressure distribution and increase the resistance. Hence, the pressure or pressure gradient recorded by a transducer attached to the catheter will differ from that of an uncatheterized artery, however small the size of the catheter may be. Therefore, it is important to study the effect due to presence of catheter in the physiological artery flows.

In recent years considerable attention has been given to the study of blood flow characteristics, due to the presence of catheters in the lumen of the artery. Several researchers have studied the flow of blood in catheterized arteries by modeling the catheter and artery as rigid coaxial circular cylinders and blood as a Newtonian or non-Newtonian fluid. Roose and Lykoudis [1] studied the fluid mechanics of the ureter with an inserted catheter by considering the peristaltic wave moving along the stationary cylinder. MacDonald [2] considered

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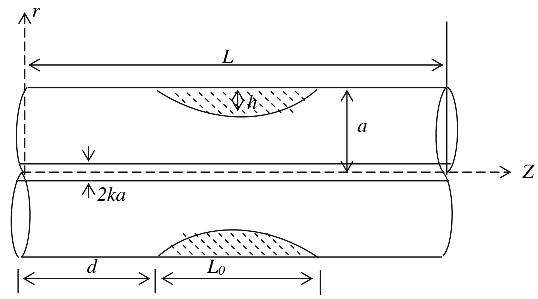

Fig. 1 Schematic diagram of catheterized stenosed artery

the pulsatile blood flow in a catheterized artery and obtained theoretical estimates for pressure gradient corrections for catheters, which are positioned eccentrically, as well as coaxially with the artery. Karahalios [3] has studied the effect of catheterization on various flow characteristics in an artery with or without stenosis. Daripa and Dash [4] have analyzed the numerical study of pulsatile blood flow in an eccentric catheterized artery using a fast algorithm treating blood as a Newtonian fluid.

It is well known that, blood being a suspension of cells, behaves like a non-Newtonian fluid at low shear rates and during its flow through narrow blood vessels. Shukla et al. [5] investigated the effects of stenosis on non-Newtonian flow of the blood in an artery. Philip and Peeyush Chandra [6] studied the flow of blood, modelling it by a simple micro fluid in the core region with a Newtonian fluid peripheral layer, in a tube in the presence of very mild stenosis. Dash et al. [7] considered the steady and pulsatile flow in a narrow artery when a catheter is inserted into it and estimated the increase in frictional resistance in the artery due to catheterization, using the Casson fluid model. Sankar et al. [8] discussed the steady flow of Herschel–bulkley fluid through a catheterized artery.

The couple stress fluid theory represents the simplest generalization of the classical viscous fluid theory that allows for polar effects in the fluids. This fluid theory shows all the important features and effects of couple stresses and results in equations that are similar to Navier–Stokes equations. The main effect of couple stresses will be to introduce a size dependent effect that is not present in the classical non-polar theories [9]. Chaturani has analyzed the problems of Poiseulle flow and pulsatile flow of couple stress fluid with application to blood flow [10,11]. An analysis of the effects of couple stresses on the blood flow through thin artery with mild stenosis has been carried out by Sinha and Singh [12]. Srivastava [13] considered the flow of couple stress fluid through stenotic blood vessels.

In this paper, the steady flow of an incompressible couple stress fluid through a catheterized artery with mild stenosis is considered. The velocity, impedance (resistance to the flow) and wall shear stress are calculated. The variation of impedance and shearing stress is analyzed for various values of couple stress fluid parameters and geometric parameters.

2 Formulation of the problem

Consider the flow of an incompressible couple stress fluid between an axisymmetric rigid tube (an artery) of radius *'a'* with a mild stenosis and a coaxial flexible tube (a catheter) of radius *'ka'* ($k \leq 1$). Assume that the flow is steady, axisymmetric and the stenosis over a length of the tube being assumed to have developed in an axisymmetric manner. The schematic diagram is shown in Fig. 1.

The equations governing the steady flow of an incompressible couple stress fluid in the absence of body force and body couple are [9]

$$\text{div}\bar{q} = 0 \tag{1}$$

$$\rho(\bar{q}.\nabla)\bar{q} = -\text{grad}\, P - \mu\,\text{curl}\,\text{curl}\bar{q} - \eta\,\text{curl}\,\text{curl}\,\text{curl}\,\text{curl}\,\bar{q} \tag{2}$$

where ρ is the density, $\bar{q}$ is the velocity vector, η is the couple stress fluid parameter, P is the fluid pressure and μ is the fluid viscosity.

The force stress tensor τ_{ij} and the couple stress tensor m_{ij} that arises in the theory are given by

$$\tau_{ij} = (-P + \lambda\text{div}\bar{q})\delta_{ij} + 2\mu d_{ij} - (1/2)\varepsilon_{ijk}\big[m_{,k} + 4\eta\omega_{k,rr} + \rho C_k)\big] \tag{3}$$

$$m_{ij} = 4\eta\omega_{j,i} + 4\eta'\omega_{i,j} \tag{4}$$

where $2\omega = \text{curl}\bar{q}$ is the spin vector, $\omega_{i,j}$ is the spin tensor, d_{ij} is the rate of deformation vector derived from the velocity vector, p is the fluid pressure and ρC_k is the body couple vector . The quantities λ and μ are the viscosity coefficients and η' and η are the constants associated with couple stresses. These material constants are considered by the inequalities.

$$\mu \geq 0, 3\lambda + 2\mu \geq 0, \eta \geq 0, \quad \eta' \leq \eta \tag{5}$$

The problem has been studied in cylindrical coordinate system (r, θ, z). Since the flow is axisymmetric all the variables are independent of θ. Hence for this flow the velocity is given by $\bar{q} = (u(r, z), 0, w(r, z))$. The stenosed wall of the artery is defined as [14]

$$\begin{aligned} r_s(z) &= a - \frac{h}{2}\left[1 + \cos\frac{2\pi}{L_0}\left(z - d - \frac{L_0}{2}\right)\right], \quad d \leq z \leq L_0 + d \\ &= a \quad \text{otherwise} \end{aligned} \tag{6}$$

where 'a' is the radius of the artery in non-stenosed portion, L_0 is the magnitude of the distance along the artery over which the stenosis is spread out and 'h' is the maximum height of the stenosis.

Introducing the following non-dimensional variables

$$r = a\tilde{r}, \quad w = w_0\tilde{w}, \quad u = \frac{aw_0\tilde{u}}{L}, \quad P = \frac{Lw_0\mu\tilde{p}}{a^2}, \quad z = L\tilde{z}, \quad d = L\tilde{d}, \quad r_s = a\tilde{r_s} \tag{7}$$

where w_0 is a typical axial velocity and L is the length of the tube, in to Eqs. (1) and (2) and dropping tildes, we get

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

$$R_e\delta^3\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \delta^2\left(\nabla_1^2 - \frac{1}{r^2}\right)u - \frac{\delta^2}{\alpha^2}\left(\nabla_1^2 - \frac{1}{r^2}\right)^2 u \tag{9}$$

$$R_e\delta\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left(\nabla_1^2 - \frac{\nabla_1^4}{\alpha^2}\right)w \tag{10}$$

$$\begin{aligned} r_s(z) &= 1 - \frac{\varepsilon}{2}\left[1 + \cos\frac{2\pi}{\gamma}\left(z - d - \frac{\gamma}{2}\right)\right], \quad d \leq z \leq d + \gamma \\ &= 1 \quad \text{otherwise} \end{aligned} \tag{11}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \delta^2\frac{\partial^2}{\partial z^2}$, $\delta = \frac{a}{L}$, $\varepsilon = \frac{h}{a}$, $\gamma = \frac{L_0}{L}$, $\text{R}_\text{e} = \frac{\rho a u_0}{\mu}$ is the Reynolds number and $\alpha = a(\mu/\eta)^{1/2}$ is the couple stress fluid parameter. Since $\sqrt{\eta/\mu}$, which is a characteristic measure of the polarity of the fluid model, has the dimensions of the length, α indicates the ratio of the tube radius to material characteristic length. In the limit $\eta \to 0$ i.e., $\alpha \to \infty$, Eqs. (9) and (10) reduce to classical Navier–Stokes equations.

With the assumption that the stenosis is very mild (which implies that the variation of all the flow characteristics except pressure along the axial direction is negligible) and the length of the tube is large compared to its radius i.e., $\delta << 1$, the equations describing the flow are given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{12}$$

$$\frac{\partial p}{\partial r} = 0 \tag{13}$$

$$-\frac{\partial p}{\partial z} + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w - \frac{1}{\alpha^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 w = 0 \tag{14}$$

The corresponding non-dimensional boundary conditions are

$$w = 0 \quad \text{at } r = r_s(z) \quad \text{and} \quad r = k \tag{15}$$

$$\left(\frac{\partial^2 w}{\partial r^2} - \frac{\sigma}{r}\frac{\partial w}{\partial r}\right) = 0 \quad \text{at } r = r_s(z) \quad \text{and} \quad r = k \tag{16}$$

where $\sigma = \eta'/\eta$ is a couple stress fluid parameter. Boundary condition (16) shows that couple stresses (4) vanish at the tube wall and catheter wall.

3 Solution of the problem

It can be noted from Eq. (13) that p is a function of z only. Equation (14) can be simplified to the form

$$E^2(E^2-\alpha^2)w=-\alpha^2\frac{\mathrm{d}p}{\mathrm{d}z} \tag{17}$$

where $E^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$.

The solution for the above equation is

$$w(r,z)=\frac{1}{\alpha^2}\big[C_1(z)I_0(\alpha r)+C_2(z)K_0(\alpha r)\big]+\left[\frac{r^2}{4}\frac{\mathrm{d}p}{\mathrm{d}z}+C_3(z)\log r\right]+C_4(z) \tag{18}$$

where $I_0(\alpha r)$ and $K_0(\alpha r)$ are the modified Bessel functions of the zeroth order, first and second kind respectively. $C_1(z)$, $C_2(z)$, $C_3(z)$, and $C_4(z)$ are arbitrary functions of z. Using the boundary conditions (15) and (16), we can obtain the values of $C_1(z)$, $C_2(z)$, $C_3(z)$, and $C_4(z)$.

The dimensionless flux, defined as $Q=\int_k^{r_s(z)}2rw\,\mathrm{d}r$ can be obtained from Eq. (18) in the following form

$$Q=\frac{2}{\alpha^3}\frac{\mathrm{d}p}{\mathrm{d}z}F[r_s(z),k] \tag{19}$$

where

$$\begin{aligned}F\big[r_s(z),k\big]&=d_1(z)\big[r_s(z)I_1(\alpha r_s(z))-kI_1(\alpha k)\big]+d_2(z)\big[r_s(z)K_1(\alpha r_s(z))-kK_1(\alpha k)\big]+\frac{\big((r_s(z))^4-k^4\big)}{8}\\&\quad+d_3(z)((r_s(z))^2\log(r_s(z))-k^2\log k)-\frac{(r_s(z))^2}{2}+\frac{k^2}{2}+d_4(z)\big[(r_s(z))^2-k^2\big]\end{aligned} \tag{20}$$

with $\frac{\mathrm{d}p}{\mathrm{d}z}\mathrm{d}_i(z)=C_i(z),\ i=1,2,3,4$

The pressure drop Δp (is p_1 at $z=0$ and p_0 at $z=L$) across the tube is obtained from Eq. (16) as

$$\Delta p=\frac{Q\alpha^3}{2}\int_0^1\frac{\mathrm{d}z}{F[r_s(z),k]} \tag{21}$$

The dimensionless resistance to the flow (resistive impedance), λ is given by

$$\lambda=\frac{\Delta p}{Q}=\frac{\alpha^3}{2}\int_0^1\frac{\mathrm{d}z}{F[r_s(z),k]} \tag{22}$$

which can be written as

$$\lambda=\frac{\alpha^3}{2}\left[\int_0^d\frac{\mathrm{d}z}{F[r_s(z),k]}+\int_d^{d+\gamma}\frac{\mathrm{d}z}{F[r_s(z),k]}+\int_{d+\gamma}^1\frac{\mathrm{d}z}{F[r_s(z),k]}\right] \tag{23}$$

Since $r_s(z)=1$, in the regions $0\le z\le d$ and $d+\gamma\le z\le 1$, the resistance to the flow (λ) simplifies to

$$\lambda=\frac{1-\gamma}{F[r_s(z),k]_{r_s(z)=1}}+\gamma\int_0^1\frac{\mathrm{d}\phi}{F[r_s(\phi),k]}. \tag{24}$$

where $\phi=(z-d)/\gamma$

The shearing stress (3) at the wall is obtained as

$$\tau_{\mathrm{rz}}=\frac{Q\alpha^3}{2\,F[r_s(z),k]}\left[\frac{r_s(z)}{2}+\frac{d_3}{r_s(z)}\right] \tag{25}$$

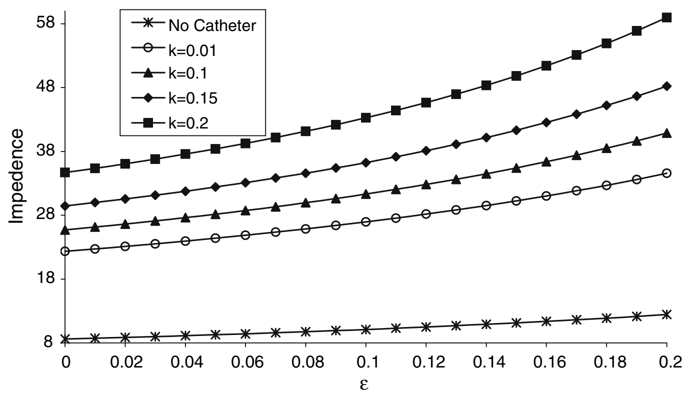

Fig. 2 Effect of k on impedance for $\gamma = 0.75, \sigma = 0.5, \alpha = 5$

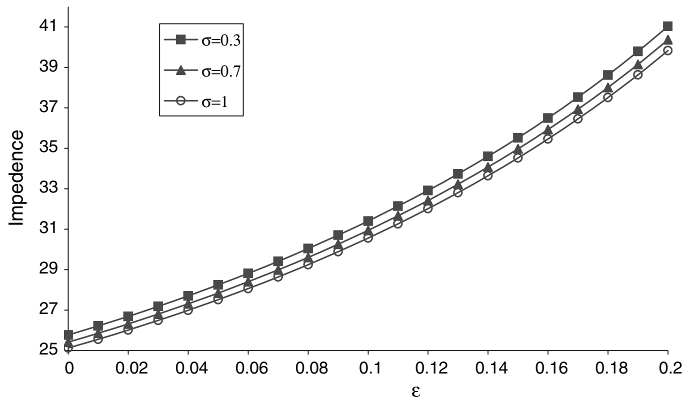

Fig. 3 Effect of σ on impedance for $\gamma = 0.75, k = 0.1, \alpha = 5$

4 Results and discussions

In the above analysis, apart from the usual non-dimensional parameters like ε (the maximum height of the constriction/stenosis), k (the ratio of radii/catheter size) and γ (the length of the constriction/stenosis), the couple stress fluid is associated with the parameters like α and σ. The effects of the above parameters on the two important physiological factors like the impedance (resistance to flow) and shear stress have been studied.

The system of equations in terms of $C_1(z)$, $C_2(z)$, $C_3(z)$, and $C_4(z)$ is obtained by using the boundary conditions (15) and (16), the dimensionless impedance (24) and the shearing stress (25) are evaluated numerically using MATHEMATICA for various values of $\varepsilon, k, \gamma, \alpha$ and σ.

The variation of impedance with ε for different values of the catheter size (k) and for fixed values of $\alpha = 5, \sigma = 0.5$ and $\gamma = 0.75$ is shown in Fig. 2. $k = 0$ corresponds to the case when there is no catheter. Here, it is observed that the presence of the catheter in a stenosed artery increases the impedance. Also, as the size of the catheter (k) increases the impedance increases. Figure 3 analyses the effect of σ on the impedance as the parameters ε varies for the fixed values of $k = 0.1, \alpha = 5$ and $\gamma = 0.75$. It can be observed that as the value of σ increases impedance decreases. $\sigma = 1$ corresponds to Newtonian fluid case. Figure 4 explains the effect of γ on the impedance for the fixed values of $k = 0.1, \sigma = 0.5$ and $\alpha = 5$. It can be seen that the impedance is almost doubled when γ increases from *0.1* to *1*. Figure 5 shows the variation of impedance for different values of the couple stress parameter α. The impedance goes on decreasing as α increases. Further it can be observed from Fig. 3, 4 and 5 that the impedance in case of couple stress fluid is more than that of a Newtonian fluid

The variation of shear stress is studied at the maximum height of the stenosis for various values of the fluid and geometric parameters. In Fig. 6, the effect of σ on the shearing stress for the fixed values of $k = 0.1$ and

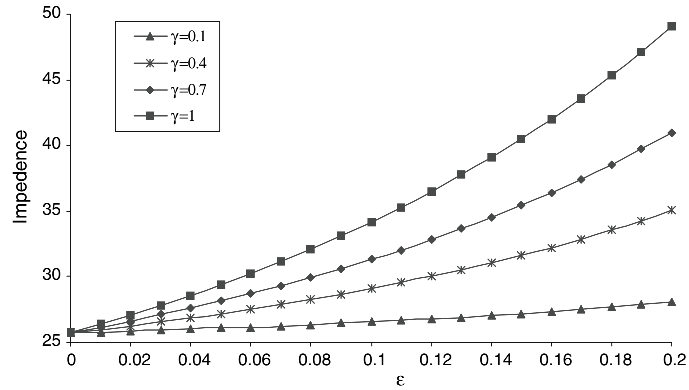

Fig. 4 Effect of γ on impedance for k $= 0.1, \sigma = 0.5, \alpha = 5$

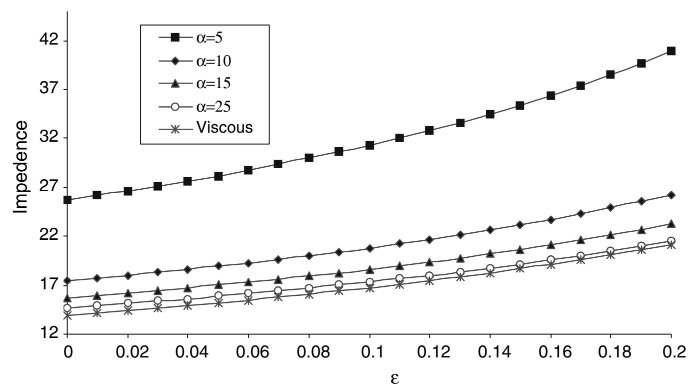

Fig. 5 Effect of α on impedance for $k = 0.1, \sigma = 0.5, \gamma = 0.75$

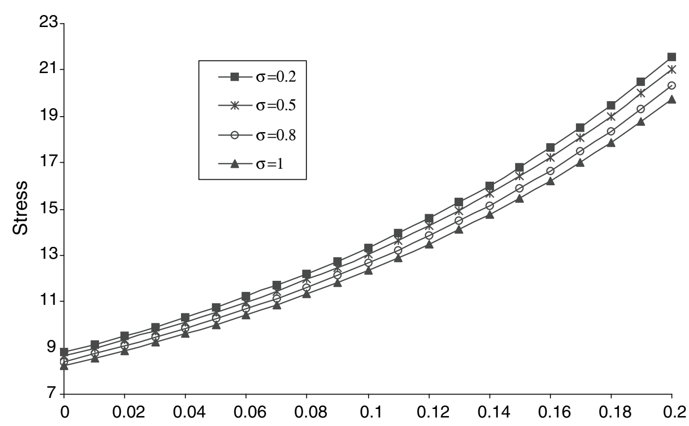

Fig. 6 Effect of σ on stress for $k = 0.1, \alpha = 5$

$\alpha = 5$ is depicted. It can be noted from Fig. 6 that the shearing stress decreases as σ increases. The effect of the catheter size on the shear stress with $\alpha = 5$ and $\sigma = 0.5$ is shown in Fig. 7. It can be observed that in the absence of the catheter, shear stress is very low and when the catheter is introduced, increase in its size increases the shear stress. Figure 8 shows the effect of α for fixed values of the parameters $k = 0.1$ and

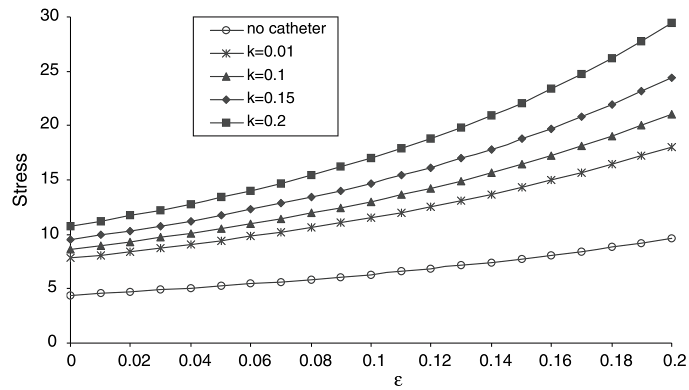

Fig. 7 Effect of k on stress for $\sigma = 0.5, \alpha = 5$

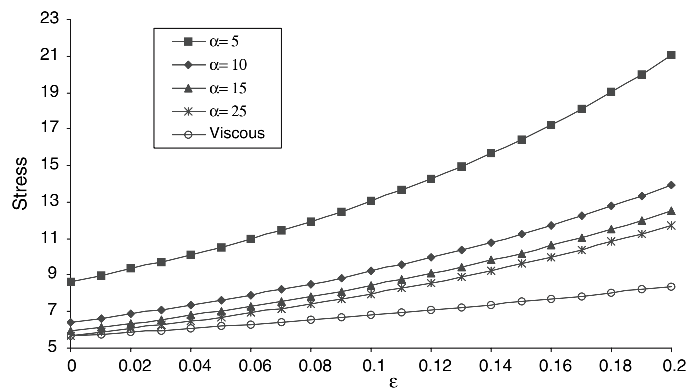

Fig. 8 Effect of α on stress for $\sigma = 0.5, k = 0.1$

$\sigma = 0.5$. The shearing stress goes on decreasing as α increases. As in the case of impedance, the shear stress is more in case of couple stress fluids when compared to Newtonian fluids.

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