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Flow and heat transfer of couple stress fluid in a porous channel with expanding and contracting walls ☆

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ABSTRACT

In this paper, an incompressible laminar flow of a couple stress fluid in a porous channel with expanding or contracting walls is considered. Assuming symmetric injection or suction along the uniformly expanding porous walls and using similarity transformations, the governing equations are reduced to nonlinear ordinary differential equations. The resulting equations are then solved numerically using quasilinearization technique. The graphs for velocity components and temperature distribution are presented for different values of the fluid and geometric parameters.

1. Introduction

The flow through channels and tubes with porous walls is of great importance both in technological as well as biophysical flows. Examples of this are found in soil mechanics, transpiration cooling, food preservation, cosmetic industry, blood flow and artificial dialysis, binary gas diffusion, filtration, ablation cooling, surface sublimation, grain regression (as in the case of combustion in solid rocket motors), and the modeling of air and blood circulation in the respiratory system. A large number of theoretical investigations dealing with steady incompressible laminar flow with either injection or suction at the boundaries have appeared during the last few decades. Several authors, to mention some [1–4] have studied the steady laminar flow of an incompressible viscous fluid in a two-dimensional channel with parallel porous walls.

The flows of fluid in a porous channel with deformable walls has also gained importance because of its applications in the modeling of pulsating diaphragms, sweat cooling or heating, isotope separation, filtration, paper manufacturing, irrigation, and the grain regression during solid propellant combustion. The viscous flow inside an impermeable tube of contracting cross section was first examined by Uchida et al. [5]. Unsteady flow of a viscous, incompressible fluid in a semi-infinite circular tube with a porous, elastic wall whose length varies with time, but whose cross section does not vary, is studied by Ohki and Morimatsu [6] considering the effect of suction or injection on the wall. In simulating the laminar flow field in cylindrical solid rocket motors, Goto et al. [7] have analyzed the laminar incompressible flow in a semi-infinite porous pipe whose radius varied with time. Goto et al. [8] made a theoretical analysis of the unsteady flow in a semi infinite expanding or contracting circular pipe into which an incompressible fluid is injected or sucked in through the wall surface. Bujurke et al. [9] obtained computer extended series solution for unsteady flow in a contracting and expanding pipe. Majdalani et al. [10] obtained exact solution to the viscous flow driven by small wall contractions and expansions of two weakly permeable walls using similarity transformations in both space and time and double perturbations in the permeation Reynolds number and the wall expansion ratio. The analysis of [7] was extended numerically using shooting method coupled with a Runge–Kutta integration scheme by Dauenhauer et al. [11] and both numerically and asymptotically for moderate to large Reynolds numbers by Majdalani and Zhou [12] to expanding or contracting channels with porous walls. Adamkowski [13] proposed a mathematical model based on the one-dimensional theory of the unsteady flow for the problem of transient flow of liquid in tapered or expanding pipes and compared with the methods available in the literature. Boutros et al. [14] considered the laminar, isothermal and incompressible flow in a rectangular domain bounded by two weakly permeable , moving porous wall, which enable the fluid to enter or exit due to successive expansions or contractions using Lie group method for determining symmetry reductions of partial differential equations followed by a double perturbation. Asghar et al. [15] extended the very restricted results obtained in [14] using various Lie point symmetries to reduce a complex system to an easy-to-handle second-order ordinary differential equation system in combination with the conservation laws that the system generates. As a particular case, they constructed exact solutions of a system modeling viscous flow between slowly expanding and contracting walls.

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Nomenclature

A	Injection coefficient (v_w/a)
$a(t)$	Distance between parallel plates
c	Specific heat at constant temperature
E	Eckert number, $\frac{\mu v_1}{\rho hc(T_2-T_1)}$
h	Distance between parallel plates
k	Thermal conductivity
m	1/3 Trace of M
M	Couple stress tensor
P	Fluid pressure
Pr	Prandtl number, $\frac{\mu c}{k}$
$\bar{q}$	Velocity vector
R	Cross flow Reynolds number, $\frac{\rho a v_w}{\mu}$
T	Dimensionless temperature, $\frac{T-T_1}{T_2-T_1}$
T_1	Temperature at the lower plate
T_2	Temperature at the upper plate
$u(x,y)$	Axial velocity component
$v(x,y)$	Velocity component in y-direction
v_w	Suction or injection velocity

Greek Letters

α^2	Dimensionless couple stress parameter, $\frac{\eta}{\mu h^2}$
β	A wall expansion ratio $\left(=\frac{\dot{a}a}{v}\right)$
λ	Dimensionless y coordinate, y/h
τ	Force stress tensor
ζ	Dimensionless axial variable, $\frac{x^2}{a^2}$
ρ	Fluid density
μ	Fluid viscosity
ρC	Body couple tensor
η, η^I	Couple stress fluid parameters

It is known that many of the industrially and technologically important fluids behave like a non-Newtonian fluid. The couple stress fluid theory developed by Stokes [16] represents the simplest generalization of the classical viscous fluid theory that sustains couple stresses and the body couples. The important feature of these fluids is that the stress tensor is not symmetric and their accurate flow behavior cannot be predicted by the classical Newtonian theory. The main effect of couple stresses will be to introduce a size dependent effect that is not present in the classical viscous theories. The fluids consisting of rigid, randomly oriented particles suspended in a viscous medium, such as blood, lubricants containing small amount of polymer additive, electro-rheological fluids and synthetic fluids are examples of these fluids. Application of the couple stress model to biomechanics problems has been proposed in the study of peristaltic transport by Srivastava [17], Shehawey and Mekheimer [18] and blood flow in the microcirculation by Dulal Pal et al. [19]. In lubrication problems many authors have investigated the couple stress effects on different lubrication problems (Chiang et al. [20], Naduvinamani et al. [21], Jian et al. [22], Lu et al. [23]).

In this paper, we study the unsteady incompressible Couple stress fluid flow in a channel of expanding walls with injection. The problem is examined numerically using quasi-linearization technique, which provides an effective computational tool for the solution of a wide class of nonlinear two-point and multipoint boundary-value problems. The effects of different parameters on velocity components and temperature distribution are studied and shown graphically.

2. Formulation of the problem

Consider the laminar incompressible couple stress fluid flow through an elongated rectangular channel exhibiting a sufficiently large aspect ratio of width w to height a. Introduce the Cartesian coordinate system with the origin through the center of the channels and the x-axis along the axial flow direction and the y-axis perpendicular to it. Both upper and lower walls are assumed to have equal permeability and expand or contract uniformly at a time-dependent rate in the transverse direction only. Hence, their separation is a function of time $a(t)$. Assume that the fluid is injected or aspirated uniformly and orthogonally through the channel walls at an absolute velocity v_w. One end of the rectangular channel ($x=0$) is closed by a solid membrane that is allowed to stretch with channel expansions or contractions (see Fig. 1). At the other end, the channel is fully open. The influence of the opening at this end can be neglected by assuming semi-infinite length despite of its finite body length [5].

The governing equations of the flow of an incompressible couple stress fluid [16] in the absence of body force and body couple and the energy equations are

$$\nabla \cdot \bar{q} = 0 \tag{1}$$

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}\right] = -\nabla P - \mu\nabla\times\nabla\times\bar{q} - \eta\nabla\times\nabla\times\nabla\times\nabla\times\bar{q} \tag{2}$$

$$\rho c\left[\frac{\partial T}{\partial t} + (\bar{q}\cdot\nabla)T\right] = \mu\left[(\nabla\bar{q}):(\nabla\bar{q})^T + (\nabla\bar{q}):(\nabla\bar{q})\right] + 4\eta\left[(\nabla\bar{\omega}):(\nabla\bar{\omega})^T\right] + 4\eta^I[(\nabla\bar{\omega}):(\nabla\bar{\omega})] + k\nabla^2 T \tag{3}$$

where ρ is the density, $\bar{q}$ is the velocity vector, P is the fluid pressure, μ is the fluid viscosity, η and η^I are the couple stress fluid parameters, k is the thermal conductivity, c is the specific heat at constant temperature, $\bar{\omega}$ is the rotation vector and T is the temperature.

The force stress tensor τ and the couple stress tensor M that arises in the theory of couple stress fluids are given by

$$\tau = (-p + \lambda_1 \operatorname{div}\bar{q})I + \mu\left[\operatorname{grad}\bar{q} + (\operatorname{grad}\bar{q})^T\right] + 1/2\ I\times[\operatorname{div}M + \rho C] \tag{4}$$

and

$$M = mI + 2\eta\operatorname{grad}(\operatorname{curl}\bar{q}) + 2\eta^I(\operatorname{grad}(\operatorname{curl}\bar{q}))^T \tag{5}$$

where m is 1/3 trace of M and ρC is the body couple tensor. The quantity λ is the material constant and η^I is the constant associated with couple stresses. The dimensions of the material constant λ_1 is that of viscosity where as the dimensions of η and η^I are those

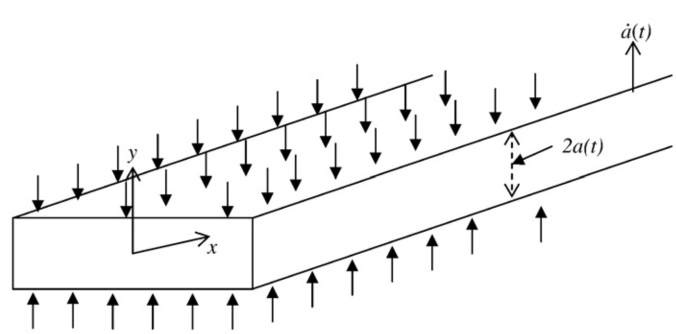

Fig. 1. Two-dimensional channel with expanding (or contracting) porous walls.

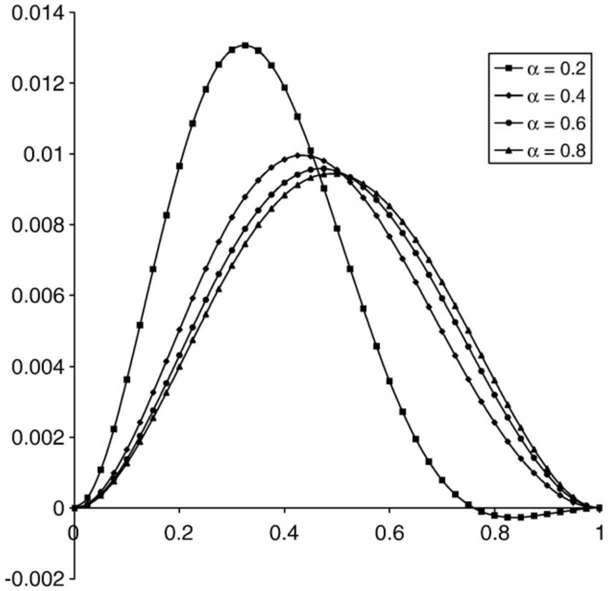

Fig. 2. The effect of α on axial velocity component.

of momentum. These material constants are considered by the inequalities,

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad \eta \geq 0, \quad \eta^I \leq \eta \tag{6}$$

The assumption of large aspect ratio (the height is smaller than the width) enables us to treat the problem as a case of two-dimensional flow. We choose the velocity vector as $\bar{q} = u(x,\ y,\ t)\hat{i} + v(x,\ y,\ t)\hat{j}$. The basic field Eqs. (1)–(4) can be expressed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial P}{\partial x} + \mu\nabla^2 u - \eta\nabla^4 u \tag{8}$$

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial \lambda}\right] = -\frac{\partial P}{\partial y} + \mu\nabla^2 v - \eta\nabla^4 v \tag{9}$$

$$\rho c\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial \lambda}\right] = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] + \eta\left[\left(\nabla^2 v\right)^2 + \left(\nabla^2 u\right)^2\right] + k\nabla^2 T \tag{10}$$

The boundary conditions on the velocity profile and temperature are

$$u(x,\lambda) = 0, \quad v(x,y) = v_w = -A\dot{\alpha}, \quad \nabla\times\bar{q} = 0, \quad T(x,y) = T_1 \text{ at } y = a(t) \tag{11}$$
$$u(x,\lambda) = 0, \quad v(x,\lambda) = 0, \quad \nabla\times\bar{q} = 0, \quad T(x,\lambda) = T_2 \text{ at } y = 0$$

At the wall, it is assumed that the fluid inflow velocity v_w is independent of position. The injection coefficient ($A \equiv v_w/a$) that appears in the above conditions is a measure of wall permeability.

Following Berman [1] and in view of the boundary conditions represented by Eqs. (9) and (10), a similarity solution with respect to x can be taken as

$$u = \frac{ux}{\rho a^2}F^I(\eta,t), \quad \upsilon = -\frac{ux}{\rho a}F(\eta,t) \tag{12}$$

where $\eta = y/a$

Substituting Eq. (12) in to Eq. (5)–(8) and eliminating pressure from the resulting equations, we get

$$\alpha^2 F^{VI} - \left(F^{IV} + 3\beta F^{II} + \beta\lambda F^{III} - RF^I F^{II} + RFF^{III} - \frac{a^2}{\nu}\frac{dF^{II}}{dt}\right) = 0 \tag{13}$$

where prime denotes differentiation with respect to η, $\beta = \beta(t) = \frac{\dot{a}a}{\nu}$ is a wall expansion ratio, $\nu = \frac{\mu}{\rho}$, $\alpha^2 = \frac{\eta}{\mu a^2}$. Eq. (10) together with Eq. (12), suggests that the form of temperature may be taken as

$$T(x,\lambda) = T_1 + \frac{\mu v_1}{ac}\left[\phi_1(\lambda) + \frac{x^2}{a^2}\phi_2(\lambda)\right] \tag{14}$$

Substituting Eq. (14) in Eq. (10), and equating the coefficients of $\frac{x^2}{a^2}$ and the terms without $\frac{x^2}{a^2}$ on both sides of the equation thus obtained, we get

$$\phi_2^{II} = Pr\left(-3\beta\phi_2 + 2F^I\phi_2 - F\phi_2^I\right) - \frac{Pr}{Re}\left(4F^{II^2} + \alpha^2 F^{III^2}\right) \tag{15}$$

$$\phi_1^{II} = -Pr\left(\beta\phi_1 + \phi_1^I\right) - \frac{Pr}{Re}\left(4F^{I^2} + \alpha^2 F^{II^2}\right) - 2\phi_2 \tag{16}$$

where $Pr = \frac{\mu c}{k}$ is the Prandtl number.

The dimensionless form of temperature from Eq. (14) can be written as

$$T = \frac{T - T_1}{T_2 - T_1} = E\left(\phi_1 + \zeta^2\phi_2\right) \tag{17}$$

where $E = \frac{\mu v_1}{\rho hc(T_2 - T_1)}$ is the Eckert number and $\zeta = \frac{x^2}{a^2}$ is the dimensionless axial variable.

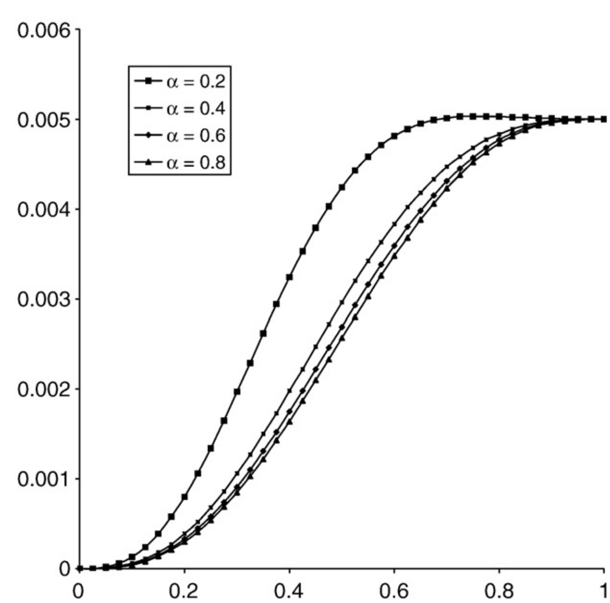

Fig. 3. The effect of α on radial velocity component.

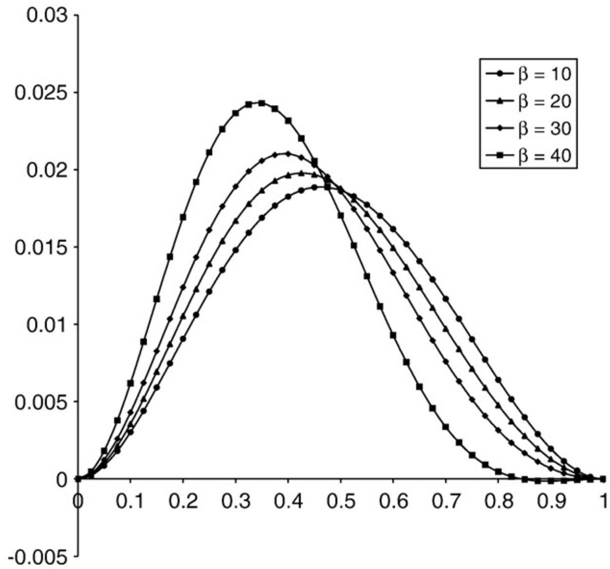

Fig. 4. The effect of β on axial velocity component.

The boundary conditions (Eq. (11)) in terms of f, ϕ_1 and ϕ_2 are

$$\begin{aligned} &f(0)=1-a, \quad f(1)=1,\\ &f^{I}(0)=0, \quad f^{I}(1)=0,\\ &f^{II}(0)=0, \quad f^{II}(1)=0,\\ &\phi_1(0)=0, \quad \phi_1(1)=0,\\ &\phi_2(0)=0, \quad \phi_2(1)=1/E=w\ (\text{say}) \end{aligned} \tag{18}$$

where $Re=\frac{\rho a v_w}{\mu}$ is cross flow Reynolds number.

A similar solution with respect to space and time can also be obtained following the transformation described by Uchida et al. [5]. This can be accomplished by considering the case for which β is constant and F is dependent on λ only. Under this assumption Eq. (13) becomes

$$\alpha^2F^{VI}-\left(F^{IV}+3\beta F^{II}+\beta\lambda F^{III}-RF^{I}F^{II}+RFF^{III}\right)=\ 0 \tag{19}$$

3. Solution of the problem

The nonlinear Eqs. (13), (15) and (16) are converted into the following system of first order differential equations by the substitution

$$\left(f,f^{I},f^{II},f^{III},f^{IV},f^{V},f^{VI},\phi_1,\phi_1^1,\phi_2,\phi_2^1\right)=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) \tag{20}$$

$$\begin{aligned} &\frac{dx_1}{d\lambda}=x_2, \quad \frac{dx_2}{d\lambda}=x_3, \quad \frac{dx_3}{d\lambda}=x_4, \quad \frac{dx_4}{d\lambda}=x_5, \quad \frac{dx_5}{d\lambda}=x_6,\\ &\frac{dx_6}{d\lambda}=\frac{1}{\alpha^2}\left[x_5+3\beta x_3+\beta\lambda x_4-x_2x_3+x_1x_4\right)\\ &\frac{dx_7}{d\lambda}=x_8, \quad \frac{dx_8}{d\lambda}=-Pr(\beta x_7+x_8)-\frac{Pr}{Re}\left(4x_2^2+\alpha^2x_3^2\right)-2x_9\\ &\frac{dx_9}{d\lambda}=x_{10}, \quad \frac{dx_{10}}{d\lambda}=Pr(-3\beta x_9+2x_2x_9-x_1x_{10})-\frac{Pr}{Re}\left(4x_3^2+\alpha^2x_4^2\right) \end{aligned} \tag{21}$$

The boundary conditions in terms of x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} are

$$\begin{aligned} &x_1(0)=1-a, \quad x_2(0)=0, \quad x_3(0)=0, \quad x_7(0)=0, \quad x_9(0)=0\\ &x_1(1)=1, \quad x_2(1)=0, \quad x_3(1)=0, \quad x_7(1)=0, \quad x_9(1)=w \end{aligned} \tag{22}$$

The system of Eq. (21) is solved numerically subject to the boundary conditions (Eq. (22)) using quasilinearization method (also known as generalized Newton's method) given by Bellman and Kalaba [24]

Let ($x_i^{(k)}$, $i=1, 2, \ldots 10$) be an approximate current solution and ($x_i^{(k+1)}$, $i=1, 2, \ldots 10$) be an improved solution of Eq. (21). By taking Taylor's series expansion around the current solution and neglecting the second and higher order derivative terms, the coupled first order system (Eq. (21)) is linearized as:

$$\begin{aligned} &\frac{dx_1^{(k+1)}}{d\lambda}=x_2^{(k+1)}, \quad \frac{dx_2^{(k+1)}}{d\lambda}=x_3^{(k+1)},\\ &\frac{dx_3^{(k+1)}}{d\lambda}=x_4^{(k+1)}, \quad \frac{dx_4^{(k+1)}}{d\lambda}=x_5^{(k+1)},\\ &\frac{dx_5^{(k+1)}}{d\lambda}=x_6^{(k+1)},\\ &\frac{dx_6^{(k+1)}}{d\lambda}=\frac{1}{\alpha^2}\left[3\beta x_3^{k+1}+\beta\lambda x_4^{k+1}-x_2^kx_3^{k+1}-x_3^kx_2^{k+1}+x_1^kx_4^{k+1}+x_4^kx_1^{k+1}+x_5^{k+1}\right]\\ &\qquad+\frac{1}{\alpha^2}\left[x_3^kx_2^k-x_4^kx_1^k\right]\\ &\frac{dx_7^{k+1}}{d\lambda}=x_8^{k+1},\\ &\frac{dx_8^{k+1}}{d\lambda}=-Pr\left(\beta x_7^{k+1}+x_8^{k+1}\right)-\frac{Pr}{Re}\left(8x_2^kx_2^{k+1}+2\alpha^2x_3^kx_3^{k+1}\right)-2x_9^{k+1}\\ &\qquad+\frac{Pr}{Re}\left(4x_2^kx_2^k+\alpha^2x_3^kx_3^k\right)\\ &\frac{dx_9^{k+1}}{d\lambda}=x_{10}^{k+1},\\ &\frac{dx_{10}^{k+1}}{d\lambda}=-Pr\left[-3\beta x_9^{k+1}+2x_2^kx_9^{k+1}+2x_9^kx_2^{k+1}-x_1^kx_{10}^{k+1}-x_{10}^kx_1^{k+1}\right]\\ &\qquad-\frac{Pr}{Re}\left[8x_3^kx_3^{k+1}+2\alpha^2x_4^kx_4^{k+1}\right]+Pr\left[-2x_9^kx_2^k+x_{10}^kx_1^k\right]+\frac{Pr}{Re}\left[4x_3^kx_3^k+\alpha^2x_4^kx_4^k\right] \end{aligned} \tag{23}$$

To solve for ($x_i^{(k+1)}$, $i=1, 2, \ldots 10$), the solution to four separate initial value problems, denoted by $x_i^{h1}(\lambda)$, $x_i^{h2}(\lambda)$, $x_i^{h3}(\lambda)$, $x_i^{h4}(\lambda)$, $x_i^{h5}(\lambda)$ (which are the solutions of the homogeneous system corresponding to Eq. (23)) and $x_i^{p}(\lambda)$ (which is the particular solution of Eq. (23)), with

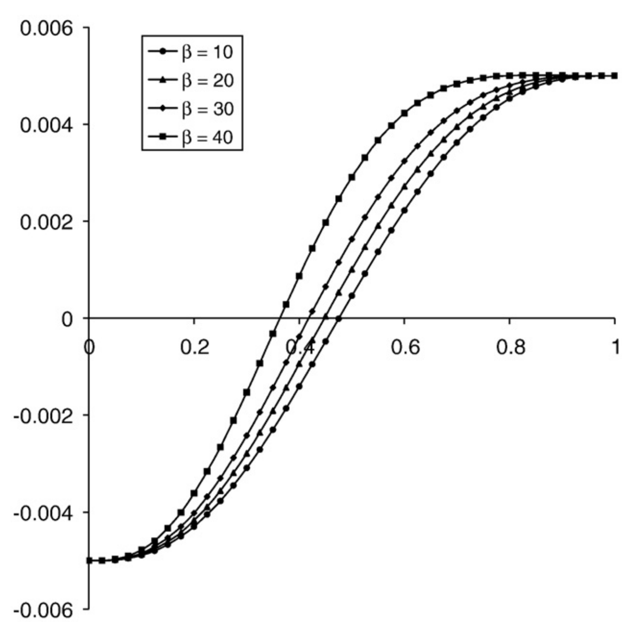

Fig. 5. The effect of β on radial velocity component.

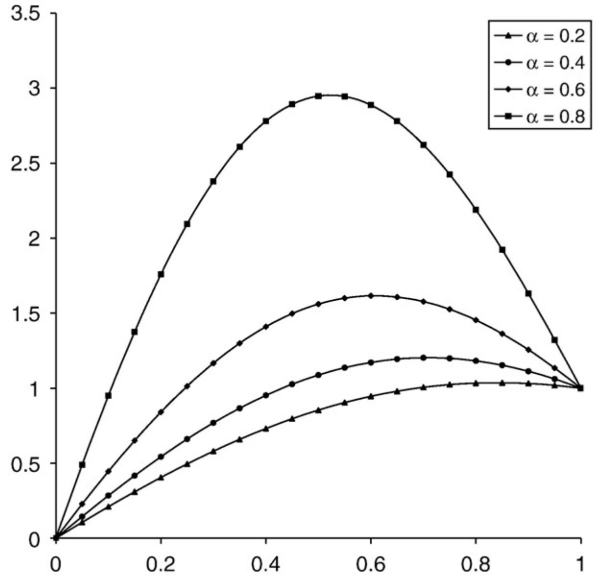

Fig. 6. The effect of α on temperature distribution.

the following initial conditions are obtained by using a Runge–Kutta method.

$$
\begin{aligned}
&x_4^{h1}(0)=1, \quad x_i^{h1}(0)=0 \text{ for } i\neq 4,\\
&x_5^{h2}(0)=1, \quad x_i^{h2}(0)=0 \text{ for } i\neq 5,\\
&x_6^{h3}(0)=1, \quad x_i^{h3}(0)=0 \text{ for } i\neq 6,\\
&x_8^{h4}(0)=1, \quad x_i^{h4}(0)=0 \text{ for } i\neq 8\\
&x_{10}^{h1}(0)=1, \quad x_i^{h5}(0)=0 \text{ for } i\neq 10\\
&x_1^{p1}(0)=1-a, \quad x_9^{p1}(0)=w\\
&x_2^{p1}(0)=x_3^{p1}(0)=x_4^{p1}(0)=x_5^{p1}(0)=x_6^{p1}(0)=x_7^{p1}(0)=x_8^{p1}(0)=x_{10}^{p1}(0)=0
\end{aligned} \tag{24}
$$

Since the differential equations are linear, the principle of superposition holds and the general solution may be written as,

$$x_i^{(k+1)}(\lambda)=C_1x_i^{h1}(\lambda)+C_2x_i^{h2}(\lambda)+C_3x_i^{h3}(\lambda)+x_i^{p}(\lambda)) \tag{25}$$

where C_1, C_2, and C_3 are the unknown constants and are determine by considering the boundary condition at $\lambda=1$. This solution ($x_i^{(k+1)}$, $i=1$, 2, …10) is then compared with solution at the previous step ($x_i^{(k)}$, $i=1$, 2, …10) and further iteration is performed if the convergence has not been achieved or greater accuracy is desired.

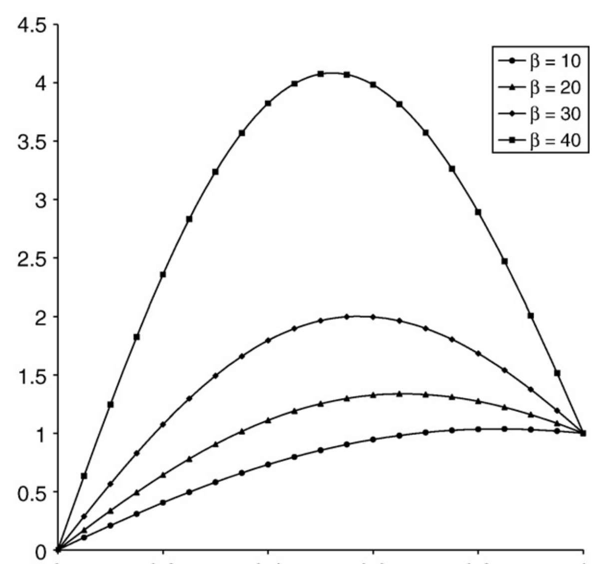

Fig. 7. The effect of β on axial temperature distribution.

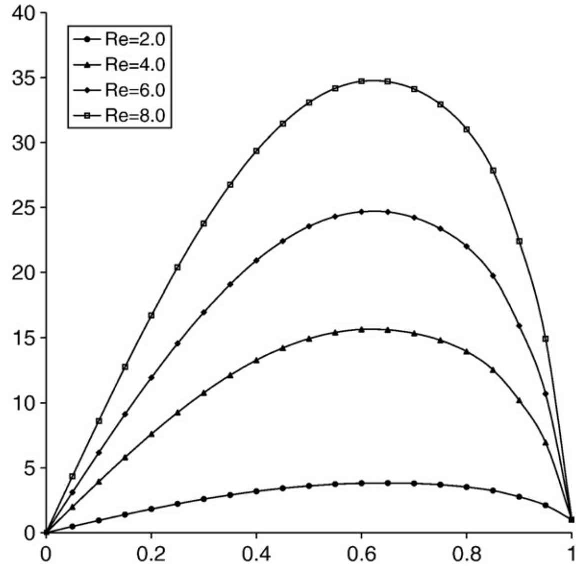

Fig. 8. The effect of suction Reynolds number on temperature distribution.

4. Results and discussion

To have a better understanding of the flow characteristics, numerical results for the velocity components and temperature distribution are calculated for different values of parameters in the domain [0,1].

The effect α on the axial and radial velocity components is shown in Figs. 2 and 3. From Fig. 2, it can be seen that the axial velocity decreases as α increases. The transverse velocity component also decrease throughout the domain for an increase in the value of α.

The effect of β on axial and transverse velocity components has been presented in Figs. 4 and 5 respectively. It can be observed that the axial velocity increases near the central plane as the value of β increases. However, this trend is reversed near walls. The transverse velocity component increase throughout the domain for an increase in the value of β. Figs. 6, 7 and 8 show that the variation temperature distribution with α, β and S. It can be observed from these figures that the temperature distribution increases for increasing values of α, β and S.

References

[1] A.S. Berman, Laminar flow in channels with porous walls, J. App. Phys. 24 (1953) 1232–1235.
[2] J.R. Sellars, Laminar flow in channels with porous walls at high suction Reynolds number, J. Appl. Phys. 26 (1955) 489–490.
[3] S.W. Yuan, Further investigation of laminar flow in channels with porous walls, J. Appl. Phys. 27 (1956) 267–269.
[4] R.M. Terril, G.M. Shrestha, Laminar flow through parallel and uniformly porous walls of different permeability, ZAMP 16 (1965) 470–482.
[5] S. Uchida, H. Aoki, Unsteady flows in a semi-infinite contracting or expanding pipe, J. Fluid Mech. 82 (2) (1977) 371–387.
[6] M. Ohki, Unsteady flow in a porous, elastic, circular tube—1 the wall contracting or expanding in an axial direction, Bull. JSME 23 (179) (1980) 679–686.
[7] M. Goto, S. Uchida, Unsteady flows in a semi-infinite expanding pipe with injection through wall, Trans. Jpn. Soc. Aeronaut. Space Sci. 33 (9) (1990) 14–27.
[8] M. Goto, S. Uchida, Unsteady flow in a semi-infinite contracting or expanding pipe with a porous wall, Theor. Appl. Mech. 40 (1991) 161–172.
[9] N.M. Bujurke, N.P. Pai, G. Jayaraman, Computer extended series solution for unsteady flow in a contracting or expanding pipe, IMA J. Appl. Math. 60 (2) (1998) 151–165.
[10] J. Majdalani, C. Zhou, C.A. Dawson, Two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, J. Biomech. 35 (10) (2002) 1399–1403.

[11] C.E. Dauenhauer, J. Majdalani, Exact self-similarity solution of the Navier–Stokes equations for a porous channel with orthogonally moving walls, Phys. Fluids 15 (6) (2003) 1485–1495.
[12] J. Majdalani, C. Zhou, Moderate-to-large injection and suction driven channel flows with expanding or contracting walls, ZAMM Z. Angew. Math. Mech. 83 (3) (2003) 181–196.
[13] A. Adamkowski, Analysis of transient flow in pipes with expanding or contracting sections, J. Fluids Eng., Trans. ASME 125 (4) (2003) 716–722.
[14] Y.Z. Boutros, M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, Lie-group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, Appl. Math. Model. 31 (6) (2007) 1092–1108.
[15] S. Asghar, M. Mushtaq, A.H. Kara, Exact solutions using symmetry methods and conservation laws for the viscous flow through expanding–contracting channels, Appl. Math. Model. 32 (12) (2008) 2936–2940.
[16] V.K. Stokes, Couple stress in fluids, Phys. Fluids 9 (1966) 1709–1715.
[17] L.M. Srivastava, Peristaltic transport of a couple-stress fluid, Rheol. Acta 25 (1986) 638–641.
[18] E.F.E. Shehawey, K.S. Mekheimer, Couple-stresses in peristaltic transport of fluids, J. Phys. D. Appl. Phys. 27 (1994) 1163–1170.
[19] D. Pal, N. Rudraiah, R. Devanathan, A couple stress model of blood flow in the microcirculation, Bull. Math. Biol. 50 (1988) 329–344.
[20] Hsiu-Lu Chiang, Cheng-Hsing Hsu, Jaw-Ren Lin, Lubrication performance of finite journal bearings considering effects of couple stresses and surface roughness, Tribol. Int. 37 (2004) 297–307.
[21] N.B. Naduvinamani, P.S. Hiremath, G. Gurubasavaraj, Effect of surface roughness on the couple-stress squeeze film between a sphere and a flat plate, Tribol. Int. 38 (2005) 451–458.
[22] Cai-Wan Jian, Chang, Chao-Kuang Chen, Chaos and bifurcation of a flexible rotor supported by porous squeeze couple stress fluid film journal bearings with non-linear suspension, Chaos, Solitons Fractals 35 (2006) 358–375.
[23] Rong-Fang Lu, Jaw-Ren Lin, A theoretical study of combined effects of non-Newtonian rheology and viscosity–pressure dependence in the sphere-plate squeeze-film system, Tribol. Int. 40 (2007) 125–131.
[24] R.E. Bellman, R.E. Kalaba, Quasilinearization and Boundary-value Problems, Elsevier publishing Co. Inc., New York, 1965.