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# Extremization of multi-objective stochastic fractional programming problem

## An application to assembled printed circuit board problem

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**Abstract** This paper addresses classes of assembled printed circuit boards, which faces certain kinds of errors during its process of manufacturing. Occurrence of errors may lead the manufacturer to be in loss. The encountered problem has two objective functions, one is fractional and the other is a non-linear objective. The manufacturers are confined to maximize the fractional objective and to minimize the non-linear objective subject to stochastic and non-stochastic environment. This problem is decomposed into two problems. A solution approach to this model has been developed in this paper. Results of some test problems are provided.

**Keywords** Stochastic programming · Fractional programming · Non-linear programming · Lexicographical

## Introduction

Stochastic programs are mathematical programs where some of the data incorporated in the objective function or constraints is uncertain. Uncertainty is usually characterized by a probability distribution on the parameters. Although uncertainty is rigorously defined, in practice it can range in detail from a few scenarios to specific and precise joint probability distributions. Stochastic programming has been applied to a wide variety of areas. Some of the specific problems are to be seen in (http://users.iems.nwu.edu/∼jrbirdge//html/dholmes/SPTSlists.html). In real life situation, there are many applications of stochastic programming. Some of these can be seen in (Charnes and Cooper, 1954; Jeeva et al., 2002, 2004) and the web site of Maarten Van Der Vlerk (http://mally.eco.rug.nl/biblio/Splist.html). Other applications are, Manufacturing/Production Planning,

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Manufacturing/ production capacity planning, Electrical generation capacity planning, Machine Scheduling, Macroeconomic modeling and planning, Timber management, Asset liability management, Portfolio selection, Traffic management, Optimal truss design, Inventory management, Human resource planning (Martel and Price, 1981).

Fractional programming is an optimization problem in which ratio of two linear functions is optimized subject to some constraints (Charnes and Cooper, 1962; Zionts, 1968). Basic concepts about stochastic fractional programming are available in (Charles et al., 2001; Charles and Dutta, 2003; Gupta and Kanti, 1979; Gupta et al., 1981). Bibliography in stochastic programming and fractional programming are provided in Im and Wets (1976) and Im (1977) respectively.

Extremization of Multi-objective Stochastic Fractional Programming (MOSFP) problem is nothing but optimizing the fractional function along with linear or non-linear functions subject to some constraints in which atleast one of the problem data is random in nature with non-negative constraints on the decision variables. Dutta et al. (1993), Eschenauer et al. (1990) and Rao (1984), Rao and Eslampour (1986) have discussed the procedures and applications of multi-objective optimization problems. We have considered a special class of multi-objective stochastic fractional programming problem, which arises in assembled printed circuit boards (PCB) at the time of manufacturing.

This paper addresses classes of assembled PCB, which faces certain kinds of errors during its processes of manufacturing. A PCB manufacturing company produces various types of assembled PCB. Each type of assembled PCB has to undergo specific number of processes. The total investment on the assembled PCB is predefined. The expenses on each type of assembled PCB are well known to the manufacturer along with its profit. During the process of assembled PCB, there are certain factors, which have deep impact on the assembled PCB that makes them defective.

The encountered problem has two objective functions, one is fractional and the other is non-linear objective function. Several methods are available to solve multi-objective optimization problems. We have adapted lexicographical method. We have found that the lexicographical method is more appropriate for the assembled printed circuit boards problem than the other methods. In this method, the manufacturer ranks the objective functions in the order of importance. The optimum solution of decision variables are then found by optimizing the objective functions starting with most important and then proceeding according to the order of preference of the objective functions. Usually, the manufacturers are confined to maximize the fractional objective function and to minimize the non-linear objective function subject to stochastic and non-stochastic environment. In our problem, the first preference is given to fractional objective function and then to non-linear objective function. This problem is decomposed into two problems, where the first problem deals with maximization of profit whereas the next deals with minimization of loss (errors). The solution of the first problem gives the upper bound of the profit as well as it helps to solve the second problem.

This paper is organized as follows; In Section 1 the multi-objective stochastic fractional programming problem model is formulated. A required algorithm to solve the model is given in Section 2. The methodology of solving the MOSFPP is presented in Section 3. Section 4 contains preliminary computational result, while Section 5 draws conclusion and indicates the future direction for research.

Assumptions

1. Process time follows Normal Distribution with known mean and variance.
2. Profit model has feasible solution.

**Table 1** Assembled PCB models' notations

| Process | Time required per unit (Hrs) PCB Type 1 | | 2 | | .... | n | | Available time per unit (Hrs) |
|---|---|---|---|---|---|---|---|---|
| 1 | $u_{11}$ | $s_{11}$ | $u_{12}$ | $s_{12}$ | .... | $u_{1n}$ | $s_{1n}$ | $b_1$ |
| 2 | $u_{21}$ | $s_{11}$ | $u_{22}$ | $s_{12}$ | .... | $u_{2n}$ | $s_{2n}$ | $b_2$ |
| .... | .... | | .... | | .... | .... | | .... |
| $m$ | $u_{m1}$ | $s_{m1}$ | $u_{m2}$ | $s_{m2}$ | .... | $u_{mn}$ | $s_{mn}$ | $b_m$ |
| MI | $c_1$ | | $c_2$ | | | $c_n$ | | |
| MP | $d_1$ | | $d_2$ | | | $d_n$ | | |

3. Prior estimation of errors is possible by R&D Department of the manufacturing company.

## 1. The assembled PCB model

Assembled PCB manufacturing problem is viewed as a Multi Objective Stochastic Fractional Programming Problem in which process times are normally distributed. An assembled PCB manufacturing company manufactures $n$ types of PCB. Further, it has to under go $m$ process. The processing time $t_{ij}$ are known to be independently distributed normal variables with estimated mean $u_{ij}$ and standard deviation $s_{ij}$. The annual average fixed cost to manufacture the assembled PCB is $\beta$. The mean capital invested (MI) is Rs. $d_j$ per assembled PCB of type $j$. Let the annual resultant mean profit (MP) be Rs. $c_j$ per assembled PCB of type $j$. The available units of time per month is known, $b = [b_1, b_2, \ldots, b_m]$ (say). The above information is presented in Table 1.

The objective is to determine the number of assembled PCB of each type that should be produced per month so as to maximize profitability (ratio of net profit and capital invested). Let us call this model as profit model.

The number of assembled PCB of type $j$ manufactured per month is denoted as $x_j$. Let $c'x$ be the profit function and $d'x + \beta$ be an investment function. The problem then can be formulated from Charles et al. (2001); Charles and Dutta (2001) as follows

$$\text{Maximize} R^{(1)}(x) = \frac{c'x}{d'x + \beta} \tag{1}$$

subject to the constraints

$$\Pr\left(\sum_{j=1}^{n} t_{ij}x_j \leq b_i\right) \geq 1 - p_i \qquad (i = 1, \ldots, m). \tag{2}$$

where $1 - p_i(0 < p_i \leq 1)$ is the least probability with which $i^{\text{th}}$ constraint is satisfied, $(i = 1, \ldots, m)$,

$$x_j \geq 0 \qquad (j = 1, \ldots, n).$$

It was identified that during the processes of assembled PCB, there are eleven factors namely nature of defects that makes them defective. They are Wrong Component Assembled (WCA), Reversal Component (RC), Component Missing (CM), Wrong Cut Done/Cut Not Done

(WCD/CND), Pattern Cut (PC), Pin Bend in IC's (PB), Dry Soldering (DS), Not Cleaned (NC), Wrong Strapping Done (WSD), Not Mounted Properly (NMP), Solder Short (SS). Nature of defects are classified into three kinds of errors namely Machine Errors, Manual Errors and Other Errors. Factors under Machine errors, Manual Errors and Other Errors are (DS, SS), (WCA, RC, CM, WCD/CND, PB, WSD) and (PC, NC, NMP) respectively.

The prior knowledge made the R&D department to estimate these errors. Each type of assembled PCB may have atmost three kinds of errors. This forms the combination FST that stands for First, Second and Third and it depicts occurrence of Machine errors, Manual Errors and Other Errors respectively. Binary values 1 and 0 are used to denote occurrence and non-occurrence of errors. We get totally seven combinations of errors, i.e. 111,110,101,011,100,010,001. Here 111-all kinds of errors occurred, 101- first and third kinds of errors occurred, 010- only second error occurred, etc. Occurrence of these errors may lead the manufacturer to be in loss. The ultimate aim of the manufacturer is to minimize these errors in order to maximize the profit.

In the following loss model, let $e_{FST}^{(j)}$ depict the number of errors of $j^{\text{th}}$ type assembled PCB. From the above discussion it is very well clear that the contribution for Machine Errors form the set (111,110,101,100), for Manual Errors form the set (111,110,011,010) and for Other Errors form the set (111,101,001,001). Obviously, $e_{111}^{(j)} + e_{110}^{(j)} + e_{101}^{(j)} + e_{100}^{(j)}$ is the total number of assembled PCB of $j^{\text{th}}$ type, which has Machine Errors. But this total is bounded by estimated errors. Let $e_l^{(j)}$ be the estimated error percent of lower limit of $j^{\text{th}}$ type assembled PCB for the above-mentioned sets. The mathematical form of loss model is as follows

$$\text{Minimize } \mathrm{R}^{(2)}(e) = \sum_{j=1}^{n} \sum_{FST=\{111,110,101,011,100,010,001\}} \left(e_{FST}^{(j)}\right)^2 \tag{3}$$

$$\frac{e_l^{(j)}}{100} x_j \le \sum_{FST=\{111,110,101,100\}} e_{FST}^{(j)} \le \frac{e_{u1}^{(j)}}{100} x_j \tag{4}$$

$$\frac{e_l^{(j)}}{100} x_j \le \sum_{FST=\{111,110,011,010\}} e_{FST}^{(j)} \le \frac{e_{u2}^{(j)}}{100} x_j \tag{5}$$

$$\frac{e_l^{(j)}}{100} x_j \le \sum_{FST=\{111,101,011,001\}} e_{FST}^{(j)} \le \frac{e_{u3}^{(j)}}{100} x_j \tag{6}$$

where $e^{(j)} = \left[e_{111}^{(j)}, e_{110}^{(j)}, e_{101}^{(j)}, e_{011}^{(j)}, e_{100}^{(j)}, e_{010}^{(j)}, e_{001}^{(j)}\right] \ge 0 \qquad (j = 1, \ldots, n)$.

The profit function after minimizing the errors is

$$\mathbf{R}^{(3)}(\mathbf{X}, \mathbf{e}) = c'[\mathrm{x} - e] \tag{7}$$

where $e = \left[\sum_{FST} e_{FST}^{(1)}, \sum_{FST} e_{FST}^{(2)}, \ldots, \sum_{FST} e_{FST}^{(n)}\right]; FST = \{111, 110, 101, 011, 100, 010, 001\}$.

## 2. The sequential linear programming (SLP) algorithm for MOSFP problem

Cheney and Goldstein (1959) and Kelly (1960) originally presented the SLP method. The concept of solving a series of linear programming problem in order to obtain the solution of the original non-linear programming problem is known as sequential linear programming problem. Each linear programming problem is generated by approximating non-linear objective and constraint functions using first order Taylor series. Though the MOSFP model is non-linear, after adopting the concepts of chance-constraints (Charles and Dutta, 2003; Charnes and Cooper, 1954), the model become partially linear, in the sense that the objective function turns out to be linear.

1. Start with an initial point $X_{int}$. The point $X_{int}$ need not be feasible. Let $X_1 = X_{int}$.
2. Linearize the non-linear objective and constraint functions about the point $X_j$ as $R(X) \approx R(X_j) + \nabla R(X_j)^T(X - X_j)$ and $g_i(X) \approx g_i\ (X_j) + \nabla\ g_i(X_j)^T(X - X_j)$
3. Formulate the approximating LSFP problem as LPP as given below

$$\text{Minimize}\, R(X_j) + \nabla R(X_j)^T(X - X_j)$$

$$\text{subject to}\, g_i(X_j) + \nabla g_i(X_j)^T(X - X_j) \leq 0; \forall X_j \geq 0 (i = 1, \ldots, m; j = 1, \ldots, n).$$

4. Solve the approximating LPP to obtain the solution vector $X_{next}$.
5. Evaluate the original constraints at $X_{next}$; i.e. Get $g_i\ (X_{next})$, $i = 1$ to $m$. If $g_i(X_{next}) \leq \xi$ for $i = 1$ to $m$, where $\xi$ is a prescribed small positive tolerance, all the original constraints can be assumed to have been satisfied. Hence stop the procedure by taking $X_{opt} \approx X_{next}$. If $g_i\ (X_{next}) > \xi$ for some $i$, find the most violated constraint , For example, as $g^*\ (X_{next}) = \max_i\ [g_i\ (X_{next})]$. Relinearize the constraint $g^*(X) \leq 0$ about the point $X_{next}$ as $g^*_{(X)} \approx g^*_{(X_{next})} + \nabla\ g^*_{(X_{next})} T(X - X_{next}) \leq 0$ and add this as the $(m+1)^{\text{th}}$ inequality constraints to the previous LSFP problem. Let $next = int + 1$. Goto step 4.

## 3. Model implementation

1. Solve the profit model using the algorithm given in Section 2 and obtain the number of assembled PCB of various types with the upper bound of profit.
2. Substitute the profit model solution in (4–6) to obtain the loss model constraints boundaries and solve the loss model using any non-linear software.
3. Use (7) to get the profit function after minimizing the errors.

## 4. Small test problem

Let us consider the data given in Table 2 and 3 for (1–2) and (3–6) respectively.

Using (Charles et al., 2001) the following non-linear programming problem is obtained.

$$\text{Max}\ R^{(1)}(X) = (17x_1 + 18x_2)/(12x_1 + 15x_2 + 10000) \tag{8}$$

**Table 2** Data for the profit model of section 4 test problem

| $u_{11}$ | $u_{12}$ | $u_{21}$ | $u_{22}$ | $s_{11}$ | $s_{12}$ | $s_{21}$ | $s_{22}$ | $c_1$ | $c_2$ | $d_1$ | $d_2$ | $\beta$ | $Kq_i$ ($q_{i=0.9}$) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 3 | 2 | 2 | 3 | 1 | 1 | 17 | 18 | 12 | 15 | 10000 | 1.28 |

subject to

$$4x_1 + 5x_2 + 1.28\sqrt{(2^2x_1^2 + 3^2x_2^2)} \leq 300 \tag{9}$$

$$3x_1 + 2x_2 + 1.28\sqrt{(1^2x_1^2 + 1^2x_2^2)} \leq 150 \tag{10}$$

$$x_1, x_2 \geq 0$$

Let us take the initial value as (1,1) and solve the objective function (8) with (9) and (10) using sequential linear programming technique of Section 2.

$$\text{Max } R^{(1)}(X) = 0.0017009x_1 + 0.0018020x_2 - 0.0000129 \tag{11}$$

Subject to

$$5.42x_1 + 8.1950x_1 \leq 300 \tag{12}$$

$$3.90x_1 + 2.9050x_2 \leq 150 \tag{13}$$

$x_1, x_2 \geq 0$
Solution of (11-13) problem is $R^{(1)}(X) = 0.0772$ at $x_1 = 22$, $x_2 = 22$.

Let us solve the loss model

$$\text{Min } R^{(2)}(e) = \sum_{j=1}^{2} \left(e_{111}^{(j)}\right)^2 + \left(e_{110}^{(j)}\right)^2 + \left(e_{101}^{(j)}\right)^2 + \left(e_{011}^{(j)}\right)^2 + \left(e_{100}^{(j)}\right)^2 + \left(e_{010}^{(j)}\right)^2 + \left(e_{001}^{(j)}\right)^2 \tag{14}$$

$$\frac{e_l^{(j)}}{100}x_j \leq e_{111}^{(j)} + e_{110}^{(j)} + e_{101}^{(j)} + e_{100}^{(j)} \leq \frac{e_{u1}^{(j)}}{100}x_j \tag{15}$$

$$\frac{e_l^{(j)}}{100}x_j \leq e_{111}^{(j)} + e_{110}^{(j)} + e_{011}^{(j)} + e_{010}^{(j)} \leq \frac{e_{u2}^{(j)}}{100}x_j \tag{16}$$

$$\frac{e_l^{(j)}}{100}x_j \leq e_{111}^{(j)} + e_{101}^{(j)} + e_{011}^{(j)} + e_{001}^{(j)} \leq \frac{e_{u3}^{(j)}}{100}x_j \tag{17}$$

where $e^{(j)} \geq 0 \qquad (j = 1, 2)$.

Using Table 3, bounded constraints (15-17) gives (18-23)

$$1.10 \leq e_{111}^{(1)} + e_{110}^{(1)} + e_{101}^{(1)} + e_{100}^{(1)} \leq 2.20 \tag{18}$$

$$1.10 \leq e_{111}^{(1)} + e_{110}^{(1)} + e_{011}^{(1)} + e_{010}^{(1)} \leq 4.40 \tag{19}$$

$$1.10 \leq e_{111}^{(1)} + e_{101}^{(1)} + e_{011}^{(1)} + e_{001}^{(1)} \leq 2.20 \tag{20}$$

**Table 3** Data for the loss model of section 4 test problem

| $i$ | $e_l^{(i)}$ | $e_{u1}^{(i)}$ | $e_{u2}^{(i)}$ | $e_{u3}^{(i)}$ |
|---|---|---|---|---|
| 1 | 5% | 10% | 20% | 10% |
| 2 | 5% | 12% | 15% | 13% |

$$1.10 \leq e^{(2)}_{111} + e^{(2)}_{110} + e^{(2)}_{101} + e^{(2)}_{100} \leq 2.64 \quad (21)$$

$$1.10 \leq e^{(2)}_{111} + e^{(2)}_{110} + e^{(2)}_{011} + e^{(2)}_{010} \leq 3.30 \quad (22)$$

$$1.10 \leq e^{(2)}_{111} + e^{(2)}_{101} + e^{(2)}_{011} + e^{(2)}_{001} \leq 2.86 \quad (23)$$

where $e^{(1)}, e^{(2)} \geq 0$.

Solving objective function (14) along with bounded constraints (18-23), we get $R^{(2)}(e) = 0.9075$ at $e^{(1)} = [0.4125, 0.2750, 0.2750, 0.2750, 0.1375, 0.1375, 0.1375]$ and $e^{(2)} = [0.4125, 0.2750, 0.2750, 0.2750, 0.1375, 0.1375, 0.1375]$. From (7), we can obtain the profit of the manufacturer after minimizing the errors which is

$$R^{(3)}(X,\ e) = 17\left[x_1 - \sum_{FST} e^{(1)}_{FST}\right] + 18\left[x_2 - \sum_{FST} e^{(2)}_{FST}\right] = 17[22 - 1.65] + 18[22 - 1.65] = 712.25$$

Now let us consider the above problem with $e^{(1)}_l = 8\%$ and $e^{(2)}_l = 10\%$.

Solving the objective function (14) along with bounded constraints (18-23), we get $R^{(2)}(e) = 2.9766$ at $e^{(1)} = [0.6600, 0.4400, 0.4400, 0.4400, 0.2200, 0.2200, 0.2200]$ and $e^{(2)} = [0.8250, 0.5500, 0.5500, 0.5500, 0.2750, 0.2750, 0.2750]$. The profit of the manufacturer after minimizing the errors is

$$R^{(3)}(X,\ e) = 17\left[x_1 - \sum_{FST} e^{(1)}_{FST}\right] + 18\left[x_2 - \sum_{FST} e^{(2)}_{FST}\right] = 17[22 - 2.64] + 18[22 - 3.3] = 665.72.$$

## 5. Conclusion & future direction

In this paper, we have formulated stochastic version of assembled PCB model. We have presented a simple sequential linear programming type approach for solving the profit model. This type of algorithm is very popular among practitioners because it uses existing linear programming techniques and has history of good performance. A solution approach to MOSFPP model has been developed and implemented using Lingo software. This paper undoubtedly assists the manufacturer to accrue the optimum profit after minimizing the loss. The development of confidence intervals of profit and loss functions and error analysis is a topic of future research. Our ultimate aim is to develop much more efficient models to make the manufacturer to attain the highest profit.

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