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A Fitted Numerov Method for Singular Perturbation Problems Exhibiting Twin Layers

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In this paper a fitted fourth-order finite difference scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at both end (left and right) points. We have introduced a fitting factor in Numerov fourth-order tridiagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the tridiagonal system. Several numerical examples are solved and compared with exact solution. It is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems, boundary layer, interior layer, finite differences, fitted method.

1 Introduction

Singular perturbation problems containing a small positive parameter ε have appeared in many fields such as fluid mechanics, chemical kinetics, elasticity, aerodynamics, plasma dynamics, magneto-hydrodynamics and other domains of the world of fluid motion. A few notable examples are boundary layer problems, WKB problems. It is well known fact that the solution of these problems exhibits a multiscale character. That is, there is a thin layer(s) where the solution varies rapidly (non uniformly), while away from the layer the solution behaves regularly (uniformly) and varies slowly. Therefore, the numerical treatment for singularly perturbed boundary value problems has always been far from trivial.

A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention Bender and Orzag [1], Doolan and Miller [2], Hemker [3], Miller [5, 6], Kevorkian and Cole [4], Nayfeh [7], O'Malley [8], Reddy [9], Reddy and Pramod Chakravarthy [10]. We shall be interested in defining numerical methods for the problem

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x)\ ,\ \ x \in [0, 1]$$

with boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ where ε is positive and very small. Moreover, we shall assume that in [0, 1], $b(x)$ and $f(x)$ are continuous and, for simplicity, $a(x)$ is differentiable. This problem has been treated by several authors in the last years. The behaviour of the solution depends, of course, on the properties of the functions $a(x)$ and $b(x)$. There are intervals of [0, 1] where the solution vary rapidly (layers). They may be localized either at the extreme points of the interval [0,1] (boundary layers) or near the roots x_i of $a(x)$, which are called turning points (interior layers). The following table essentially taken from the above differential equation summarizes these facts.

$a(x)\neq 0$; $0 \leq x \leq 1$	$a(x) < 0$ boundary layer at $x = 0$ $a(x) > 0$ boundary layer at $x = 1$
$a(x)=0$	$b(x) > 0$ boundary layers at x=0 and x=1 $b(x) < 0$ rapidly oscillatory solution $b(x)$ changes sign (turning points)
$a'(x_i) \neq 0$, $a(x_i) = 0$	$a'(x_i) > 0$no boundary layers, interior layer at x_i $a'(x_i) < 0$ possible boundary layers, no interior layer at x_i

In this paper a fitted fourth-order finite difference scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at both end (left and right) points i.e., $a(x) = 0$ and $b(x) > 0$ in the above differential equation. We have introduced a fitting factor in Numerov fourth-order tridiagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the tridiagonal system. Several numerical examples are solved and compared with exact solution. It is observed that the present method approximates the exact solution very well.

2 Description of the Method

We consider the second order linear differential equation

$$-\varepsilon y''(x) + b(x)y(x) = f(x); \qquad x \in [0, 1] \tag{2.1}$$

with the boundary conditions

$$y(0) = \alpha \text{ and } y(1) = \beta.$$

The solution of the reduced problem is

$$y_0(x) = \frac{f(x)}{b(x)}, \tag{2.2}$$

which does not satisfy both the boundary conditions. The solution of Eq. (2.1) will be of the form

$$y(x) = y_0 + v_0 + w_0, \tag{2.3}$$

where v_0 is the left boundary layer function (or solution) and w_0 is the right boundary layer function (or solution). v_0 and w_0 satisfy the differential equations

$$\frac{-d^2 v_0(\tau)}{d\tau^2} + b(0)v_0(\tau) = 0 \; ; \quad \tau \in (0,\ \infty) \tag{2.4}$$

$$\frac{-d^2 w_0(\eta)}{d\eta^2} + b(1)w_0(\eta) = 0 \; ; \quad \eta \in (0,\ \infty) \tag{2.5}$$

with

$$v_0(\tau = 0) + w_0(\eta = 1/\sqrt{\varepsilon}) = \alpha - y_0(0),$$
$$v_0(\tau = 1/\sqrt{\varepsilon}) + w_0(\eta = 0) = \beta - y_0(1),$$
$$v_0(\tau = \infty) = w_0(\eta = \infty) = 0,$$

where $\tau = x/\sqrt{\varepsilon}$ and $\eta = (1-x)/\sqrt{\varepsilon}$.

Solutions of Eq. (2.4) and Eq. (2.5) are given by

$$v_0(\tau) = Ae^{-\sqrt{b(0)}\tau}, \tag{2.6}$$

and

$$w_0(\eta) = Be^{-\sqrt{b(1)}\eta}. \tag{2.7}$$

Therefore, solution of Eq. (2.1) becomes

$$y(x) = y_0(x) + Ae^{-\sqrt{b(0)/\varepsilon}\ x} + Be^{-\sqrt{b(1)/\varepsilon}\ (1-x)}, \tag{2.8}$$

where *A* and *B* are given by

$$A = \frac{(\beta - y_0(1)) - (\alpha - y_0(0))\, e^{-\sqrt{b(0)/\varepsilon}}}{1 - e^{-\left(\sqrt{b(0)}+\sqrt{b(1)}\right)/\sqrt{\varepsilon}}}, \tag{2.9}$$

$$B = \frac{(\alpha - y_0(0)) - (\beta - y_0(1))\, e^{-b(1)/\varepsilon}}{1 - e^{-\left(\sqrt{b(0)}+\sqrt{b(1)}\right)/\sqrt{\varepsilon}}}. \tag{2.10}$$

We rearrange the differential equation $-\varepsilon y'' + b(x)y(x) = f(x)$ as $\varepsilon y''(x) = g(x, y)$ where $g(x, y) = b(x)y(x) - f(x)$.

Now we divide the interval [0,1] into N equal parts with constant mesh length h. let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, \ldots, N$. We choose n such that $x_n = 1/2$. In the interval $[0, 1/2]$ the boundary layer will be in the left hand side i.e., at $x = 0$ and in the interval $[1/2, 1]$ the boundary layer will be in the right hand side i.e., at $x = 1$. At $x = x_i$ the above differential equation can be written as

$$\varepsilon y_i''(x) = g(x_i, y_i) \;\text{ where } g(x_i, y_i) = b(x_i)y(x_i) - f(x_i).$$

By Numerov method, we have

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) = \frac{1}{12}\left(g_{i-1} + 10g_i + g_{i+1}\right),$$

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) = \frac{1}{12}\left(b_{i-1}y_{i-1} - f_{i-1} + 10b_iy_i - 10f_i + b_{i+1}y_{i+1} - f_{i+1}\right),$$

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) = \frac{1}{12}\left(b_{i-1}y_{i-1} + 10b_iy_i + b_{i+1}y_{i+1} - f_{i-1} - 10f_i - f_{i+1}\right). \tag{2.11}$$

In the interval $[0, 1/2]$, we introduce a fitting factor σ in the above difference scheme as

$$\varepsilon\sigma\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) - \frac{1}{12}\left(b_{i-1}y_{i-1} + 10b_iy_i + b_{i+1}y_{i+1}\right) = \frac{-1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right). \tag{2.12}$$

for $i = 1, 2, \ldots, n-1$.

To find σ on the left boundary layer we use the asymptotic solution

$$v_0(x_i) = y_i = Ae^{-\sqrt{b(0)/\varepsilon}\, x_i} \tag{2.13}$$

and A is given by Eq. (2.9). We assume that solution converges uniformly to the solution of Eq. (2.1), then $f_{i-1} + 10f_i + f_{i+1}$ is bounded. As $h \to 0$ equation Eq. (2.12) becomes

$$\lim_{h\to 0} \frac{\sigma}{\rho^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) = \frac{b(0)}{12} \lim_{h\to 0}\left(y_{i-1} + 10y_i + y_{i+1}\right), \tag{2.14}$$

where $\rho = h/\sqrt{\varepsilon}$. Substituting Eq. (2.13) in Eq. (2.14) and simplifying, we get the fitting factor as

$$\sigma = \frac{\rho^2 b(0)\left(e^{\sqrt{b(0)}\rho} + e^{-\sqrt{b(0)}\rho} + 10\right)}{48\text{Sinh}^2\left(\sqrt{b(0)}\rho/2\right)}, \tag{2.15}$$

which is a constant fitting factor. This will be the fitting factor in the interval $[0, 1/2]$.

Substituting the fitting factor Eq. (2.15) in Eq. (2.12), and rearranging, we get the three term recurrence relation as

$$\left(\frac{\varepsilon\sigma}{h^2}-\frac{b_{i-1}}{12}\right)y_{i-1}-\left(\frac{2\varepsilon\sigma}{h^2}+\frac{10}{12}b_i\right)y_i+\left(\frac{\varepsilon\sigma}{h^2}-\frac{b_{i+1}}{12}\right)y_{i+1}=\frac{-1}{12}\left(f_{i-1}+10f_i+f_{i+1}\right) \tag{2.16}$$

for $i = 1, 2, \ldots, n-1$.

We solved the above tridiagonal system by Thomas algorithm. The value of $y_n = y\,(x = 1/2)$ is obtained by the solution of the reduced problem i.e., $y_0(x)$.

In the interval $[1/2, 1]$, the boundary layer will be in the right hand side, i.e., at $x = 1$. We introduce a fitting factor σ_1 in the difference scheme Eq. (2.12) as

$$\varepsilon\sigma_1\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)-\frac{1}{12}\left(b_{i-1}y_{i-1}+10b_iy_i+b_{i+1}y_{i+1}\right)=\frac{-1}{12}\left(f_{i-1}+10f_i+f_{i+1}\right) \tag{2.17}$$

for $i = n+1, n+2, \ldots, N-1$.

To find σ_1 on the left boundary layer we use the asymptotic solution

$$w_0(x_i) = y_i = Be^{-\sqrt{b(1)/\varepsilon}\;(1-x_i)}, \tag{2.18}$$

where B is given by Eq. (2.10). Assume that solution converges uniformly to the solution of Eq. (2.1), then $f_{i-1} + 10f_i + f_{i+1}$ is bounded. As $h \to 0$ equation Eq. (2.17) becomes

$$\lim_{h\to 0}\frac{\sigma_1}{\rho^2}\left(y_{i-1}-2y_i+y_{i+1}\right)=\frac{b(1)}{12}\lim_{h\to 0}\left(y_{i-1}+10y_i+y_{i+1}\right), \tag{2.19}$$

where $\rho = h/\sqrt{\varepsilon}$.

Substituting Eq. (2.18) in Eq. (2.19), and simplifying, we get the fitting factor as

$$\sigma_1=\frac{\rho^2 b(1)\left(e^{\sqrt{b(1)}\rho}+e^{-\sqrt{b(1)}\rho}+10\right)}{48\text{Sinh}^2\left(\sqrt{b(1)}\rho/2\right)}, \tag{2.20}$$

which is a constant fitting factor. This will be the fitting factor in the interval $[1/2, 1]$.

From Eq. (2.17) we have the three term recurrence relation

$$\left(\frac{\varepsilon\sigma_1}{h^2}-\frac{b_{i-1}}{12}\right)y_{i-1}-\left(\frac{2\varepsilon\sigma_1}{h^2}+\frac{10}{12}b_i\right)y_i+\left(\frac{\varepsilon\sigma_1}{h^2}-\frac{b_{i+1}}{12}\right)y_{i+1}=\frac{-1}{12}\left(f_{i-1}+10f_i+f_{i+1}\right)$$

for $i = n+1, n+2, \ldots N-1$.

We solved the above tridiagonal system by Thomas algorithm. The value of $y_n = y\,(x = 1/2)$ is obtained by the solution of the reduced problem i.e., $y_0(x)$.

Remark 2.1. When *b(0) = b(1)* , both the fitting factors become equal and the constant fitting factor is $\sigma = \rho^2 b(0)\left(e^{\sqrt{b(0)}\rho} + e^{-\sqrt{b(0)}\rho} + 10\right)/\left(48\text{Sinh}^2\left(\sqrt{b(0)}\rho/2\right)\right)$.

3 Numerical Examples

To demonstrate the applicability of the method we have applied it to four linear singular perturbation problems with two boundary layers. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example 3.1. Consider the following non-homogeneous singular perturbation problem $\varepsilon y''(x) - y(x) = \cos^2 \pi x + 2\varepsilon\pi^2 \cos 2\pi x;\ x \in [0, 1]$ with $y(0) = 0$ and $y(1) = 0$.

The exact solution is given by

$$y(x) = \frac{e^{\left(-(1-x)/\sqrt{\varepsilon}\right)} + e^{\left(-x/\sqrt{\varepsilon}\right)}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2 \pi x.$$

The numerical results are given in tables 1(a), 1(b) for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 3.2. Consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) - y(x) = -1; \quad x \in [0, 1] \quad \text{with } y(0) = 0 \text{ and } y(1) = 0.$$

The exact solution is given by

$$y(x) = 1 - e^{-x/\sqrt{\varepsilon}} - e^{-(1-x)/\sqrt{\varepsilon}}.$$

The numerical results are given in tables 2(a), 2(b) for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 3.3. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y(x) = 0; \quad x \in [0, 1] \quad \text{with } y(0) = 1 \text{ and } y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{\varepsilon^{(-2x+1)/2h} + \varepsilon^{(2x-1)/2h}}{\varepsilon^{1/2h} + \varepsilon^{-1/2h}},$$

where $\varepsilon = e^{h/\sqrt{\varepsilon}}$.

The numerical results are given in tables 3(a) and 3(b) for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 3.4. Consider the following variable coefficient singular perturbation problem

$$\varepsilon y''(x) - (2 - x^2)y(x) = -1; \quad x \in [-1, 1] \quad \text{with } y(-1) = 0 \text{ and } y(1) = 0.$$

The exact solution is given by

$$y(x) = \frac{1}{2 - x^2} - e^{-(1+x)/\sqrt{\varepsilon}} - e^{-(1-x)/\sqrt{\varepsilon}}.$$

The numerical results are given in tables 4(a) and 4(b) for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 1(a) Numerical Results of Example 3.1 with $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
0.00	0.0000000	0.0000000
0.01	-0.2701197	-0.2701199
0.02	-0.4647713	-0.4647717
0.05	-0.7697869	-0.7697875
0.10	-0.8621783	-0.8621789
0.20	-0.6527151	-0.6527154
0.30	-0.3454138	-0.3454137
0.40	-0.0954872	-0.0954866
0.50	-0.0000005	0.0000003
0.60	-0.0954914	-0.0954908
0.70	-0.3454206	-0.3454204
0.80	-0.6527218	-0.6527221
0.90	-0.8621823	-0.8621831
0.92	-0.8584825	-0.8584834
0.95	-0.7697890	-0.7697898
0.98	-0.4647722	-0.4647733
0.99	-0.2701202	-0.2701215
1.00	0.0000000	0.0000000

Table 1(b) Numerical Results of Example 3.1 with $\varepsilon = 10^{-4}$ and $h = 10^{-2}$

X	Approximate Solution	Exact solution
0.00	0.0000000	0.0000000
0.01	-0.6311290	-0.6311339
0.02	-0.8607154	-0.8607221
0.05	-0.9687830	-0.9687902
0.10	-0.9044564	-0.9044627
0.20	-0.6545048	-0.6545072
0.30	-0.3454919	-0.3454895
0.40	-0.0954962	-0.0954898
0.50	-0.0000079	0.0000000
0.60	-0.0955004	-0.0954940
0.70	-0.3454988	-0.3454963
0.80	-0.6545116	-0.6545138
0.90	-0.9044606	-0.9044669
0.92	-0.9378142	-0.9378210
0.95	-0.9687852	-0.9687924
0.98	-0.8607163	-0.8607235
0.99	-0.6311294	-0.6311362
1.00	0.0000000	0.0000000

Table 2(a) Numerical Results of Example 3.2 with $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
0.00	0.0000000	0.0000000
0.01	0.2711065	0.2711066
0.02	0.4687143	0.4687144
0.05	0.7942593	0.7942593
0.10	0.9576708	0.9576707
0.20	0.9982083	0.9982082
0.30	0.9999240	0.9999242
0.40	0.9999966	0.9999968
0.50	0.9999995	0.9999997
0.60	0.9999965	0.9999968
0.70	0.9999238	0.9999242
0.80	0.9982079	0.9982082
0.90	0.9576705	0.9576708
0.92	0.9203265	0.9203269
0.95	0.7942591	0.7942594
0.98	0.4687143	0.4687151
0.99	0.2711065	0.2711077
1.00	0.0000000	0.0000000

Table 2(b) Numerical Results of Example 3.2 with $\varepsilon = 10^{-4}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
0.00	0.0000000	0.0000000
0.01	0.6321206	0.6321205
0.02	0.8646648	0.8646647
0.05	0.9932621	0.9932621
0.10	0.9999547	0.9999546
0.20	1.0000000	1.0000000
0.30	1.0000000	1.0000000
0.40	1.0000000	1.0000000
0.50	1.0000000	1.0000000
0.60	1.0000000	1.0000000
0.70	1.0000000	1.0000000
0.80	1.0000000	1.0000000
0.90	0.9999548	0.9999546
0.92	0.9996647	0.9996645
0.95	0.9932622	0.9932621
0.98	0.8646648	0.8646653
0.99	0.6321206	0.6321224
1.00	0.0000000	0.0000000

Table 3(a) Numerical Results of Example 3.3 with $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
0.00	1.0000000	1.0000000
0.01	0.7288935	0.7288934
0.02	0.5312856	0.5312856
0.05	0.2057407	0.2057407
0.10	0.0423292	0.0423292
0.20	0.0017918	0.0017918
0.30	0.0000758	0.0000758
0.40	0.0000032	0.0000032
0.50	0.0000003	0.0000003
0.60	0.0000032	0.0000032
0.70	0.0000758	0.0000758
0.80	0.0017918	0.0017918
0.90	0.0423292	0.0423292
0.92	0.0796732	0.0796731
0.95	0.2057406	0.2057406
0.98	0.5312856	0.5312849
0.99	0.7288934	0.7288923
1.00	1.0000000	1.0000000

Table 3(b) Numerical Results of Example 3.3 with $\varepsilon = 10^{-4}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
0.00	1.0000000	1.0000000
0.01	0.3678795	0.3678795
0.02	0.1353353	0.1353353
0.05	0.0067380	0.0067379
0.10	0.0000454	0.0000454
0.20	0.0000000	0.0000000
0.30	0.0000000	0.0000000
0.40	0.0000000	0.0000000
0.50	0.0000000	0.0000000
0.60	0.0000000	0.0000000
0.70	0.0000000	0.0000000
0.80	0.0000000	0.0000000
0.90	0.0000454	0.0000454
0.92	0.0003355	0.0003355
0.95	0.0067380	0.0067379
0.98	0.1353353	0.1353347
0.99	0.3678795	0.3678776
1.00	1.0000000	1.0000000

Table 4(a) Numerical Results of Example 3.4 with $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
-1.00	0.0000000	0.0000000
-0.98	0.4421755	0.4306225
-0.96	0.6615047	0.6450354
-0.92	0.8016422	0.7871784
-0.90	0.8096560	0.7980069
-0.80	0.7367131	0.7335024
0.00	0.5002511	0.5000000
0.20	0.5104928	0.5102041
0.40	0.5439147	0.5434783
0.50	0.5720217	0.5714284
0.60	0.6106211	0.6097528
0.70	0.6635978	0.6621758
0.80	0.7367129	0.7335023
0.90	0.8096560	0.7980069
0.92	0.8016422	0.7871785
0.96	0.6615047	0.6450359
0.98	0.4421755	0.4306243
1.00	0.0000000	0.0000000

Table 4(b) Numerical Results of Example 3.4 with $\varepsilon = 10^{-4}$ and $h = 10^{-2}$

x	Approximate solution	Exact solution
-1.00	0.0000000	0.0000000
-0.98	0.8308195	0.8265729
-0.96	0.9113764	0.9089841
-0.92	0.8671620	0.8665162
-0.90	0.8407799	0.8402907
-0.80	0.7355374	0.7352941
0.00	0.5000264	0.5000000
0.20	0.5102345	0.5102041
0.40	0.5435241	0.5434783
0.50	0.5714906	0.5714286
0.60	0.6098462	0.6097561
0.70	0.6623931	0.6622516
0.80	0.7355373	0.7352941
0.90	0.8407798	0.8402907
0.92	0.8671620	0.8665161
0.96	0.9113762	0.9089841
0.98	0.8308194	0.8265743
1.00	0.0000000	0.0000000

4 Conclusion

We have presented a fitted fourth-order finite difference method for solving singularly perturbed two-point boundary value problems with boundary layer at both (left and right) end points. We have implemented the present method on standard test problems. Numerical results are presented in tables and compared with exact solution. It is observed from the results that the present method approximate the exact solution very well.

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