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Fitted fourth-order tridiagonal finite difference method for singular perturbation problems

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Abstract

In this paper, a fitted fourth-order tridiagonal finite difference scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. We have taken a fourth-order tridiagonal finite difference scheme by M.M. Chawla [A fourth-order tridiagonal finite difference method for general non-linear two-point boundary value problems with mixed boundary conditions, J. Inst. Maths Appl. 21 (1978) 83–93] and introduced a fitting factor. The fitting factor is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system. To demonstrate the applicability of the present method, we have solved five linear problems (three with left end and two with right end boundary layers). Solutions of these problems using the present fitted method are compared with Chawla's solutions. From the results, it is observed that the present method is stable and has better approximation to the exact solution.

Keywords: Singular perturbation problems; Finite differences; Fitted method

1. Introduction

The numerical solution of singular perturbation problems is currently a field in which active research is going on. Singular perturbation problems are of common occurrence in fluid mechanics and other branches of Applied Mathematics. For detailed analytical discussion on singular perturbation problems, one can refer to Bender and Orsazag [1], Kevorkian and Cole [2], Nayfeh [7,8], O'Mally [9] and Van Dyke [14]. More discussions on fitted and some other numerical methods for singular perturbation problems can be referred in [3,6,10,12,13].

In this paper a fourth-order tridiagonal finite difference scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. We have introduced a fitting factor in Chawla's [5] fourth-order tridiagonal finite difference scheme. For further discussion of the fourth-order tridiagonal finite difference method one can refer Chawla [5] and Jain [4, pp. 200–202].

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Thomas Algorithm is used to solve the system. To demonstrate the applicability of the method, we have solved five linear problems (three with left end and two with right end boundary layers). Solutions of these problems using the present fitted method are compared with solutions by Chawla's method. From the results, it is observed that the present method is stable and has better approximation to the exact solution.

2. Fitted fourth-order scheme

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1] \tag{1}$$

$$\text{with } y(0) = \alpha, \tag{2a}$$

$$\text{and } y(1) = \beta, \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $b(x) \leqslant 0$, $a(x) \geqslant M > 0$ throughout the interval $[0,1]$, where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$ which in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

From the theory of singular perturbations it is known that the solution of (1) and (2) is of the form (cf. [9, pp. 22–26])

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{-\int_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right) dx} + O(\varepsilon), \tag{3}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(1) = \beta.$$

By taking the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '0' and restricting to their first terms, (3) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x} + O(\varepsilon). \tag{4}$$

Now we divide the interval $[0,1]$ into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, 2, \ldots, N$.

From (4) we have

$$\begin{aligned} & y(x_i) = y_0(x_i) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x_i} + O(\varepsilon), \\ & \text{i.e.,} \quad y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)ih} + O(\varepsilon), \\ & \therefore \quad \lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)} i\rho, \end{aligned} \tag{5}$$

where $\rho = \frac{h}{\varepsilon}$.

Now let us rewrite Eq. (1) in the form

$$\varepsilon y''(x) = f(x) - a(x)y'(x) - b(x)y(x) = g(x, y, y') \tag{6}$$

$$\text{with } y(0 = \alpha,$$

$$\text{and } y(1) = \beta.$$

Now we divide the interval $[0,1]$ into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots\ldots x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, 2, \ldots, N$.

Let us denote the exact solution $y(x)$ at the grid points x_i by y_i; similarly, $y_i' = y'(x_i)$.
For $i = 1, 2, \dots, N-1$, let

$$\bar{y}_i' = \frac{y_{i+1} - y_{i-1}}{2h}, \tag{7a}$$

$$\bar{y}_{i+1}' = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, \tag{7b}$$

$$\bar{y}_{i-1}' = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}, \tag{7c}$$

$$\bar{\bar{y}}_{i-1}' = \bar{y}_i' - \frac{h}{20}(\bar{g}_{i+1} - \bar{g}_{i-1}). \tag{7d}$$

Then for each x_i, $i = 1, 2, \dots, N-1$, (6) can be described as

$$\frac{\varepsilon}{h^2}\delta^2 y_i = \frac{1}{12}(\bar{g}_{i+1} + 10\bar{\bar{g}}_i + \bar{g}_{i-1}), \tag{8}$$

where $\bar{\bar{g}}_i = g(x_i, y_i, \bar{\bar{y}}_i')$, (9a)

and $\bar{g}_{i\pm1} = g(x_{i\pm1}, y_{i\pm1}, \bar{y}_{i\pm1}')$. (9b)

Using (7) and (9), terms of the right hand side expressions of (8) can be simplified as

$$\frac{1}{12}\bar{g}_{i+1} = \frac{f_{i+1}}{12} - \left(\frac{a_{i+1}}{8h} + \frac{b_{i+1}}{12}\right)y_{i+1} + \frac{a_{i+1}}{6h}y_i - \frac{a_{i+1}}{24h}y_{i-1}, \tag{10a}$$

$$\begin{aligned}\frac{10}{12}\bar{\bar{g}}_i = {} & \left(\frac{10a_i}{24h} - \frac{a_i a_{i+1}}{48} - \frac{a_i a_{i-1}}{16} + \frac{ha_i b_{i-1}}{24}\right)y_{i-1} + \left(\frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12}\right)y_i \\ & + \left(-\frac{10a_i}{24h} - \frac{a_i a_{i+1}}{16} - \frac{ha_i b_{i+1}}{24} - \frac{a_i a_{i-1}}{48}\right)y_{i+1} + \frac{10}{12}f_i + \frac{ha_i}{24}f_{i+1} - \frac{ha_i}{24}f_{i-1},\end{aligned} \tag{10b}$$

$$\frac{1}{12}\bar{g}_{i-1} = \frac{f_{i-1}}{12} + \frac{a_{i-1}}{24h}y_{i+1} - \frac{a_{i-1}}{6h}y_i + \left(\frac{a_{i-1}}{8h} - \frac{b_{i-1}}{12}\right)y_{i-1}. \tag{10c}$$

Now substituting (10a)–(10c) in (8) we get

$$\begin{aligned}\frac{\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = {} & \left(\frac{-a_{i+1} + 10a_i}{24h} - \frac{a_i a_{i+1}}{48} - \frac{a_i a_{i-1}}{16} + \frac{ha_i b_{i-1}}{24} + \frac{a_{i-1}}{8h} - \frac{b_{i-1}}{12}\right)y_{i-1} \\ & + \left(\frac{a_{i+1}}{6h} + \frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12} - \frac{a_{i-1}}{6h}\right)y_i \\ & + \left(-\frac{a_{i+1}}{8h} - \frac{b_{i+1}}{12} - \frac{10a_i}{24h} - \frac{a_i a_{i+1}}{16} - \frac{ha_i b_{i+1}}{24} - \frac{a_i a_{i-1}}{48} + \frac{a_{i-1}}{24h}\right)y_{i+1} \\ & + \left(\frac{f_{i+1} + 10f_i + f_{i-1}}{12} + \frac{ha_i(f_{i+1} - f_{i-1})}{24}\right).\end{aligned} \tag{11}$$

Introducing fitting factor $\sigma(\rho)$ in to Eq. (11), we get

$$\begin{aligned}\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = {} & \left(\frac{-a_{i+1} + 10a_i}{24h} - \frac{a_i a_{i+1}}{48} - \frac{a_i a_{i-1}}{16} + \frac{ha_i b_{i-1}}{24} + \frac{a_{i-1}}{8h} - \frac{b_{i-1}}{12}\right)y_{i-1} \\ & + \left(\frac{a_{i+1}}{6h} + \frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12} - \frac{a_{i-1}}{6h}\right)y_i \\ & + \left(-\frac{a_{i+1}}{8h} - \frac{b_{i+1}}{12} - \frac{10a_i}{24h} - \frac{a_i a_{i+1}}{16} - \frac{ha_i b_{i+1}}{24} - \frac{a_i a_{i-1}}{48} + \frac{a_{i-1}}{24h}\right)y_{i+1} \\ & + \left(\frac{f_{i+1} + 10f_i + f_{i-1}}{12} + \frac{ha_i(f_{i+1} - f_{i-1})}{24}\right), \quad 1 \leqslant i \leqslant N-1.\end{aligned} \tag{12}$$

Multiplying (12) by h and taking limit as $h \to 0$, we get

$$\lim_{h\to0}\frac{\sigma}{\rho}(y((i-1)h) - 2y(ih) + y((i-1)h)) = \frac{a(0)}{2}\lim_{h\to0}(y((i-1)h) - y((i+1)h)). \tag{13}$$

By substituting (5) in to (13) we get

$$\lim_{h\to 0}\frac{\sigma}{\rho}=\frac{1}{4}a(0)\frac{\sinh\left(\left(\frac{a(0)^2-\varepsilon b(0)}{a(0)}\right)\rho\right)}{\left[\sinh\left(\left(\frac{a(0)^2-\varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right)\right]^2}. \tag{14}$$

$\therefore$ We have

$$\sigma=\frac{\rho}{4}a(0)\frac{\sinh\left(\left(\frac{a(0)^2-\varepsilon b(0)}{a(0)}\right)\rho\right)}{\left[\sinh\left(\left(\frac{a(0)^2-\varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right)\right]^2}. \tag{15}$$

σ given by (15) is the constant fitting factor.

From Eq. (12) we get the recurrence relation of the form

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i;\quad i=1,2,3,\ldots,N-1, \tag{16}$$

where

$$\begin{aligned}
E_i&=\frac{\varepsilon\sigma}{h^2}+\frac{a_{i+1}-10a_i}{24h}+\frac{a_ia_{i+1}}{48}+\frac{a_ia_{i-1}}{16}-\frac{ha_ib_{i-1}}{24}-\frac{a_{i-1}}{8h}+\frac{b_{i-1}}{12},\\
F_i&=\frac{2\varepsilon\sigma}{h^2}+\frac{a_{i+1}-a_{i-1}}{6h}+\frac{a_i(a_{i+1}+a_{i-1})-10b_i}{12},\\
G_i&=\frac{\varepsilon\sigma}{h^2}+\frac{a_{i+1}}{8h}+\frac{b_{i+1}}{12}+\frac{10a_i-a_{i-1}}{24h}+\frac{ha_ib_{i+1}}{24}+\frac{a_ia_{i+1}}{16}+\frac{a_ia_{i-1}}{48},\\
H_i&=\frac{f_{i+1}+10f_i+f_{i-1}}{12}+\frac{ha_i(f_{i+1}-f_{i-1})}{24}.
\end{aligned}$$

To solve the tridiagonal system (16), we used Thomas Algorithm.

3. Numerical examples

In this section, to demonstrate the applicability of the present method we have chosen three linear singular perturbation problems with left-end boundary layer which are widely discussed in literature. The approximate solutions of these problems are used for comparison. The approximate solution is compared with the exact solution.

Example 3.1. Consider the following homogeneous singular perturbation problem from Bender and Orsazag [1, p. 480; problem 9.17 with $\alpha=0$]

$$\varepsilon y''(x)+y'(x)-y(x)=0;\quad x\in[0,1]$$

with $y(0)=1$ and $y(1)=1$.

The exact solution is given by

$$y(x)=\frac{[(e^{m_2}-1)e^{m_1x}+(1-e^{m_1})e^{m_2x}]}{[e^{m_2}-e^{m_1}]},$$

where $m_1=(-1+\sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2=(-1-\sqrt{1+4\varepsilon})/(2\varepsilon)$.

The numerical results are given in Table 1a and b for $\varepsilon=10^{-2}$ and 10^{-3}, respectively.

Example 3.2. Let us consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt [11, example 2]

$$\varepsilon y''(x)+y'(x)=1+2x;\quad x\in[0,1]$$

with $y(0)=0$ and $y(1)=1$.

The exact solution is given by $y(x)=x(x+1-2\varepsilon)+\frac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$.

Table 1
Numerical results of Example 3.1

X	$Y(X)$ (fitted)	Chawla's solution	Exact solution
Panel a: $\varepsilon = 10^{-2}$, $h = 10^{-2}$			
0.00	1.0000000	1.0000000	1.0000000
0.01	0.6209159	0.6008961	0.6041336
0.02	0.4424902	0.4238589	0.4623245
0.03	0.3595471	0.3464715	0.4130807
0.04	0.3220527	0.3138159	0.3975782
0.05	0.3062052	0.3012550	0.3943904
0.06	0.3006817	0.2977365	0.3957127
0.07	0.3000962	0.2983025	0.3987026
0.08	0.3018866	0.3007292	0.4023249
0.09	0.3048346	0.3040194	0.4062027
0.10	0.3083609	0.3077259	0.4101991
0.20	0.3509845	0.3505355	0.4528673
0.30	0.4000615	0.3996136	0.5000052
0.40	0.4560010	0.4555634	0.5520498
0.50	0.5197623	0.5193467	0.6095114
0.60	0.5924392	0.5920602	0.6729541
0.70	0.6752784	0.6749543	0.7430004
0.80	0.7697008	0.7694545	0.8203378
0.90	0.8773260	0.8771855	0.9057249
1.00	1.0000000	1.0000000	1.0000000
Panel b: $\varepsilon = 10^{-3}$, $h = 10^{-2}$			
0.00	1.0000000	1.0000000	1.0000000
0.01	0.3736567	−0.1525618	0.3719724
0.02	0.2884285	0.5115076	0.3756784
0.03	0.2795464	0.1378099	0.3794502
0.04	0.2814718	0.3570651	0.3832599
0.05	0.2849635	0.2376433	0.3871079
0.06	0.2887190	0.3117861	0.3909945
0.07	0.2925550	0.2754246	0.3949201
0.08	0.2964464	0.3022745	0.3988851
0.09	0.3003902	0.2930932	0.4028900
0.10	0.3043866	0.3045790	0.4069350
0.20	0.3473942	0.3448339	0.4496879
0.30	0.3964786	0.3939106	0.4969323
0.40	0.4524982	0.4499849	0.5491403
0.50	0.5164331	0.5140416	0.6068335
0.60	0.5894015	0.5872170	0.6705877
0.70	0.6726800	0.6708092	0.7410401
0.80	0.7677249	0.7663009	0.8188942
0.90	0.8761991	0.8753862	0.9049277
1.00	1.0000000	1.0000000	1.0000000

The numerical results are given in Table 2a and b for $\varepsilon = 10^{-2}$ and 10^{-3}, respectively.

Example 3.3. Consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [2, p. 33; Eqs. (2.3.26) and (2.3.27) with $\alpha = -1/2$]

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [8, p. 148; Eq. (4.2.32)]) as our 'exact' solution

Table 2
Numerical results of Example 3.2

X	$Y(X)$ (fitted)	Chawla's solution	Exact solution
Panel a: $\varepsilon = 10^{-2}$, $h = 10^{-2}$			
0.00	0.0000000	0.00000000	0.0000000
0.01	−0.8245463	−0.8760144	−0.6095782
0.02	−1.2178808	−1.2670670	−0.8273715
0.03	−1.4017174	−1.4373831	−0.9009086
0.04	−1.4837021	−1.5071491	−0.9212507
0.05	−1.5160987	−1.5310333	−0.9218968
0.06	−1.5242823	−1.5339022	−0.9151708
0.07	−1.5205728	−1.5270667	−0.9056064
0.08	−1.5109519	−1.5156716	−0.8948712
0.09	−1.4983236	−1.5020572	−0.8835791
0.10	−1.4840981	−1.4872879	−0.8719556
0.20	−1.3158038	−1.3181360	−0.7440000
0.30	−1.1196603	−1.1217003	−0.5960001
0.40	−0.8968498	−0.8985986	−0.4280000
0.50	−0.6473734	−0.6488308	−0.2400000
0.60	−0.3712310	−0.3723968	−0.0320001
0.70	−0.0684225	−0.0692968	0.1960000
0.80	0.2610522	0.2604693	0.4439999
0.90	0.6171930	0.6169016	0.7119999
1.00	1.0000000	1.0000000	1.0000000
Panel b: $\varepsilon = 10^{-3}$, $h = 10^{-2}$			
0.00	0.0000000	0.0000000	0.0000000
0.01	−1.3985754	−2.6077983	−0.9878747
0.02	−1.5879061	−1.0837667	−0.9776400
0.03	−1.6032392	−1.9400475	−0.9671600
0.04	−1.5933403	−1.4243878	−0.9564800
0.05	−1.5795872	−1.6988108	−0.9456000
0.06	−1.5650519	−1.5175732	−0.9345200
0.07	−1.5501759	−1.5984638	−0.9232400
0.08	−1.5350226	−1.5278976	−0.9117600
0.09	−1.5196011	−1.5441794	−0.9000800
0.10	−1.5039128	−1.5099998	−0.8882000
0.20	−1.3323622	−1.3437098	−0.7584000
0.30	−1.1341459	−1.1441001	−0.6086000
0.40	−0.9092641	−0.9177976	−0.4388001
0.50	−0.6577170	−0.6648292	−0.2490000
0.60	−0.3795043	−0.3851951	−0.0392001
0.70	−0.0746263	−0.0788951	0.1905999
0.80	0.2569171	0.2540707	0.4403999
0.90	0.6151260	0.6137024	0.7102000
1.00	1.0000000	1.0000000	1.0000000

$$y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-(x-x^2/4)/\varepsilon}.$$

The numerical results are given in Table 3a and b for $\varepsilon = 10^{-2}$ and 10^{-3}, respectively.

4. Right-end boundary layer problems

Now let us discuss our present method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form

Table 3
Numerical results of Example 3.3

X	$Y(X)$ (fitted)	Chawla's solution	Exact solution
Panel a: $\varepsilon = 10^{-2}$, $h = 10^{-2}$			
0.00	0.0000000	0.0000000	0.0000000
0.01	0.1990452	0.2095716	0.3181124
0.02	0.2964433	0.3060890	0.4367028
0.03	0.3450344	0.3515598	0.4821542
0.04	0.3701083	0.3738933	0.5006725
0.05	0.3838162	0.3857052	0.5092342
0.06	0.3920188	0.3927225	0.5141078
0.07	0.3975543	0.3975565	0.5176194
0.08	0.4018025	0.4014028	0.5206365
0.09	0.4054370	0.4048115	0.5234846
0.10	0.4087878	0.4080370	0.5262867
0.20	0.4416590	0.4407397	0.5555555
0.30	0.4788066	0.4778685	0.5882353
0.40	0.5215091	0.5205625	0.6250000
0.50	0.5709993	0.5700607	0.6666667
0.60	0.6288814	0.6279783	0.7142857
0.70	0.6972787	0.6964532	0.7692308
0.80	0.7790488	0.7783682	0.8333333
0.90	0.8781180	0.8776891	0.9090909
1.00	1.0000000	1.0000000	1.0000000
Panel b: $\varepsilon = 10^{-3}$, $h = 10^{-2}$			
0.00	0.0000000	0.0000000	0.0000000
0.01	0.3222527	0.5827496	0.5024893
0.02	0.3711521	0.2533934	0.5050505
0.03	0.3806959	0.4456725	0.5076142
0.04	0.3845032	0.3405810	0.5102041
0.05	0.3874857	0.4045970	0.5128205
0.06	0.3903733	0.3728183	0.5154639
0.07	0.3932768	0.3952281	0.5181347
0.08	0.3962137	0.3871669	0.5208333
0.09	0.3991872	0.3962768	0.5235602
0.10	0.4021985	0.3958319	0.5263158
0.20	0.4345494	0.4292463	0.5555555
0.30	0.4715418	0.4661086	0.5882353
0.40	0.5141674	0.5086559	0.6250000
0.50	0.5637082	0.5582095	0.6666667
0.60	0.6218457	0.6165099	0.7142857
0.70	0.6908247	0.6858976	0.7692308
0.80	0.7737027	0.7695873	0.8333333
0.90	0.8747349	0.8721039	0.9090909
1.00	1.0000000	1.0000000	1.0000000

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1] \tag{17}$$

$$\text{with } y(0) = \alpha, \tag{18a}$$

$$\text{and } y(1) = \beta, \tag{18b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \leqslant M < 0$ throughout the interval $[0,1]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

From the theory of singular perturbations the solution of (17) and (18) is of the form

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right) dx} + O(\varepsilon), \tag{19}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(0) = \alpha. \tag{20}$$

By taking first terms of the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '1', (19) becomes,

$$y(x) = y_0(x) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-x)} + O(\varepsilon). \tag{21}$$

Now we divide the interval $[0, 1]$ into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih$; $i = 0, 1, 2, \ldots, N$.

From (24) we have

$$y(ih) = y_0(ih) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-ih)} + O(\varepsilon).$$

Therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))e^{\left(\frac{a^2(1) - \varepsilon b(1)}{a(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)}, \tag{22}$$

where $\rho = \frac{h}{\varepsilon}$.

Now, we consider the fourth-order finite difference scheme (16) and introduce the fitting factor $\sigma(\rho)$

$$\begin{aligned}
\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) &= \left(\frac{-a_{i+1} + 10a_i}{24h} - \frac{a_i a_{i+1}}{48} - \frac{a_i a_{i-1}}{16} + \frac{h a_i b_{i-1}}{24} + \frac{a_{i-1}}{8h} - \frac{b_{i-1}}{12}\right) y_{i-1} \\
&\quad + \left(\frac{a_{i+1}}{6h} + \frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12} - \frac{a_{i-1}}{6h}\right) y_i \\
&\quad + \left(-\frac{a_{i+1}}{8h} - \frac{b_{i+1}}{12} - \frac{10a_i}{24h} - \frac{a_i a_{i+1}}{16} - \frac{h a_i b_{i+1}}{24} - \frac{a_i a_{i-1}}{48} + \frac{a_{i-1}}{24h}\right) y_{i+1} \\
&\quad + \left(\frac{f_{i+1} + 10f_i + f_{i-1}}{12} + \frac{h a_i (f_{i+1} - f_{i-1})}{24}\right), \quad 1 \leqslant i \leqslant N-1
\end{aligned} \tag{23}$$

Multiplying (23) by h and taking limit as $h \to 0$, we get

$$\lim_{h\to 0} \frac{\sigma}{\rho}(y((i-1)h) - 2y(ih) + y((i-1)h)) = \frac{a(0)}{2} \lim_{h\to 0}(y((i-1)h) - y((i+1)h)) \tag{24}$$

By substituting (22) in to (24) we get

$$\lim_{h\to 0} \frac{\sigma}{\rho} = \frac{1}{4} a(0) \frac{\sinh\left(\left(\frac{a(1)^2 - \varepsilon b(1)}{a(1)}\right)\rho\right)}{\left[\sinh\left(\left(\frac{a(1)^2 - \varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right)\right]^2}. \tag{25}$$

$\therefore$ We have

$$\sigma = \frac{1}{4} a(0) \frac{\sinh\left(\left(\frac{a(1)^2 - \varepsilon b(1)}{a(1)}\right)\rho\right)}{\left[\sinh\left(\left(\frac{a(1)^2 - \varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right)\right]^2} \tag{26}$$

σ given by (26) is the constant fitting factor.

From Eq. (23) we get the recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; \quad i = 1, 2, 3, \ldots, N-1, \tag{27}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} + \frac{a_{i+1} - 10a_i}{24h} + \frac{a_i a_{i+1}}{48} + \frac{a_i a_{i-1}}{16} - \frac{h a_i b_{i-1}}{24} - \frac{a_{i-1}}{8h} + \frac{b_{i-1}}{12},$$
$$F_i = \frac{2\varepsilon\sigma}{h^2} + \frac{a_{i+1} - a_{i-1}}{6h} + \frac{a_i(a_{i+1} + a_{i-1}) - 10b_i}{12},$$
$$G_i = \frac{\varepsilon\sigma}{h^2} + \frac{a_{i+1}}{8h} + \frac{b_{i+1}}{12} + \frac{10a_i - a_{i-1}}{24h} + \frac{h a_i b_{i+1}}{24} + \frac{a_i a_{i+1}}{16} + \frac{a_i a_{i-1}}{48},$$
$$H_i = \frac{f_{i+1} + 10f_i + f_{i-1}}{12} + \frac{h a_i (f_{i+1} - f_{i-1})}{24}.$$

To solve the tridiagonal system (27), we used Thomas Algorithm.

Table 4
Numerical results of Example 5.1

X	$Y(X)$ (fitted)	Chawla's solution	Exact solution
Panel a: $\varepsilon = 10^{-2}$, $h = 10^{-2}$			
0.00	1.0000000	1.0000000	1.0000000
0.10	0.9999987	0.9999990	1.0000000
0.20	0.9999979	0.9999978	1.0000000
0.30	0.9999968	0.9999967	1.0000000
0.40	0.9999957	0.9999956	1.0000000
0.50	0.9999945	0.9999945	1.0000000
0.60	0.9999934	0.9999934	1.0000000
0.70	0.9999922	0.9999923	1.0000000
0.80	0.9999905	0.9999911	1.0000000
0.90	0.9992620	0.9996110	0.9999546
0.91	0.9984906	0.9991565	0.9998766
0.92	0.9969020	0.9981574	0.9996645
0.93	0.9936300	0.9959612	0.9990881
0.94	0.9868909	0.9911329	0.9975212
0.95	0.9730108	0.9805182	0.9932621
0.96	0.9444227	0.9571823	0.9816844
0.97	0.8855410	0.9058793	0.9502131
0.98	0.7642649	0.7930915	0.8646653
0.99	0.5144774	0.5451316	0.6321224
1.00	0.0000000	0.0000000	0.0000000
Panel b: $\varepsilon = 10^{-3}$, $h = 10^{-2}$			
0.00	1.0000000	1.0000000	1.0000000
0.10	1.0000007	0.9999998	1.0000000
0.20	1.0000014	0.9999994	1.0000000
0.30	1.0000021	0.9999992	1.0000000
0.40	1.0000029	0.9999991	1.0000000
0.50	1.0000036	0.9999991	1.0000000
0.60	1.0000042	0.9999991	1.0000000
0.70	1.0000049	0.9999990	1.0000000
0.80	1.0000056	0.9999828	1.0000000
0.90	1.0000063	0.9959657	1.0000000
0.91	1.0000063	1.0069994	1.0000000
0.92	1.0000063	0.9878500	1.0000000
0.93	1.0000052	1.0210843	1.0000000
0.94	0.9999978	0.9634053	1.0000000
0.95	0.9999454	1.0635089	1.0000000
0.96	0.9995803	0.8897761	1.0000000
0.97	0.9970394	1.1912943	1.0000000
0.98	0.9793568	0.6680008	1.0000000
0.99	0.8563052	1.5761919	0.9999546
1.00	0.0000000	0.0000000	0.0000000

5. Examples with right-end boundary layer

Here we considered two singularly perturbed two-point boundary value problems with right-end boundary layer and demonstrated the applicability of the present method. The approximate solution is compared with the exact solution.

Example 5.1. Consider the following singular perturbation problem:

$$\varepsilon y''(x) - y'(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at $x = 1$. i.e.; at the right end of the underlying interval.

Table 5
Numerical results of Example 5.2

X	$Y(X)$ (fitted)	Chawla's solution	Exact solution
Panel a: $\varepsilon = 10^{-2}$, $h = 10^{-2}$			
0.00	1.0000000	1.0000000	1.0000000
0.10	1.0000000	0.9999990	0.9048374
0.20	1.0000000	0.9999978	0.8187308
0.30	1.0000000	0.9999967	0.7408183
0.40	1.0000000	0.9999956	0.6703200
0.50	1.0000000	0.9999945	0.6065307
0.60	1.0000000	0.9999934	0.5488117
0.70	1.0000000	0.9999923	0.4965853
0.80	1.0000001	0.9999913	0.4493290
0.90	1.0002503	1.0001296	0.4066108
0.91	1.0005190	1.0002966	0.4026370
0.92	1.0010762	1.0006640	0.3988287
0.93	1.0022316	1.0014719	0.3954040
0.94	1.0046275	1.0032481	0.3929622
0.95	1.0095955	1.0071530	0.3931504
0.96	1.0198973	1.0157380	0.4004903
0.97	1.0412594	1.0346119	0.4273985
0.98	1.0855559	1.0761057	0.5079660
0.99	1.1774099	1.1673287	0.7357938
1.00	1.3678794	1.3678794	1.3678794
Panel b: $\varepsilon = 10^{-3}$, $h = 10^{-2}$			
0.00	1.0000000	1.0000000	1.0000000
0.10	1.0000008	0.9999998	0.9048374
0.20	1.0000014	0.9999994	0.8187308
0.30	1.0000021	0.9999992	0.7408183
0.40	1.0000029	0.9999992	0.6703200
0.50	1.0000036	0.9999992	0.6065307
0.60	1.0000043	0.9999992	0.5488117
0.70	1.0000050	0.9999992	0.4965853
0.80	1.0000057	1.0000051	0.4493290
0.90	1.0000063	1.0014830	0.4065697
0.91	1.0000064	0.9974239	0.4025242
0.92	1.0000066	1.0044686	0.3985191
0.93	1.0000070	0.9922423	0.3945537
0.94	1.0000099	1.0134614	0.3906278
0.95	1.0000292	0.9766351	0.3867410
0.96	1.0001637	1.0405481	0.3828929
0.97	1.0010985	0.9296253	0.3790830
0.98	1.0076034	1.1221349	0.3753111
0.99	1.0528704	0.7880290	0.3716217
1.00	1.3678794	1.3678794	1.3678794

The exact solution is given by $y(x) = \frac{\left(e^{(x-1)/\varepsilon}-1\right)}{(e^{-1/\varepsilon}-1)}$.

The numerical results are given in Table 4a and b for $\varepsilon = 10^{-2}$ and 10^{-3}, respectively.

Example 5.2. Now we consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0; \quad x \in [0,1]$$

with $y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$ and $y(1) = 1 + 1/e$.

Clearly this problem has a boundary layer at $x = 1$. The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$.

The numerical results are given in Table 5a and b for $\varepsilon = 10^{-2}$ and 10^{-3}, respectively.

6. Discussion and conclusions

We have presented a fitted fourth-order tridiagonal finite difference method for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. To demonstrate the applicability of the method, we have solved several linear and nonlinear problems. Solutions of these problems using the present fitted method are compared with Chawla's [5]. From the results, it is observed that the present method is stable and has better approximation to the exact solution.

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