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An exponentially fitted special second-order finite difference method for solving singular perturbation problems

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Abstract

An exponentially fitted special second-order finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. A fitting factor is introduced in a tri-diagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and its stability is investigated. To demonstrate the applicability of the method, we have solved several linear and non-linear problems. From the results, it is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems; Finite differences; Fitted method

1. Introduction

Singularly perturbed second-order two-point boundary value problems occur very frequently in fluid mechanics and other branches of Applied Mathematics. These problems depend on a small positive parameter in such away that the solution varies rapidly (called boundary layer region) in some part and varies slowly in some other parts. The numerical treatment of singular perturbation problems is far from the trivial because of the boundary layer behavior of the solution. There are a wide variety of techniques for solving singular perturbation problems (cf. [1–13]).

In this paper, an exponentially fitted special second-order finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. A fitting factor is introduced in a tri-diagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and its stability is investigated. To demonstrate the applicability of the method, we have solved several linear and non-linear problems. From the results, it is observed that the present method approximates the exact solution very well.

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2. Special second-order finite difference method

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y''(x) + b(x)y(x) = f(x), \quad x \in [0,1], \tag{1}$$

$$\text{with } y(0) = \alpha \tag{2a}$$

$$\text{and } y(1) = \beta; \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Further more, we assume that $b(x) \leqslant 0$, $a(x) \geqslant M > 0$ throughout the interval $[0,1]$, where M is some positive constant.

A finite difference scheme is often a convenient choice for the numerical solution of two point boundary value problems. Through out the discussion the symbols μ, δ denote the usual central difference operators and E and D denote the shift (displacement) and the differential operators, respectively. We divide the interval $[0,1]$ in to N equal subintervals of uniform mesh size h. Consider a typical pivotal point in the mesh, at $x = x_i + gh$. The following expression can be written for y, y', y'':

$$y_g = y(x_i + gh) = E^g y(x_i), \tag{3}$$

$$y'_g = Dy_g, \tag{4}$$

$$y''_g = D^2 y_g. \tag{5}$$

The shift operator

$$E = e^{hD} \tag{6}$$

can be related to the central difference operators μ, δ by using the following expressions:

$$hD = \mu\delta - \frac{1}{6}\mu\delta^3 + \frac{1}{30}\mu\delta^5 + \cdots, \tag{7}$$

$$h^2D^2 = \delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 + \cdots, \tag{8}$$

$$h^3D^3 = \mu\delta^3 - \frac{1}{4}\mu\delta^5 + \cdots, \tag{9}$$

$$h^4D^4 = \delta^4 - \frac{1}{6}\delta^6 + \cdots. \tag{10}$$

By substituting (6)–(10) into (3)–(5), we get

$$y_g = \left[1 + g\mu\delta + \frac{g^2}{2}\delta^2 + \frac{1}{6}g(g^2-1)\mu\delta^3 + \frac{g^2(g^2-1)}{24}\delta^4 + \cdots\right]y_i, \tag{11}$$

$$y'_g = \frac{1}{h}\left[\mu\delta + g\delta^2 + \frac{(3g^2-1)}{6}\mu\delta^3 + g(2g^2-1)\frac{\delta^4}{12} + \cdots\right]y_i, \tag{12}$$

$$y''_g = \frac{1}{h^2}\left[\delta^2 + g\mu\delta^3 + \frac{6g^2-1}{12}\delta^4 + g(2g^2-3)\frac{\mu\delta^5}{6} + \cdots\right]y_i. \tag{13}$$

By substituting Eqs. (11)–(13) with $g = 1/2$ in to Eq. (1) we get:

$$\frac{\varepsilon}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_{i+1/2}}{h}(y_{i+1} - y_i) + b_{i+1/2}\left(\frac{3y_{i+1} + 6y_i - y_{i-1}}{8}\right) = f_{i+1/2}; \quad 1 \leqslant i \leqslant N-1 \tag{14}$$

3. Exponentially fitted special second-order finite difference scheme

A difference scheme with a fitting factor containing exponential functions is known as *exponentially fitted difference scheme*.

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1], \tag{1}$$

$$\text{with } y(0) = \alpha, \tag{2a}$$

$$\text{and } y(1) = \beta; \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $b(x) \leqslant 0$, $a(x) \geqslant M > 0$ throughout the interval $[0,1]$, where M is some positive constant. Under these assumptions, Eq. (1) has a unique solution y(x) which in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

From the theory of singular perturbations it is known that the solution of (1) and (2) is of the form (cf. O'Malley [8, pp. 22–26])

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{-\int_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon), \tag{15}$$

where $y_0(x)$is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), y_0(1) = \beta. \tag{16}$$

By taking first terms of the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '0', (15) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x} + O(\varepsilon). \tag{17}$$

Now we divide the interval $[0,1]$ into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih; \ i = 0, 1, 2, \ldots, N$.

From (17) we have

$$y(x_i) = y_0(x_i) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x_i} + O(\varepsilon)$$

i.e. $y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)ih} + O(\varepsilon)$.

Therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)i\rho} \tag{18}$$

where $\rho = \frac{h}{\varepsilon}$.

Now, we consider the special second-order finite difference scheme (14) and introduce the fitting factor $\sigma(\rho)$:

$$\frac{\varepsilon\sigma(\rho)}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \left(\frac{a_{i+1/2}}{h}\right) + b_{i+1/2}\left(\frac{3y_{i+1} + 6y_i - y_{i-1}}{8}\right) = f_{i+1/2};$$
$$1 \leqslant i \leqslant N - 1; \tag{19}$$

$y_0 = \alpha; \ y_N = \beta$; where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (19) converges uniformly to the solution of (1) and (2).

Multiplying (19) by h and taking the limit as $h \to 0$; we get

$$\lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y_{i+1} - 2y_i + y_{i-1}) + a_{i+1/2}(y_{i+1} - y_i)\right] = 0$$

if $f_{i+1/2} - b_{i+1/2}\left(\frac{3y_{i+1}+6y_i-y_{i-1}}{8}\right)$ is bounded.

$$\therefore \quad \lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y(ih+h)-2y(ih)+y(ih-h))+a(ih+h/2)(y(ih+h)-y(ih))\right]=0, \tag{20}$$

$$\lim_{h\to 0}\frac{\sigma}{\rho}=\frac{1}{4}a(0)\frac{\left[1-\mathrm{e}^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)\rho}\right]}{\left[\sinh\left(\left(\frac{a(0)^2-\varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right)\right]^2}. \tag{21}$$

$$\therefore \quad \text{We have} \quad \sigma=\frac{\rho}{4}a(0)\frac{\left[1-\mathrm{e}^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)\rho}\right]}{\left[\sinh\left(\left(\frac{a(0)^2-\varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right)\right]^2}, \tag{22}$$

σ given by (22) is the constant fitting factor.

From (19) we have

$$\frac{\varepsilon\sigma(\rho)}{h^2}(y_{i+1}-2y_i+y_{i-1})+\frac{a_{i+1/2}}{h}(y_{i+1}-y_i)+b_{i+1/2}\left(\frac{3y_{i+1}+6y_i-y_{i-1}}{8}\right)=f_{i+1/2};\quad i=1,2,\dots,N-1$$

$$\left(\frac{\varepsilon\sigma}{h^2}-\frac{b_{i+1/2}}{8}\right)y_{i-1}-\left(\frac{2\varepsilon\sigma}{h^2}+\frac{a_{i+1/2}}{h}-\frac{6b_{i+1/2}}{8}\right)y_i+\left(\frac{\varepsilon\sigma}{h^2}+\frac{3b_{i+1/2}}{8}\right)y_{i+1}=f_{i+1/2};\quad i=1,2,\dots,N-1 \tag{23}$$

where the fitting factor σ is given by (22).

The equivalent three term recurrence relation of Eq. (23) is given by:

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i;\quad i=1,2,3,\dots,N-1 \tag{24}$$

where

$$E_i=\frac{\varepsilon\sigma}{h^2}-\frac{b(x_i+h/2)}{8}, \tag{25a}$$

$$F_i=\frac{2\varepsilon\sigma}{h^2}+\frac{a(x_i+h/2)}{h}-\frac{6b(x_i+h/2)}{8}, \tag{25b}$$

$$G_i=\frac{\varepsilon\sigma}{h^2}+\frac{3b(x_i+h/2)}{8}, \tag{25c}$$

$$H_i=f(x_i+h/2). \tag{25d}$$

This gives us the tri-diagonal system which can be solved easily by Thomas Algorithm described in the next section.

4. Thomas algorithm

A brief discussion on solving the tri-diagonal system using Thomas algorithm is presented as follows:

Consider the scheme:

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i;\quad i=1,2,3,\dots,N-1 \tag{26}$$

subject to the boundary conditions

$$y_0=y(0)=\alpha; \tag{27a}$$

$$y_N=y(1)=\beta. \tag{27b}$$

We set

$$y_i=W_iy_{i+1}+T_i\quad \text{for } i=N-1,\ N-2,\dots 2,1. \tag{28}$$

where $W_i=W(x_i)$ and $T_i=T(x_i)$ which are to be determined.

From (28), we have

$$y_{i-1}=W_{i-1}y_i+T_{i-1} \tag{29}$$

By substituting (29) in (26), we get

$$E_i(W_{i-1}y_i + T_{i-1}) - F_iy_i + G_iy_{i+1} = H_i.$$

$$\therefore \quad y_i = \left(\frac{G_i}{F_i - E_iW_{i-1}}\right)y_{i+1} + \left(\frac{E_iT_{i-1} - H_i}{F_i - E_iW_{i-1}}\right). \tag{30}$$

By comparing (30) and (28), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_iW_{i-1}}\right), \tag{31a}$$

$$T_i = \left(\frac{E_iT_{i-1} - H_i}{F_i - E_iW_{i-1}}\right). \tag{31b}$$

To solve these recurrence relations for $i = 0, 1, 2, 3, \ldots, N-1$, we need the initial conditions for W_0 and T_0. For this we take$y_0 = \alpha = W_0y_1 + T_0$. We choose $W_0 = 0$ so that the value of $T_0 = \alpha$. With these initial values, we compute W_i and T_i for $i = 1, 2, 3, \ldots, N-1$ from (31) in forward process, and then obtain y_i in the backward process from (28) and (27b).

5. Stability analysis

We will now show that the algorithm is computationally stable. By stability, we mean that the effect of an error made in one stage of the calculation is not propagated into larger errors at later stages of the calculations. Let us now examine the recurrence relation given by (31a). Suppose that a small error e_{i-1} has been made in the calculation of W_{i-1}; then, we have

$\overline{W}_{i-1} = W_{i-1} + e_{i-1}$ and we are actually calculating

$$\overline{W}_i = \left(\frac{G_i}{F_i - E_i\overline{W}_{i-1}}\right). \tag{32}$$

From (32) and (31a), we have

$$e_i = \left(\frac{G_i}{F_i - E_i(W_{i-1} + e_{i-1})}\right) - \left(\frac{G_i}{F_i - E_iW_{i-1}}\right)$$
$$= \left(\frac{G_iE_ie_{i-1}}{(F_i - E_i(W_{i-1} + e_{i-1}))(F_i - E_iW_{i-1})}\right) = \left(\frac{W_i^2E_i}{G_i}\right)e_{i-1} \tag{33}$$

under the assumption that the error is small initially. From the assumptions made earlier that $a(x) > 0$ and $b(x) \leqslant 0$, we have

$$F_i \geqslant E_i + G_i; \quad i = 1, 2, 3, \ldots, N-1$$

Form (31a) we have

$$W_1 = \frac{G_1}{F_1} < 1, \quad \text{since } F_1 > G_1$$
$$W_2 = \frac{G_2}{F_2 - E_2W_1} < \frac{G_2}{F_2 - E_2}; \quad \text{since } W_1 < 1,$$
$$< \frac{G_2}{E_2 + G_2 - E_2} = 1; \quad \text{since } F_2 \geqslant E_2 + G_2$$

successively, it follows that

$$|e_i| = |W_i|^2\left|\frac{E_i}{G_i}\right||e_{i-1}| < |e_{i-1}| \quad \text{since } |E_i| \leqslant |G_i|.$$

Therefore the recurrence relation (31a) is stable. Similarly we can prove that the recurrence relation (31b) is also stable. Finally the convergence of the Thomas Algorithm is ensured by the condition $|W_i| < 1, i = 1, 2, 3, \ldots, N-1$.

6. Numerical examples

In this section, to demonstrate the applicability of the present method we have chosen three linear singular perturbation problems with left-end boundary layer which are widely discussed in literature. The approximate solutions of these problems are used for comparison. The approximate solution is compared with the exact solution.

Example 6.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag ([1, pp. 480]; problem 9.17 with $\alpha = 0$)

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]},$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon)$.

The numerical results are given in Tables 1a and 1b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 6.2. Let us consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt ([10], example 2)

$$\varepsilon y''(x) + y'(x) = 1 + 2x; x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{(1 - e^{-1/\varepsilon})}.$$

The numerical results are given in Tables 2a and 2b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 6.3. Consider the following variable coefficient singular perturbation problem from Kevorkian and Cole ([4, pp. 33]; Eqs. (2.3.26) and (2.3.27) with $\alpha = -1/2$)

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right)y'(x) - \frac{1}{2}y(x) = 0; x \in [0, 1]$$

Table 1a
Numerical results of Example 6.1, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.01	0.3723905	0.3719724
0.02	0.3753125	0.3756784
0.04	0.3828933	0.3832599
0.06	0.3906282	0.3909945
0.08	0.3985194	0.3988851
0.10	0.4065700	0.4069350
0.20	0.4493294	0.4496879
0.30	0.4965857	0.4969323
0.40	0.5488120	0.5491403
0.50	0.6065310	0.6068335
0.60	0.6703203	0.6705877
0.70	0.7408184	0.7410401
0.80	0.8187310	0.8188942
0.90	0.9048376	0.9049277
1.00	1.0000000	1.0000000

Table 1b
Numerical results of Example 6.1, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.01	0.3723643	0.3716134
0.02	0.3753144	0.3753479
0.04	0.3828952	0.3829296
0.06	0.3906302	0.3906645
0.08	0.3985214	0.3985557
0.10	0.4065720	0.4066062
0.20	0.4493313	0.4493649
0.30	0.4965876	0.4966201
0.40	0.5488138	0.5488445
0.50	0.6065326	0.6065609
0.60	0.6703218	0.6703468
0.70	0.7408197	0.7408404
0.80	0.8187319	0.8187470
0.90	0.9048380	0.9048464
1.00	1.0000000	1.0000000

Table 2a
Numerical results of Example 6.2, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.01	−0.9898533	−0.9878747
0.02	−0.9795987	−0.9776400
0.04	−0.9583987	−0.9564800
0.06	−0.9363987	−0.9345200
0.08	−0.9135987	−0.9117600
0.10	−0.8899987	−0.8882000
0.20	−0.7599989	−0.7584000
0.30	−0.6099989	−0.6086000
0.40	−0.4399991	−0.4388001
0.50	−0.2499992	−0.2490000
0.60	−0.0399993	−0.0392001
0.70	0.1900005	0.1905999
0.80	0.4400004	0.4403999
0.90	0.7100002	0.7102000
1.00	1.0000000	1.0000000

with $y(0) = 0$ and $y(1) = 1$.

We have chosen to use uniformly valid approximation which is obtained by the method given by Nayfeh ([7, pp. 148], equation 4.2.32) as our 'exact' solution:

$$y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-(x-x^2/4)/\varepsilon}.$$

The numerical results are given in Tables 3a and 3b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

7. Non-linear problems

Non-linear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using Quasilinearization (Replacing the non-linear problem by a sequence of linear problems) method. The outer solution (the solution of the given problem by putting $\varepsilon = 0$) is taken to be the initial approximation.

Table 2b
Numerical results of Example 6.2, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.01	−0.9898996	−0.9897020
0.02	−0.9795996	−0.9794040
0.04	−0.9583996	−0.9582080
0.06	−0.9363996	−0.9362120
0.08	−0.9135996	−0.9134160
0.10	−0.8899996	−0.8898200
0.20	−0.7599996	−0.7598400
0.30	−0.6099995	−0.6098601
0.40	−0.4399996	−0.4398801
0.50	−0.2499996	−0.2499000
0.60	−0.0399997	−0.0399201
0.70	0.1900003	0.1900600
0.80	0.4400002	0.4400398
0.90	0.7100001	0.7100199
1.00	1.0000000	1.0000000

Table 3a
Numerical results of Example 6.3, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.01	0.5021808	0.5024893
0.02	0.5050551	0.5050505
0.04	0.5102089	0.5102041
0.06	0.5154687	0.5154639
0.08	0.5208381	0.5208333
0.10	0.5263206	0.5263158
0.20	0.5555605	0.5555555
0.30	0.5882403	0.5882353
0.40	0.6250049	0.6250000
0.50	0.6666715	0.6666667
0.60	0.7142904	0.7142857
0.70	0.7692348	0.7692308
0.80	0.8333368	0.8333333
0.90	0.9090930	0.9090909
1.00	1.0000000	1.0000000

The approximate solution is compared with the exact solution.

8. Non-linear examples

We considered three non-linear singular perturbation problems with left-end boundary layer to demonstrate the applicability of the present method.

Example 8.1. Consider the following singular perturbation problem from Bender and Orszag ([1, pp. 463]; equations: 9.7.1)

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \quad x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 0$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right].$$

Table 3b
Numerical results of Example 6.3, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.01	0.5022034	0.5025126
0.02	0.5050550	0.5050505
0.04	0.5102088	0.5102041
0.06	0.5154686	0.5154639
0.08	0.5208380	0.5208333
0.10	0.5263205	0.5263158
0.20	0.5555604	0.5555555
0.30	0.5882401	0.5882353
0.40	0.6250048	0.6250000
0.50	0.6666714	0.6666667
0.60	0.7142903	0.7142857
0.70	0.7692348	0.7692308
0.80	0.8333365	0.8333333
0.90	0.9090929	0.9090909
1.00	1.0000000	1.0000000

We have chosen to use Bender and Orszag's uniformly valid approximation ([1, pp. 463], equation: 9.7.6) for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)\mathrm{e}^{-2x/\varepsilon}.$$

For this example, we have boundary layer of thickness $\mathrm{O}(\varepsilon)$ at $x = 0$ (cf. Bender and Orszag [1]).

The numerical results are given in Tables 4a and 4b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 8.2. Let us consider the following singular perturbation problem from Kevorkian and Cole ([4, pp. 56], Eq. (2.5.1))

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0; \quad x \in [0,1]$$

with $y(0) = -1$ and $y(1) = 3.9995$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995.$$

Table 4a
Numerical results of Example 8.1, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.01	0.6840518	0.6831968
0.02	0.6733397	0.6733446
0.04	0.6539229	0.6539265
0.06	0.6348748	0.6348783
0.08	0.6161829	0.6161861
0.10	0.5978339	0.5978370
0.20	0.5108232	0.5108256
0.30	0.4307811	0.4307829
0.40	0.3566736	0.3566750
0.50	0.2876811	0.2876821
0.60	0.2231429	0.2231436
0.70	0.1625185	0.1625189
0.80	0.1053602	0.1053605
0.90	0.0512932	0.0512933
1.00	0.0000000	0.0000000

Table 4b
Numerical results of Example 8.1, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	0.0000000	0.0000000
0.01	0.6840518	0.6831968
0.02	0.6733397	0.6733446
0.04	0.6539229	0.6539265
0.06	0.6348748	0.6348783
0.08	0.6161829	0.6161861
0.10	0.5978339	0.5978370
0.20	0.5108232	0.5108256
0.30	0.4307811	0.4307829
0.40	0.3566736	0.3566750
0.50	0.2876811	0.2876821
0.60	0.2231429	0.2231436
0.70	0.1625185	0.1625189
0.80	0.1053603	0.1053605
0.90	0.0512932	0.0512933
1.00	0.0000000	0.0000000

We have chosen to use the Kivorkian and Cole's uniformly valid approximation ([4, pp. 57–58], Eqs. (2.5.5), (2.5.11) and (2.5.14)) for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right),$$

where $c_1 = 2.9995$ and $c_2 = (1/c_1)\log_e[(c_1 - 1)/(c_1 + 1)]$.

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. Kevorkian and Cole [4, pp. 56–66]).

The numerical results are given in Tables 5a and 5b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 8.3. Consider the following singular perturbation problem from O'Malley ([8, pp. 9], Equation (1.10) case 2):

$$\varepsilon y''(x) - y(x)y'(x) = 0; \quad x \in [-1, 1].$$

With $y(-1) = 0$ and $y(1) = -1$.

Table 5a
Numerical results of Example 8.2, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	−1.0000000	−1.0000000
0.01	3.0095010	3.0095000
0.02	3.0195010	3.0195000
0.04	3.0395010	3.0395000
0.06	3.0595009	3.0595000
0.08	3.0795009	3.0795000
0.10	3.0995009	3.0994999
0.20	3.1995008	3.1995001
0.30	3.2995007	3.2995000
0.40	3.3995006	3.3994999
0.50	3.4995005	3.4995000
0.60	3.5995004	3.5994999
0.70	3.6995003	3.6995001
0.80	3.7995002	3.7995000
0.90	3.8995001	3.8994999
1.00	3.9995000	3.9995000

Table 5b
Numerical results of Example 8.2, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	−1.0000000	−1.0000000
0.01	3.0095010	3.0095000
0.02	3.0195010	3.0195000
0.04	3.0295010	3.0295000
0.06	3.0395010	3.0395000
0.08	3.0495009	3.0495000
0.10	3.0595009	3.0595000
0.20	3.0695009	3.0695000
0.30	3.0795009	3.0795000
0.40	3.0895009	3.0895000
0.50	3.0995009	3.0994999
0.60	3.1995008	3.1995001
0.70	3.2995007	3.2995000
0.80	3.3995006	3.3994999
0.90	3.4995005	3.4995000
1.00	3.5995004	3.5994999
0.70	3.6995003	3.6995001
0.80	3.7995002	3.7995000
0.90	3.8995001	3.8994999
1.00	3.9995000	3.9995000

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0.$$

We have chosen to use O'Malley's approximate solution ([8, pp. 9–10], Eqs. (1.13) and (1.14)) for comparison,

$$y(x) = -\frac{\left(1 - e^{-(x+1)/\varepsilon}\right)}{\left(1 + e^{-(x+1)/\varepsilon}\right)}.$$

Table 6a
Numerical results of Example 8.3, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
−1.00	0.0000000	0.0000000
−0.90	−1.0000000	−1.0000000
−0.80	−1.0000000	−1.0000000
−0.70	−1.0000000	−1.0000000
−0.60	−1.0000000	−1.0000000
−0.50	−1.0000000	−1.0000000
−0.40	−1.0000000	−1.0000000
−0.30	−1.0000000	−1.0000000
−0.20	−1.0000000	−1.0000000
−0.10	−1.0000000	−1.0000000
0.00	−1.0000000	−1.0000000
0.10	−1.0000000	−1.0000000
0.20	−1.0000000	−1.0000000
0.30	−1.0000000	−1.0000000
0.40	−1.0000000	−1.0000000
0.50	−1.0000000	−1.0000000
0.60	−1.0000000	−1.0000000
0.70	−1.0000000	−1.0000000
0.80	−1.0000000	−1.0000000
0.90	−1.0000000	−1.0000000
1.00	−1.0000000	−1.0000000

Table 6b
Numerical results of Example 8.3, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
−1.00	0.0000000	0.0000000
−0.90	−1.0000000	−1.0000000
−0.80	−1.0000000	−1.0000000
−0.70	−1.0000000	−1.0000000
−0.60	−1.0000000	−1.0000000
−0.50	−1.0000000	−1.0000000
−0.40	−1.0000000	−1.0000000
−0.30	−1.0000000	−1.0000000
−0.20	−1.0000000	−1.0000000
−0.10	−1.0000000	−1.0000000
0.00	−1.0000000	−1.0000000
0.10	−1.0000000	−1.0000000
0.20	−1.0000000	−1.0000000
0.30	−1.0000000	−1.0000000
0.40	−1.0000000	−1.0000000
0.50	−1.0000000	−1.0000000
0.60	−1.0000000	−1.0000000
0.70	−1.0000000	−1.0000000
0.80	−1.0000000	−1.0000000
0.90	−1.0000000	−1.0000000
1.00	−1.0000000	−1.0000000

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = -1$ (cf. O'Malley [8, pp. 9–10], Eqs. (1.10), (1.13), (1.14), case 2).

The numerical results are given in Tables 6a and 6b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

9. Right-end boundary layer problems

Now let us discuss our present method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1] \tag{34}$$

With $y(0) = \alpha$, (35a)

and $y(1) = \beta$, (35b)

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \leqslant M < 0$ throughout the interval $[0,1]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

From the theory of singular perturbations the solution of (34) and (35) is of the form

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon), \tag{36}$$

where $y_0(x)$is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(0) = \alpha. \tag{37}$$

By taking first terms of the Taylor's series expansion for $a(x)$ and $b(x)$ about the point '1', (36) becomes,

$$y(x) = y_0(x) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-x)} + O(\varepsilon). \tag{38}$$

Now we divide the interval $[0,1]$ into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots x_N = 1$ be the mesh points. Then we have $x_i = ih;\ i = 0,1,2,\ldots,N$.

From (38) we have

$$y(ih) = y_0(ih) + (\beta - y_0(1))\mathrm{e}^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-ih)} + \mathrm{O}(\varepsilon).$$

Therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))\mathrm{e}^{\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)(\frac{1}{\varepsilon}-i\rho)} \tag{39}$$

where $\rho = \frac{h}{\varepsilon}$.

Now, we consider the special second-order finite difference scheme (14) and introduce the fitting factor $\sigma(\rho)$:

$$\frac{\varepsilon\sigma(\rho)}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_{i+1/2}}{h}(y_{i+1} - y_i) + b_{i+1/2}\left(\frac{3y_{i+1} + 6y_i - y_{i-1}}{8}\right) = f_{i+1/2}; \quad 1 \leqslant i \leqslant N-1; \tag{40}$$

$y_0 = \alpha; y_N = \beta$; where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (40) converges uniformly to the solution of (35) and (36).

Multiplying (40) by h and taking the limit as $h \to 0$; we get

$$\lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y_{i+1} - 2y_i + y_{i-1}) + a_{i+1/2}(y_{i+1} - y_i)\right] = 0,$$

if $f_{i+1/2} - b_{i+1/2}\left(\frac{3y_{i+1}+6y_i-y_{i-1}}{8}\right)$ is bounded.

$$\therefore \quad \lim_{h\to 0}\left[\frac{\sigma(\rho)}{\rho}(y(ih+h) - 2y(ih) + y(ih-h)) + a(ih+h/2)(y(ih+h) - y(ih))\right] = 0. \tag{41}$$

Substituting (39) in (41) and simplifying, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho} = \frac{1}{4}a(0)\frac{\left[1 - \mathrm{e}^{-\left(\frac{a^2(1)-\varepsilon b(1)}{a(0)}\right)\rho}\right]}{\left[\sinh\left(\left(\frac{a(1)^2-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right)\right]^2}. \tag{42}$$

We have:

$$\sigma = \frac{\rho}{4}a(0)\frac{\left[1 - \mathrm{e}^{-\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\rho}\right]}{\left[\sinh\left(\left(\frac{a(1)^2-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right)\right]^2}, \tag{43}$$

σ given by (43) is the constant fitting factor.

From (40) we have

$$\frac{\varepsilon\sigma(\rho)}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_{i+1/2}}{h}(y_{i+1} - y_i) + b_{i+1/2}\left(\frac{3y_{i+1} + 6y_i - y_{i-1}}{8}\right) = f_{i+1/2}; \quad i = 1,2,\ldots,N-1$$

$$\left(\frac{\varepsilon\sigma}{h^2} - \frac{b_{i+1/2}}{8}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma}{h^2} + \frac{a_{i+1/2}}{h} - \frac{6b_{i+1/2}}{8}\right)y_i + \left(\frac{\varepsilon\sigma}{h^2} + \frac{3b_{i+1/2}}{8}\right)y_{i+1} = f_{i+1/2}; \quad i = 1,2,\ldots,N-1 \tag{44}$$

where the fitting factor σ is given by (43).

An equivalent three term recurrence relation for Eq. (40) is:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; \ i = 1,2,3,\ldots,N-1, \tag{45}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} - \frac{b(x_i + h/2)}{8},$$
$$F_i = \frac{2\varepsilon\sigma}{h^2} + \frac{a(x_i + h/2)}{h} - \frac{6b(x_i + h/2)}{8},$$
$$G_i = \frac{\varepsilon\sigma}{h^2} + \frac{3b(x_i + h/2)}{8},$$
$$H_i = f(x_i + h/2).$$

Thomas Algorithm is used to solve the tri-diagonal system (45).

10. Examples with right-end boundary layer

Here we considered two singularly perturbed two point boundary value problems with right-end boundary layer and demonstrated the applicability of the present method. The approximate solution is compared with the exact solution.

Table 7a
Numerical results of Example 10.1, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.10	1.0000005	1.0000000
0.20	1.0000011	1.0000000
0.30	1.0000017	1.0000000
0.40	1.0000023	1.0000000
0.50	1.0000029	1.0000000
0.60	1.0000035	1.0000000
0.70	1.0000041	1.0000000
0.80	1.0000046	1.0000000
0.90	1.0000052	1.0000000
0.92	1.0000054	1.0000000
0.94	1.0000055	1.0000000
0.96	1.0000056	1.0000000
0.98	1.0000057	1.0000000
0.99	0.9999605	0.9999546
1.00	0.0000000	0.0000000

Table 7b
Numerical results of Example 10.1, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.10	1.0000000	1.0000000
0.20	1.0000000	1.0000000
0.30	1.0000000	1.0000000
0.40	1.0000000	1.0000000
0.50	1.0000000	1.0000000
0.60	1.0000000	1.0000000
0.70	1.0000000	1.0000000
0.80	1.0000000	1.0000000
0.90	1.0000000	1.0000000
0.92	1.0000000	1.0000000
0.94	1.0000000	1.0000000
0.96	1.0000000	1.0000000
0.98	1.0000000	1.0000000
0.99	1.0000000	1.0000000
1.00	1.0000000	1.0000000

Example 10.1. Consider the following singular perturbation problem:

$$\varepsilon y''(x) - y'(x) = 0; \quad x \in [0,1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at $x = 1$. i.e., at the right end of the underlying interval.

The exact solution is given by $y(x) = \frac{(e^{(x-1)/\varepsilon}-1)}{(e^{-1/\varepsilon}-1)}$.

The numerical results are given in Tables 7a and 7b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Example 10.2. Now we consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0; \quad x \in [0,1]$$

with $y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$; and $y(1) = 1 + 1/e$.

Clearly this problem has a boundary layer at $x = 1$.

The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$.

The numerical results are given in Tables 8a and 8b for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

Table 8a
Numerical results of Example 10.2, $\varepsilon = 10^{-3}$, $h = 10^{-2}$

x	$y(x)$	Exact solution
0.00	1.0000000	1.0000000
0.10	0.9056407	0.9048374
0.20	0.8201851	0.8187308
0.30	0.7427930	0.7408183
0.40	0.6727035	0.6703200
0.50	0.6092277	0.6065307
0.60	0.5517415	0.5488117
0.70	0.4996795	0.4965853
0.80	0.4525301	0.4493290
0.90	0.4098296	0.4065697
0.92	0.4017858	0.3985191
0.94	0.3938998	0.3906278
0.96	0.3861686	0.3828929
0.98	0.3786025	0.3753111
0.99	0.3712000	0.3716217
1.00	1.3678794	1.3678794

Table 8b
Numerical results of Example 10.2, $\varepsilon = 10^{-4}$, $h = 10^{-2}$

x	$y(x)$	?>Exact solution
0.00	1.0000000	1.0000000
0.10	0.9057209	0.9048374
0.20	0.8203305	0.8187308
0.30	0.7429905	0.7408183
0.40	0.6729421	0.6703200
0.50	0.6094978	0.6065307
0.60	0.5520348	0.5488117
0.70	0.4999894	0.4965853
0.80	0.4528508	0.4493290
0.90	0.4101565	0.4065697
0.92	0.4021133	0.3985191
0.94	0.3942279	0.3906278
0.96	0.3864971	0.3828929
0.98	0.3789316	0.3753111
0.99	0.3714887	0.3716217
1.00	1.3678794	1.3678794

11. Discussion and conclusions

We have presented an exponentially fitted special second-order finite difference method for solving singularly perturbed two-point boundary value problems. We have implemented the present method on three linear examples, three non-linear examples, with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. Numerical results are presented in tables and compared with the exact solutions. It can be observed from the tables that the present method approximates the exact solution very well.

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