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A seventh order numerical method for singular perturbation problems

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Abstract

In this paper, a seventh order numerical method is presented for solving singularly perturbed two-point boundary value problems with a boundary layer at one end point. The two-point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a seventh order compact difference scheme is presented for the first order system. An asymptotically equivalent first order equation of the original singularly perturbed two-point boundary value problem is obtained from the theory of singular perturbations. It is used in the seventh order compact difference scheme to get a two term recurrence relation and is solved. Several linear and nonlinear singular perturbation problems have been solved and the numerical results are presented to support the theory.

Keywords: Singular perturbation problems; Two-point boundary value problems; Ordinary differential equations; Boundary layer; Seventh order compact difference scheme

1. Introduction

Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto-hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary

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layer behavior. Thus more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems.

A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these, we mention Bender and Orszag [1], Kevorkian and Cole [7], Nayfeh [8], O' Malley [9], Hemker and Miller [5], Roberts [11], Kadalbajoo and Reddy [6].

In fact, some numerical techniques employed for solving singularly-perturbed boundary value problems in ordinary differential equations are based on the idea of replacing a two-point boundary value problem by two suitable initial value problems. For example, Gasparo and Macconi [3] considered a semilinear ordinary differential equation which was integrated to obtain a first-order ordinary differential equation, and considered both the inner and outer solutions. The outer solution corresponds to the reduced problem, i.e., that obtained by setting the small perturbation parameter to zero. A similar matching idea combining the reduced problem and a WKB approximation for the full problem has also been employed by Gasparo and Macconi [4] for linear and semilinear, singularly-perturbed BVP in ordinary differential equations. These matching ideas are based on the method of asymptotic expansions and on the work of Roberts [11] who considered the matching between inner, i.e., boundary layer, and outer solutions at an unknown location which was determined iteratively, and referred to his method as a boundary value technique. Robert's idea has been extended by Valanarasu and Ramanujam [12] for boundary value problems of singularly-perturbed systems of odes; these authors used exponentially-fitted methods for solving the singularly-perturbed initial value problem. Reddy and Chakravarthy [10] considered the full ordinary differential equation in the inner and outer regions, albeit they determined the boundary condition at the matching point from the solution of the reduced problem. Higher order Numerical method for two-point boundary value problems was presented by Peng [2].

In this paper, a seventh order numerical method is presented for solving singularly perturbed two-point boundary value problems with a boundary layer at one end point. The two-point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a seventh order compact difference scheme is presented for the first order system. An asymptotically equivalent first order equation of the original singularly perturbed two-point boundary value problem is obtained from the theory of singular perturbations. It is used in the seventh order compact difference scheme to get a two term recurrence relation and is solved. Several linear and nonlinear singular perturbation problems have been solved and the numerical results are presented to support the theory.

2. Numerical method

We Consider the two-point boundary value problem

$$y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{1}$$

with the boundary conditions $y(0) = \alpha$ and $y(1) = \beta$.

The first order linear system corresponding to the above boundary value problems is

$$Y' = A(x)Y + R(x), \quad x \in [0, 1] \tag{2}$$

with the boundary conditions are given by $B_1Y(0) + B_2Y(1) = D$, where A, B_1 and B_2 are second order matrices and Y, R, D are two dimensional vectors.

Now we divide the interval $[0, 1]$ into N equal parts with constant mesh length H. Let $0 = x_0,\ x_1,\ x_2, \ldots, x_N = 1$ be the mesh points. Again we subdivide each interval $[x_{\rm i}, x_{\rm i+1}]$ into six equal smaller subintervals. Let $t_1, t_2, \ldots, t_7$ are the grids in the subinterval $[x_{\rm i}, x_{\rm i+1}]$ and corresponding values of the variables and its derivatives are $Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6,\ Y_7$ and $Y'_1,\ Y'_2,\ Y'_3,\ Y'_4,\ Y'_5,\ Y'_6,\ Y'_7$.

By considering Taylor's expansions of $Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6,\ Y_7$ at the fractional grid t_4 (Ref. Dianyun Peng [2]), we have

$$\frac{h^{n+1}}{(n+1)!}Y_4^{(n+1)} = \sum_{j=1}^{7} a_j^n Y_j + a_8^n Y'_4 + O(h^8 Y_4^{(8)}), \tag{3}$$

where $h = \frac{(x_{i+1}-x_i)}{6}$, $n = 1,2,3,4,5,6$ and the coefficients a_j^n are given by

$$\begin{aligned}
&a_3^1 = a_5^1 = \frac{3}{4},\quad a_4^1 = \frac{-49}{36},\quad a_8^1 = 0,\quad a_1^1 = a_7^1 = \frac{1}{180},\quad a_2^1 = a_6^1 = \frac{-3}{40},\\
&a_1^2 = -a_7^2 = -\frac{1}{540},\quad a_2^2 = -a_6^2 = \frac{3}{80},\quad a_3^2 = -a_5^2 = \frac{-3}{4},\quad a_4^2 = 0,\quad a_8^2 = \frac{-49}{36}\\
&a_1^3 = a_7^3 = \frac{-1}{144},\quad a_2^3 = a_6^3 = \frac{1}{12},\quad a_3^3 = a_5^3 = \frac{-13}{48},\quad a_4^3 = \frac{7}{18},\quad a_8^3 = 0,\\
&a_1^4 = -a_7^4 = \frac{1}{432},\quad a_2^4 = -a_6^4 = \frac{-1}{24},\quad a_3^4 = -a_5^4 = \frac{13}{48},\quad a_4^4 = 0,\quad a_8^4 = \frac{7}{18},\\
&a_1^5 = a_7^5 = \frac{1}{720},\quad a_2^5 = a_6^5 = \frac{-1}{120},\quad a_3^5 = a_5^5 = \frac{1}{48},\quad a_4^5 = \frac{-1}{36},\quad a_8^5 = 0,\\
&a_1^6 = -a_7^6 = \frac{-1}{2160},\quad a_2^6 = -a_6^6 = \frac{1}{240},\quad a_3^6 = -a_5^6 = \frac{-1}{48},\quad a_4^6 = 0,\quad a_8^6 = \frac{-1}{36}.
\end{aligned} \tag{4}$$

By taking the Taylor's series expansions of Y_1', Y_2', Y_3', Y_4', Y_5', Y_6', Y_7' at the grid point t_4 and substituting (3), we get

$$Y_k' = \frac{1}{h}\sum_{j=1}^{7} b_j^k Y_j + b_8^k Y_4' + O(h^7 Y_4^{(8)}) \quad \text{for } k = 1,2,3,5,6,7, \tag{5}$$

where

$$\begin{aligned}
&b_j^1 = -6a_j^1 + 27a_j^2 - 108a_j^3 + 405a_j^4 - 1458a_j^5 + 5103a_j^6 + \mathrm{Sgn}(j-8),\\
&b_j^2 = -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 - 192a_j^5 + 448a_j^6 + \mathrm{Sgn}(j-8),\\
&b_j^3 = -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 - 6a_j^5 + 7a_j^6 + \mathrm{Sgn}(j-8),\\
&b_j^5 = 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + 6a_j^5 + 7a_j^6 + \mathrm{Sgn}(j-8),\\
&b_j^6 = 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + 192a_j^5 + 448a_j^6 + \mathrm{Sgn}(j-8),\\
&b_j^7 = 6a_j^1 + 27a_j^2 + 108a_j^3 + 405a_j^4 + 1458a_j^5 + 5103a_j^6 + \mathrm{Sgn}(j-8),\quad j = 1,2,\ldots,8
\end{aligned}$$

and $\mathrm{Sgn}(x) = \left\{\begin{matrix} 1, & x \geqslant 0 \\ 0, & x < 0 \end{matrix}\right\}$. The variable Y and its derivative Y' at grids $t_1, t_2, \ldots, t_7$ are subject to equations

$$Y_j' = A_j Y_j + R_j, \quad j = 1,2,3,4,5,6,7, \tag{6}$$

where A_j and R_j are values of A and R at grids t_j.

Substituting (6) in (5), we get six linear algebraic equations with respect to seven unknown variables Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7.

By eliminating Y_2, Y_3, Y_4, Y_5, Y_6, a relation between Y_1 and Y_7 can be obtained as follows:

$$\frac{1}{h}S_i Y_i + \frac{1}{h}T_i Y_{i+1} = F_i \quad \text{for } i = 1,2,\ldots,N-1, \tag{7}$$

where S_i and T_i are second order matrices and F_i is a two dimensional vector. By assuming

$$\begin{aligned}
&c_1 = b_5^7 b_3^1 - b_5^1 b_3^7,\\
&c_2 = (b_6^7 b_5^1 - b_6^1 b_5^7)/c_1,\\
&c_3 = (b_2^7 b_5^1 - b_2^1 b_5^7)/c_1,\\
&W_1 = ((b_7^7 b_5^1 - b_7^1 b_5^7)I - h b_5^1 A_7)/c_1,\\
&W_2 = ((b_1^7 b_5^1 - b_1^1 b_5^7)I + h b_5^7 A_1)/c_1,\\
&W_3 = ((b_4^7 b_5^1 - b_4^1 b_5^7)I + h(b_8^7 b_5^1 - b_8^1 b_5^7)A_4)/c_1,\\
&G_1 = (b_5^7 R_1 - b_5^1 R_7 + (b_8^7 b_5^1 - b_8^1 b_5^7)R_4)/c_1,
\end{aligned}$$

$$c_4 = (b_6^1 b_3^7 - b_6^7 b_3^1)/c_1,$$
$$c_5 = (b_2^1 b_3^7 - b_2^7 b_3^1)/c_1,$$
$$W_4 = ((b_7^1 b_3^7 - b_7^7 b_3^1)I + hb_3^1 A_7)/c_1,$$
$$W_5 = ((b_1^1 b_3^7 - b_1^7 b_3^1)I - hb_3^7 A_1)/c_1,$$
$$W_6 = ((b_4^1 b_3^7 - b_4^7 b_3^1)I + h(b_8^1 b_3^7 - b_8^7 b_3^1)A_4)/c_1,$$
$$G_2 = (b_3^1 R_7 - b_3^7 R_1 + (b_3^7 b_8^1 - b_3^1 b_8^7)R_4)/c_1,$$
$$c_6 = b_2^6 + b_3^6 c_3 + b_5^6 c_5,$$
$$W_7 = b_7^6 I + b_3^6 W_1 + b_5^6 W_4,$$
$$W_8 = b_1^6 I + b_3^6 W_2 + b_5^6 W_5,$$
$$W_9 = b_4^6 I + b_3^6 W_3 + b_5^6 W_6 + hb_8^6 A_4,$$
$$W_{10} = (b_3^6 c_2 + b_5^6 c_4 + b_6^6)I - hA_6,$$
$$G_3 = R_6 - b_8^6 R_4 - b_3^6 G_1 - b_5^6 G_2,$$
$$c_7 = b_6^2 + b_3^2 c_2 + b_5^6 c_4,$$
$$W_{11} = b_3^2 W_1 + b_5^2 W_4 + b_7^2 I,$$
$$W_{12} = b_1^2 I + b_3^2 W_2 + b_5^2 W_5,$$
$$W_{13} = b_4^2 I + b_3^2 W_3 + b_5^2 W_6 + hb_8^2 A_4,$$
$$W_{14} = (b_2^2 + b_3^2 c_3 + b_5^2 c_5)I - hA_2,$$
$$G_4 = R_2 - b_8^2 R_4 - b_3^2 G_1 - b_5^2 G_2,$$
$$W_{15} = b_3^5 W_1 + b_5^5 W_4 + b_7^5 I - hW_4 A_5,$$
$$W_{16} = b_3^5 W_3 + b_5^5 W_5 + b_1^5 I - hW_5 A_5,$$
$$W_{17} = b_3^5 W_3 + b_5^5 W_6 + b_4^5 I + h(b_8^5 A_4 - W_6 A_5),$$
$$W_{18} = b_3^5 c_2 I + b_5^5 c_4 I + b_6^5 I - hc_4 A_5,$$
$$W_{19} = b_3^5 c_3 I + b_5^5 c_5 I + b_2^5 I - hc_5 A_5,$$
$$G_5 = R_5 - b_8^5 R_4 - b_3^5 G_1 - b_5^5 G_2 + hA_5 G_2,$$
$$W_{20} = b_3^3 W_1 + b_5^3 W_4 + b_7^3 I - hW_1 A_3,$$
$$W_{21} = b_3^3 W_2 + b_5^3 W_5 + b_1^3 I - hW_2 A_3,$$
$$W_{22} = b_3^3 W_3 + b_5^3 W_6 + b_4^3 I + h(b_8^3 A_4 - W_3 A_3),$$
$$W_{23} = (b_3^3 c_2 + b_5^3 c_4 + b_6^3)I - hc_2 A_3,$$
$$W_{24} = (b_3^3 c_3 + b_5^3 c_5 + b_2^3)I - hc_3 A_3,$$
$$G_6 = R_3 - b_8^3 R_4 - (b_3^3 - hA_3)G_1 - b_5^3 G_2,$$
$$W_{28} = W_{10} W_{14} - c_6 c_7 I,$$
$$W_{25} = W_{28}^{-1}(c_6 W_{11} - W_7 W_{14}),$$
$$W_{26} = W_{28}^{-1}(c_6 W_{12} - W_8 W_{14}),$$
$$W_{27} = W_{28}^{-1}(c_6 W_{13} - W_9 W_{14}),$$
$$G_7 = W_{28}^{-1}(c_6 G_4 - G_3 W_{14}),$$
$$W_{29} = -(W_{10} W_{25} + W_7)/c_6,$$
$$W_{30} = -(W_{10} W_{26} + W_8)/c_6,$$
$$W_{31} = -(W_{10} W_{27} + W_9)/c_6,$$

$$G_8 = -(G_3 + W_{10}G_7)/c_6,$$
$$W_{32} = W_{19}W_{29} + W_{18}W_{25} + W_{15},$$
$$W_{33} = W_{19}W_{30} + W_{18}W_{26} + W_{16},$$
$$W_{34} = W_{19}W_{31} + W_{18}W_{27} + W_{17},$$
$$G_9 = G_5 + W_{19}G_8 + W_{18}G_7,$$
$$W_{35} = W_{24}W_{29} + W_{23}W_{25} + W_{20},$$
$$W_{36} = W_{24}W_{30} + W_{23}W_{26} + W_{21},$$
$$W_{37} = W_{24}W_{31} + W_{23}W_{27} + W_{22},$$
$$G_{10} = G_6 + W_{24}G_8 + W_{23}G_7,$$
we get $S_i = W_{36} - W_{37}W_{33}W_{34}^{-1}$,
$$T_i = W_{35} - W_{37}W_{32}W_{34}^{-1},$$
$$F_i = G_{10} - W_{37}G_9W_{34}^{-1}.$$

Now we consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y(x) = f(x), \quad x \in [0,1], \tag{8}$$
$$\text{with } y(0) = \alpha, \tag{9a}$$
$$\text{and } y(1) = \beta, \tag{9b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Further more, we assume that $a(x) \geqslant M > 0$ throughout the interval $[0,1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$.

First we obtain the reduced problem by setting $\varepsilon = 0$ in Eq. (8) and solve it for the solution with an appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem

$$[a(x)y(x)]' + b(x)y(x) = f(x) \quad \text{with} \quad y_0(1) = \beta.$$

We now set up the approximation equation to given Eq. (8) as follows:

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y_0(x) = f(x), \tag{10}$$

where we simply replaced $y(x)$ by $y_0(x)$ in the last term of left hand side of Eq. (8). Now we rewrite Eq. (10) in the form

$$\varepsilon y''(x) + [a(x)y(x)]' = H(x), \tag{11}$$

where $H(x) = f(x) - b(x)y_0(x)$.

By integrating (11), we obtain

$$\varepsilon y'(x) + a(x)y(x) = P(x) + K, \tag{12}$$

where $P(x) = \int H(x)\mathrm{d}x$ and K is a constant to be determined.

In order to determine K, we introduce the condition that the reduced equation of (12) should satisfy the boundary condition $y(1) = \beta$.

$$\begin{aligned} &\text{i.e., } y(1) = \frac{1}{a(1)}[P(1) + K] = \beta, \\ &\therefore K = a(1)\beta - P(1). \end{aligned} \tag{13}$$

Remark. This choice of K ensure that the solution of the reduced equation of (8) and (9) satisfies the reduced equation of (12). Hence, Eq. (12) is a first order equation which is asymptotically equivalent to the linear singularly perturbed two-point boundary value problem (8).

It is used in the seventh order compact difference scheme (7) to get a two term recurrence relation. The system (7) i.e., $\frac{1}{h}S_iY_i + \frac{1}{h}T_iY_{i+1} = F_i$ gives the following equations:

$$S_{11}y_i + S_{12}y'_i + T_{11}y_{i+1} + T_{12}y'_{i+1} = hf_1, \tag{14}$$

$$S_{21}y_i + S_{22}y'_i + T_{21}y_{i+1} + T_{22}y'_{i+1} = hf_2. \tag{15}$$

By eliminating y'_{i+1} from the above equations we have,

$$(S_{11}T_{22} - T_{12}S_{21})y_i + (S_{12}T_{22} - T_{12}S_{22})y'_i + (T_{11}T_{22} - T_{12}T_{21})y_{i+1} = h(T_{22}f_1 - T_{12}f_2). \tag{16}$$

From the asymptotic boundary condition (12) we have

$$y'(x) = \frac{1}{\varepsilon}(P(x) + k - a(x)y(x)).$$

Substituting this in (16) and simplifying, we get a two term recurrence relation as follows:

$$y_{i+1} = \frac{1}{\varepsilon(T_{22}T_{11} - T_{12}T_{21})}[h\varepsilon(T_{22}f_1 - T_{12}f_2) - (P_i + k)(T_{22}S_{12} - S_{22}T_{12}) - (\varepsilon(T_{22}S_{11} - S_{21}T_{12}) - a_i(T_{22}S_{12} - T_{12}S_{22}))y_i]. \tag{17}$$

The condition $y_0 = \alpha$, is used to solve the above two term recurrence relation in forward process.

3. Numerical examples

To demonstrate the applicability of the method we have applied it to two linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The approximate solution is compared with the exact solution.

Example 3.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [1, p. 480; problem (9.17) with $\alpha = 0$]:

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1$ and $y(1) = 1$. The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]},$$

where $m_1 = \left(-1 + \sqrt{1 + 4\varepsilon}\right)/(2\varepsilon)$ and $m_2 = \left(-1 - \sqrt{1 + 4\varepsilon}\right)/(2\varepsilon)$.

The numerical results are given in Tables 1(a) and 1(b) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 1(a)
Numerical results of Example 3.1, $\varepsilon = 10^{-3}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	1.000000000000000e+000	1.000000000000000e+000
1.000000000000000e−003	6.007917971118448e−001	6.009586917270412e−001
5.000000000000000e−003	3.743262782418618e−001	3.739979292656276e−001
1.000000000000000e−002	3.719723959330329e−001	3.716055656467813e−001
5.000000000000000e−002	3.871078683269507e−001	3.867409917855329e−001
1.000000000000000e−001	4.069350064869349e−001	4.065696264479281e−001
2.000000000000000e−001	4.496878534047455e−001	4.493289273231297e−001
3.000000000000000e−001	4.969323412245205e−001	4.965852631276493e−001
4.000000000000000e−001	5.491403645555466e−001	5.488115911536213e−001
5.000000000000000e−001	6.068333955506272e−001	6.065306100458046e−001
6.000000000000000e−001	6.705876925538863e−001	6.703199114530440e−001
7.000000000000000e−001	7.410400559723790e−001	7.408181600185163e−001
8.000000000000000e−001	8.188941888631804e−001	8.187306860347755e−001
9.000000000000000e−001	9.049277257677435e−001	9.048373439417579e−001
1.000000000000000e+000	1.000000000000000e+000	9.999999181132426e−001

Table 1(b)
Numerical results of Example 3.1, $\varepsilon = 10^{-4}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	1.000000000000000e+000	1.000000000000000e+000
1.000000000000000e−003	3.683129549003972e−001	3.722612628741235e−001
5.000000000000000e−003	3.697602265008225e−001	3.700563137660214e−001
1.000000000000000e−002	3.716134715795036e−001	3.719112287536567e−001
5.000000000000000e−002	3.867777582501955e−001	3.870892139299358e−001
1.000000000000000e−001	4.066062453397311e−001	4.069357023234568e−001
2.000000000000000e−001	4.493649046843792e−001	4.497335037345733e−001
3.000000000000000e−001	4.966200590285670e−001	4.970323892117160e−001
4.000000000000000e−001	5.488445591957479e−001	5.493057418984449e−001
5.000000000000000e−001	6.065609809398519e−001	6.070767310781289e−001
6.000000000000000e−001	6.703468540124427e−001	6.709235482279791e−001
7.000000000000000e−001	7.408404411178888e−001	7.414851937536867e−001
8.000000000000000e−001	8.187471245827021e−001	8.194678723202614e−001
9.000000000000000e−001	9.048464646461863e−001	9.056520607856806e−001
1.000000000000000e+000	1.000000000000000e+000	1.000900319475613e+000

Example 3.2. Now consider the following non-homogeneous singular perturbation problem:

$$\varepsilon y''(x) + y'(x) = 1 + 2x; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$.

The numerical results are given in Tables 2(a) and 2(b) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

4. Nonlinear problems

Nonlinear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using Newton method of Quasilinearization. The outer solution (the solution of the given problem by putting $\varepsilon = 0$) is taken to be the initial approximation. The approximate solution is compared with the exact solution. To demonstrate the applicability of the method, we have applied it on a nonlinear singular perturbation problems with left-end boundary layer.

Table 2(a)
Numerical results of Example 3.2, $\varepsilon = 10^{-3}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	0	0
1.000000000000000e−003	−6.298573177109006e−001	−6.303846373794365e−001
5.000000000000000e−003	−9.862605288949127e−001	−9.870862518408403e−001
1.000000000000000e−002	−9.878746908700971e−001	−9.887018256048586e−001
5.000000000000000e−002	−9.456000000000000e−001	−9.463937308072652e−001
1.000000000000000e−001	−8.882000000000000e−001	−8.889519284784425e−001
2.000000000000000e−001	−7.584000000000000e−001	−7.590683238207974e−001
3.000000000000000e−001	−6.086000000000000e−001	−6.091847191631525e−001
4.000000000000000e−001	−4.388000000000000e−001	−4.393011145055076e−001
5.000000000000000e−001	−2.490000000000000e−001	−2.494175098478627e−001
6.000000000000000e−001	−3.920000000000001e−002	−3.953390519021775e−002
7.000000000000000e−001	1.906000000000003e−001	1.903496994674275e−001
8.000000000000000e−001	4.404000000000001e−001	4.402333041250724e−001
9.000000000000000e−001	7.101999999999999e−001	7.101169087827173e−001
1.000000000000000e+000	1.000000000000000e+000	1.000000513440362e+000

Table 2(b)
Numerical results of Example 3.2, $\varepsilon = 10^{-4}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	0	0
1.000000000000000e−003	−9.987538091502235e−001	−9.905776013165379e−001
5.000000000000000e−003	−9.947760000000000e−001	−9.922645199239503e−001
1.000000000000000e−002	−9.897020000000000e−001	−9.871895775238370e−001
5.000000000000000e−002	−9.473100000000000e−001	−9.447900382738046e−001
1.000000000000000e−001	−8.898200000000001e−001	−8.872906142112640e−001
2.000000000000000e−001	−7.598400000000001e−001	−7.572917660861831e−001
3.000000000000000e−001	−6.098600000000001e−001	−6.072929179611019e−001
4.000000000000000e−001	−4.398800000000001e−001	−4.372940698360208e−001
5.000000000000000e−001	−2.499000000000000e−001	−2.472952217109397e−001
6.000000000000000e−001	−3.991999999999996e−002	−3.729637358585854e−002
7.000000000000000e−001	1.900600000000003e−001	1.927024745392228e−001
8.000000000000000e−001	4.400400000000002e−001	4.427013226643038e−001
9.000000000000000e−001	7.100199999999999e−001	7.127001707893848e−001
1.000000000000000e+000	1.000000000000000e+000	1.002699018914466e+000

Example 4.1. Consider the following singular perturbation problem from Bender and Orszag [1, p. 463; Eqs. (9.7.1)]:

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 0$. The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right].$$

We have chosen to use Bender and Orszag's uniformly valid approximation [1, p. 463; Eq. (9.7.6)] for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)e^{-2x/\varepsilon}.$$

For this example, we have boundary layer of thickness $O(\varepsilon)$ at $x = 0$ [cf. Bender and Orszag [1]].
The numerical results are given in Tables 3(a) and 3(b) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Table 3(a)
Numerical results of Example 4.1, $\varepsilon = 10^{-3}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	0	0
1.000000000000000e−003	5.983404102211222e−001	5.984379697942955e−001
5.000000000000000e−003	6.881281702155939e−001	6.881230939411335e−001
1.000000000000000e−002	6.831968482780944e−001	6.831924583398247e−001
5.000000000000000e−002	6.443570163905132e−001	6.443594751549894e−001
1.000000000000000e−001	5.978370007556204e−001	5.978479580207977e−001
2.000000000000000e−001	5.108256237659907e−001	5.108532931516190e−001
3.000000000000000e−001	4.307829160924542e−001	4.308268342327517e−001
4.000000000000000e−001	3.566749439387324e−001	3.567345957803070e−001
5.000000000000000e−001	2.876820724517809e−001	2.877569252446644e−001
6.000000000000000e−001	2.231435513142098e−001	2.232330741743283e−001
7.000000000000000e−001	1.625189294977747e−001	1.626226045335766e−001
8.000000000000000e−001	1.053605156578264e−001	1.054778440717806e−001
9.000000000000000e−001	5.129329438755048e−002	5.142379954828764e−002
1.000000000000000e+000	0	1.432286796641612e−004

Table 3(b)
Numerical results of Example 4.1, $\varepsilon = 10^{-4}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	0	0
1.000000000000000e−003	6.921476787981791e−001	6.506543048555499e−001
5.000000000000000e−003	6.881596390489064e−001	6.879738129098669e−001
1.000000000000000e−002	6.831968497067772e−001	6.830137473746494e−001
5.000000000000000e−002	6.443570163905132e−001	6.441912027005219e−001
1.000000000000000e−001	5.978370007556204e−001	5.976920141280261e−001
2.000000000000000e−001	5.108256237659907e−001	5.107199231303178e−001
3.000000000000000e−001	4.307829160924542e−001	4.307137097390938e−001
4.000000000000000e−001	3.566749439387324e−001	3.566398055551009e−001
5.000000000000000e−001	2.876820724517809e−001	2.876788748200377e−001
6.000000000000000e−001	2.231435513142098e−001	2.231704148691400e−001
7.000000000000000e−001	1.625189294977747e−001	1.625741818843904e−001
8.000000000000000e−001	1.053605156578264e−001	1.054426595432628e−001
9.000000000000000e−001	5.129329438755048e−002	5.140098154994006e−002
1.000000000000000e+000	0	1.320102511843774e−004

5. Right-end boundary layer problems

Finally, we discuss our method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1] \tag{18}$$

$$\text{with } y(0) = \alpha \tag{19a}$$

$$\text{and } y(1) = \beta, \tag{19b}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[0,1]$. Further more, we assume that $a(x) \leqslant M < 0$ throughout the interval $[0,1]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

From the asymptotic boundary condition we have

$$\varepsilon y'(x) + a(x)y(x) = P(x) + k,$$

where $P(\mathrm{x}) = \int H(x)\,\mathrm{d}x$, $k = a(0)\beta - P(0)$ and $H(x) = f(x) - b(x)y_0(x)$.

Substituting this in (16) and simplifying, we get the two term recurrence relation as follows:

$$y_i = \frac{h\varepsilon(T_{22}f_1 - T_{12}f_2) - (P_i + k)(T_{22}S_{12} - S_{22}T_{12}) - \varepsilon(T_{22}T_{11} - T_{21}T_{12})y_{i+1}}{\varepsilon(S_{11}T_{22} - S_{21}T_{12}) - a_i(S_{12}T_{22} - S_{22}T_{12})}. \tag{20}$$

The condition $y_n = \beta$ is used to solve the above two term recurrence relation in backward process.

6. Examples with right-end boundary layer

To illustrate the method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval we considered two examples. The approximate solution is compared with the exact solution.

Example 6.1. Consider the following singular perturbation problem:

$$\varepsilon y''(x) - y'(x) = 0, \quad x \in [0,1]$$

$$\text{with } y(0) = 1 \quad \text{and} \quad y(1) = 0.$$

Clearly, this problem has a boundary layer at $x = 1$. i.e.; at the right end of the underlying interval.
The exact solution is given by $y(x) = \frac{(e^{(x-1)/\varepsilon}-1)}{(e^{-1/\varepsilon}-1)}$.

The numerical results are given in Tables 4(a) and 4(b) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

Example 6.2. Now we consider the following singular perturbation problem:

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0, \quad x \in [0,1]$$
$$\text{with} \quad y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon) \quad \text{and} \quad y(1) = 1 + 1/e.$$

Clearly this problem has a boundary layer at $x = 1$. The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$ The numerical results are given in Tables 5(a) and 5(b) for $\varepsilon = 10^{-3}$ and 10^{-4} respectively.

7. Discussion and conclusions

A seventh order numerical method is presented for solving singularly perturbed two-point boundary value problems with a boundary layer at one end point. A two term recurrence relation is obtained and is solved.

Table 4(a)
Numerical results of Example 6.1, $\varepsilon = 10^{-3}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	1.000000000000000e+000	1.000000000000000e+000
1.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
2.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
3.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
4.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
5.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
6.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
7.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
8.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
9.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
9.900000000000000e−001	9.999546000702375e−001	9.999546000678188e−001
9.950000000000000e−001	9.932620530009145e−001	9.932620528214147e−001
9.990000000000000e−001	6.321205588285580e−001	6.321205568684837e−001
1.000000000000000e+000	0	0

Table 4(b)
Numerical results of Example 6.1, $\varepsilon = 10^{-4}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	1.000000000000000e+000	1.000000000000000e+000
1.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
2.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
3.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
4.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
5.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
6.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
7.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
8.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
9.000000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
9.900000000000000e−001	1.000000000000000e+000	1.000000000000000e+000
9.950000000000000e−001	1.000000000000000e+000	9.999999999938426e−001
9.990000000000000e−001	9.999546000702375e−001	9.942736058544357e−001
1.000000000000000e+000	0	0

Table 5(a)
Numerical results of Example 6.2, $\varepsilon = 10^{-3}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	1.000000000000000e+000	9.990004182346399e−001
1.000000000000000e−001	9.048374180359596e−001	9.038378362705996e−001
2.000000000000000e−001	8.187307530779819e−001	8.177311713126219e−001
3.300000000000000e−001	7.189237334319262e−001	7.179241516665662e−001
4.000000000000000e−001	6.703200460356393e−001	6.693204642702791e−001
5.000000000000000e−001	6.065306597126334e−001	6.055310779472733e−001
6.000000000000000e−001	5.488116360940265e−001	5.478120543286664e−001
7.000000000000000e−001	4.965853037914095e−001	4.955857220260496e−001
8.000000000000000e−001	4.493289641172216e−001	4.483293823518615e−001
9.000000000000000e−001	4.065696597405991e−001	4.055700779752390e−001
9.900000000000000e−001	3.716216392149593e−001	3.706223875442146e−001
9.950000000000000e−001	3.764277858922879e−001	3.754561392792220e−001
9.990000000000000e−001	7.357592502223562e−001	7.353595249630720e−001
1.000000000000000e+000	1.367879441171442e+000	1.367879441171442e+000

Table 5(b)
Numerical results of Example 6.2, $\varepsilon = 10^{-4}$, $h = 10^{-3}$

X	Exact solution	Approximate solution
0	1.000000000000000e+000	9.999000941512859e−001
1.000000000000000e−001	9.048374180359596e−001	9.047375121872455e−001
2.000000000000000e−001	8.187307530779819e−001	8.186308472292678e−001
3.000000000000000e−001	7.408182206817180e−001	7.407183148330039e−001
4.000000000000000e−001	6.703200460356393e−001	6.702201401869251e−001
5.000000000000000e−001	6.065306597126334e−001	6.064307538639194e−001
6.000000000000000e−001	5.488116360940265e−001	5.487117302453125e−001
7.000000000000000e−001	4.965853037914095e−001	4.964853979426955e−001
8.000000000000000e−001	4.493289641172216e−001	4.492290582685075e−001
9.000000000000000e−001	4.065696597405991e−001	4.064697538918851e−001
9.900000000000000e−001	3.715766910220457e−001	3.714767851735604e−001
9.950000000000000e−001	3.697234445440590e−001	3.696240169244649e−001
9.990000000000000e−001	3.682928591661881e−001	4.225930889950365e−001
1.000000000000000e+000	1.367879441171442e+000	1.367879441171442e+000

The advantage of this two term recurrence relation is it can be solvable by forward or backward process. There is no need of applying any analytical or numerical method to solve the system of equations. Several linear and nonlinear singular perturbation problems have been solved and the numerical results are presented to support the theory.

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