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Slow Steady Flow of an Idealized Elastico-Viscous Incompressible Fluid of Oldroyd Type Through a Straight Tube with an Arbitrary Cross-section

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In der Arbeit wird die langsame stationäre Strömung einer idealisierten elastisch-viskosen Flüssigkeit durch ein gerades Rohr untersucht. Die Bewegungs- und Zustandsgleichungen werden mit Hilfe der Störungsrechnung linearisiert und dann mit funktionentheoretischen Hilfsmitteln gelöst. Unter dem Einfluß eines konstanten Druckgradienten längs des Rohres ist die Strömung eine Parallelströmung, welcher eine Sekundärströmung in den Ebenen senkrecht zur Rohrachse überlagert ist. Man erhält die Lösung für beliebig geformte Rohrquerschnitte; als Beispiel wird ein Rohr behandelt, dessen Querschnitt eine Pascalsche Schnecke ist.

In this paper, slow steady flow of an idealized elastico-viscous liquid through a straight tube is discussed. Perturbation technique is employed to linearize the equations of motion and state. We observe that the flow, under the influence of a constant pressure gradient down the axis of the tube, is a rectilinear flow over which a secondary flow is superposed in planes perpendicular to the length of the tube. The basic equations derived are solved with the function theoretic method. The solution for any arbitrary cross-section of the tube is obtained. The case of a tube with an elliptic-limacon cross-section is given as an example.

В работе исследуется медленное установившееся течение идеализированной упруго-вязкой жидкости в прямой трубе. Уравнения движения и состояния линеаризуются при помощи метода возмущений, а решения находят по теории функций. Под влиянием постоянного градиента давления, направленного вдоль трубы, поток становится параллельным течением, на которое в плоскости перпендикулярно к оси трубы накладывается вторичное течение. Решение возможно для сечения трубы любой формы. В качестве примера рассматривается труба с сечением в виде улитки Паскаля.

1. Introduction

Due to the non-linearity of the differential equations in the theory of elastico-viscous liquids of Oldroyd type [9] only a few problems with a high degree of symmetry have been solved in an exact way. A few attempts have been made to solve the equations for flow problems even in an approximate manner [10]—[12]. In the present paper, we examine the slow steady flow of an elastico-viscous liquid through a tube of arbitrary cross-section by employing a method of successive approximation based on the recursive approach proposed by Langlois [5] and Langlois and Rivlin [7]. It is observed that a uniform pressure gradient down the tube can produce a rectilinear flow over which a certain secondary flow is superposed in the cross-section. This is noticeable only when terms of the order S^4 are retained in the expansion. However, considering terms of third order in S only, we observe the normal stress variation on the boundary different from zero unlike in the Newtonian case. This is in confirmity with Kearsly's experimental results reported by Pipkin and Rivlin [13]. Earlier, several authors [1], [3], [6], [7] considered the slow steady motion of non-Newtonian fluids through straight tubes with a special reference to an elliptic cross-section. Recently, the secondary flow formation of non-Newtonian liquids of Reiner-Rivlin type has been examined for the cross-section bounded by (*i*) two eccentric circles [4] and (*i i*) two confocal ellipses [2]. In this paper, the basic equations in the successive approximations for the flow through a tube of arbitrary cross-section are solved by the function theoretic method developed by Mushkhelishvili [8]. The case of an elliptic limacon is given as an example.

2. Basic Equations of the Slow Motion of Elastico-viscous Liquids

The stress-tensor τ_{ik} and the rate of deformation tensor $E_{ik} = \frac{1}{2}(U_{i,k} + U_{k,i})$, for a class of isotropic liquids characterized by Oldroyd [9] are related by

$$\tau_{ik} = -P\,\delta_{ik} + S_{ik} \tag{2.1}$$

with

$$\begin{cases} S_{ik} + \lambda_1 \dfrac{DS_{ik}}{DT} + \mu_0 S_{jj} E_{ik} - \mu_1 (S_{ij} E_{jk} + S_{jk} E_{ij}) + \nu_1 S_{jl} E_{jl} \delta_{ik} \\ \quad = 2\eta \left[E_{ik} + \lambda_2 \dfrac{DE_{ik}}{DT} - 2\mu_2 E_{ij} E_{jk} + \nu_2 E_{jl} E_{jl} \delta_{jk}\right], \end{cases} \tag{2.2}$$

U_i being the velocity in the i-th direction, P is the hydrostatic mean pressure, S_{ik} the deviatoric stress-tensor, η the kinematic coefficient of viscosity and λ_1, λ_2; μ_0; μ_1, μ_2; ν_1, ν_2 are scalar physical coefficients (exhibiting the elastico-viscous properties of the liquid) each with the dimension of time T, restricted by the inequalities

$$\sigma_1 > \sigma_2 \geqq \sigma_1/9\,, \tag{2.3}$$

where

$$\sigma_q = \lambda_1 \lambda_q + \mu_0 \left(\mu_q - \frac{3}{2}\nu_q\right) - \mu_1 (\mu_q - \nu_q)\,. \tag{2.4}$$

Also D/DT indicates a corotational derivative

$$\frac{D}{DT} S_{ik} = \frac{\partial}{\partial T} S_{ik} + U_j S_{ik,j} + W_{ij} S_{kj} + W_{kj} S_{ij} \tag{2.5}$$

which, following the typical element, takes into account the linear and angular motion of the element measured by the velocity vector U_i and the vorticity tensor $W_{ik} = \frac{1}{2}(U_{k,i} - U_{i,k})$.

We consider such an idealized elastico-viscous liquid at rest filling a long straight tube. Taking the Z-axis of a system of rectangular Cartesian coordinates (x, y, Z) along the axis of the tube, let the cross-section of the tube be given by the profile Γ whose equation is $f(x, y) = 0$. The state of rest is disturbed by imposing a small uniform pressure gradient $\partial P/\partial Z = S\,G$ where S is non-dimensional and small. This parameter S may be taken as the REYNOLD's number characteristic of slow motion depending purely upon a characteristic length of the cross-section, pressure gradient and the kinematic coefficient of viscosity but independent of any of the elastico-viscous constants.

Following LANGLOIS [5] and LANGLOIS and RIVLIN [7], we assume that the velocity and in fact any physical quantity X such as the deviatoric stress-components, pressure, stream function etc., can be expanded in an absolute convergent series in the real parameter S in some range $|S| \leqq S_0$ and for relevant values of x and y. We thus write

$$X = S\,X^{(1)} + S^2 X^{(2)} + S^3 X^{(3)} + S^4 X^{(4)} + \ldots \tag{2.6}$$

and assume that the first and second order derivatives with respect to the space variables x and y can be obtained term wise differentiation and the resulting series be absolutely convergent.

The equations of motion are

$$\varrho\, U_j\, U_{i,j} = \tau_{ij,j}\,, \tag{2.7}$$

ϱ being the density of the liquid, together with the incompressibility condition

$$U_{j,j} = 0\,. \tag{2.8}$$

Let the components of the velocity (u, v, w) be given by

$$u = \frac{\partial \Psi}{\partial y}\,, \qquad v = \frac{-\,\partial \Psi}{\partial x}\,, \qquad w = w(x, y)\,. \tag{2.9}$$

The function Ψ may be interpreted as the stream function giving the flow pattern in the cross-section of the tube. The expansions of the type (2.6) are substituted in the consititutive relations (2.1), (2.2) and the equations of motion (2.7) and like powers of S are grouped. Thus the equations for the stresses and motion can be formulated in various stages which may be called the equations in the 1st, 2nd, 3rd, 4th . . . order approximations respectively. The boundary conditions, in each order of approximation may be written as

$$u^{(n)} = 0, \quad v^{(n)} = 0, \quad w^{(n)} = 0 \tag{2.10}$$

the first two of which being equivalent to

$$\Psi_x^{(n)} = 0, \quad \Psi_y^{(n)} = 0 \tag{2.11}$$

on Γ. We thus have at any stage a system of linear differential equations, with homogeneous, boundary conditions, making use of the results of all the previous stages. The equations determining the flow in the successive orders of approximation reduce, after simplification, to the following:

First Order Approximation:

(2.12), (2.13), (2.14) $\quad \nabla^2 w^{(1)} = G/\eta\ (= 4\,q), \quad \nabla^4 \Psi^{(1)} = 0, \quad P^{(1)} = G\,z + \text{constant}\,.$

Second Order Approximation:

(2.15), (2.16) $$\nabla^2 w^{(2)} = 0, \qquad \nabla^4 \Psi^{(2)} = 0\,,$$

(2.17) $$P^{(2)} = \frac{\eta}{2}(-2\nu - \lambda + \mu)\, I + (-\lambda + \mu)\, G\, w^{(1)} + \text{constant}$$

where

(2.18) $$\lambda = \lambda_1 - \lambda_2\,, \qquad \mu = \mu_1 - \mu_2\,, \qquad \nu = \nu_1 - \nu_2\,,$$

(2.19) $$I = [w_x^{(1)}]^2 + [w_y^{(1)}]^2\,.$$

Third Order Approximation:

(2.20) $$\nabla^2 w^{(3)} = (\sigma_1 - \sigma_2)\left[G \frac{I}{\eta} + w_x^{(1)} I_x + w_y^{(1)} I_y\right],$$

(2.21), (2.22) $$\nabla^4 \Psi^{(3)} = 0\,, \qquad P^{(3)} = \text{constant}\,.$$

Fourth Order Approximation:

(2.23), (2.24) $$\nabla^2 w^{(4)} = 0\,, \qquad \nabla^4 \Psi^{(4)} = (\mu_2 - \lambda_2)\, \frac{\partial}{\partial(x, y)}\, (w^{(1)}, \nabla^2 w^{(3)})\,.$$

These equations subject to the boundary conditions (2.10) yield the results

(2.25) $$\Psi^{(1)} = 0, \quad w^{(2)} = 0, \quad \Psi^{(2)} = 0, \quad \Psi^{(3)} = 0, \quad w^{(4)} = 0$$

throughout the cross-section and $w^{(1)}$, $w^{(3)}$ and $\Psi^{(4)}$ can be determined as functions of x and y.

The secondary flow, characterized by $\Psi^{(4)}$, is not observed until the terms of the order S^4 are retained in the expansions (2.6). Further, this secondary flow depends on the elastico-viscous coefficients. For the models with $\mu_2 = \lambda_2$ or $\sigma_1 = \sigma_2$ the equation (2.24) reduces to $\nabla^4 \Psi^{(4)} = 0$ which yields $\Psi^{(4)} = 0$ in the entire cross-section and hence upto this order, a purely rectilinear flow of such models through straight tubes is possible.

3. Normal Stress Distribution on the Boundary

Motivated by the experimental results of KEARSLY reported by RIVLIN and PIPKIN [9], we examine the normal stress variation on the boundary. Retaining terms upto the third order approximation only, we calculate the normal stress $N = \tau_{ij}\, n_i\, n_j$ on the boundary:

(3.1) $$N = -S\,G\,Z + \frac{S^2 \eta\, (-\lambda + \mu)}{2\,(f_x^2 + f_y^2)}\, [(f_x^2 - f_y^2)\,(w_x^{(1)^2} - w_y^{(1)^2}) + 4\, f_x\, f_y\, w_x^{(1)}\, w_y^{(1)}]\,.$$

This normal stress is seen much before the onset of the secondary flow which appeared in a later (4th) approximation. The variation of this normal stress along Γ is thus mainly responsible for the formation of the secondary flow in the planes perpendicular to the axis. In the NEWTONian flow, N is a constant along Γ and hence the flow is purely rectilinear for all cross-sections of the tube.

We adopt the function theoretic procedure for solving the equations (2.12), (2.20), (2.24) for an arbitrary cross-section which is simply connected so that by a suitable choise of the mapping function, the cross-section of the tube can be conformally mapped onto a unit circle [8].

4. Transformation of Coordinate Variables

Let us introduce the complex variable $z = x + i\,y$ and its conjugate $\bar{z} = x - i\,y$. The equation (2.12) then reduces to

$$\frac{\partial^2 w^{(1)}}{\partial z\, \partial \bar{z}} = \frac{G}{4\,\eta} = q \qquad (\text{say})\,,$$

the solution of which can be written as

(4.2) $$w^{(1)} = f_1(z) + \overline{f_1(z)} + q\, z\, \bar{z}$$

where $f_1(z)$ is analytic inside Γ and is so chosen that

(4.3) $$f_1(z) + \overline{f_1(z)} = -\,q\, z\, \bar{z}$$

on Γ. Also $\overline{f_1(z)}$ represents $\bar{f}_1(\bar{z})$.

The pressure distribution (2.17) obtained in the second approximation can be written as

$$p^{(2)} = 2\eta(-2\nu-\lambda+\mu)(f_1'(z)+q\bar{z})(\overline{f_1'(z)}+qz) + \\ + 4q\eta(-\lambda+\mu)(f_1(z)+\overline{f_1(z)}+qz\bar{z}) \tag{4.4}$$

plus an arbitrary constant.

The equation (2.10) for $w^{(3)}$ can now be written as

$$\frac{\partial^2 w^{(3)}}{\partial z\,\partial\bar{z}} = 2(\sigma_1-\sigma_2)\frac{\partial^2}{\partial z\,\partial\bar{z}}[l(z,\bar{z})] \tag{4.5}$$

where

$$l(z,\bar{z}) = g'''(z)\overline{h'(z)} + 2q\,g''(z)\,\overline{\bar{z}g'''(z)} + \\ + q^2\bar{z}\{z^2g'''(z)+2z\,g''(z)-2g'(z)\} + \\ + \frac{1}{2}q^3z^2\bar{z}^2 + \text{conjugate expression} \tag{4.6}$$

together with

$$g''(z) = f_1(z), \qquad h''(z) = f_1'^2(z)\,. \tag{4.7}$$

The solution of (4.5) can be written as

$$w^{(3)} = 2(\sigma_1-\sigma_2)(f_3(z)+\overline{f_3(z)}+l(z,\bar{z})) \tag{4.8}$$

where $f_3(z)$ is analytic inside Γ and satisfies the condition:

$$f_3(z)+\overline{f_3(z)} = -\,l(z,\bar{z}) \quad \text{on } \Gamma\,. \tag{4.9}$$

Similarly, the equation (2.24) for $\Psi^{(4)}$, characterizing the secondary flow, reduces to

$$\frac{\partial^4\psi^{(4)}}{\partial z^2\,\partial\bar{z}^2} = -\,i(\sigma_1-\sigma_2)(\mu_2-\lambda_2)\frac{\partial^4 m(z,\bar{z})}{\partial z^2\,\partial\bar{z}^2} \tag{4.10}$$

where

$$m(z,\bar{z}) = k(z) + 3q\,h(z)\,\bar{z}\,\overline{g'''(z)} + 3q^2g'(z)\,\bar{z}^2\,\overline{g'''(z)} + \\ + \tfrac{1}{2}q^3z^2\{\bar{z}^3\overline{g'''(z)} - 6\bar{z}\,\overline{g(z)} + 12\overline{g(z)}\} - \text{conjugate expression} \tag{4.11}$$

with

$$k''(z) = f'^3(z)\,. \tag{4.12}$$

The boundary conditions $\Psi_x^{(4)} = 0$, $\Psi_y^{(4)} = 0$ can easily be verified to be identical with $\Psi^{(4)}(z,\bar{z}) = 0$ and $\Psi_z^{(4)} = 0$ or $\Psi_{\bar{z}}^{(4)} = 0$ on Γ. The solution of (4.10) can now be written as

$$\Psi^{(4)} = -\,i(\sigma_1-\sigma_2)(\mu_2-\lambda_2)[f_4(z)-\overline{f_4(z)}+\bar{z}\varphi_4(z)-z\overline{\varphi_4(z)}+m(z,\bar{z})] \tag{4.13}$$

where the functions $f_4(z)$ and $\varphi_4(z)$ are analytic inside Γ and satisfy the conditions on Γ:

$$f_4(z)-\overline{f_4(z)}+\bar{z}\varphi_4(z)-z\overline{\varphi_4(z)} = -\,m(z,\bar{z})\,, \tag{4.14}$$

$$-\overline{f_4'(z)}+\varphi_4(z)-z\overline{\varphi_4'(z)} = -\frac{\partial m}{\partial\bar{z}} = -\,n(z,\bar{z})\,. \tag{4.15}$$

The normal stress distribution (3.1) on the boundary, when the terms of the third order perturbation are retained, is given by

$$N = -\,SGZ - S^2\eta(\lambda-\mu)\left[\frac{dz}{d\bar{z}}\{f_1'(z)+q\bar{z}\}^2 + \frac{d\bar{z}}{dz}\{\overline{f_1'(z)}+qz\}^2\right] \tag{4.16}$$

calculated at points of Γ.

5. Use of Conformal Mapping

Let the mapping function

$$z = \Omega(\zeta)\,, \qquad \zeta = \beta\sigma\,, \qquad \sigma = e^{i\varphi} \tag{5.1}$$

map the region inside Γ in the z plane onto the unit circle γ: $|\zeta| < 1$ in the ζ plane. Further, we denote the images of the respective functions in the ζ plane by the corresponding capital letters, i. e.,

$$\left\{ \begin{array}{l} f_p(z) = F_p(\zeta), \quad p = 1, 3, 4, \quad \varphi_4(z) = \Phi_4(\zeta), \quad l(z, \bar{z}) = L(\bar{\zeta}, \zeta), \\ m(z, \bar{z}) = M(\zeta, \bar{\zeta}), \quad n(z, z) = N(\zeta, \bar{\zeta}). \end{array} \right. \tag{5.2}$$

Determination of $F_1(\zeta)$: The boundary condition (4.2) can be written as

$$F_1(\sigma) + \overline{F_1(\sigma)} = - q\, \Omega(\sigma)\, \overline{\Omega(\sigma)}. \tag{5.3}$$

Multiplying by $\frac{1}{2}\pi i\,(\sigma - \zeta)$ and integrating along γ: $|\zeta| = 1$, we get

$$F_1(\zeta) + \overline{F_1(0)} = \frac{-q}{2\pi i} \int_\gamma \frac{\Omega(\sigma)\, \overline{\Omega(\sigma)}}{\sigma - \zeta}\, d\sigma. \tag{5.4}$$

The pressure in the second approximation can be written as

$$p^{(2)} = 2\eta\,(-2\nu - \lambda + \mu)\left[\frac{F_1'(\zeta)}{\Omega'(\zeta)} + q\, \overline{\Omega(\zeta)}\right]\left[\frac{\overline{F_1'(\zeta)}}{\overline{\Omega'(\zeta)}} + q\, \Omega(\zeta)\right] + \\ + 4 q \eta\,(-\lambda + \mu)\,[F_1(\zeta) + \overline{F_1(\zeta)} + q\, \Omega(\zeta)\, \overline{\Omega(\zeta)}]. \tag{5.5}$$

Determination of $F_3(\zeta)$: The boundary condition (4.9) can be written in the ζ plane as

$$F_3(\sigma) + \overline{F_3(\sigma)} = - L(\sigma, 1/\sigma) \tag{5.6}$$

from which we obtain as above,

$$F_3(\zeta) + \overline{F_3(0)} = - \frac{1}{2\pi i} \int \frac{L(\sigma, 1/\sigma)}{\sigma - \zeta}\, d\sigma \tag{5.7}$$

determining $F_3(\zeta)$.

Determination of $\Phi_4(\zeta)$: The condition (4.15) can now be written as

$$\frac{-\overline{F_4'(\sigma)}}{\overline{\Omega'(\sigma)}} + \Phi_4(\sigma) - \frac{\Omega(\sigma)}{\overline{\Omega'(\sigma)}}\, \overline{\Phi_4'(\sigma)} = - N(\sigma, 1/\sigma). \tag{5.8}$$

Multiplying (5.8) by $1/2\pi i\,(\sigma - \zeta)$ and integrating along γ, we get

$$-\overline{X(0)} + \Phi_4(\zeta) - \frac{1}{2\pi i} \int \frac{\Omega(\sigma)}{\overline{\Omega'(\sigma)}}\, \overline{\Phi_4'(\sigma)}\, \frac{d\sigma}{\sigma - \zeta} = - \frac{1}{2\pi i} \int N(\sigma, 1/\sigma)\, \frac{d\sigma}{\sigma - \zeta}. \tag{5.9}$$

The function $\Phi_4(\zeta)$ can be obtained as a solution of this integral equation. Here $X(0) = F_1'(0)/\Omega'(0)$.

Determination of $F_4(\zeta)$: Having determined $\Phi_4(\zeta)$ in the above manner, the boundary condition (4.14) can now be written as

$$F_4(\sigma) - \overline{F_4(\sigma)} = - M(\sigma, 1/\sigma) - \overline{\Omega(\sigma)}\, \Phi_4(\sigma) + \Omega(\sigma)\, \overline{\Phi_4(\sigma)} = P(\sigma, 1/\sigma) \quad \text{(say)}. \tag{5.10}$$

Then, as before

$$F_4(\zeta) - \overline{F_4(0)} = \frac{1}{2\pi i} \int_\gamma \frac{P(\sigma, 1/\sigma)}{\sigma - \zeta}\, d\sigma, \tag{5.11}$$

from which the function F_4 can be determined.

The determination of $\Phi_4(\zeta)$ and $F_4(\zeta)$ can be facilitated by assuming power series expansion for the functions, the constants of which can be obtained by comparing like powers of ζ in the equations (5.9) and (5.11) respectively.

The normal stress distribution responsible for the secondary flow can be obtained as

$$N = - S G Z - S^2 \eta\,(\lambda - \mu)\left[\frac{\Omega'(\sigma)}{\overline{\Omega'(\sigma)}} \left\{\frac{F_1'(\sigma)}{\Omega'(\sigma)} + q\, \overline{\Omega(\sigma)}\right\} + \text{conjugate expression}\right]. \tag{5.12}$$

6. Flow through a Tube of Arbitrary Cross-section

Let the mapping function

(6.1) $$Z = \Omega(\zeta) = \sum_{r=1}^{n} \omega_r \zeta^r$$

transforms conformally the interior of Γ in the z plane, into the unit circle γ: $|\zeta| = 1$ in the ζ plane. For generality, we regard the coefficients ω_r to be complex. Then

(6.2), (6.3) $$\Omega(\sigma)\,\overline{\Omega(\sigma)} = \sum_{r=-(n-1)}^{n+1} e_r \sigma^r, \qquad e_r = \sum_{j=1}^{n-r} \bar{\omega}_j \omega_{r+j}.$$

Determination of $F_1(\zeta)$: From (5.4), we get after integrating,

(6.4) $$F_1(\zeta) + \overline{F_1(0)} = -q \sum_{r=0}^{n-1} e_r \zeta^r.$$

Since $e_0 = \sum_{j=1}^{n} \bar{\omega}_j \omega_j$ is real, we can take

(6.5) $$F_1(0) = \overline{F_1(0)} = -q\, e_0/2$$

but for an imaginary constant the presence of which can be avoided as it does not effect the final result. Hence we have

(6.6) $$F_1(\zeta) = -q\left[e_0/2 + \sum_{r=1}^{n-1} e_r \zeta^r\right]$$

and the velocity field in the first approximation:

(6.7) $$w^{(1)} = -q\left[e_0 + \sum_{r=1}^{n-1}(e_r \zeta^r + \bar{e}_r \bar{\zeta}^r) - \sum_{j=1}^{n} \omega_j \zeta^j \cdot \sum_{j=1}^{n} \bar{\omega}_j \bar{\zeta}^j\right].$$

Determination of $F_3(\zeta)$: Let

(6.8) $$L(\sigma, 1/\sigma) = l_0 + \sum_{r=1}^{\infty}(l_r \sigma^r + \bar{l}_r/\sigma^r).$$

Then the equation (5.7), after integration, yields

(6.9) $$F_3(\zeta) + \overline{F_3(0)} = -\sum_{r=0}^{\infty} l_r \zeta^r.$$

Taking $F_3(0) = \overline{F_3(0)} = -l_0/2$ (real), we obtain

(6.10) $$F_3(\zeta) = -\left[\frac{1}{2} l_0 + \sum_{r=1}^{\infty} l_r \zeta^r\right]$$

and hence

(6.11) $$w^{(3)} = -2(\sigma_1 - \sigma_2)\left[l_0 + \sum_{r=1}^{\infty}(l_r \zeta^r + \bar{l}_r \bar{\zeta}^r) - L(\zeta, \bar{\zeta})\right]$$

Determination of $\Phi_4(\zeta)$: Let

(6.12), (6.13) $$N(\sigma, 1/\sigma) = \sum_{r=-\infty}^{\infty} n_r \sigma^r, \qquad \frac{\Omega(\sigma)}{\overline{\Omega'(\sigma)}} = \sum_{r=-\infty}^{n} c_r \sigma^r.$$

We shall also assume

(6.14) $$\Phi_4(\zeta) = \sum_{r=0}^{\infty} a_r \zeta^r$$

and obtain

(6.15) $$\frac{\Omega(\sigma)}{\overline{\Omega'(\sigma)}}\,\overline{\Phi_4'(\sigma)} = \sum_{r=0}^{n} K_r \sigma^r + \sum_{j=1}^{\infty} K_{-j}/\sigma^j,$$

where

(6.16) $$K_r = \sum_{s=r}^{n} (s - r + 1)\, c_s\, \bar{a}_{s-r+1}$$

for all $r \geqq 0$. We do not write out the expression for K_{-j}, $j > 0$, as they are not required in the calculation of $\Phi_4(\zeta)$. We then get after substituting these expressions in (5.9) and comparing the like powers of ζ

(6.17) $$-\overline{X(0)} + a_0 - K_0 = -n_0\,, \quad a_r - K_r = -n_r \text{ for } 0 < r \leqq n\,, \quad a_r = -n_r \text{ for } r > n\,.$$

This set of equations determines constants a_n in terms of $\overline{X(0)}$ and n_r.

Determination $F_4(\zeta)$: Let

(6.18) $$\Omega(\sigma)\, \Phi_4(\sigma) = d_0 + \sum_{r=1}^{\infty} d_r \sigma^r + \sum_{r=1}^{n} d_{-r}/\sigma^r\,,$$

where

(6.19) $$d_r = \sum_{s=1}^{n} \bar{\omega}_s\, a_{s+r}\,, \qquad d_{-r} = \sum_{s=r}^{n} \bar{\omega}_s\, a_{s-r}\,.$$

Also let

(6.20), (6.21) $$M(\sigma, 1/\sigma) = \sum_{r=0}^{\infty} m_r\, \sigma^r\,, \qquad F_4(\zeta) = \sum_{r=0}^{\infty} b_r\, \zeta^r\,.$$

The boundary condition (5.10) can now be written as

(6.22) $$F_4(\sigma) - \overline{F_4(\sigma)} = -\left[(d_0 - \bar{d}_0) + \sum_{r=1}^{\infty} (d_r\, \sigma^r - \bar{d}_r/\sigma^r) + \sum_{r=1}^{n} \left(\frac{d_{-r}}{\sigma^r} - \bar{d}_{-r}\, \sigma^r\right) + \sum_{r=-\infty}^{+\infty} m_r\, \sigma^r\right].$$

Multiplying by $\frac{1}{2}\pi\, i\, (\sigma - \zeta)$ and integrating along γ, we obtain

(6.23) $$F_4(\zeta) - \overline{F_4(0)} = -\left[(d_0 - \bar{d}_0) + \sum_{r=1}^{\infty} d_r\, \zeta^r - \sum_{r=1}^{n} \bar{d}_{-r}\, \zeta^r + \sum_{r=0}^{\infty} m_r\, \zeta^r\right].$$

Substituting from (6.21) yields

(6.24) $$\begin{cases} b_0 - \bar{b}_0 = -(d_0 - \bar{d}_0 + m_0)\,, \quad b_r = -(d_r - \bar{d}_{-r} + m_r) & \text{for } 0 < r \leqq n\,, \\ b_r = -(d_r + m_r) & \text{for } r > n\,. \end{cases}$$

Thus determining the coefficients b's. Using all the above results, we can now write the stream function:

(6.25)

$$\Psi^{(4)} = -i\,(\sigma_1 - \sigma_2)\,(\mu_2 - \lambda_2)\left[\sum_{r=0}^{\infty} (b_r\, \zeta^r - \bar{b}_r\, \bar{\zeta}^r) + \sum_{r=1}^{n} \bar{\omega}_r\, \bar{\zeta}^r \cdot \sum_{r=0}^{\infty} a_r\, \zeta^r - \sum_{r=1}^{n} \omega_r\, \zeta^r \cdot \sum_{r=0}^{\infty} \bar{a}_r\, \bar{\zeta}^r + M(\zeta, \bar{\zeta})\right]$$

which characterizes the secondary flow pattern.

Further, the normal stress distribution (4.16) on the boundary can be obtained as

(6.26) $$N = -S\,G\,Z - \frac{S^2\, q^2\, \eta\, (\lambda - \mu)}{U}\left[\left\{-e_0/2 - \sum_{r=1}^{n-1} e_r\, \sigma^r + \sum_{r=1}^{n} \left(\sum_{j=1}^{n-r} j\, \omega_j\, \bar{\omega}_{j+r-1}/\sigma^r\right) + \right.\right.$$
$$\left.\left. + \sum_{r=0}^{n-2} \left(\sum_{j=1}^{n+1} (j + r + 1)\, \bar{\omega}_j\, \omega_{j+r+n}\, \sigma^r\right)\right\}^2 + \text{Conjugate}\right],$$

where

$$U = \sum_{r=-(n-1)}^{n+1} \sum_{j=1}^{n-r} j\,(j + r)\, \bar{\omega}_j\, \omega_{(j+r)}\, \sigma^r\,.$$

7. Application: Flow through an Elliptic Limacon

The function

(7.1) $$z = \Omega(\zeta) = b\,(\zeta + m\, \zeta^2)\,, \quad 0 \leqq m < 1/2$$

maps the region inside the elliptic limacon on to the region bounded by the unit circle γ in the ζ plane. The results in this case are mentioned here under.

First Order Solution: The velocity in this approximation is given by

(7.2) $$w^{(1)} = -q\, b^2\, (1 - \beta^2)\, [1 + m\, (1 + \beta^2) + 2\, m\, \beta \cos \varphi]\,.$$

Second Order Approximation: The pressure field noticed in the second approximation is

(7.3) $$p^{(2)} = 2\, q^2\, b^2\, \eta\, (-2\, \nu - \lambda + \mu)\, [\beta^2\, (1 + 2\, m\, \beta \cos \varphi + m^2\, \beta^2) +$$
$$+ \{m^2\, (1 - 4\, m^2) - 2\, m\, \beta\, (1 + 2\, m^2\, \beta^2) \cos \varphi - m^2\, \beta^2 \cos 2\, \varphi\}/H] +$$
$$+ 4\, \eta\, q^2\, b^2\, (-\lambda + \mu)\, (1 - \beta^2)\, \{1 + m^2\, (1 + \beta^2) + 2\, m\, \beta \cos \Phi\}$$

where $H = 1 + 4\, m\, \beta \cos \Phi + 4\, m^2\, \beta^2$.

Third Order Solution: The effect of the non-Newtonian coefficients in this approximation is the rectilinear flow characterized

$$w^{(3)} = \frac{(\sigma_1 - \sigma_2)\, q^3\, b^4}{3}\,[3\, m^2\, w_1^{(3)} + 2\, w_2^{(3)}] \tag{7.4}$$

where

$$w_1^{(3)} = 4\, m\,(1 - \beta^2)\,[\{\log(1 - 4\, m^2) - A\}\cdot\{2\, m\,(1 + \beta^2) + (1 + 4\, m^2)\,\beta\cos\varphi\} - B\,(1 - 4\, m^2)\,\beta\sin\varphi]/D \tag{7.5a}$$

and

$$\begin{aligned} w_2^{(3)} = &- (1 - \beta^2)\,[\{3\,(1 + 4\, m^2 + m^4) + (3 + 12\, m^2 + 43\, m^4 + 12\, m^6)\,\beta^2 + \\ &+ m^2\,(48 + 19\, m^2 + 12\, m^4)\,\beta^4 + 3\, m^4\,(25 + 4\, m^2)\,\beta^6 + 12\, m^6\,\beta^8\} + \\ &+ 4\, m\,\{(3 + 11\, m^2 + 3\, m^4) + (6 + 3\, m^2 + 8\, m^4)\,\beta^2 + \\ &+ 2\, m^2\,(15 + 4\, m^2)\,\beta^4 + 15\, m^4\,\beta^6\}\,\beta\cos\varphi + \\ &+ 2\, m^2\,\{(1 + 5\, m^2) + 3\,(5 + m^2)\,\beta^2 + 24\, m^2\,\beta^4\}\,\beta^2\cos 2\varphi + \\ &+ 12\, m^3\,(1 + \beta^2)\,\beta^3\cdot\cos 3\varphi]/H \end{aligned} \tag{7.5b}$$

with

$$\begin{aligned} A &= (1/2)\log H\,, \\ B &= \tan^{-1}\{(2\, m\,\beta\sin\varphi)/(1 + 2\, m\,\beta\cos\varphi)\}\,, \\ D &= 4\, m^2 + (1 + 4\, m^2)^2\,\beta^2 + 4\, m^2\,\beta^4 + 4\, m\,(1 + 4\, m^2)\,(1 + \beta^2)\cos\varphi + 8\, m^2\,\beta^2\cos 2\varphi\,. \end{aligned}$$

Fourth Order Solution: The secondary flow, in planes perpendicular to the tube-axis is characterized by the stream function Ψ:

$$\begin{aligned} &\Psi^{(4)}/[2\, q^4\, b^6\,(\sigma_1 - \sigma_2)\,(\mu - \lambda_2)] \\ &= \{n_0 + (2 - 4\, m^2 - \beta^2)\, n_2 + (3 - \beta^4)\, m\, n_3\}\,\beta\sin\varphi + \\ &+ \sum_{r=2}^{\infty}\{- m_r + (n_{r+1} + m\, n_{r+2}\,[1 + \beta^2])\,(1 - \beta^2)\}\,\beta^r\sin r\varphi + \\ &+ (3\, m/8\, H)\,\{A\,[(1 - 6\, m^2\,\beta^2 - 8\, m^2\,\beta^4)\,\beta\sin\varphi + m\,(1 - 8\, m^2\,\beta^2)\,\beta^2\sin 2\varphi] - \\ &\quad - B\,[6\, m\,(1 + 2\, m^2\,\beta^2)\,\beta^2 + (1 + 18\, m^2\,\beta^2 + 8\, m^4\,\beta^4)\,\beta\cos\varphi + m\,(1 + 8\, m^2\,\beta^2)\cdot\beta^2\cos 2\varphi]\} + \\ &+ (m/80\, H)\,\{[- 5\,(3 + 8\, m^2) - 30\,(4 + m^2)\,\beta^2 - 40\, m^2\,(8 + 9\, m^2)\,\beta^4 - \\ &\quad - 80\, m^2\,(2 + 3\, m^4)\,\beta^6 - 280\, m^4\,\beta^8 - 48\, m^6\,\beta^{10}]\,\beta\sin\varphi + \\ &+ [- 5\, m\,(3 + 16\, m^2) - 120\, m\,(2 + m^2)\,\beta^2 - 40\, m\,(1 + 4\, m^2 + 9\, m^4)\,\beta^4 - \\ &\quad - 288\, m^3\,\beta^6 - 120\, m^5\,\beta^8]\,\beta^2\sin 2\varphi + \\ &+ [- 120\, m^2\,(1 + m^2)\,\beta^2 - 72\, m^2\,\beta^4 - 96\, m^4\,\beta^6]\,\beta^3\sin 3\varphi - 24\, m^3\,\beta^8\sin 4\varphi\}\,. \end{aligned} \tag{7.6}$$

The coefficients n_r are given by

$$n_0 = \left[\frac{-3\, m}{4}\,(1 - 8\, m^2 + 28\, m^4)\log(1 - 4\, m^2) + \frac{m}{16\,(1 - 4\, m^2)}\,(51 - 276\, m^2 + 440\, m^4 - 480\, m^6)\right],$$

$$n_1 = \left[\frac{3}{32}\,(1 - 8\, m^2 + 64\, m^4)\log(1 - 4\, m^2) + \frac{m^2}{8\,(1 - 4\, m^2)}\,(27 - 126\, m^2 + 40\, m^4)\right],$$

$$n_2 = - n_1/2\, m\,,$$

$$n_3 = \left[\frac{3}{64\, m^2}\,(1 - 4\, m^2 + 8\, m^4 + 64\, m^6)\log(1 - 4\, m^2) + \frac{1}{80\,(1 - 4\, m^2)}\,(15 - 170\, m^2 + 216\, m^4 + 576\, m^6)\right],$$

$$\begin{aligned} n_4 = \Bigg[&- \frac{3}{64\, m^3}\,(1 - 4\, m^2 + 4\, m^4 + 128\, m^8)\log(1 - 4\, m^2) - \\ &- \frac{1}{80\, m\,(1 - 4\, m^2)}\,(15 - 90\, m^2 + 244\, m^4 - 776\, m^6 + 480\, m^8)\Bigg], \end{aligned}$$

and for $r \geq 5$,

$$\begin{aligned} n_r = \Bigg[&- \frac{3\, m}{32}\left(- \frac{1}{2\, m}\right)^r\{(4 - 3\, r + r^2) - 8\,(8 - 5\, r + r^2)\, m^2 + \\ &+ 16\,(14 - 7\, r + r^2)\, m^4\}\,\{H_r + \log(1 - 4\, m^2)\} - \frac{3\, m}{8}\,(- 2\, m)^r\log(1 - 4\, m^2) - \\ &- \frac{(- 2\, m)^r}{32\, m\, r\,(r - 1)\,(r - 2)\,(1 - 4\, m^2)}\,\{2\, r\,(r - 1)\,(r - 2) - 3\,(8 + 10\, r - 15\, r^2 + 5\, r^3)\, m^2 + \\ &+ 12\,(40 - 38\, r + 18\, r^2 - 6\, r^3 + r^4)\, m^4 + 48\,(r - 2)\,(30 - 33\, r + 13\, r^2 - 2\, r^3)\, m^6 \\ &+ 192\,(r - 1)\,(r - 2)\,(14 - 7\, r + r^2)\, m^8\}\Bigg]\,. \end{aligned}$$

Also m are given by

$$m_0 = 0\,,$$

$$m_1 = \left[\frac{-3}{64\,m}(1-4\,m^2)^3\log(1-4\,m^2) - \frac{m}{40}(75+25\,m^2+56\,m^4)\right],$$

$$m_2 = \left[\frac{3}{128\,m^2}(1-16\,m^4)(1-4\,m^2)^2\log(1-4\,m^2) + \frac{1}{160}(15-140\,m^2-768\,m^4+160\,m^6)\right],$$

$$m_3 = \left[-\frac{3}{256\,m^3}(1-64\,m^6)(1-4\,m^2)^2\log(1-4\,m^2) + \frac{1}{320\,m}(-15+90\,m^2-648\,m^4+2160\,m^6-640\,m^8)\right],$$

and for $r \geqq 4$,

$$m_r = \left[\frac{3}{32}\left(-\frac{1}{2\,m}\right)^r (1-4\,m^2)^2\{H_r + (1-4^r\,m^{2r})\log(1-4\,m^2)\} + \right.$$
$$\left. + \frac{(-2\,m)^r}{64\,r\,(r^2-1)}\{13\,r\,(r^2-1) - 4\,(24-7\,r-6\,r^2+13\,r^3)\,m^2 - 16\,(r^2-1)\,(r-6)\,m^4\}\right].$$

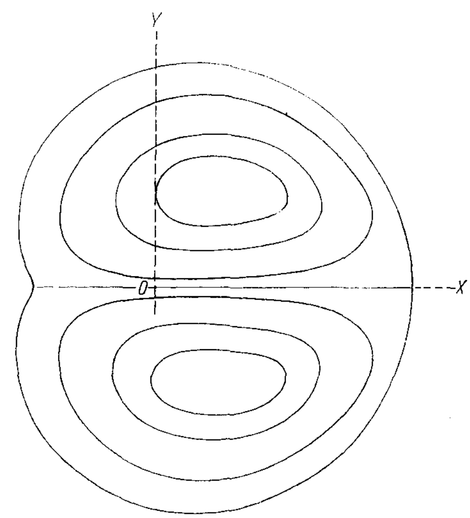
In the above expressions, H_r represents $\sum_{k=1}^{r}\{(4\,m^2)^k/k\}$.

Form the structure of (7.6), it can be noticed that the flow is symmetrical about the line $\mathrm{Re}\,(\zeta) = 0$, which forms a dividing stream line. Hence, the secondary flow is composed of two regions of circulatory flow with opposite directions in the two symmetrical halves of the cross-section. The flow pattern in the cross-section has been illustrated in the figure.

When $m = 0$, the cross-section of the tube reduces to a circle in which case, we get $\Psi^{(4)} = 0$, i. e., a purely rectilinear flow is sustained without the formation of secondary flow [9].

The normal stress, noticed much before the onset of the secondary flow (7.6) is given in the non-dimensional form:

$$\Delta N^* = \frac{N + S\,G\,Z}{2\,S^2\,q^2\,b^2\,(-\lambda+\mu)} = - \frac{\cos 2\,\varphi\,(1 + 2\,m^2 + 2\,m\cos\varphi)^2}{1 + 4\,m + 4\,m\cos\varphi}\,. \tag{2.7}$$



Secondary flow pattern in a tube whose cross-section is an elliptic limacon

The following table gives the values of ΔN^* along one half of the limacon, for various values of m. The variation in the other half can be got from symmetry. The cases of a circle ($m = 0$) and cardioid ($m = 1/2$) are included in the table for reference. It is observed that as the aspect ratio m of the limacon increases, N^* also increases. The departure of ΔN^* for an elliptic limacon from its value in the circular section increases with m. Within the range $45° < \varphi < 135°$, ΔN^* is very close to its value for the circular case. The variation in the range $135° < \varphi < 180°$ is much larger than that in $0° < \varphi < 45°$ for $m \neq 0$. This non-uniformity in the variation of ΔN^* and its departure from the circular cases are responsible for the secondary flow. It is also worth mentioning that for $m = 1/2$. ΔN^* becomes infinite at $\varphi = 180°$ i. e., at the cusp of the cardioid, a case which is excluded from our present discussion.

Variation of ΔN^* along One Half of the Boundary

φ \ m	0.0 Circle	0.1	0.2	0.3	0.4	0.5 Cardioid
0°	-1.0000	-1.0336	-1.1176	-1.2377	-1.3872	-1.5625
15°	-0.86603	-0.89372	-0.96360	-1.0645	-1.1908	-1.3395
30°	-0.50000	-0.51382	-0.54945	-0.60243	-0.67000	-0.75054
45°	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60°	+0.50000	+0.50512	+0.52443	+0.55804	+0.60545	+0.66582
75°	+0.86603	+0.86944	+0.88689	+0.92385	+0.98270	+1.0637
90°	+1.0000	+1.0004	+1.0055	+1.0239	+1.0626	+1.1252
105°	+0.86603	+0.86765	+0.86721	+0.86724	+0.87582	+0.90089
120°	+0.50000	+0.50518	+0.51083	+0.51080	+0.50514	+0.50138
135°	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
150°	-0.50000	-0.51511	-0.57380	-0.67680	-0.70914	-0.74582
165°	-0.86603	-0.90459	-1.0743	-1.5505	-2.7298	-3.5941
180°	-1.00000	-1.0506	-1.2844	-2.1025	-6.7599	$-\infty$

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