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Slow Steady Flow of an Idealized Elastico-viscous Liquid through a Cone with a Source/Sink at the Vertex

By N. CH. PATTABHI RAMACHARYULU

Durch Linearisierung wurden die Gleichungen gelöst, die für eine idealisierte elastisch-viskose Flüssigkeit von Oldroyd-Typ gelten, wenn diese langsam und stationär durch ein konisches Rohr strömt, wobei sich in der Spitze des Kegels eine Quelle bzw. Senke befindet. Die nicht-newtonschen Effekte werden ausführlich untersucht. Getrennt von einer rein radialen Strömung in größeren Entfernungen, treten in der Nähe der Spitze Gebiete zirkulierender Strömung auf.

Employing a linearization technique, the momentum equations and the general constitutive relations for idealized elastico-viscous liquids of the Oldroyd type have been solved for the slow steady flow through a conical channel with a source or sink at the vertex. The non-Newtonian effects have been discussed in detail. We notice regions of circulatory flow near the vertex separated from a purely radial flow at large distances.

При помощи линеаризации в данной работе решаются уравнения для идеальных эластично-вязких жидкостей типа Олдройда в том случае, когда жидкость медленно и стационарно течёт по конической трубе, а в острии находится источник или сток. Подробно исследуются не — ньютоновские эффекты. Вблизи острия возникают области циркуляции течения, которые находятся на большом расстоянии от чисто радиального течения.

1. Introduction

The equations of classical hydrodynamics are non-linear and in the case of general viscous liquids, even the constitutive relations are non-linear. The non-linearity of these equations presents intractable difficulties in solving specific flow problems. However, some attempts have been made to obtain solutions by approximating and truncating the equations involved. Recently, LANGLOIS [2] proposed a technique of linearizing the momentum equations and constitutive relations under the assumption that the steady state of a slow flow field is a perturbation from the state of rest and that the flow variables are expandable in powers of a suitable small non-dimensional parameter S, the smallness of which characterizes the degree of slowness of the flow. The first order solution corresponds to the so called creeping flow of a Newtonian liquid, which we may call as the primary flow. The second and higher order solutions give the effect of the nonlinear terms and throw light upon the formation of secondary flows superposed on the primary one. This method had been successfully employed by the present author in solving some non-Newtonian flow problems: (i) Slow steady rotation of a sphere [6], (ii) slow steady flow through a tube of non-circular cross-section under a constant pressure gradient [7], (iii) Slow steady helical flow through an anulus [8]. The constitutive relations of the non-Newtonian model of our choice are the more general relations (2.1)—(2.2) given by OLDROYD [5] for a class of idealized elastico-viscous liquids. A similar procedure had been employed by LESLIE [4] in discussing the slow steady non-Newtonian stream past a sphere with $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$ in the equations (2.1)—(2.2). The perturbation parameter chosen by LESLIE depends on the relaxation time λ_1, one of the non-Newtonian coefficients. The smallness of this parameter has therefore a strong restriction on the type of the non-Newtonian fluid or on the size of the sphere in question.

In the present paper, we employ the linearization technique mentioned above to examine the flow of an idealized elastico-viscous liquid of OLDROYD-type [5] in a conical region due to a weak source or sink at the vertex. Such a problem has been discussed earlier by LANGLOIS and RIVLIN [1] by neglecting inertial terms in the momentum equations for a class of slightly visco-elastic models of the RIVLIN-ERICKSEN type [9]. These authors obtained an approximate solution of the problem by a different method, adopting the scheme followed by LANGLOIS [3] in which the non-Newtonian flow between rotating spheres is examined. But LANGLOIS and RIVLIN [1] in their work neglected the inertia of liquid. They noticed a purely radial flow for a Newtonian liquid. The non-Newtonian effect is a transverse velocity superposed on the radial flow in the meridian plane. This effect is negligibly small at large distances from the vertex.

As the inertial terms are more predominant in the vicinity of the vertex, they have a strong influence on the primary flow by generating a secondary flow even in the case of classical viscous liquids. The structure of this secondary flow is of the same type as mentioned before and is composed of two regions of circulatory motion touching the cone-wall in the case of source: fig. 4 and one such region around the axis for a sink: fig. 9. Outside these regions, the flow is approximately radial at large distances from the vertex. This observation is being supported by the conclusions of ROBERT C. ACKERBERG [10] in discussing the flow of a Newtonian viscous liquid in a cone due to a sink at the vertex. Such an effect is not noticed by Langlois and Rivlin [1] when their analysis is applied for a classical viscous liquid as they neglected inertial terms completely.

In our present investigation, the momentum and the constitutive equations have been taken without any approximations and truncations. A perturbation parameter S is chosen, independent of any of the elastico-viscous coefficients. We observe that up to the first order of S, the flow is purely radial as in the Newtonian creeping flow. But when the second order terms are retained, the radial flow is deviated due to the superposed secondary flow. Thus the effects of non-linear terms is a secondary flow composed of both radial and transverse velocities in the meridian plane. We observe that this secondary flow pattern is characterized by one or more loops near the wall and/or near the cone-axis. This effect is predominant near the vertex only and at large distances from it, the flow is purely radial coinciding with the creeping flow of a Newtonian liquid. As an example, the flow pattern is illustrated in figs. (1)—(10) for various values of a certain non-Newtonian parameter K and when the semi-angle of the cone is 60°.

2. Basic equations

Oldroyd [5] considered a class of isotropic and incompressible fluids, with elastico-viscous properties, characterized by a set of general equations of state, relating the stress-tensor τ_{ik} and the rate of strain-tensor $E_{ik} = (U_{i,k} + U_{k,i})/2$:

(2.1) $$\tau_{ik} = - P\, g_{ik} + S_{ik}$$

with

(2.2) $$S^{ik} + \lambda_1 \frac{D}{DT} S^{ik} + \mu_0 E^{ik} S^j_j - \mu_1 (S^i_j E^{jk} + S^k_j E^{ij}) + \nu_1 E^{jl} S_{jl}\, g^{ik}$$
$$= 2\eta \left[E^{ik} + \lambda_2 \frac{D}{DT} E^{ik} - 2\mu_2 E^i_j E^{jk} + \nu_2 E^{jl} E_{jl}\, g^{ik} \right],$$

where U_i is the velocity in the direction of the i-th coordinate X_i, g_{ik} is metric tensor and P is an isotropic pressure, η is the kinematic viscosity, λ_1, λ_2 are the relaxation and retardation times and μ_0, μ_1, μ_2; ν_1, ν_2 are constants having the dimensions of time T. These constants λ_1, λ_2; μ_0, μ_1, μ_2; ν_1, ν_2 characterize the elastico-viscous properties of the fluid and are restricted by the inequalities:

(2.3) $$\sigma_1 > \sigma_2 \geqq \frac{1}{9}\sigma_1,$$

where

$$\sigma_q = \lambda_1 \lambda_q + \mu_0 \left(\mu_q - \frac{3}{2}\nu_q\right) - \mu_1 (\mu_q - \nu_q), \qquad q = 1, 2.$$

Also D/DT indicates a total material derivative:

(2.4) $$\frac{D}{DT} B^{ik} = \frac{\partial}{\partial T} B^{ik} + U^j B^{ik}_{,j} + W^i_m B^{mk} + W^k_m B^{im}$$

which following the typical element, takes into account the linear and angular motion of the element measured by the velocity vector U_i and the vorticity tensor: $W_{ik} = (U_{k,i} - U_{i,k})/2$.

The equations governing the steady flow of an incompressible fluid, in the absence of external forces are (2.5) together with the continuity equation (2.6)

(2.5) $$\varrho\, U^k U^i_{,k} = \tau^{ik}_{,k},$$

(2.6) $$E^i_i = 0$$

ϱ being the density of the liquid.

We consider the slow steady laminar flow of an idealized elastico-viscous liquid characterized by the constitutive relations (2.1)—(2.2) in a right cone of semi-angle α with a source or sink at the vertex. Let u, v, w be the physical components of the velocity in the directions of the spherical polar coordinates r, θ, φ where r is the distance from the vertex, $\theta = 0$ represents the axis of the cone and φ is the azimuth. All the tensor quantities in the above equations are expressed in terms of their physical components referred to this choice of spherical polar coordinates. We introduce a stream function ψ defined by the equations:

(2.7) $$u = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}, \qquad v = \frac{-1}{r \sin\theta} \frac{\partial\psi}{\partial r},$$

so as to satisfy the continuity equation (2.6).

The boundary conditions of the problem are

(2.8) $$u = 0\,, \quad v = 0\,, \quad w = 0$$

on $\theta = \alpha$ for all r and the condition of constant mass flux around the vertex:

(2.9) $$\varrho \int_0^\alpha 2\,\pi\, r^2 \sin\theta u\, d\theta = Q$$

which is positive for a source and negative for a sink.

Following Langlois [2], we take for slow motion:

(2.10) $$Q = \xi\, q$$

and assume that any physical quantity X such as velocity, pressure, deviatoric stress, stream function etc., are expressed as

(2.11) $$X = \xi\, X^{(1)} + \xi^2\, X^{(2)} + \xi^3\, X^{(3)} + \cdots,$$

where ξ is a suitable non-dimensional parameter characteristic of the slow motion. We assume that ξ is sufficiently small so as to validate this expansion technique. This slowness parameter has later been related to another non-dimensional parameter S by the relation (5.2) which has a physical bearing over the problem. These expansions (2.11) for all the flow variables in question are substituted in the constitutive relations (2.1)—(2.2), the dynamical equations (2.5)—(2.6) and the boundary conditions (2.8)—(2.9) and the coefficients of ξ, ξ^2, ξ^3 ... are grouped together. This leads to the equations determining the flow in the lst, 2nd, 3rd, ... approximations respectively. At each stage of the analysis, the system of equations form a linear set making explicit use of the results of the previous stages. The equations in the first order are the same as those obtained for a creeping flow of a Newtonian liquid (i.e., the equations for the Stokes' flow in which the inertial terms are neglected).

3. First order solution

The deviatoric stresses in the first order are

(3.1) $$S_{ik}^{(1)} = 2\,\eta\, E_{ik}^{(1)}$$

for all i and k. The equations of motion now reduce to

(3.2), (3.3) $$0 = -\frac{\partial p^{(1)}}{\partial r} + \frac{\eta}{r^2 \sin\theta}\frac{\partial}{\partial\theta} E^2\,\psi^{(1)}\,, \qquad 0 = -\frac{1}{r}\frac{\partial p^{(1)}}{\partial\theta} - \frac{\eta}{r\sin\theta}\frac{\partial}{\partial r} E^2\,\psi^{(1)}$$

and

(3.4) $$0 = \nabla^2\, w^{(1)} - \frac{w^{(1)}}{r^2 \sin^2\theta}$$

where

(3.5), (3.6) $$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right), \qquad \nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

Elimating $p^{(1)}$ from (3.2)—(3.3), we obtain the equations for $\psi^{(1)}$

(3.7) $$E^4\,\psi^{(1)} = 0$$

with the boundary conditions:

(3.8a) $$\frac{\partial\psi^{(1)}}{\partial\theta} = 0\,, \qquad \frac{\partial\psi^{(1)}}{\partial r} = 0\,, \qquad w^{(1)} = 0 \qquad \text{on } \theta = \alpha \text{ for all } r\,,$$

(3.8b), (3.8c) $$\lim_{r\to\infty}\frac{\psi^{(1)}}{r^2} = 0\,, \qquad \psi^{(1)}(r,\alpha) - \psi^{(1)}(r,0) = \frac{q}{2\,\pi\,\varrho}.$$

The solution of the equations (3.4) and (3.7) can be obtained as

(3.9), (3.10) $$w^{(1)} = 0\,, \qquad \psi^{(1)} = -\frac{N}{3}(\cos\theta - \cos\alpha)^2\,(2\cos\alpha + \cos\theta)$$

where

(3.11) $$N = \frac{3\,q}{2\,\pi\,\varrho\,(1-\cos\alpha)^2\,(1+2\cos\alpha)}.$$

Using these results, we obtain the pressure from the equations (3.2)—(3.3):

(3.12) $$p^{(1)} = \frac{2\,N\,\eta}{3\,r^3}\,(1 - 3\cos^2\theta) + \text{constant}\,.$$

We thus realize, in the first approximation, a purely radial flow field:

(3.13) $$u^{(1)} = \frac{N f(\theta)}{r^2}, \quad v^{(1)} = 0, \quad w^{(1)} = 0,$$

where

(3.14) $$f(\theta) = \cos^2\theta - \cos^2\alpha .$$

4. Second order solution

In this order of approximation, we have the deviatoric stresses:

(4.1) $$\begin{cases} S_{rr}^{(2)} = 2\eta E_{rr}^{(2)} - \dfrac{\eta\lambda N^2}{r^6}(18 f^2 - f'^2) + \dfrac{\eta\mu N^2}{r^6}(12 f^2 + f'^2) + X^{(2)}, \\ S_{\theta\theta}^{(2)} = 2\eta E_{\theta\theta}^{(2)} - \dfrac{\eta(\lambda-\mu) N^2}{r^6} f'^2 + X^{(2)}, \qquad S_{\varphi\varphi}^{(2)} = 2\eta E_{\varphi\varphi}^{(2)} + X^{(2)}, \\ S_{r\theta}^{(2)} = 2\eta E_{r\theta}^{(2)} + \dfrac{2\eta(3\lambda-\mu) N^2}{r^6} f f', \qquad S_{\theta\varphi}^{(2)} = 2\eta E_{\theta\varphi}^{(2)}, \qquad S_{\varphi r}^{(2)} = 2\eta E_{\varphi r}^{(2)}, \end{cases}$$

where

(4.2) $$X^{(2)} = \frac{2\eta(3\lambda + 2\mu - 6\nu) N^2}{r^6} f^2 - \frac{\eta\nu N^2}{r^6} f'^2$$

and

(4.3) $$\lambda = \lambda_1 - \lambda_2, \quad \mu = \mu_1 - \mu_2, \quad \nu = \nu_1 - \nu_2 .$$

Also, the acceleration components are

(4.4) $$a_r^{(2)} = -\frac{2 N^2 f^2}{r^5}, \quad a_\theta^{(2)} = 0, \quad a_\varphi^{(2)} = 0 .$$

The equations of motion now reduce to

(4.5)
$$-\frac{2\varrho N^2 f^2}{r^5} = -\frac{\partial}{\partial r}[P^{(2)} - X^{(2)}] + \frac{\eta}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}E^2\psi^{(2)} + \frac{3\eta\lambda N^2}{r^7}[24 f^2 + f'^2 + 2 f f'' + 2 f f'\cot\theta] -$$
$$-\frac{\eta\mu N^2}{r^7}[48 f^2 + 7 f'^2 + 2 f f'' + 2 f f'\cot\theta],$$

(4.6) $$0 = -\frac{\partial}{\partial\theta}[P^{(2)} - X^{(2)}] - \frac{\eta}{\sin\theta}\frac{\partial}{\partial r}E^2\psi^{(2)} - \frac{\eta\lambda N^2}{r^6}[18 f + 2 f'' + f'\cot\theta] f' +$$
$$+\frac{\eta\mu N^2}{r^6}[6 f + 2 f'' + f'\cot\theta] f'$$

and

(4.7) $$0 = \nabla^2 w^{(2)} - \frac{w^{(2)}}{r^2\sin^2\theta}$$

in the directions of r, θ, φ respectively. Eliminating $P^{(2)}$ from (4.5)—(4.6), we obtain the equation for $\psi^{(2)}$:

(4.8) $$E^4\psi^{(2)} = \frac{8\varrho N^2}{\eta r^5}(\cos^2\theta - \cos^2\alpha)\sin^2\theta\cos\theta + \frac{16\mu N^2}{r^7}[(6\cos^2\alpha - 1) - 5\cos^2\theta]\sin^2\theta\cos\theta .$$

The boundary conditions for (4.7)—(4.8) are

(4.9) $$\begin{cases} w^{(2)}(r,\alpha) = 0, \qquad \left.\dfrac{\partial\psi^{(2)}}{\partial r}\right|_{\theta=\alpha} = 0, \qquad \left.\dfrac{\partial\psi^{(2)}}{\partial\theta}\right|_{\theta=\alpha} = 0, \\ \lim\limits_{r\to\infty}\dfrac{\psi^{(2)}(r,\theta)}{r^2} = 0, \qquad \psi^{(2)}(r,\alpha) - \psi^{(2)}(r,0) = 0 . \end{cases}$$

Further, we notice symmetry of the flow about the axis of the cone. The equations (4.7) and the first of (4.9) yield the solution

(4.10) $$w^{(2)} = 0$$

which shows that no flow is induced in the φ-direction by the non-linear terms. The solution of (4.8) can be expressed in the form:

$$\psi^{(2)} = \frac{8\,\varrho\, N^2}{\eta\, r} F_1(\theta) + \frac{16\,\mu\, N^2}{r^3} F_3(\theta), \tag{4.11}$$

where $F_1(\theta)$ and $F_3(\theta)$ satisfy the equations:

$$(L+2)(L+12)\, F_1(\theta) = (\cos^2\theta - \cos^2\alpha)\sin^2\theta\cos\theta\,, \tag{4.12}$$

$$(L+12)(L+30)\, F_3(\theta) = [(6\cos^2\alpha - 1) - 5\cos^2\theta]\sin^2\theta\cos\theta \tag{4.13}$$

with

$$L \equiv \frac{d^2}{d\theta^2} - \cot\theta\,\frac{d}{d\theta}\,. \tag{4.14}$$

The boundary conditions are

$$F_i(\theta) = 0\,,\qquad F_i'(\theta) = 0\,,\qquad (i = 1, 3)\quad \text{at}\quad \theta = 0 \text{ and } \alpha\,. \tag{4.15}$$

The equations (4.12)—(4.13) yield the solutions:

$$F_1(\theta) = \sin^2\theta\,(\cos\theta - \cos\alpha)^2\,(3 - 5\cos^2\alpha + 2\cos\theta\cos\alpha)/[288\cos\alpha]\,, \tag{4.16}$$

$$F_3(\theta) = \sin^2\theta\,(\cos\theta - \cos\alpha)^2\cdot g(\theta)/[288\cos\alpha\cdot h(\alpha)]\,, \tag{4.17}$$

where

$$\begin{aligned} g(\theta) = {} & 21\,(-15\cos^4\alpha + 58\cos^2\alpha + 5)\cos^2\theta + \\ & + 12\,(105\cos^4\alpha + 140\cos^2\alpha + 46)\cos\alpha\cos\theta + \\ & + 567\cos^4\alpha + 372\cos^2\alpha - 95\,, \end{aligned} \tag{4.18}$$

$$h(\alpha) = 105\cos^4\alpha - 42\cos^2\alpha + 19\,. \tag{4.19}$$

The pressure $P^{(2)}$ can be obtained by a straight-forward integration of the equations (4.5) to (4.6). The stream function $\psi^{(2)}$ given by (4.11) together with (4.16)—(4.19), characterizes the secondary flow superposed on the primary motion obtained in the first approximation. We thus notice that the non-linear terms in the equations (2.1), (2.2) and (2.5) induce a secondary flow only in the meridian planes. This flow is not prominent at large distances from the cone-vertex.

5.

The non-Newtonian effects are thus noticed when the terms of the order ξ^2 are retained in the equations of motion and the constitutive relations. Upto this order, the resultant stream function can be written as

$$\psi = \xi\,\psi^{(1)} + \xi^2\,\psi^{(2)} \tag{5.1}$$

in which $Q = \xi\, q$ may be employed in eliminating ξ to which no physical meaning is attached hitherto excepting that it is taken as sufficiently small so as to validate the perturbation technique. However, to facilitate the comparison of the results with those in the Newtonian case, a non-dimensional number S can be chosen which is purely dependent upon the kinematic viscosity η of the liquid, a characteristic (geometrical) length L and also upon the agency responsible for the flow, but independent of any of the elastico-viscous coefficients occuring in the constitutive relations. This number S may as well be preferred to ξ as a perturbation parameter throughout the above analysis. In our present case, we take the strength Q of the source or sink as this agency and construct

$$S = Q/8\,\pi\,\eta\, L = \xi\, q/8\,\pi\,\eta\, L \tag{5.2}$$

which represents the REYNOLDS' number characteristic of the slow motion. We further assume that $|S| \ll 1$ as a necessary condition for slow motion so as to validate the analysis. This would give

$$\frac{|Q|}{L} \ll 8\,\pi\,\eta \tag{5.3}$$

an approximate upper limit for the strength of the (weak) source or sink.

For analysing the result numerically, we introduce the following non-dimensional quantities r^*, K, Ψ defined by the scheme:

$$\begin{aligned} r = L\, r^*, \qquad K = \mu\,\eta/\varrho\, L^2\,, \\ \psi = Q\,\Psi/[2\,\pi\,\varrho\,(1 - \cos\alpha)^2\,(1 + 2\cos\alpha)] \end{aligned} \tag{5.4}$$

and express the non-Newtonian effects in terms of the parameter K.

The flow pattern can be realized from the stream function Ψ in the non-dimensional form:

$$(5.5)\quad \Psi = (\cos\theta - \cos\alpha)^2 \Bigg[-(\cos\theta + 2\cos\alpha) + \frac{S\sin^2\theta}{\cos\alpha\,(1-\cos\alpha)^2\,(1+2\cos\alpha)} \left\{ \frac{3 - 5\cos^2\alpha + 2\cos\alpha\cos\theta}{r^*} + \frac{2K\,g(\theta)}{h(\alpha)\,r^{*3}} \right\} \Bigg].$$

6. Observations

The stream function (5.5) shows that the flow at large distances is purely radial: divergent in the case of a source and convergent for a sink. As the vertex is approached, the intertial and non-Newtonian effect (contributed by the last two terms respectively in (5.5)), is to produce a velocity in the transverse direction. The stream lines (in the meridian plane) are therefore bent, from the radial pattern either, towards the wall or the axis of the cone. Since these terms are of order $1/r^*$ and $1/r^{*3}$, respectively, the non-Newtonian terms have a stronger influence near the vertex than the intertial terms in bending down the stream lines.

The total flux across any arbitrary surface around the vertex

$$(6.1)\quad 2\pi\varrho \int_0^\infty r^2 \sin\theta\, u\, d\theta = Q\,[\Psi(r^*, \alpha) - \Psi(r^*, 0)]/[(1 + 2\cos\alpha)(1-\cos\alpha)^2] = Q$$

which is independent of r^*. Hence upto the order of approximation taken, (5.5) represents the flow satisfying the kinematic condition of constant flux for a source/sink at the vertex.

The stream line $\Psi = 0$ is composed of two branches: the cone wall ($\theta = \alpha$) and the dividing stream line in the axial plane represented by the curve:

(6.2)

$$\frac{2Kg(\theta)}{h(\alpha)\,r^{*3}} + \frac{3 - 5\cos^2\alpha + 2\cos\alpha\cos\theta}{r^*} - \frac{\cos\alpha\,(1-\cos\alpha)^2\,(1+2\cos\alpha)\,(2\cos\alpha + \cos\theta)}{S\sin^2\theta} = 0\,.$$

This is the boundary of the region of circulatory flow near the cone wall. The intersections of this line with the cone wall can be obtained from the cubic:

(6.3)

$$\frac{2K}{h(\alpha)\,r^{*3}}(945\cos^6\alpha + 3465\cos^4\alpha + 1029\cos^2\alpha - 95) + \frac{3\sin^2\alpha}{r^*} - \frac{3\cot^2\alpha}{S}(1-\cos\alpha)^2(1+2\cos\alpha) = 0.$$

The stream line $\Psi = \Psi_a = -(1 + 2\cos\alpha)(1 - \cos\alpha)^2$ is composed of the cone-axis $\theta = 0$ and the dividing stream line in the axial plane whose equation is given by

(6.4)

$$\frac{2Kg(\theta)}{h(\alpha)\,r^{*3}} + \frac{3 - 5\cos^2\alpha + 2\cos\alpha\cos\theta}{r^*} + \frac{\cos\alpha(1-\cos\alpha)^2(1+2\cos\alpha)(1+\cos\theta+\cos^2\theta-3\cos^2\alpha)}{S\,(1+\sin\theta)\,(\cos\theta-\cos\alpha)^2} = 0$$

which gives the boundary of the region of circulatory flow around the axis and its intersections with it are obtained from the cubic:

$$(6.5)\quad \frac{4K}{h(\alpha)\,r^{*3}}(630\cos^5\alpha + 126\cos^4\alpha + 840\cos^3\alpha + 795\cos^2\alpha + 276\cos\alpha + 5) + \frac{(3+5\cos\alpha)(1-\cos\alpha)}{r^*} + \frac{3\cos\alpha\sin^2\alpha\,(1+2\cos\alpha)}{S} = 0\,.$$

By varying the values of K and S suitably, the equations (6.3) and (6.4) yield one or more real positive roots for any assigned value of α. We also note that these equations have two equal positive roots when K takes two particular values $K_1(\alpha, S)$ and $K_2(\alpha, S)$. These are found to be negative when $\alpha = 45°$, 60° (c.f: Table 1).

Table 1
Values of $K_1(\alpha, S)$, $K_2(\alpha, S)$

α	K_1/S^2	K_2/S^2
45°	−.01119	−.5031
60°	−.1292	−.01933

The flow pattern is illustrated by taking $\alpha = 60°$ for a source $S = +.1$ and sink $S = -.1$: figs. 1–10.

Flow Pattern for a source: ($S > 0$). The curve (6.2) meets the cone wall in no points, two different points or only one point according as $K < K_1(\alpha, S)$, $K_1(\alpha, S) < K < 0$

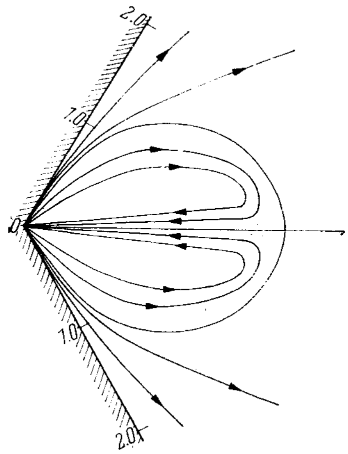

Fig. 1. Flow pattern when $K = -1.0$ with the source: $S = +0.1$

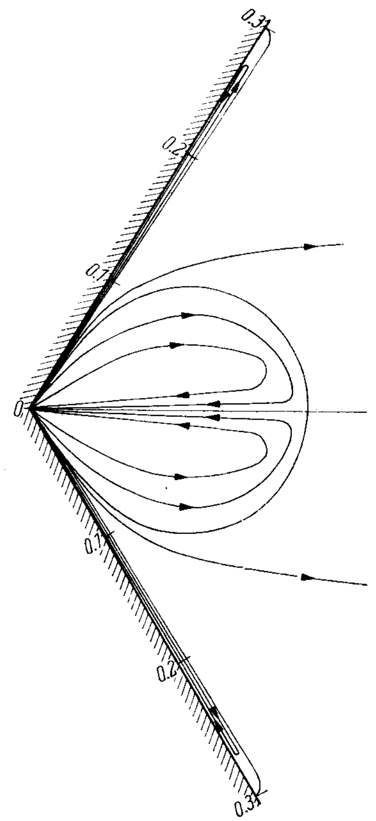

Fig 2. Flow pattern when $K = -0.001292$ with the source: $S = +0.1$

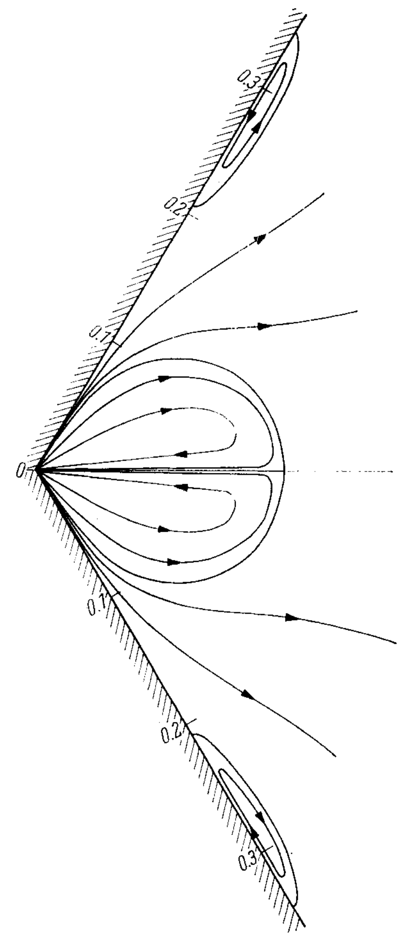

Fig. 3. Flow pattern when $K = -0.001$ with the source: $S = +0.1$

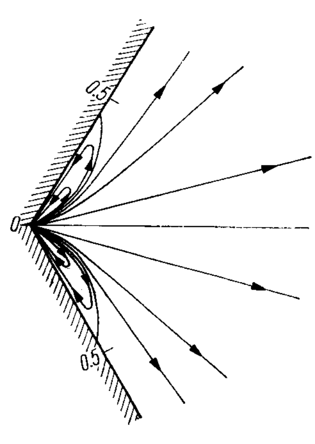

Fig. 4. Flow pattern when $K=0$ (Newtonian liquid) with the source: $S=+0.1$

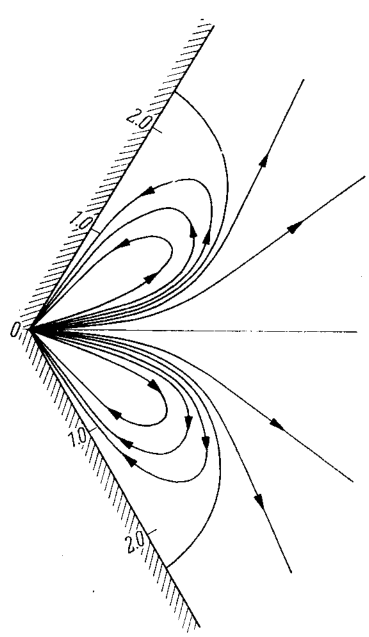

Fig. 5. Flow pattern when $K=+1.0$ with the source: $S=+0.1$

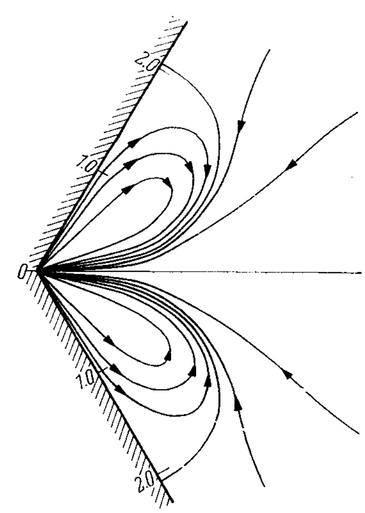

Fig. 6. Flow pattern when $K=$ 1.0 with the sink: $S=-0.1$

or $0 < K$ respectively. Also when $K = K_1(\alpha, S)$, the dividing stream line (6.2) hits the wall at only one point. Further, the curve (6.4) meets the axis at one point only for $K < 0$ and in no point when $K \geqq 0$. Hence in the case of a source, the flow pattern (in the axial plane) near the vertex consists of

(i) only one region of circulatory flow symmetrically placed around the axis when $K < K_1$: fig. 1,

(ii) one region of circulatory flow near the axis and two such regions near the wall when $K = K_1$: fig. 2,

(iii) one region of circulatory flow near the axis with the boundary passing through the vertex and two such regions near the wall, but with boundaries not passing through the vertex when $K_1 < K <$: 0 fig. 3,

(iv) two regions of circulatory flow near the wall with their boundaries passing through the vertex when $0 \leqq K$: figs. 4 and 5.

Flow pattern for a sink: $S < 0$. The curve (6.2) meets the cone wall at only one point for $K < 0$ and in no point $K \geqq 0$. Further, the curves (6.4) meets the axis in no points, two points or only one point according as $K < K_2(\alpha, S)$, $K_2(\alpha, S) < K < 0$, or $0 \leqq K$. Also when $K = K_2(\alpha, S)$, we get only one intersection. We thus have the flow pattern (in the axial plane) near the vertex in this case:

(v) two regions of circulatory flow near the wall when $K < K_2(\alpha, S)$: fig. 6,

(vi) one region of circulatory flow round the axis and two such regions near the wall when $K = K_2(\alpha, S)$: fig. 7,

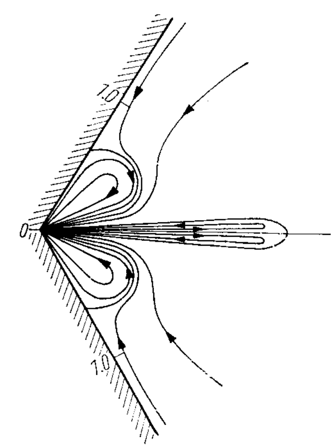

Fig. 7. Flow pattern when $K = -0.0001933$ with the sink: $S = -0.1$

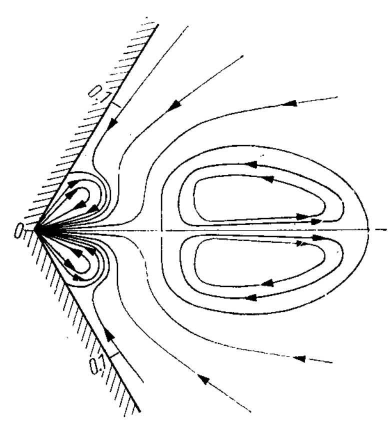

Fig. 8. Flow pattern when $K = -0.0001$ with the sink: $S = -0.1$

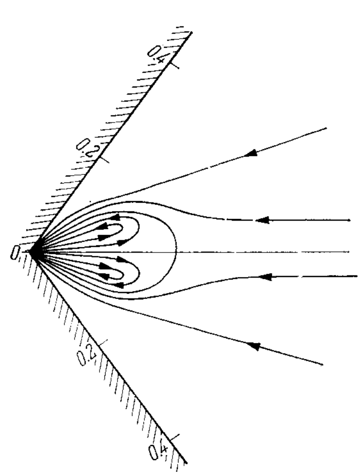

Fig. 9. Flow pattern when $K = 0$ (Newtonian liquid) with the sink: $S = -0.1$

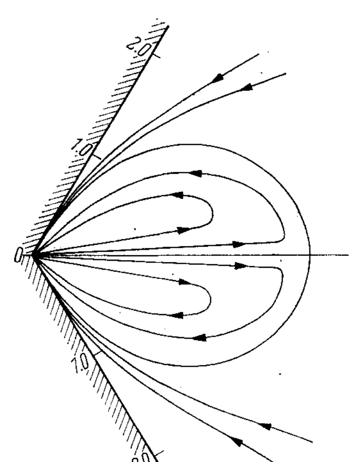

Fig. 10. Flow pattern when $K = +1.0$ with the sink: $S = -0.1$

(vii) one region of circulatory flow round the axis whose boundary does not pass through the vertex and two regions of circulatory flow near the wall with their boundaries passing through the vertex when $K_2(\alpha, S) < K < 0$: fig. 8,

(viii) only one region of circulatory flow round the axis when $0 \leqq K$: figs. 9 and 10.

The paths of the particles within these regions form closed loops which have a common node at the vertex of the cone whenever the boundary of the region passes through it.

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Anschrift: N. Ch. Pattabhi Ramacharyulu, Department of Mathematics, Regional Engineering College, Warangal — 4 (A.P.), India