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Steady flow of a viscous liquid in a porous elliptic tube

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Abstract. Flow of a viscous-liquid in a porous tube of elliptic cross-section is studied using the generalized momentum equation. As a particular case, flow of the liquid in a tube of a circular cross-section is obtained. It is observed that the classical Darcian effect is realized only in a core very near to the axis of the tube while the non-Darcian phenomenon is felt predominantly near the boundary of the tube.

Keywords. Porous; highly porous; permeability; Mathieu function.

1. Introduction

Flow of a viscous liquid in a porous medium is of importance in the study of percolation through soils in hydrology, petroleum industry and in agricultural engineering. Henry Darcy had observed while studying flow of water through sand filters that the flow rate of water is proportional to the difference in head of water across the filter. Subsequently, many experiments were conducted to study the flow of various fluids through different types of porous solids.

Brinkman (1947), proposed a generalized Darcy's law to study flow through highly porous media.

$$0 = -\nabla \mathbf{p} - (\mu/k)\mathbf{V} + \mu\nabla^2\mathbf{V}$$

where **V**, **p** represent velocity and pressure fields, μ is viscosity coefficient of fluid and k is the permeability of the medium. Later Tam (1969) derived analytically the same equation to study flow past spherical particles at low Reynold's number.

The generalized law is found useful in the study of flow in highly porous media such as pappus of dendelion and fibres. Yamamoto (1971, 1973) examined flow past porous bodies applying the generalized law.

We observe that the classical Darcian effect is seen only in a core very near to the axis of the tube and at the boundary the non-Darcian effect is felt predominantly.

2. Basic equations and solution

Brinkman (1947) proposed the basic equation for the slow steady flow of a viscous liquid in a porous medium as a generalization to the classical Darcy's law.

$$0 = -\nabla p + \mu\nabla^2\mathbf{v} - (\mu/k)\mathbf{v} \tag{1}$$

together with the equation of continuity

$$\operatorname{div} \mathbf{v}=0 \tag{2}$$

Let (x, y, z) be the rectangular co-ordinate system such that Z-axis lying along the length of the tube with impermeable boundary T. The velocity components in x, y, z directions are taken to be 0, 0, $w\ (x, y)$ respectively. The equation of continuity is satisfied with the choice of the velocity and the equation of motion becomes

$$0=-\frac{\partial p}{\partial z}+\mu \nabla^2 w-\frac{\mu}{k} w \tag{3}$$

with $w=0$ on T, the boundary of the tube. (4)

If the pressure gradient $-\partial p/\partial z=\rho a$, is a constant, we re-write eq. (3) as

$$\nabla^2 \phi-\frac{\phi}{k}=0 \tag{5}$$

where

$$\phi=w-\frac{ak}{\nu},\ \nu=\mu/\rho \tag{6}$$

and $$\phi=-\frac{ak}{\nu} \text{ on } T. \tag{7}$$

The transformation of eq. (5) into elliptic co-ordinates (ξ, η) can be done by taking $x+iy=C \cosh (\xi+i\eta)$ as in Mc Lachlan (1964, pp. 170–173), which gives

$$\frac{\partial^2 \phi}{\partial \xi^2}+\frac{\partial^2 \phi}{\partial \eta^2}-2q\,(\cosh 2\xi-\cos 2\eta)\phi=0 \tag{8}$$

where $q=(1/k)C^2/4$ and $C^2=a^2-b^2$ with $a=C \cosh \xi$ and $b=C \sinh \xi$ as the half of the major and minor axes of the elliptic cross-section T.

Let $\phi(\xi, \eta)=f(\xi)\, g(\eta)$ in the above equation then we get,

$$\frac{d^2 f}{d\xi^2}-[\lambda^{(2n)}+2q \cosh 2\xi] f=0 \tag{9}$$

$$\frac{d^2 g}{d\eta^2}+[\lambda^{(2n)}+2q \cos 2\eta]\, g=0. \tag{10}$$

Here $\lambda^{(2n)}$ is a separation constant. The flow is symmetric about the axes of the ellipse and is periodic with period π in η. The solutions of eqs (9) and (10) are given by Mc Lachlan (1964, pp. 21, 27)

as $g=Ce_{2n}(\eta, -q)$

and $f=Ce_{2n}(\xi, -q)$.

Hence ϕ is given by

$$\phi=\sum_{n=0}^{\infty} P_{2n} Ce_{2n}(\xi, -q)\, Ce_{2n}(\eta, -q) \tag{11}$$

where $Ce_{2n}(\eta, -q)=(-1)^n \sum_{n=0} (-1)^r A_{2r}^{(2n)} \cos 2r\eta.$ (12)

$$Ce_{2n}(\xi, -q)=(-1)^n \sum_{n=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cosh 2\xi r \tag{13}$$

and the coefficients $A_{2r}^{(2n)}$ are functions of q. The constants P_{2n} are determined using the boundary condition on $\xi=\xi_0$.

$$-\alpha \frac{k}{\nu} = \Sigma P_{2n}\, Ce_{2n}(\xi_0, -q)\, Ce_{2n}(\eta, -q). \tag{14}$$

Multiplying both sides of eq. (14) by $Ce_{2n}(\eta, -q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation of Mc Lachlan (1964) we get

$$P_{2n} = \frac{\frac{\alpha k}{\nu} (-1)^{n+1}\, 2\pi\, A_0^{2n}}{Ce_{2n}^2(\xi_0, -q)\, I_{2n}} \tag{15}$$

where

$$I_{2n} = \int_0^{2\pi} Ce_{2n}^2 (\eta, -q)\, d\eta \tag{16}$$

Hence $w = \phi + \dfrac{\alpha k}{\nu}$

$$= \frac{\alpha k}{\nu}\left[1+\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 2\pi\, A_0^{(2n)}}{Ce_{2n}(\xi, -q) I_{2n}} Ce_{2n}(\xi, -q)\, Ce_{2n}(\eta, q)\right] \tag{17}$$

Case I

when $k\to\infty$, $q\to 0$ (Mc Lachlan 1964 p. 15)

$$Ce_0(\eta, -q) 1+(\tfrac{1}{2})q \cos 2\eta \tag{18}$$

$$Ce_{2n}(\eta, -q) \cos 2\eta+\{(-\tfrac{1}{4})(1/12)\cos 4\eta)\}\, q \tag{19}$$

$$Ce_{2n}(\xi, -q) = Ce_{2n}(i\xi, -q) \tag{20}$$

$$A_0^{(0)} = 1,\ A_0^{(2)} = 0\ A_0^{(2n)} = 0 \text{ for all } n > 2 \tag{21}$$

$$P_0 = \frac{aK}{\nu}(1-(\tfrac{1}{2})q \cosh 2\xi_0) \tag{22}$$

$$P_2 = \frac{aK}{\nu}\left(\frac{q}{2\cosh 2\xi_0}\right), p_{2n} = 0: \ n > 2 \tag{23}$$

and velocity w becomes

$$w = \frac{a}{2\nu}\,\frac{a^2b^2}{(a^2+b^2)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{24}$$

which is the same as the velocity of a viscous liquid in an elliptic tube when no resistance is offered by the medium.

Case II

when k is very small q is very large

$$A_0^{(0)} = 1, \ A_0^{(2n)} \simeq 0 \text{ for } n > 1 \tag{25}$$

and from Mc Lachlan (1964 p. 230)

$$Ce_{2n}(\xi, -q) \simeq \frac{(-1)^n C_{2n}}{2^{2n-\frac{1}{2}}(i \sinh)^{\frac{1}{2}}} \cosh\Big[2 q^{\frac{1}{2}} \cosh \xi$$

$$-(4n+1)\tanh^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{i\xi}{2}\right)\right\}\Big] \tag{26}$$

$$C_{2n} = \frac{(-1)^n\, 2^{2n-\frac{1}{2}}}{A_0^{(2n)}\,(\pi/\sqrt{q})^{\frac{1}{2}}}\, Ce_{2n}(0)\, Ce_{2n}(\pi/2);$$

$$C_{2n} \simeq 0 \text{ for } n > 1 \text{ for large } |q| \tag{27}$$

$$\phi \simeq P_0\, Ce_0(\xi, -q)\, Ce_0(\eta, -q)$$

$$\simeq -\frac{ak}{\nu}\,\frac{\cosh(\frac{1}{2}\xi_0)}{\cosh(\frac{1}{2}\xi)} \exp(-2/q^{\frac{1}{2}})(\cosh \xi_0 - \cosh \xi). \quad Ce_0(\eta, -q). \tag{28}$$

Hence w is given by (Mc Lachlan 1964, p. 388)

$$w = \frac{ak}{\nu}\left[1 - \frac{\cosh \frac{1}{2}\xi_0}{\cosh \frac{1}{2}\xi_0}\, Ce_0(\eta, -q) \exp(-d/k^{1/2})\right] \tag{29}$$

where $d = c(\cosh \xi_0 - \cosh \xi)$

Case III: When the tube is of circular cross-section

In the case of a circle $c \to 0$, $\xi_0 \to \infty$ such that $a = \cosh \xi_0$.
The ellipse turns to a circle of radius a and also (Mc Lachlan 1964 p. 367).

$$Ce_0(\eta, -q) \to 2^{-1/2} \tag{30}$$

$$Ce_{2n}(\eta - q) \to \cos 2n\,\eta,\ n > 1 \tag{31}$$

$$Ce_{2n}(\xi, -q) \to P'_{2n}.I_{2n}(K_1\,\eta):\ n > 0 \tag{32}$$

$$Ce_{2n}(\xi_0 - q) \to P'_{2n}\,I_{2n}(K_1\,a):\ n > 0 \tag{33}$$

$$A_0^{(0)} \to 2^{-1/2} \tag{34}$$

$$A_0^{(2n)} \to 0 : n > 1 \tag{35}$$

and $I_{2n} \to \pi : n > 0$ (36)

where $K_1 = k^{-1/2}$, $r = \cosh \xi = (x^2 + y^2)^{1/2}$ (37)

with the help of these limiting values we get

$$w = \frac{\alpha k}{\nu}\left[1 - \frac{I_0\,(r/k^{1/2})}{I_0\,(a/k^{1/2})}\right] \tag{38}$$

which is the same as that obtained by Pattabhiramacharyulu (1976).

3. Conclusion

We notice that the second term in eq. (36) damps out as d, the distance from the wall of the tube increases while the first term remains almost constant $\alpha k/\nu$, which is the same as that obtained under the classical Darcy's law (Mushakat 1937).

$$0 = -\nabla \mathbf{p} - \frac{\mu}{k}\mathbf{V}.$$

Hence the non-Darcian effect is seen to exist predominantly near the boundary of the tube and the classical Darcian effect is realized only in a core near to the axis of the tube.

A similar phenomenon is observed by Sexl (1930) in the study of pulsating flow of a classical viscous fluid in a circular tube and by Pattabhiramacharyulu (1976) for the flow in porous circular tube.

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