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The Slow Stationary Flow of a Micropolar Liquid Past a Sphere

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SUMMARY

The paper is concerned with the slow stationary flow of a micropolar incompressible fluid past a sphere. Adopting the Stokesian approach of neglecting the inertial terms in the momentum equation and the bilinear terms in the balance of first stress moments, the equations are integrated and the flow parameters determined. The drag on the sphere is seen to be more in the present case than that in the case of non-polar fluids. It is found that in spite of the couple stress in the fluid, there is no resultant action by it on the sphere. Numerical work shows that the streamlines in the polar case have greater deflection towards the sphere than in the non-polar (or classical) case.

1. Introduction

The theory of micro-polar fluids introduced by A. C. Eringen [1] deals with a class of fluids which respond to certain microscopic effects arising from the presence of micro-structure and are influenced by the spin inertia. A simplified case of this theory has also been discussed recently by A. C. Eringen [2]. An interesting feature of this class of fluids is the sustainance of couple stresses. Some anisotropic fluids such as animal blood and liquid crystals made up of bar-like or dumb-bell shaped molecules seem to fall within the scope of this theory. Apart from the usual quantities ρ (mass density), $\boldsymbol{q}$ (fluid velocity vector) and t_{ij} (stress tensor), we have in the present theory the following additional quantities: micro-stress average (s_{ij}) and the first stress moment λ_{ijm}. In the theory of micropolar fluids, the constitutive equation is linear, the micro-inertia moments have an isotropic distribution, the gyration tensor v_{ij} is antisymmetric and the first stress moment λ_{ijm} is antisymmetric in the last two indices. Fluid particles contained in a small volume element have besides the usual rigid rotation, also rotation about the centroid of the volume element in an average sense, and the vector $\boldsymbol{v}$ defined by the anti-symmetric tensor v_{ij} describes this rotation. There is no micro-stretch of the particles in this theory, since the tensor v_{ij} is antisymmetric. The field equations are then presentable in terms of the fluid velocity vector $\boldsymbol{q}$ and the micro-rotation vector $\boldsymbol{v}$.

In this paper we examine the slow stationary flow of an incompressible micro-polar fluid past a sphere. As is usual with the classical investigations of the problem, as a first step the inertial terms of the momentum equation and the bilinear terms in the balance of first stress moments are neglected and the flow is obtained over the space outside the body under the above approximation. Explicit calculations are given for the velocity and micro-rotation and the stresses as well as couple stresses. The drag on the body is determined. It is seen that in the present theory the drag is more than in the classical case. We find that the body as a whole does not experience any couple.

2. Basic Equations

The field equations of the micro-polar fluid dynamics are [1]

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{q}) = 0 \tag{1}$$

$$\rho \frac{D\boldsymbol{q}}{Dt} = (\lambda_1 + 2\mu + k) \operatorname{grad} \operatorname{div} \boldsymbol{q} - (\mu + k) \operatorname{curl} \operatorname{curl} \boldsymbol{q} + k \operatorname{curl} \boldsymbol{v} - \operatorname{grad} p + \rho \boldsymbol{f} \tag{2}$$

$$\rho j \frac{Dv}{Dt} = (\alpha+\beta+\gamma) \text{ grad div } \boldsymbol{v} - \gamma \text{ curl curl } \boldsymbol{v} + k \text{ curl } \boldsymbol{q} - 2k\boldsymbol{v} + \rho \boldsymbol{l} \tag{3}$$

in which $\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{f}, \boldsymbol{l}$ are respectively the velocity, micro-rotation, body force and body couple vector. The constants ρ and j are the density and gyration parameter, while (λ_1, μ, k) and (α, β, γ) are material constants, which are governed by certain inequalities. The stress tensor t_{ij} and the couple stress tensor $m_{ij} = -\varepsilon_{jpq}\lambda_i^{pq}$ are given by

$$t_{ij} = (-p + \lambda_1 \text{ div } \boldsymbol{q})\delta_{ij} + (2\mu+k)d_{ij} + k\varepsilon_{ijm}(\omega^m - v^m) \tag{4}$$

and

$$m_{ij} = (\alpha \text{ div } \boldsymbol{v})\delta_{ij} + \beta v_{i,j} + \gamma v_{j,i} \tag{5}$$

where v_i and $2\omega_i$ are the components of the micro-rotation vector and vorticity vector respectively, d_{ij} denote the rate of strain components and comma denotes covariant differentiation.

3. Slow Stationary Flow past a Sphere

Let $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$ be the unit base vectors of the spherical polar system r, θ, ϕ. The flow is past the sphere $r=a$ and is a uniform stream at infinity. The flow of the fluid is in the meridian plane and all physical quantities are independent of ϕ. We choose the velocity vector in the form

$$\boldsymbol{q} = u(r,\theta)\boldsymbol{e}_r + v(r,\theta)\boldsymbol{e}_\theta \tag{6}$$

and in view of the incompressibility condition div $\boldsymbol{q}=0$, we have

$$u(r,\theta) = \frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}\,;\; v(r,\theta) = \frac{-1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \tag{7}$$

where Ψ is the stream function. Since the vorticity has its only component perpendicular to the meridian plane, we take the micro-rotation vector $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = v(r,\theta)\boldsymbol{e}_\phi\,. \tag{8}$$

It is then obvious that

$$\text{div } \boldsymbol{v} = 0\,. \tag{9}$$

Under these conditions, the equations (2) and (3) can be put in the form

$$-(\mu+k) \text{ curl curl } \boldsymbol{q} + k \text{ curl } \boldsymbol{v} - \text{grad } p = 0\,, \tag{10}$$

$$-\gamma \text{ curl curl } \boldsymbol{v} + k \text{ curl } \boldsymbol{q} - 2k\boldsymbol{v} = 0\,. \tag{11}$$

From these we see that

$$\boldsymbol{v} = \tfrac{1}{2} \text{ curl } \boldsymbol{q} - \frac{\gamma(\mu+k)}{2k^2} \text{ curl curl curl } \boldsymbol{q} \tag{12}$$

and pressure is to be determined from the equation

$$\text{grad } p = -\tfrac{1}{2}(2\mu+k) \text{ curl curl } \boldsymbol{q} - \frac{\gamma(\mu+k)}{2k} \text{ curl curl curl curl } \boldsymbol{q}\,. \tag{13}$$

The velocity vector satisfies the equation

$$\text{curl curl curl curl curl } \boldsymbol{q} + \frac{\lambda^2}{a^2} \text{ curl curl curl } \boldsymbol{q} = 0\,, \tag{14}$$

where

$$\frac{\lambda^2}{a^2} = \frac{k(2\mu+k)}{\gamma(\mu+k)}\,. \tag{15}$$

These equations can be rewritten in terms of the stream function Ψ. If

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} \tag{16}$$

we see that

$$E^4\left(E^2\Psi - \frac{\lambda^2}{a^2}\Psi\right) = 0 \tag{17}$$

and

$$\nu(r,\theta) = \frac{-1}{2r\sin\theta}\left\{E^2\Psi + \frac{\gamma(\mu+k)}{k^2}E^4\Psi\right\}. \tag{18}$$

The solution of the problem consists in solving the equations (17) and (18) subject to the following conditions:

(i) adherence of the fluid to the solid boundary, which means that in this problem we have u, v, $\nu = 0$ on $r = a$ conforming to the conditions of non-slip and non-spin on the boundary.

(ii) at infinity the flow approaches a uniform stream of speed U, parallel to the axis of symmetry.

To have the uniform stream at infinity it is essential that

$$\Psi \sim \tfrac{1}{2}Ur^2\sin^2\theta \tag{19}$$

for large values of r. We therefore seek the solution for Ψ in the form

$$\Psi = f(r)\sin^2\theta\,. \tag{20}$$

The function $f(r)$ is determined by

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2\left(\frac{d^2}{dr^2} - \frac{2}{r^2} - \frac{\lambda^2}{a^2}\right)f(r) = 0 \tag{21}$$

and the solution of this is

$$f(r) = Ar^4 + Br^2 + Cr + \frac{D}{r} + \sqrt{r}\left\{EI_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right) + FK_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right)\right\} \tag{22}$$

involving six constants and the functions $I_{\frac{3}{2}}(\ldots)$ and $K_{\frac{3}{2}}(\ldots)$ denote the modified Bessel functions. For the flow to be regular at infinity and equal to the uniform stream as indicated in (10), we must discard the constants A and E in (22) and choose

$$B = \tfrac{1}{2}U\,. \tag{23}$$

The remaining three constants in the solution C, D, F are determined by the adherence conditions, *viz.*,

$$u(a,\theta) = 0;\; v(a,\theta) = 0 \quad \text{and} \quad \nu(a,\theta) = 0\,. \tag{24}$$

The velocity and the micro-rotation are then found to be

$$u(r,\theta) = \left[U + \frac{2C}{r} + \frac{2D}{r^3} + \frac{2F}{r^{\frac{3}{2}}}K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right)\right]\cos\theta \tag{25}$$

$$v(r,\theta) = \left[-U - \frac{C}{r} + \frac{D}{r^3} + \frac{F}{r^{\frac{3}{2}}}\left\{K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right) + \frac{\lambda r}{a}K_{\frac{1}{2}}\left(\frac{\lambda r}{a}\right)\right\}\right]\sin\theta \tag{26}$$

$$\nu(r,\theta) = \left[\frac{C}{r^2} - F\frac{\mu+k}{k}\frac{\lambda^2}{a^2}\frac{1}{\sqrt{r}}K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right)\right]\sin\theta \tag{27}$$

where

$$C = -\frac{3aU(\lambda+1)(\mu+k)}{2[2(\mu+k)\lambda + (2\mu+k)]} \tag{28}$$

$$D = \frac{a^3 U\left[\mu(\lambda+1)+k\left(\lambda+2+\frac{3}{\lambda}+\frac{3}{\lambda^2}\right)\right]}{2[2(\mu+k)\lambda+(2\mu+k)]} \tag{29}$$

$$F = -\frac{3Uka^{\frac{3}{2}}}{2\lambda[2(\mu+k)\lambda+(2\mu+k)]}. \tag{30}$$

The pressure is found from the equation (13). We see that

$$\operatorname{grad} p = -(2\mu+k)\frac{C}{r^3}[2\cos\theta \boldsymbol{e}_r+\sin\theta \boldsymbol{e}_\theta] \tag{31}$$

and hence

$$p = (2\mu+k)\frac{C}{r^3}\cos\theta+p_\infty. \tag{32}$$

4. Stress Tensor

The stress tensor t_{ij} is defined in (4). Taking the suffixes r, θ, ϕ corresponding to indices 1, 2, 3 we have the physical components of the strain velocity given by

$$d_{rr} = -2d_{\theta\theta} = -2d_{\phi\phi} = -2\left[\frac{C}{r^2}+\frac{3D}{r^4}+F\frac{\lambda}{a}\frac{1}{r^{\frac{3}{2}}}K_{\frac{5}{2}}\left(\frac{\lambda r}{a}\right)\right]\cos\theta \tag{33}$$

$$d_{r\theta} = -\left[\frac{3D}{r^4}+\frac{F\lambda}{ar^{\frac{3}{2}}}K_{\frac{5}{2}}\left(\frac{\lambda r}{a}\right)\right]\sin\theta \tag{34}$$

and

$$d_{r\phi} = d_{\theta\phi} = 0. \tag{35}$$

Hence the stress tensor has the physical components

$$t_{rr} = -p-2(2\mu+k)\left\{\frac{C}{r^2}+\frac{3D}{r^4}+F\frac{\lambda}{a}\frac{1}{r^{\frac{3}{2}}}K_{\frac{5}{2}}\left(\frac{\lambda r}{a}\right)\right\}\cos\theta \tag{36}$$

$$t_{\theta\theta} = t_{\phi\phi} = -p+(2\mu+k)\left\{\frac{C}{r^2}+\frac{3D}{r^4}+F\frac{\lambda}{a}\frac{1}{r^{\frac{3}{2}}}K_{\frac{5}{2}}\left(\frac{\lambda r}{a}\right)\right\}\cos\theta \tag{37}$$

$$t_{r\theta} = (2\mu+k)\left\{-\frac{3D}{r^4}-F\frac{\lambda}{a}\frac{1}{r^{\frac{3}{2}}}K_{\frac{5}{2}}\left(\frac{\lambda r}{a}\right)\right\}\sin\theta \tag{38}$$

$$t_{\theta r} = (2\mu+k)\left[-\frac{3D}{r^4}-\frac{F}{r^{\frac{3}{2}}}\frac{\lambda}{a}\left\{K_{\frac{5}{2}}\left(\frac{\lambda r}{a}\right)+\frac{\lambda r}{a}K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right)\right\}\right]\sin\theta \tag{39}$$

$$t_{r\phi} = t_{\phi r} = t_{\theta\phi} = t_{\phi\theta} = 0. \tag{40}$$

The stress vector on the surface $r=a$ is

$$t_{rr}\boldsymbol{e}_r+t_{r\theta}\boldsymbol{e}_\theta+t_{r\phi}\boldsymbol{e}_\phi, \tag{41}$$

and it is found to be

$$-(2\mu+k)\frac{C}{a^2}(\cos\theta \boldsymbol{e}_r-\sin\theta \boldsymbol{e}_\theta). \tag{42}$$

Thus we see that the stress vector is everywhere parallel to the axis of the symmetry. The drag on the sphere is now found to be

$$\frac{6\pi aU(\lambda+1)(\mu+k)(2\mu+k)}{2(\mu+k)\lambda+2\mu+k}. \tag{43}$$

We recover the expression for the drag in Stokes solution by taking the limit as $k \to 0$ in (43).

Let D_λ denote the drag on the sphere in the micropolar liquid and let $D_0(=6\pi a U\mu)$ be the drag in the non-polar liquid.

We have

$$D_\lambda / D_0 = \frac{(1+\lambda)(\mu+k)(2\mu+k)}{\{2(\mu+k)\lambda+(2\mu+k)\}\mu} . \tag{44}$$

Since the parameters μ, k, λ are all >0, we easily see that

$$1 + \frac{k}{2\mu} < D_\lambda/D_0 < 1 + k/\mu . \tag{45}$$

Thus the drag on the sphere is greater in the micropolar liquid than in the ordinary non-polar liquid.

5. Couple Stress

The couple stress m_{ij} is given by (5). In the present case the physical components of the tensor are

$$m_{rr} = m_{\theta\theta} = m_{\phi\phi} = m_{r\theta} = m_{\theta r} = 0 , \tag{46}$$

$$m_{r\phi} = \left[(\beta+2\gamma) \left\{ -\frac{C}{r^3} + F \frac{\mu+k}{k} \frac{\lambda^2}{a^2} \frac{1}{r^{\frac{3}{2}}} K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right) \right\} + \gamma F \frac{\mu+k}{k} \frac{\lambda^3}{a^3} \frac{1}{\sqrt{r}} K_{\frac{1}{2}}\left(\frac{\lambda r}{a}\right) \right] \sin\theta \tag{47}$$

$$m_{\phi r} = \left[(2\beta+\gamma) \left\{ -\frac{C}{r^3} + F \frac{\mu+k}{k} \frac{\lambda^2}{a^2} \frac{1}{r^{\frac{3}{2}}} K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right) \right\} + \beta F \frac{\mu+k}{k} \frac{\lambda^3}{a^3} \frac{1}{\sqrt{r}} K_{\frac{1}{2}}\left(\frac{\lambda r}{a}\right) \right] \sin\theta \tag{48}$$

$$m_{\theta\phi} = -m_{\phi\theta} = \frac{(\gamma-\beta)}{r} \left[\frac{C}{r^2} - F \frac{\mu+k}{k} \frac{\lambda^2}{a^2} \frac{1}{\sqrt{r}} K_{\frac{3}{2}}\left(\frac{\lambda r}{a}\right) \right] \cos\theta . \tag{49}$$

The couple stress vector on the sphere $r=a$ is hence seen to be

$$[m_{r\phi}]_{(r=a)} e_\phi \tag{50}$$

and this reduces to

$$-\frac{3Uk(2\mu+k)}{2[2(\mu+k)\lambda+(2\mu+k)]} \sin\theta e_\phi . \tag{51}$$

The resultant couple vector on the sphere is therefore

$$\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} [m_{r\phi}]_{r=a}\, a^2 \sin\theta e_\phi \, d\theta \, d\phi \tag{52}$$

and this is seen to be zero. Thus we find that there is no resultant action by the couple stress on the body as a whole and it experiences only a drag even as in the case of non-polar viscous liquids.

Figures 1 to 6 indicate the stream lines, micro-rotation, shear stress difference and couple stress components, for the values $k/\mu=5$, $\lambda=1$ and $\beta/\gamma=0.5$. It is seen that the stream lines in the polar case have greater deflection towards the sphere than in the non-polar (or classical) case.

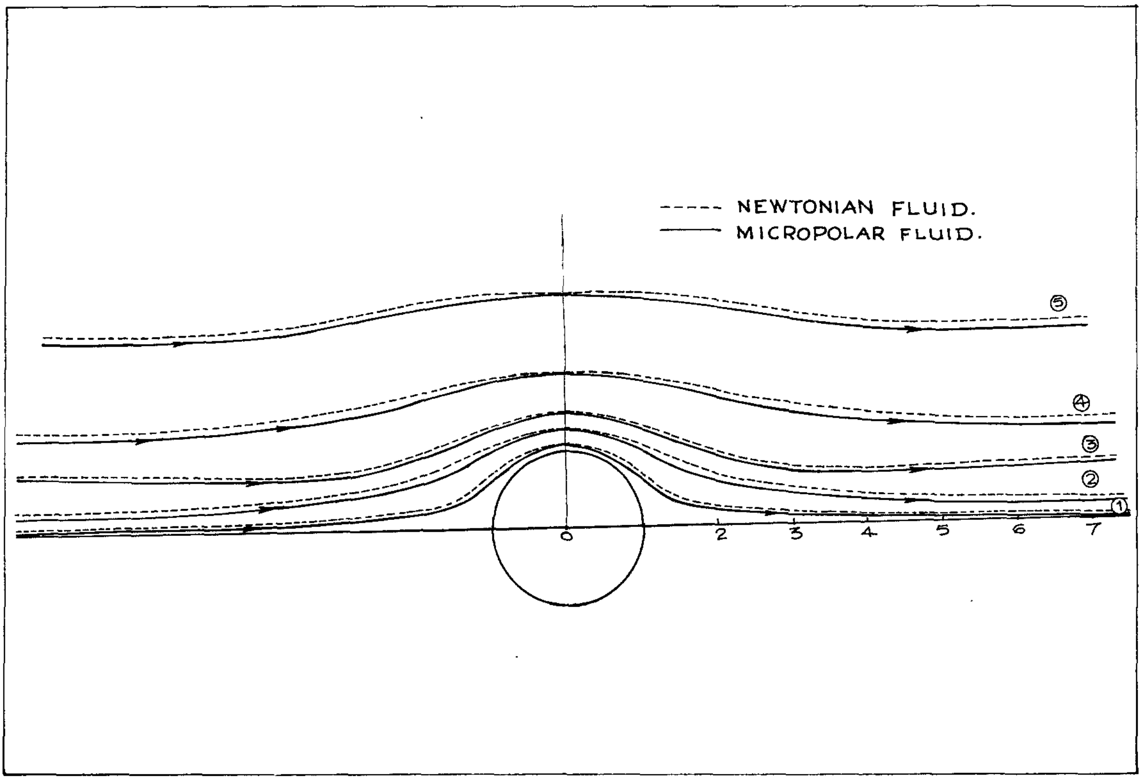

Fig. 1. Stream lines.

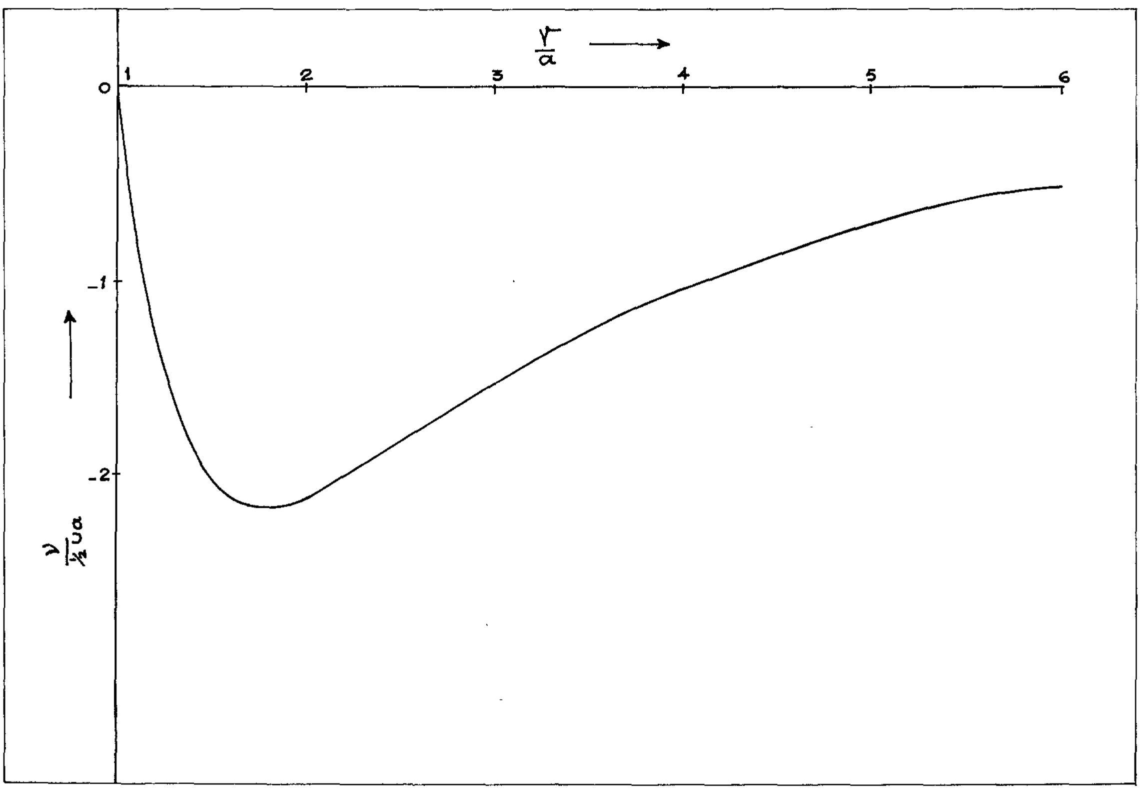

Fig. 2. Micro-rotation.

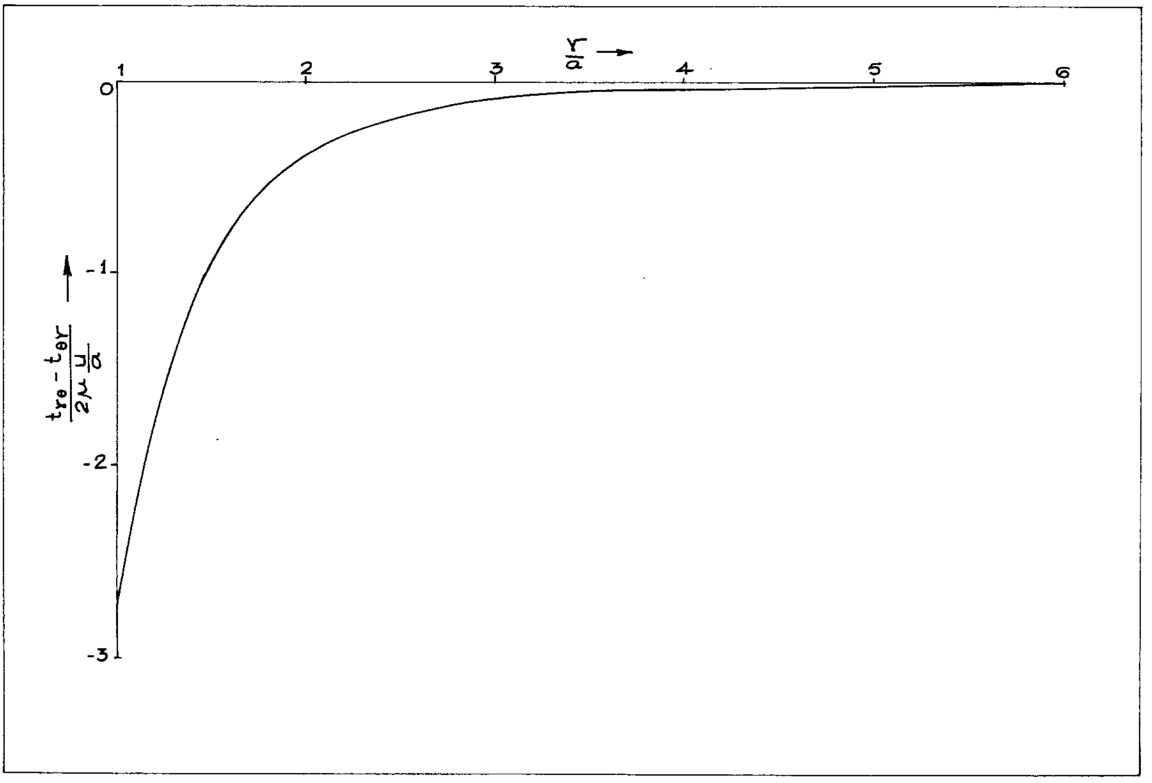

Fig. 3. Stress difference $t_{r\theta}-t_{\theta r}$.

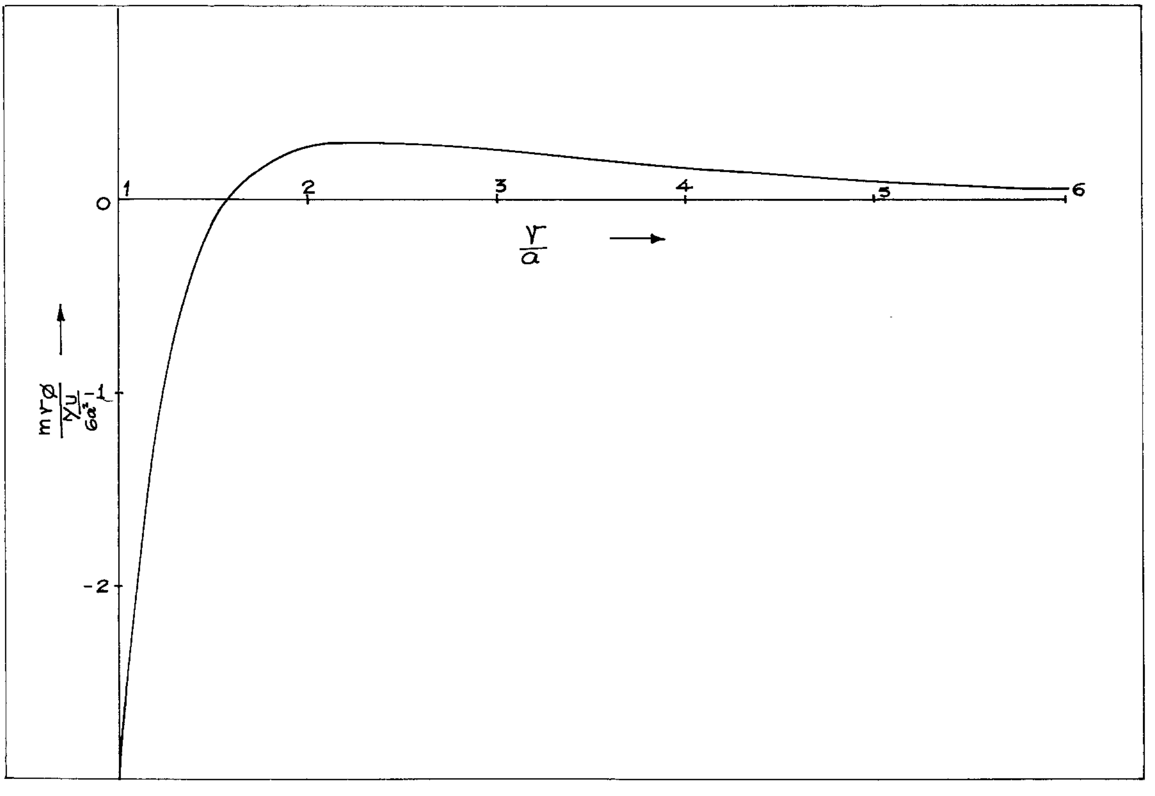

Fig. 4. Couple stress $m_{r\phi}$.

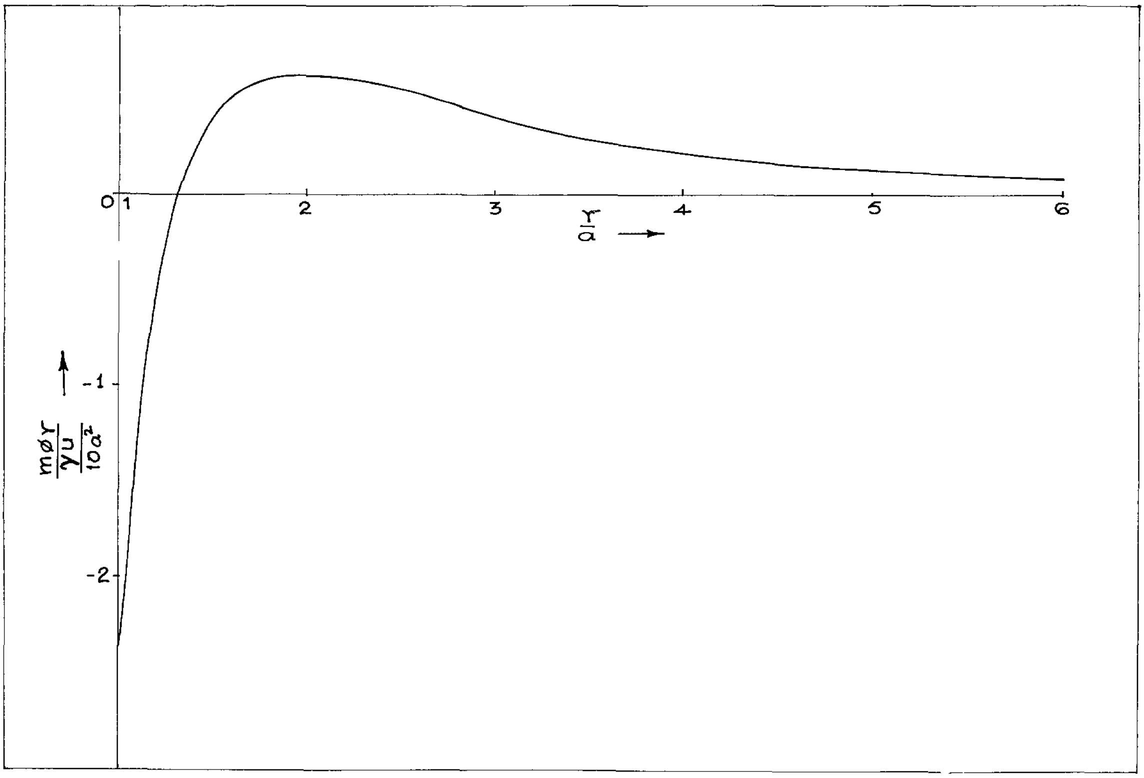

Fig. 5. Couple stress $m_{\phi r}$.

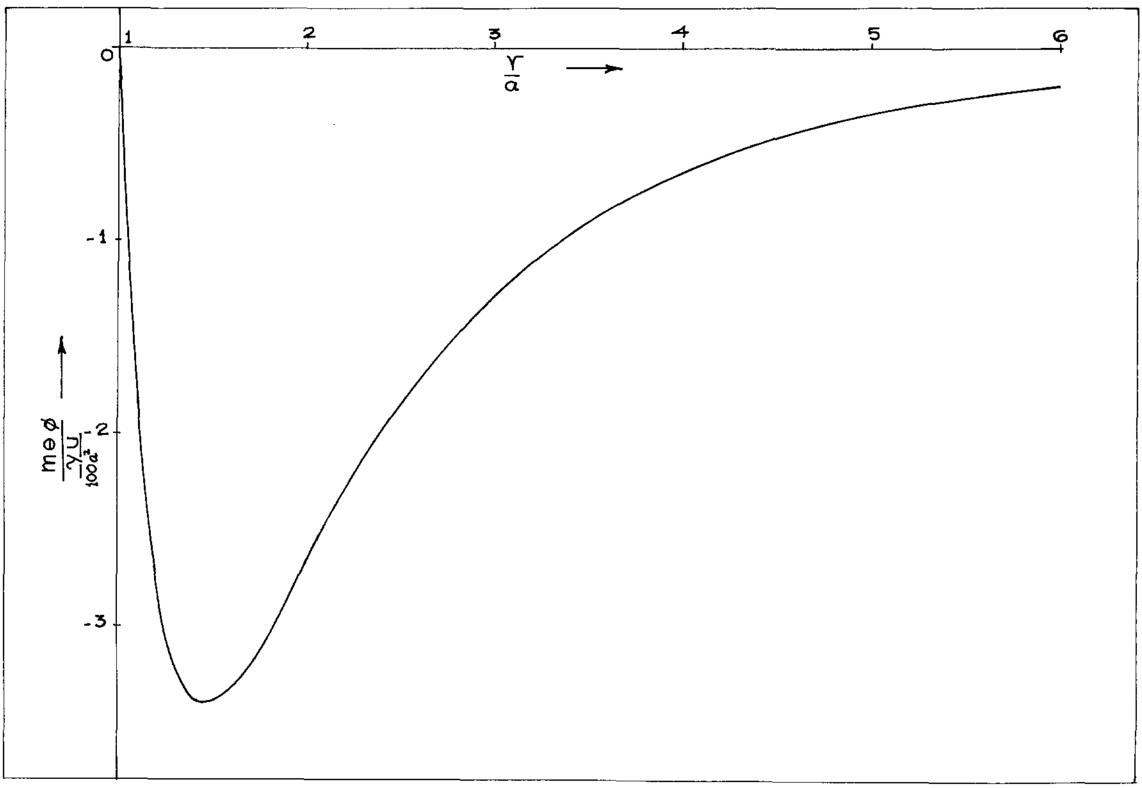

Fig. 6. Couple stress $m_{\theta\phi}$.

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