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LAMINAR FORCED CONVECTION IN ELLIPTIC DUCTS

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Abstract

The problem of laminar forced convection heat transfer in short elliptical ducts with (i) uniform wall temperature and (ii) prescribed wall heat flux is examined in detail with the well known Lévêque theory of linear velocity profile near the wall. Moreover, consideration is given to the variation of the slope of the linear velocity profile with the position on the duct wall. A correction factor for the temperature dependent viscosity is included. Expressions for the local and average Nusselt numbers and wall temperatures are obtained. For the case of constant heat flux the Nusselt numbers are higher than for constant wall temperature.

The results corresponding to the classical Graetz and Purday problems are deduced as special cases.

Nomenclature

a, b	semiaxes of ellipse, $b < a$
A_{h}	area of heat transfer surface
$c = ae$	distance between focus and centre of the ellipse
C	heat capacity of the fluid
D_{e}	equivalent diameter, (18)
e	eccentricity of the elliptical duct
$E(e)$	complete elliptic integral
g	Laplace transform of T
g_{w}	Laplace transform of T_{w}
G_z^0	Graetz number (local), $Re\ Pr\ D_{\mathrm{e}}/z$
$\overline{Gz}$	Graetz number (average), $Re\ Pr\ D_{\mathrm{e}}/Z$
$h_{i\theta}^0$	local heat transfer coefficient
$J_n(x)$	Bessel function of order n
K	thermal conductivity of the fluid
$\mathscr{L}[X]$	Laplace transform of X
N_u^0	local Nusselt number, $h_{i\theta}^0 D_{\mathrm{e}}/K$

$\overline{N_u^0}$	perimeter average Nusselt number
$\overline{Nu}$	overall average Nusselt number
Nu_W	wall Nusselt number
Nu_∞	Nusselt number at large distance from the inlet
p	Laplace transform parameter
Pr	Prandtl number, $C\mu_a/K$
Re	Reynolds number, $D_e\bar{u}\rho/\mu_a$
T	temperature of the fluid
T_1, T_W	inlet and wall temperatures, respectively
u_z	local isothermal velocity along the axis of the duct
$\bar{u}$	average fluid velocity
x, y, z	Cartesian coordinates, z-axis parallel to the axis of the duct ($z = 0$ at duct inlet)
Z	length of the duct
α	thermal diffusivity, $K/\rho C$
β^*	correction factor for the temperature dependent viscosity
$\Gamma(x)$	gamma function
η	coordinate measured normal to the wall of the duct
μ_a, μ_W	viscosity of fluid at average and wall temperatures
ξ, θ, z	elliptic cylindrical coordinates
ρ	density of fluid
$\phi(z)$	heat flux

§ 1. Introduction

Heat transfer in noncircular conduits has become important in recent years since – due to the fact that they possess a high ratio of surface area to core volume –, such conduits are employed in nuclear reactors, compact heat exchangers, and other high heat load systems. The problem of heat transfer in laminar flow in elliptical ducts with uniform wall temperature has been investigated earlier by Dunwoody [1] using a numerical method. Recently, Schenk and Bong Swy Han [2] considered the same problem with prescribed external or wall Nusselt numbers. As the wall Nusselt number approaches infinity, isothermal conditions are attained on the duct wall.

Lévêque [3] examined the problem of heat transfer in flow over a flat plate assuming a linear velocity profile near the wall and Krishnamurty et al [4–8] obtained exact solutions of the energy equation for various noncircular conduits with uniform wall temperature using the Lévêque theory. In the present investigation the problem of laminar forced convection heat transfer in short elliptical ducts with (i) uniform wall temperature and (ii) prescribed heat flux has been studied. The function $F(\theta)$ (cf. eq. (5)) introduced in this work, takes care of the variation of the slope of the linear velocity profile along the perimeter. Further, a Sieder-Tate type of correction factor β^* has been introduced to account for the temperature dependent viscosity. The present results are applicable to short tubes, high fluid velocities, and low thermal conductivities. Since the transport mechanism of heat is essentially the same as that for mass, the results of the present work can be employed successfully for similar cases of mass transfer.

§ 2. Analysis

For fully developed laminar isothermal steady unidirectional flow of an incompressible Newtonian fluid through an elliptic duct of constant cross section, the velocity distribution is

$$u_z = 2\bar{u}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{1}$$

The equation for energy transfer without heat generation and with the assumption that the longitudinal conduction is negligible in comparison with convective transport, is

$$c^2(\cosh^2 \xi - \cos^2 \theta)u_z \frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \theta^2}\right), \tag{2}$$

where $x = c \cosh \xi \cos \theta$, $y = c \sinh \xi \sin \theta$ and $z = z$.

The elliptic wall is represented by $\xi = \xi_0$. In the region around a point on the duct wall at $\eta = \xi - \xi_0 \approx 0$ and $\theta = \theta$, the term $\partial^2 T/\partial \theta^2$ may be neglected in comparison with $\partial^2 T/\partial \xi^2$. Since the velocity near the wall is small, the term $c^2(\cosh^2 \xi - \cos^2 \theta)u_z$ can be approximated by the first nonvanishing term of its Taylor expansion at $\xi = \xi_0$. These approximations agree with the assumptions in [3–9]. Equation (2) now reduces to

$$\frac{\partial^2 T}{\partial \eta^2} - \frac{\eta}{m} \frac{\partial T}{\partial z} = 0, \tag{3}$$

where

$$m = \frac{\alpha}{4\beta^* \bar{u} ab F(\theta)} \tag{4}$$

and

$$F(\theta) = \left(\frac{a}{b}\sin^2\theta + \frac{b}{a}\cos^2\theta\right)^2. \tag{5}$$

The factor β^* is the velocity distortion factor and accounts for distortion of the isothermal velocity profile due to non-isothermal conditions. In our investigation β^* is taken as

$$\beta^{*\frac{1}{3}} = 1.15\left(\frac{\mu_\mathrm{a}}{\mu_\mathrm{w}}\right)^{0.14}, \tag{6}$$

as suggested by Marshall and Pigford [10] based on experimental observations.

Introducing the Laplace transform of $T(\eta, z)$ as

$$\mathscr{L}[T(\eta, z)] = g(\eta, p) = p\int_0^\infty \mathrm{e}^{-pz}\, T(\eta, z)\, \mathrm{d}z \tag{7}$$

(3) transforms to the Stokes equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\eta^2} - \frac{p\eta}{m}(g - T_1) = 0, \tag{8}$$

the general solution of which can be obtained as

$$g = T_1 + B_1\sqrt{\eta}\, J_{\frac{1}{3}}\left[\frac{2\mathrm{i}}{3}\left(\frac{p}{m}\eta^3\right)^{\frac{1}{2}}\right] + B_2\sqrt{\eta}\, J_{-\frac{1}{3}}\left[\frac{2\mathrm{i}}{3}\left(\frac{p}{m}\eta^3\right)^{\frac{1}{2}}\right], \tag{9}$$

where B_1 and B_2 are constants to be determined from the boundary conditions.

The equation

$$\frac{\mathrm{d}g}{\mathrm{d}\eta} = \mathscr{L}\left[\frac{\partial T}{\partial \eta}\right] \to 0 \text{ as } \eta \to \infty \tag{10}$$

can be used as a boundary condition for both prescribed wall temperature and prescribed wall heat flux. This condition yields

$$B_1 = B_2 \mathrm{i}^{\frac{4}{3}}. \tag{11}$$

2.1. *Case of constant wall temperature.* This situation arises when the second fluid on the other side of the wall changes phase and when the duct wall thickness and thermal conductivity are large.

For this case ($T_{\mathrm{w}} = \text{constant}$), the equation

$$g(0, p) = g_{\mathrm{w}} = \mathscr{L}[T_{\mathrm{w}}] = T_{\mathrm{w}} \tag{12}$$

is the second boundary condition, using it we obtain

$$B_2 = (g_{\mathrm{w}} - T_1)3^{-\frac{1}{3}}\Gamma(\tfrac{2}{3})\, \mathrm{i}^{\frac{1}{3}}\left(\frac{p}{m}\right)^{\frac{1}{6}}. \tag{13}$$

Local heat transfer coefficient and Nusselt number:
The local heat transfer coefficient, $h^{\circ}_{i\theta}$ (at a point ξ_0, θ on the wall), based on the inlet temperature drop $T_{\mathrm{w}} - T_1$, may be computed from the heat balance

$$-\left[\frac{K}{\sqrt{ab}\{F(\theta)\}^{\frac{1}{4}}}\right]\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0} = h^{\circ}_{i\theta}(T_{\mathrm{w}} - T_1). \tag{14}$$

In terms of the Laplace transform, (14) can be written as

$$-\left[\frac{K}{\sqrt{ab}F(\theta)^{\frac{1}{4}}}\right]\left(\frac{\mathrm{d}g}{\mathrm{d}\eta}\right)_{\eta=0} = \mathscr{L}[h^{\circ}_{i\theta}(T_{\mathrm{w}} - T_1)]. \tag{15}$$

From (9), (11), and (13) we can show

$$\left(\frac{\mathrm{d}g}{\mathrm{d}\eta}\right)_{\eta=0} = \frac{(g_{\mathrm{w}} - T_1)\Gamma(\frac{2}{3})(p/m)^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma(\frac{4}{3})}. \tag{16}$$

From (15) and (16) we obtain, after inversion,

$$h^{\circ}_{i\theta} = \frac{K}{\Gamma(\frac{4}{3})}\left(\frac{4}{9}\right)^{\frac{1}{3}}\left(\frac{\beta^* \bar{u}}{\alpha}\right)^{\frac{1}{3}} \frac{F(\theta)^{1/12}}{(ab)^{\frac{1}{6}}}\, z^{-\frac{1}{3}}. \tag{17}$$

For elliptical ducts the equivalent diameter D_{e} (defined as four times the ratio of cross section to wetted perimeter), is given by

$$D_{\mathrm{e}} = \frac{\pi a(1 - e^2)^{\frac{1}{2}}}{E(e)}. \tag{18}$$

Hence, the local Nusselt number, $\overset{\circ}{Nu} = h^{\circ}_{i\theta}D_{\mathrm{e}}/K$, can be obtained as

$$\overset{\circ}{Nu} = 0.983(\overset{\circ}{Gz})^{\frac{1}{3}}\left(\frac{\mu_{\mathrm{a}}}{\mu_{\mathrm{w}}}\right)^{0.14}\left(\frac{D_{\mathrm{e}}}{\sqrt{ab}}\right)^{\frac{1}{3}} F(\theta)^{1/12}. \tag{19}$$

Average Nusselt number:

The perimeter average Nusselt number is given by

$$\overline{\overset{\circ}{Nu}} = 1.44\psi(e)(\overset{\circ}{Gz})^{\frac{1}{3}}\left(\frac{\mu_a}{\mu_w}\right)^{0.14} \tag{20}$$

and the overall average Nusselt number is

$$\overline{Nu} = 2.16\psi(e)(\overline{Gz})^{\frac{1}{3}}\left(\frac{\mu_a}{\mu_w}\right)^{0.14}, \tag{21}$$

where

$$\psi(e) = E(e)^{-\frac{4}{3}} \int_0^{\pi/2} (1 - e^2 \cos^2 \theta)^{\frac{2}{3}} \, d\theta. \tag{22}$$

Table I gives the function $\psi(e)$.

TABLE I

e	0 (circle)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\psi(e)$	0.86	0.8614	0.8622	0.8736	0.8843	0.8997	0.9249	0.9548
e	0.8	0.9	0.99	0.992	0.994	0.996	0.998	1.000 (slit)
$\psi(e)$	1.0010	1.0740	1.226	1.316	1.284	1.267	1.254	0.9111

Behaviour of the function $\psi(e)$.

2.2. *Prescribed wall heat flux.* A prescribed wall heat flux $\phi(z)$ arises in electrical and nuclear heating systems, where the surface temperature adjusts itself to the heat flux. For the present case the second boundary condition is

$$-\left[\frac{K}{\sqrt{ab}F(\theta)^{\frac{1}{4}}}\right]\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0} = \phi(z). \tag{23}$$

The wall temperature follows from

$$T_w - T_1 = \frac{\sqrt{ab}}{K} 3^{\frac{2}{3}} \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})} F(\theta)^{\frac{1}{4}} \mathscr{L}^{-1}\left[\left(\frac{p}{m}\right)^{-\frac{1}{3}} \mathscr{L}[\phi(z)]\right]. \tag{24}$$

Constant heat flux:

When $\phi(z)$ is a constant ϕ_0, the wall temperature distribution can be obtained from (24) as

$$T_{\mathrm{w}} - T_1 = \frac{\phi_0\sqrt{ab}}{K} \frac{3^{\frac{2}{3}}}{\Gamma(\frac{2}{3})} F(\theta)^{\frac{1}{4}} m^{\frac{1}{3}} z^{\frac{1}{3}}. \tag{25}$$

The Nusselt number based on the local heat transfer coefficient, $h_{i\theta}^{\mathrm{o}} = \phi_0/(T_{\mathrm{w}} - T_1)$, is given by

$$\overset{\mathrm{o}}{Nu} = 1.19(\overset{\mathrm{o}}{Gz})^{\frac{1}{3}}\left(\frac{\mu_{\mathrm{a}}}{\mu_{\mathrm{w}}}\right)^{0.14}\left(\frac{D_{\mathrm{e}}}{\sqrt{ab}}\right)^{\frac{1}{3}} F(\theta)^{1/12}. \tag{26}$$

The average Nusselt numbers are calculated as

$$(Nu)_{\mathrm{avg}} = \int \overset{\mathrm{o}}{Nu}\, \mathrm{d}A_{\mathrm{h}} / \int \mathrm{d}A_{\mathrm{h}}. \tag{27}$$

The perimeter average Nusselt number is

$$\overline{\overset{\mathrm{o}}{Nu}} = 1.74\psi(e)(\overset{\mathrm{o}}{Gz})^{\frac{1}{3}}\left(\frac{\mu_{\mathrm{a}}}{\mu_{\mathrm{w}}}\right)^{0.14}. \tag{28}$$

The overall average Nusselt number is

$$\overline{Nu} = 2.61\psi(e)(\overline{Gz})^{\frac{1}{3}}\left(\frac{\mu_{\mathrm{a}}}{\mu_{\mathrm{w}}}\right)^{0.14}. \tag{29}$$

Other flux distributions:

Using (24) we can obtain wall temperature distributions for other flux distributions. Expressions for the wall temperature drop $T_{\mathrm{w}} - T_1$ involve hypergeometric functions in the case of cosine and chopped cosine flux distributions and then the expressions for Nusselt numbers will be complicated.

§ 3. Discussion

(a) It is noticed from (5) and (19) that the local Nusselt number $\overset{\mathrm{o}}{Nu}$ has a maximum at the ends of the minor axis and a minimum at the ends of the major axis. This effect increases with the eccentricity e. At the ends of the major axis the slope of the velocity profile will have its minimum value and this results in increased thermal resistance. The opposite happens at the ends of the minor axis. The general trend of the variation of $\overset{\mathrm{o}}{Nu}$ around the perimeter obtained in the present work checks well with the observations of Schenk and Bong Swy Han [2].

(b) The results of Table I indicate that $\psi(e)$ has a maximum when $e = 0.992$. This trend agrees with the observation of [2] that the established Nusselt number Nu_{∞} has a maximum for e near 0.968.

(c) The Nusselt numbers for the case of constant heat flux are found to be greater than the values for constant wall temperature at any given Graetz number.

(d) The present results reduce to expressions for circular tubes when $e \to 0$ (Graetz problem).

(e) When $e \to 1$ and $a \to \infty$ keeping b finite, we may reduce the present problem to the classical problem of Purday, discussed in [5]. In this case the separation between the plates equals $2b$ and $D_e = \pi b$.

TABLE II

Gz	$\overline{\overset{0}{Nu}}$	
	[2]	present work
20	3.73	3.143
30	3.81	3.598
40	4.03	3.960
50	4.22	4.266
60	4.32	4.533
70	4.44	4.772
80	4.58	4.990
90	4.72	5.189
100	4.90	5.375
120	5.01	5.712
140	5.13	6.013
160	5.26	6.287
180	5.43	6.539
200	5.80	6.772

Comparison of the results of Schenk et al [2] with the present results.

(f) Table II shows the comparison of the present results for $\beta^* = 1$ with those of [2] for $Nu_W = \infty$ for the Graetz number range of 20 to 200 and $e = 0.6$. For $Gz = 20$ our results are 15.5 percent lower and at $Gz = 30$, our results are only 5.5 percent lower. Beyond $Gz = 45$, our values of Nusselt numbers are higher. The deviation from $Gz = 120$ to 200 is around 12 percent. For Gz above 200, data of [2] are not available for comparison. It is, therefore, concluded that the present results are applicable for Graetz numbers above 25 within a reasonable percentage error that occurs in experimental investigations. The present equations (19) to (21) and (26) to (29) for the Nusselt numbers are more convenient and simple for engineering applications.

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REFERENCES

[1] DUNWOODY, N. T., J. Appl. Mech. **29** (1962) 165.
[2] SCHENK, J. and BONG SWY HAN, Appl. Sci. Res. **17** (1967) 96.
[3] LÉVÊQUE, J., Ann. d. Mines **13** (1928) 201, 305, 381.
[4] VENKATA RAO, C., C. SYAMALA RAO, and V. V. G. KRISHNAMURTY, Indian J. Technol. **5** (1967) 164.
[5] KRISHNAMURTY, V. V. G. and C. VENKATA RAO, Indian J. Technol. **5** (1967) 166.
[6] KRISHNAMURTY, V. V. G., Indian J. Technol. **5** (1967) 167.
[7] KRISHNAMURTY, V. V. G. and N. V. SAMBASIVA RAO, Indian J. Technol. **5** (1967) 331.
[8] KRISHNAMURTY, V. V. G., C. SYAMALA RAO, and N. V. SAMBASIVA RAO, Indian J. Technol. **5** (1967) 364.
[9] BIRD, R. B., W. E. STEWART, and E. N. LIGHTFOOT, Transport Phenomena, 104, 141, Wiley, New York 1958.
[10] MARSHALL JR., W. R. and R. L. PIGFORD, Applications of differential equations to chemical engineering problems, 142, Univ. of Delaware Press, Newark (Del.) 1947.