KLEINE MITTEILUNGEN

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Self-Modelling Flows of Non-Newtonian Viscous Liquids

0. An interesting class of exact solutions of the equations of motion of viscous liquids pertains to the case of flows with axial symmetry so that the derivatives of the functions with respect to one of the coordinates vanish. If r, θ, φ are the spherical polar coordinates of a point with θ measured from the axis of symmetry, one way of seeking exact solutions is to assume that each of the dependent functions such as the velocity components and pressure can be represented as a product of a power of the radius r and a function of the angle θ. Following L. A. Vulis and V. P. Kaskarov [1], we may refer to such flows as 'self-modelling flows'.

In the present paper we seek the self-modelling solutions of the equations of motion of non-Newtonian viscous liquids of the Reiner-Rivlin type for which the constitutive relation connecting the stress and rate of deformation tensors is

(1) $$t_{ij} = -p\,\delta_{ij} + 2\mu\, d_{ij} + 2\mu_c\, d_{i\alpha}\, d_{\alpha j}\,.$$

As in [1] the problem reduces to the solution of ordinary differential equations for the function depending on θ. We find that the only possible solutions of the form $\psi = r^n f(\theta)$ arise when $n = 2$ or 4. Both these are also solutions of the Navier-Stokes equation governing the motion of linear (Newtonian) viscous liquids. In the latter case there is yet another self-modelling solution corresponding to $n = 1$ (cf. [2]), but there is no analogous solution when non-linear viscous terms are included in the equations of motion.

1. If $(u, v, 0)$ are the velocity components in the directions of the spherical polar coordinates r, θ, φ and are dependent only on r and θ, the equations of steady motion of non-Newtonian viscous liquids of the Reiner-Rivlin type are given by

(2) $$\begin{aligned} u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} &= -\frac{1}{\varrho}\frac{\partial p}{\partial r} + \nu\left(\Delta u - \frac{2}{r^2}u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{2\cot\theta}{r^2}v\right)\\ &+ \nu_c\left\{\frac{\partial}{\partial r}\left[2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2\right]\right.\\ &+ \frac{1}{r}\left[4\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2\right]\\ &- \frac{1}{r}\frac{\partial}{\partial \theta}\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\left(\frac{u}{r} + \frac{\cot\theta}{r}v\right)\right]\\ &- \frac{\cot\theta}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\left(\frac{u}{r} + \frac{\cot\theta}{r}v\right)\\ &\left.- \frac{2}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 - \frac{2}{r}\left(\frac{u}{r} + \frac{\cot\theta}{r}v\right)^2\right\},\end{aligned}$$

(3) $$\begin{aligned} u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} &= -\frac{1}{\varrho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{\operatorname{cosec}^2\theta}{r^2}v\right)\\ &+ \nu_c\left\{-\frac{\partial}{\partial r}\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\left(\frac{u}{r} + \frac{\cot\theta}{r}v\right)\right]\right.\\ &- \frac{3}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\left(\frac{u}{r} + \frac{\cot\theta}{r}v\right)\\ &+ \frac{1}{2r}\frac{\partial}{\partial \theta}\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2 + 4\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2\right]\\ &+ \frac{\cot\theta}{2r}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2\\ &\left.- \frac{2\cot\theta}{r}\frac{\partial u}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{\cot\theta}{r}v\right)\right\},\end{aligned}$$

where

$$\nu = \mu/\varrho\,,\qquad \nu_c = \mu_c/\varrho \qquad \text{and}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}$$

is the Laplacian operator. From the continuity equation viz.

(4) $$\frac{\partial u}{\partial r} + \frac{2}{r}u + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\cot\theta}{r}v = 0$$

we can write the velocity components in terms of the Stokes's stream function $\psi(r, \theta)$ in the form

(5) $$u = -\frac{\operatorname{cosec}\theta}{r^2}\frac{\partial \psi}{\partial \theta}\,,\qquad v = \frac{\operatorname{cosec}\theta}{r}\frac{\partial \psi}{\partial r}\,.$$

On eliminating the pressure p from the equations of motion (2), (3) we obtain the following equation for the stream function:

(6) $$\begin{aligned}\sin\theta\,\frac{\partial\left(\psi, \frac{1}{r^2}\operatorname{cosec}^2\theta\, D^2\psi\right)}{\partial(r, \theta)} &= \nu\, D^2 D^2 \psi\\ &+ \nu_c\left\{-\frac{\operatorname{cosec}^2\theta}{r^2}\left(\cos\theta\frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \psi}{\partial \theta}\right) D^2 D^2\psi\right.\\ &+ \frac{2\operatorname{cosec}\theta}{r^2}\,\frac{\partial\left(\psi, \frac{1}{r^2}\operatorname{cosec}^2\theta\, D^2\psi\right)}{\partial(r, \theta)}\\ &- \frac{2}{r}\,\frac{\partial\left(\sin\theta\frac{\partial \psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \psi}{\partial \theta}, \frac{1}{r^2}\operatorname{cosec}^2\theta\, D^2\psi\right)}{\partial(r, \theta)}\\ &- \frac{\operatorname{cosec}^2\theta}{r^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(D^2\psi)^2\\ &+ \frac{2}{r}\operatorname{cosec}\theta\left(\cos\theta\frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \psi}{\partial \theta}\right)\times\\ &\left.\times\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\operatorname{cosec}^2\theta}{r^2}D^2\psi\right)\right\},\end{aligned}$$

where

(7) $$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\,.$$

2. To seek the self-modelling solutions of the equations (2), (3) we put

(8) $$\psi = \nu\, r^n f(\theta)$$

in (6) and after some calculation we have

(9) $$r^{2n-5} f_1(\theta) = r^{n-4} f_2(\theta) + \nu_c\, r^{2n-7} f_3(\theta)\,,$$

where

(10a) $$\begin{aligned} f_1(\theta) = \sin\theta\Big\{&n\, f(\theta)\frac{d}{d\theta}[\operatorname{cosec}^2\theta\,\times\\ &\times(f''(\theta) - \cot\theta\, f'(\theta) + n(n-1)\, f(\theta))]\\ &- (n-4)\operatorname{cosec}^2\theta\, f'(\theta)\,[f''(\theta) - \cot\theta\, f'(\theta)\\ &+ n(n-1)\, f(\theta)]\Big\}\,,\end{aligned}$$

(10b) $$\begin{aligned} f_2(\theta) = &\left(\frac{d^2}{d\theta^2} - \cot\theta\frac{d}{d\theta} + (n-2)(n-3)\right)\times\\ &\times(f''(\theta) - \cot\theta\, f'(\theta) + n(n-1)\, f(\theta))\,,\end{aligned}$$

and

$$(10c)\quad f_3(\theta) = -\operatorname{cosec}^2\theta\,(n\cos\theta\, f(\theta) - \sin\theta\, f'(\theta)) + 2\operatorname{cosec}^2\theta\, f_1(\theta) - 2\Big[(n-1)\,(n\sin\theta\, f(\theta) + \cos\theta\, f'(\theta))\frac{d}{d\theta}[\operatorname{cosec}^2\theta\,(f''(\theta) - \cot\theta\, f'(\theta) + n\,(n-1) f(\theta)] - (n-4)\operatorname{cosec}^2\theta\,(f''(\theta) - \cot\theta\, f'(\theta) + n\,(n-1)\, f(\theta))\frac{d}{d\theta}(n\sin\theta\, f(\theta) + \cos\theta\, f'(\theta))\Big] + - 2\,(n-2)\operatorname{cosec}\theta\cot\theta\,[f''(\theta) - \cot\theta\, f'(\theta) + n\,(n-1)\, f(\theta)]^2 + \operatorname{cosec}\theta\frac{d}{d\theta}[f''(\theta) - \cot\theta\, f'(\theta) + n\,(n-1)\, f(\theta)]^2 + 2\operatorname{cosec}\theta\,(n\cos\theta\, f(\theta) - \sin\theta\, f'(\theta))\Big[(n-4)\operatorname{cosec}\theta\,(f''(\theta) - \cot\theta\, f(\theta) + n\,(n-1)\, f(\theta)) + \cos\theta\frac{d}{d\theta}(\operatorname{cosec}^2\theta\,(f''(\theta) - \cot\theta\, f'(\theta) + n\,(n-1)\, f(\theta))\Big].$$

In order that the function in (8) may be compatible with the equations of motion (2), (3) the equation(9) must be identically satisfied and we consider the following cases.

(a) n arbitrary, and $f(\theta)$ is a solution common to

$$f_1(\theta) = 0\,,\qquad f_2(\theta) = 0\,,\qquad f_3(\theta) = 0;$$

(b) $n = 1$, and $f(\theta)$ is a solution common to

$$f_1(\theta) - f_2(\theta) = 0\,,\qquad f_3(\theta) = 0;$$

(c) $n = 3$, and $f(\theta)$ is a solution common to

$$f_1(\theta) = 0\,,\qquad f_2(\theta) + \nu_c\, f_3(\theta) = 0\,.$$

The equation $f_2(\theta) = 0$ gives

$$\left(\frac{d^2}{d\theta^2} - \cot\theta\frac{d}{d\theta} + (n-2)\,(n-3)\right)\left(\frac{d^2}{d\theta^2} - \cot\theta\frac{d}{d\theta} + n\,(n-1)\right) f(\theta) = 0$$

and has the solution

$$(11)\quad f(\theta) = \sin^2\theta\,\{c_1\, P'_{n-1}(\cos\theta) + c_2\, Q'_{n-1}(\cos\theta) + c_3\, P'_{n-3}(\cos\theta) + c_4\, Q'_{n-3}(\cos\theta)\}$$

with some modification when $n = 0, 1, 2, 3$. We may easily see that $f(\theta)$ in (11) satisfies the equations $f_1(\theta) = 0$, $f_3(\theta) = 0$ also if and only if $n = 4$ and then we have the solution

$$(12)\quad \psi = \nu\, r^4\sin^2\theta\,[c_1\, P'_3(\cos\theta) + c_2\, Q'_3(\cos\theta) + \alpha]\,.$$

From the above analysis it is clear that this is also a solution of the NAVIER-STOKES equations of linear viscous motion (cf. [3]). We thus see that the stream function (12) is compatible with the equations of motion whether the fluid is inviscid or linearly viscous or is of the REINER-RIVLIN type. It is also noteworthy that this solution is self-additive in all these three cases; i. e., the motion determined by this solution is superposable on itself.

3. This raises the interesting question of determing the totality of steady flows of the axially symmetric type, the streamline patterns of which are common to the three distinct types of flows viz., inviscid flows, viscous flows and non-Newtonian viscous flows. To determine such a class of flows, we have to consider the following equations (cf. (6))

$$(13a, b)\quad D^2\psi = r^2\sin^2\theta\, F(\psi)\,,\quad D^2[r^2\sin^2\theta\, F(\psi)] = 0\,,$$

and

$$(13c)\quad -\frac{2}{r}\,\frac{\partial\left(\sin\theta\frac{\partial\psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\psi}{\partial\theta},\ F(\psi)\right)}{\partial(r,\theta)} - \frac{\operatorname{cosec}^2\theta}{r^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(r^2\sin^2\theta\, F(\psi))^2 + \frac{2}{r}\operatorname{cosec}\theta\left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)\times\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)F(\psi) = 0\,.$$

All these equations are automatically satisfied when we take $F(\psi) = a$ constant, and the solution (12) corresponds to this choice. It is not clear whether there is any other choice of $F(\psi)$ compatible with the equations (13a, b, c).

4. When $n = 1$, $\psi = \nu\, r\, f(\theta)$ and we want a solution $f(\theta)$ common to the equations

$$f_1(\theta) - f_2(\theta) = 0\,,\qquad f_3(\theta) = 0\,.$$

The former equation has the solution (cf. [2])

$$(14)\quad f(\theta) = \alpha\,(1 + x) + \beta\,(1 - x) + \left\{2\,(1 - x^2)\,\frac{(1+x)^\beta}{(1-x)^\alpha}\Bigg/\left[a - \int_1^x \frac{(1+x)^\beta}{(1-x)^\alpha}\,dx\right]\right\}$$

where $x = \cos\theta$ and α, β, a are constants. This is not compatible with the equation $f_3(\theta) = 0$ and hence the conclusion that the equations of motion of non-linear viscous liquids (of the REINER-RIVLIN type) do not admit a solution of the form $\psi = r\, f(\theta)$; i. e., there is no solution corresponding to the laminar axially symmetric jet in REINER-RIVLIN fluids.

5. When $n = 3$ the equation $f_1(\theta) = 0$ can be integrated once, leading to the relation

$$(15)\quad (f''(\theta) - \cot\theta\, f'(\theta) + 6\, f(\theta))^3\, f(\theta) = c_1\,(\sin\theta)^6\,,$$

and further integration is not simple. We can however check that $f(\theta) = (\sin\theta)^3$ is one solution of (15) and this makes $f_3(\theta) = 0$ also, but does not fit into the equation $f_2(\theta) = 0$. We conclude that there is no elementary solution common to the equations $f_1(\theta) = 0$ and $f_2(\theta) + \nu_c\, f_3(\theta) = 0$.

6. From eq. (6) it is obvious that the stream function determined by the equation $D^2\psi = 0$ fits into the equations (2) and (3) also. We may readily determine the function $f(\theta)$ in this case and this corresponds to $n = 2$ and gives irrotational flow.

References

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