KLEINE MITTEILUNGEN

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On the Motion of an Infinite Cylinder in Rotating 'Non-Newtonian' Viscous Fluid

The simple shear flow of pseudo-plastic and dilatant fluids is expressed to a considerable degree of accuracy by the Waele-Ostwald's formula

$$(1) \qquad \frac{du}{dy} = \tau^n/\mu_{psu} ,$$

where τ is the shearing stress, du/dy is the velocity gradient, μ_{psu} corresponds to the viscosity coefficient and has the dimensions (mass)^n / (length)^n (time)$^{2n-1}$. The parameter n occurring in the formula (1) is a constant, denoting the rheological constant for the fluid and is usually a non-integral quantity. In the special case of $n = 1$ the relation (1) is linear and this situation corresponds to the case of Newtonian fluids. The flow is non-Newtonian for other values of n and is called pseudo-plastic if $n > 1$ and is called dilatant if $n < 1$. The constitutive relation connecting the stress and rate of strain components is non-linear in these cases and a mathematical study of the hydrodynamical properties of such fluids has been initiated by Y. Tomita [1] by adopting the law

$$(2) \qquad t_{ij} = -p\,\delta_{ij} + \eta\, e_{ij}$$

between the stress tensor t_{ij} and the rate of deformation tensor $e_{ij} = (u_{i,j} + u_{j,i})/2$. Here p is a scalar quantity which can be identified with the mean pressure and η is an arbitrary scalar function of the invariants of the rate of strain matrix e_{ij}. For incompressible fluids the first invariant $I_1 = \operatorname{div} \vec{q} = 0$ and in plane motions the third invariant $I_3 = \det e_{ij}$ also vanishes. In [1] Y. Tomita has shown that the equations of motion for non-Newtonian fluids governed by the constitutive relation (2) can, under some simplifying assumptions, be obtained by means of a variation principle and this procedure gives an approximate method of solution of the problem of flow of such fluids past bodies of different shapes.

In the present note we examine the two-dimensional motion of an infinite cylinder in such pseudo-plastic or dilatant fluids governed by the constitutive relation

$$(3) \qquad t_{ij} = -p\,\delta_{ij} + 2\,\mu_{psu}^{1/n}\,\Theta\, e_{ij}$$

with

$$(4) \qquad \Theta = \{2\,(e_{xx}^2 + e_{yy}^2) + 4\,e_{xy}^2\}^{\frac{1-n}{2n}}$$

$$(4\text{a}) \qquad = \left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\}^{\frac{1-n}{2n}}$$

and show that if a uniform rotation is imposed on the whole system, the motion of the cylinder is not altered. This property was first established for inviscid fluids by Sir Geoffrey Taylor [2] and was later shown to remain valid also in the case of viscous fluids by W. R. Dean [3].

Using the constitutive relation (3) we obtain the equations of motion in the form

$$(5\text{a}) \qquad \varrho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + 2\,\mu_{psu}^{1/n} \times \left\{\frac{\partial}{\partial x}[\Theta\, e_{xx}] + \frac{\partial}{\partial y}[\Theta\, e_{xy}]\right\},$$

$$(5\text{b}) \qquad \varrho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + 2\,\mu_{psu}^{1/n} \times \left\{\frac{\partial}{\partial x}[\Theta\, e_{xy}] + \frac{\partial}{\partial y}[\Theta\, e_{yy}]\right\}.$$

We compare the two-dimensional motion of an infinite rigid cylinder in rotating viscous fluid with a second motion derived from the former by the addition on the whole system of a rotation with constant angular velocity ω. If ψ_1 is the stream function of the first motion with the velocity distribution $u_1 = -\partial\psi_1/\partial y$, $v_1 = \partial\psi_1/\partial x$ and mean pressure p, we have from (5a) and (5b)

$$(6\text{a}) \qquad \varrho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{dx} + v_1\frac{\partial u_1}{\partial y}\right) = -\frac{\partial p_1}{\partial x} + \mu_{psu}^{1/n} \times \left\{\Theta_1\,\Delta u_1 + 2\frac{\partial u_1}{\partial x}\frac{\partial \Theta_1}{\partial x} + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)\frac{\partial \Theta_1}{\partial y}\right\},$$

$$(6\text{b}) \qquad \varrho\left(\frac{\partial v_1}{\partial t} + u_1\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y}\right) = -\frac{\partial p_1}{\partial y} + \mu_{psu}^{1/n} \times \left\{\Theta_1\,\Delta v_1 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)\frac{\partial \Theta_1}{\partial x} + 2\frac{\partial v_1}{\partial y}\frac{\partial \Theta_1}{\partial y}\right\},$$

where

$$(7) \qquad \Theta_1 = \left\{2\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial v_1}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)^2\right\}^{\frac{1-n}{2n}}.$$

At a point P of the boundary of the cylinder the velocity of the fluid is equal to that of the cylinder. In particular, we get

$$(8) \qquad (\vec{q}_1)_n = (u_1, v_1)_n = -\frac{\partial \psi_1}{\partial s}$$

giving the velocity component along the outward normal n to the contour of the cylinder, s denoting the direction of the tangent to the boundary. We may obtain the second motion from the first by the addition on the whole system a rotation about the origin O with constant angular velocity ω. The liquid particle at the point (x, y) at time t in the first motion is at the same point in the second motion when referred to rotating axes. The velocity and acceleration are $(\dot{x}, \dot{y})$ and $(\ddot{x}, \ddot{y})$ in the former case and $(\dot{x} - \omega\, y,\ \dot{y} + \omega\, x)$, $(\ddot{x} - 2\,\omega\,\dot{y} - \omega^2 x,\ \ddot{y} + 2\,\omega\,\dot{x} - \omega^2 y)$ in the latter case in terms of Lagrangian components. If $u_2 = -\partial\psi_2/\partial y$, $v_2 = \partial\psi_2/\partial x$ are the (Eulerian) components of velocity in the second motion and (f_1, g_1), (f_2, g_2) are the acceleration components in the first and second motions, we have

$$(9\text{a}) \qquad -\frac{\partial \psi_2}{\partial y} = u_2 = u_1 - \omega\, y = -\frac{\partial \psi_1}{\partial y} - \omega\, y,$$

$$(9\text{b}) \qquad -\frac{\partial \psi_2}{\partial x} = v_1 = v_1 + \omega\, x = \frac{\partial \psi_1}{\partial x} + \omega\, x,$$

$$(10\text{a}) \qquad f_2 = f_1 - 2\,\omega\,\dot{y} - \omega^2 x = f_1 - 2\,\omega\frac{\partial \psi_1}{\partial x} - \omega^2 x,$$

$$(10\text{b}) \qquad g_2 = g_1 + 2\,\omega\,\dot{x} - \omega^2 y = g_1 - 2\,\omega\frac{\partial \psi_1}{\partial y} - \omega^2 y.$$

The equations of the second motion are obtained from (6a), (6b) on changing u_1, v_1, p_1, Θ_1 to u_2, v_2, p_2, Θ_2 where p_2 is the mean pressure of the second motion and Θ_2 has the value

$$(11) \qquad \left\{2\left(\frac{\partial u_2}{\partial x}\right)^2 + 2\left(\frac{\partial v_2}{\partial y}\right)^2 + \left(\frac{\partial u_2}{\partial y} + \frac{dv_2}{\partial x}\right)^2\right\}^{\frac{1-n}{2n}}.$$

From (9a), (9b) we see that $\Theta_1 = \Theta_2$ and obtain by subtraction

$$(12\text{a}) \qquad \frac{\partial}{\partial x}(p_2 - p_1) = \varrho\frac{\partial}{\partial x}\left[2\,\omega\,\psi_1 + \frac{1}{2}\,\omega^2(x^2 + y^2)\right],$$

$$(12\text{b}) \qquad \frac{\partial}{\partial y}(p_2 - p_1) = \varrho\frac{\partial}{\partial y}\left[2\,\omega\,\psi_1 + \frac{1}{2}\,\omega^2(x^2 + y^2)\right],$$

as in the case of inviscid fluids though the present relations are between the mean pressures. If $(t_{xx})_1$, $(t_{xy})_1$, $(t_{yy})_1$ are the stress components in the first motion we have

$$(t_{xx})_1 = - p_1 + 2\,\mu_{psu}^{1/n}\,\Theta_1 \frac{\partial u_1}{\partial x}\,,$$

$$(t_{xy})_1 = \mu_{psu}^{1/n}\,\Theta_1 \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right),$$

$$(t_{yy})_1 = - p_1 + 2\,\mu_{psu}^{1/n}\,\Theta_1 \frac{\partial v_1}{\partial y}\,.$$

If θ denotes the angle between the outward normal n to the cylinder and the x-axis, the components of the force exerted on an element ds of the boundary L of the cylinder in the first motion are

$$\{(t_{xx})_1 \cos\theta + (t_{xy})_1 \sin\theta\,,\quad (t_{xy})_1 \cos\theta + (t_{yy})_1 \sin\theta\}\,ds$$

per unit length of the cylinder. In view of the relations

$$(t_{xx})_2 = (t_{xx})_1 - (p_2 - p_1)\,,$$

$$(t_{yy})_2 = (t_{yy})_1 - (p_2 - p_1)\,, \quad (t_{xy})_2 = (t_{xy})_1$$

we find that the differences between the two stress systems arise only from the difference in the mean pressures. If (F_1, G_1) is the force exerted on unit length of the cylinder and H_1 is the moment of the liquid pressures about the mass-centre of the cross section of the cylinder in the first motion and F_2, G_2, H_2 denote the similar quantities in the second motion, we have

$$F_2 - F_1 = - \int_L (p_2 - p_1)\,dy$$

$$= - 2\,\varrho\,\omega \int_L \psi_1\,dy - \frac{1}{2}\,\varrho\,\omega^2 \int_L (x^2 + y^2)\,dy\,,$$

$$G_2 - G_1 = \int_L (p_2 - p_1)\,dx$$

$$= 2\,\varrho\,\omega \int_L \psi_1\,dx + \frac{1}{2}\,\varrho\,\omega^2 \int_L (x^2 + y^2)\,dx\,,$$

$$H_2 - H_1 = \int_L (p_2 - p_1)\,[(y - y_0)\cos\theta - (x - x_0)\sin\theta]\,ds,$$

where (x_0, y_0) is the mass-centre of the cross section. The evaluation of the line integrals is exactly similar to the case of inviscid fluids (cf. [2], [3]) and we obtain

$$(13) \qquad F_2 - F_1 = - \varrho\,A\,(\omega^2 x_0 + 2\,\omega\,\dot{y}_0)\,,$$

$$(14) \qquad G_2 - G_1 = - \varrho\,A\,(\omega^2 y_0 - 2\,\omega\,\dot{x}_0)\,,$$

$$(15) \qquad H_2 - H_1 = 0\,,$$

where A denotes the area of cross section of the cylinder. We thus see that the forces due to the stresses on the cylinder in the second motion are the resultant of the forces that act in the first motion and a force

$$(- \varrho\,A\,\omega^2\,x_0 - 2\,\varrho\,A\,\omega\,\dot{y}_0\,,\ - \varrho\,A\,\omega^2\,y_0 + 2\,\varrho\,A\,\omega\,\dot{x}_0)$$

thus confirming that TAYLOR's result for inviscid fluids holds also in the case of pseudo-plastic and dilatant fluids.

References

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Hydromagnetic Couette's Flow with Suction and Injection

1. Basic Equations and their Solution

The problem of the flow between two parallel plates has been investigated by the various authors. In hydrodynamics it was first successfully attempted by COUETTE. Later HARTMANN and LAZARUS [1] in 1937 investigated it under a transverse magnetic field. REGIRER [2] has attempted the same problem for variable viscosity. Recently A. S. GUPTA [3] has studied the POISEUILLE flow including suction and injection and under a transverse magnetic field.

In this paper the problem has been investigated for the case when one plate is moving parallel to itself with a constant velocity while the other is at rest. Here μ, ν, ϱ and σ denote the magnetic permeability, kinematic viscosity, density and electrical conductivity of the fluid respectively and $\vec{V}(u_x, v, 0)$, $\vec{H}(H_x, H_0, 0)$, $\vec{E}(0, 0, E_z)$, $\vec{J}(0, 0, J_z)$ stand for fluid velocity, magnetic field and current field vectors, p is the pressure and T is temperature at a point. Let x-axis be parallel to the plates and y-axis normal to them. It is assumed that an external magnetic field H_0 is acting normal to the plates i. e. along y-axis and the fluid is being injected at a constant velocity at the lower plate and is being sucked at the same rate at the upper plate. It is also assumed that the motion is steady therefore all the dependent variables are independent of t and are function of y only except p. We take the pressure gradient along x-axis i. e. $\partial p/\partial x$ to be a constant. Subjected to these assumptions basic equations are reduced to

$$(1) \qquad v\,\frac{\partial u_x}{\partial y} = - \frac{1}{\varrho}\,\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u_x}{\partial y^2} + \frac{\mu\,H_0}{4\,\pi\,\varrho}\,\frac{\partial H_x}{\partial y}\,,$$

$$(2) \qquad 0 = - \frac{1}{\varrho}\,\frac{\partial p}{\partial y} - \frac{\mu\,H_x}{4\,\pi\,\varrho}\,\frac{\partial H_x}{\partial y}\,,$$

$$(3) \qquad 0 = \frac{1}{4\,\pi\,\sigma\,\mu}\,\frac{\partial^2 H_x}{\partial y^2} + H_0\,\frac{\partial u_x}{\partial y} - v\,\frac{\partial H_x}{\partial y}\,.$$

Integrating equation (2) we find that

$$(4) \qquad p + \frac{\mu\,H_x^2}{8\,\pi} = \text{const}\,.$$

This shows that the sum of hydrostatic pressure and magnetic pressure is constant through out the medium.

Now we transform the equations to non-dimensional form with the help of relations:

$$H_x = H_0\,H\,, \quad u_x = u_0\,u\,, \quad v = m\,u_0\,, \quad y = \eta\,L\,,$$

$$R_m = 4\,\pi\,\sigma\,\mu\,u_0\,L\,,$$

$$(5) \qquad R = \frac{u_0\,L}{\nu}\,, \qquad M = \mu\,H_0\,L\,\sqrt{\frac{\sigma}{\varrho\,\nu}} \qquad \text{and}$$

$$P = - \frac{L}{\varrho\,u_0^2}\,\frac{\partial p}{\partial x}\,.$$

We get

$$(6) \qquad m\,\frac{du}{d\eta} = P + \frac{1}{R}\,\frac{d^2 u}{d\eta^2} + \frac{M^2}{R\,R_m}\,\frac{dH}{d\eta}\,,$$

$$(7) \qquad \frac{du}{d\eta} = m\,\frac{dH}{d\eta} - \frac{1}{R_m}\,\frac{d^2 H}{d\eta^2}\,.$$

Eliminating u between (6) and (7) we obtain,

$$(8) \qquad \frac{d^3 H}{d\eta^3} - m\,(R + R_m)\,\frac{d^2 H}{d\eta^2} - (M^2 - m^2\,R\,R_m)\,\frac{dH}{d\eta} = P\,R\,R_m\,.$$

The solution of this differential equation is

$$(9) \qquad H = c_1\,e^{\alpha\,\eta} + c_2\,e^{\beta\,\eta} - \frac{P\,R\,R_m}{M^2 - m^2\,R\,R_m}\,\eta + c_3\,,$$